**Lösningar till tentamensskrivning i Kombinatorik pk 2007-12-17**

1. *Svaren* på deluppgifterna är:
   a) och bb): 23; c): 10 d): 7.
   *Fullständiga lösningar* kan för varje deluppgift ges antingen medelst falluppräkning, eller medelst genererande formella potensserier. Vidare är det tillåtet att använda att svaren i a) och b) måste vara lika på grund av argumenten om transposition av ferrergraferna i slutet av avsnitt 9.3 i Grimaldi; se särskilt figur 9.2 på sidan 435.
   Exempelvis är nedanstående fullt tillräckligt:

   b) The answer is the coefficient for $x^{10}$ in $\prod_{i=1}^{4}(1-x^i)^{-1}$ (one factor per permitted part size), which we may calculate rapidly $\pmod{x^{10}}$: $(1-x^2)^{-1}\,(1-x^3)^{-1}\,(1-x^4)^{-1} \equiv (1+x^2+x^4+x^6+x^8+x^{10})\,(1+x^3+x^6+x^9)\,(1+x^4+x^8) \equiv 1+x^2+x^3+2x^4+x^5+3x^6+2x^7+4x^8+3x^9+5x^{10}$; multiplying with $(1-x)^{-1} \equiv 1+x+x^2+x^3+\ldots+x^{10}$ yields the $x^{10}$ coefficient $1+1+1+2+1+3+2+4+3+5 = \underline{23}$.

   a) $\underline{23}$, by b), and since according to Grimaldi there are as many partitions of $n$ into at most 4 parts as into parts of sizes at most 4.

   c) $10 = 9+1 = 7+3 = 7+1+1+1 = 5+5 = 5+3+1+1 = 5+1+1+1+1+1 = 3+3+3+1 = 3+3+1+1+1+1 = 3+1+1+1+1+1+1+1 = 1+1+1+1+1+1+1+1+1+1$, i.e., there are $\underline{10}$ partitions of 10 into odd sized parts.

   d) $10 = 8+2 = 6+4 = 6+2+2 = 4+4+2 = 4+2+2+2 = 2+2+2+2+2$, i.e., there are $\underline{7}$ partitions of 10 into even sized parts; not forgetting the trivial partition into a single part.

2. Också här är olika kombinationer av räkning med i detta fall exponentiella genererande formella potensserier och/eller fallstudier tänkbara; exempelvis följande:

   a) and c): There are the singleton letters **A**, **S**, and **T**, and the doubletons **K**, **L**, and **O**, yielding the generating exponential formal power series $(1+x)^3\,(1+x+\frac{1}{2}x^2)^3 \equiv (1+3x+3x^2+x^3)\,(1+3x+\frac{9}{2}x^2+4x^3+\frac{9}{4}x^4) \equiv 1+6x+\frac{33}{2}x^2+\frac{55}{2}x^3+\frac{123}{4}x^4 \pmod{x^5}$, whence the number of words of lengh 3 is the $\frac{1}{6}x^3$ coefficient, i.e., $6\cdot\frac{55}{2} = \underline{165}$, and similarly the number of words of length 4 is $24\cdot\frac{123}{4} = \underline{738}$.

   b) Two equal adjacent letters may be achieved in $2\cdot3\cdot5 = 30$ ways, since there are two possible positions for the adjacent letters (either xx* or *xx), three choices for the doubleton letter, and 5 choices of the singleton. Subtracting this from the total number of 3 letter words, we get $165-30 = \underline{135}$ words without adjacent letters.

   d) There are $6\cdot5\cdot4\cdot3 = 360$ words of length 4 without repetition of letters, by the principle of multiplication; whence subtraction yields the answer $738-360 = \underline{378}$.

3. Det finns ett antal olika sätt att lösa denna uppgift på. Här följer ett av dessa:
   First, consider (but do not calculate) the generating formal power series $A(x) := \sum_{n=0}^{\infty} a_n x^n$. The recursive formula may be reformulated as

$$a_n - 5a_{n-1} + 6a_{n-2} = -2^{n-1}\,; \qquad (1)$$

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whence $A(x) = \frac{p(x)}{q(x)}$ for some polynomials $p$ and $q$; and the denominator $q(x) = (1 - 5x + 6x^2) \cdot (1 - 2x)$, where the first and the second factors correspond to the right hand side and the left hand side of (1), respectively. Solving an equation of degree 2 an collecting factors yields $q(x) = (1-2x)(1-3x)(1-2x) = (1-2x)^2(1-3x)$. Thus, if we did calculate $p(x)$, too, afterwards we could have rewritten $A(x)$ as

$$A(X) = \frac{B}{1-2x} + \frac{c}{(1-2x)^2} + \frac{d}{1-3x}\,, \tag{2}$$

for some constants $B$, $c$, and $d$. Hence, $a_n = (b+cn)\cdot 2^n + d3^n$, where $b = B+c$. Thus, we may skip the intermediate calculations, and instead directly employ that we know that $b + d = a_0 = 2$, $2b + 2c + 3d = a_1 = 7$, and $4b + 8c + 9d = a_2 = 5a_1 - 6a_0 - 2^1 = 35 - 12 - 2 = 21$; yielding $b = c = d = 1$; i.e., that $a_n = \underline{(n+1)2^n + 3^n}$, $n \geq 0$

4. *One out of a number of good solutions:* Recall that $\binom{n}{k} = \binom{n}{n-k}$, whence the l.h.s. may be rewritten as
$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}.$$
Consider a fixed $2n$-set $S$ and a fixed partition of $S$ into two $n$-sets $S_1$ and $S_2$, and count the number of $n$-subsets of $S$ in two ways. The term $\binom{n}{k}\binom{n}{n-k}$ counts the number of these $n$-subsets with exactly $k$ elements in $S_1$ (and hence the remaining $n-k$ elements in $S_2$); whence the l.h.s. counts the total number of $n$-subsets of $S$. So does the r.h.s., whence they must be equal.

5. Only your imagination – and potentionally a vagueness in your understanding of the concepts – limit your answers. There are infinitely many correct ones; and they will not be enumerated here.

6. a) $n!\binom{8}{n}^2 = \dfrac{8!^2}{(8-n)!^2\, n!}$

   b) $|E_n| = n\,(8-n) \cdot |V_n| = \dfrac{8!^2}{(8-n)!\,(7-n)!\,(n-1)!}$

   c) Each vertex is connected with e.g. the one with castles in the first $n$ diagonal positions, since you may move there by first emptying the $i$'th horizontal line, before you move a castle into position the $i$'th vertical line.

   d) For $1 \leq n \leq 7$.

   e) $\chi(G_1) = 8$.

7. a) Any MST must contain the edges *AB*, *BE*, *CF*, *DG*, *EF*, *FG*, *FH*, *IM*, *IK*, *KN*, *ML*, *NO*, and *OP*, and exactly one of the four edges *FI*, *FK*, *GI*, and *HI*. (Thus, the total weight of any MST is 33; however, this number was not asked for.)

   b) Any shortest path from A to P has total weight 17. *ACFIMP* is one of the correct answers.

   c) (No answer here. Grimaldi explains Dijkstra's shortest-path algorithm in chapter 13.1.)