MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Instructor: Benjamin Nill

Combinatorics II
January 11, 2014
9:00 - 14:00

SOLUTIONS

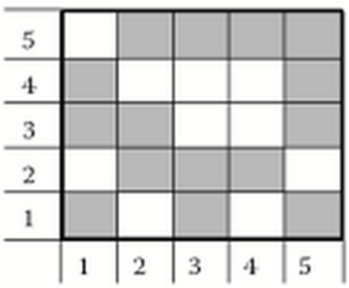
1. (2,5 points) Alice is upset, because she forgot her codeword! She knows that it has length 5 and consists of a permutation of the digits $1, \ldots, 5$. Moreover, after some thinking she remembers that at least one of the following conditions is true:

- the first digit is 2 or 4,
- the second digit is 1 or 5,
- the third digit is 3 or 4,
- the fourth digit is 2, 3, or 4,
- the fifth digit is 1.

Alice wonders how long it might need her to check out every possible codeword in order to find the correct one. Luckily, the Red King's rook tells her that

$$1 + 10x + 33x^2 + 43x^3 + 20x^4 + 3x^5$$

is the rook polynomial of the board consisting of the following **white** cells:



How many possible codewords are there?

Hint: You can use the rook polynomial without having to prove that it is correct.

Solution: This is essentially Example 8.17 in the book. We use the notation from the book (Section 8.1). Let S be the set of injective functions $\{1, \ldots, 5\} \to \{1, \ldots, 5\}$. Let $c_1, \ldots, c_5$ denote the conditions stated. Then we want the number of functions in S that satisfy at least one of the conditions $c_1, \ldots, c_5$. This equals the number of all functions in S minus the number of functions that satisfy none of the conditions. This equals by inclusion-exclusion in the notation from the book

$$N - N(\bar{c_1}, \ldots, \bar{c_5}) = S_1 - S_2 + S_3 - S_4 + S_5,$$

where $S_1 = 10 \cdot 4!$, $S_2 = 33 \cdot 3!$, $S_3 = 43 \cdot 2!$, $S_4 = 20 \cdot 1!$, $S_5 = 3 \cdot 0!$.

To see this, read Section 8.5 again. For instance, $S_2 = \sum_{i<j} N(c_i, c_j)$. This is $33 \cdot 3!$, since the rook polynomial tells us that for an injective function there are 33 possibilities to place two non-taking rooks on the white board (so that the function satisfies two conditions), and for each such choice there are $3 \cdot 2 \cdot 1$ possibilities at the other three values (since the function should be injective).

Hence, the answer is

$$10 \cdot 4! - 33 \cdot 3! + 43 \cdot 2! - 20 \cdot 1! + 3 \cdot 0! = 111.$$

2. (4,5 points) Let's check your intuition on graphs. Let G be a loop-free, undirected, connected graph on a given set of vertices.

In the following you will find three (not quite mathematically rigorous) statements about G. Clearly mark which of the choices is in your opinion closest to the truth. Moreover, for full credits, you have to give a short justification, the more precise the better.

(a) (1,5 points) If G has 'many' edges, then G
- is planar
- is not planar
- can't say

(b) (1,5 points) If G has 'many' edges, then G
- has an Eulerian circuit
- does not have an Eulerian circuit
- can't say

(c) (1,5 points) If G has 'many' edges, then G
- has a Hamiltonian cycle
- does not have a Hamiltonian cycle
- can't say

Solution:
a) One can assume that G has at least five vertices (otherwise G is always planar). Then we learned in class as a consequence of Euler's formula that G has at most $3n - 6$ edges, where n is the number of vertices of G. So, the best answer is 'not planar'.

b) G has an Euler circuit if and only if every vertex has even degree. Its existence cannot be directly deduced from the graph having many edges (since being odd/even has nothing to do with the size of a number). For instance, K_n (the complete graph on n vertices) has an Euler circuit if and only if n is odd. So, the best answer is 'can't say'.

c) Note that K_n has a Hamilton cycle for every n. More precisely, in the book the following result was shown (Cor. 11.6): If G has $n \geq 3$ vertices and at least $\binom{n-1}{2} + 2$ edges, then G has a Hamilton cycle. So, the best answer is 'has a Hamilton cycle'.

3. (4 points) Let a_n be the sequence that satisfies $a_0 = 2$, $a_1 = 7$, and for $n \geq 2$ the recursion relation

$$a_n - 4a_{n-1} + 4a_{n-2} = 2^{n+2} + 3.$$

Give a closed formula for a_n.

Clearly present every step of your computation.

Solution:
1) Find the general homogeneous solution $a_n^{(n)}$:

The characteristic equation is $r^2 - 4r + r = 0$, which has a double root $r = 2$. Therefore, we get

$$a_n^{(n)} = c_1 2^n + c_2 n 2^n.$$

2) Find a particular solution $a_n^{(p)}$:

Looking at the right side $2^{n+2} + 3$, we guess

$$a_n^{(p)} = An^2 2^n + B.$$

Plugging this in the recursion relation we get:

$$An^2 2^n + B - 4(A(n-1)^2 2^{n-1} + B) + 4(A(n-2)^2 2^{n-2} + B) = 2^{n+2} + 3.$$

Doing the algebra on the left side simplifies this to

$$A2^{n+1} + B = 2^{n+2} + 3,$$

so we have $A = 2$ and $B = 3$. Hence,

$$a_n^{(p)} = n^2 2^{n+1} + 3.$$

3) Find the general solution a_n:

Putting everything together we get

$$a_n = a_n^{(h)} + a_n^{(p)} = c_1 2^n + c_2 n 2^n + n^2 2^{n+1} + 3$$

Plugging in the initial conditions gives

$$2 = a_0 = c_1 + 3,$$
$$7 = a_1 = 2c_1 + 2c_2 + 7.$$

This implies $c_1 = -1$ and $c_2 = 1$. Hence, we have the final answer

$$a_n = -2^n + n2^n + n^2 2^{n+1} + 3 = (2n^2 + n - 1)2^n + 3.$$

4. (2,5 points) Let $S = \{1, \ldots, k\}$. Let G_k be the undirected graph

- whose vertices are the subsets of S of size 3,
- and where two vertices are adjacent, if the corresponding subsets of S do not intersect.

Find the number of *edges* of G_k for any $k \geq 3$.

Solution: G_k has $\binom{k}{3}$ many vertices. Fix one vertex (i.e., a subset A of S of size 3). Then another subset B of S of size 3 has no intersection with A if and only if B is a subset of the complement of A. This shows $\deg(A) = \binom{k-3}{3}$. Therefore,

$$|E(G)| = \frac{1}{2} \sum_{A \in V(G)} \deg(A) = \frac{1}{2} \sum_{A \in V(G)} \binom{k-3}{3} = \frac{1}{2}\binom{k}{3}\binom{k-3}{3}.$$

5. (2,5 points) How many ways are there to distribute 50 different books among three classes A,B,C, when B and C get each an *even* number of books?

Solution: Using exponential generating functions we get that the answer is the coefficient of $x^{50}/50!$ in

$$(1 + x + x^2/2! + x^3/3! + \cdots)(1 + x^2/2! + x^4/4! + \cdots)^2 = e^x \left(\frac{e^x + e^{-x}}{2}\right)^2$$

$$= \frac{e^{3x} + 2e^x + e^{-x}}{4} = \frac{1}{4} \sum_{k=0}^{\infty} \frac{1}{k!}(3^k + 2 + (-1)^k)x^k.$$

Note that the coefficient of x^{50} is $\frac{3^{50}+2+(-1)^{50}}{4\cdot 50!}$. Therefore, the answer is

$$\frac{3^{50}+3}{4}.$$

6. (3 points) In how many ways can one distribute 50 balls among 4 boxes such that *each box contains at least one ball*,

(a) (1,5 points) when balls and boxes are numbered?
(b) (1,5 points) when the balls are numbered, but the boxes are identical?

Solution:
a) This is the number of surjective functions $\{1,\ldots,50\} \to \{1,\ldots,4\}$. By the inclusion-exclusion-formula this equals

$$4^{50} - \binom{4}{1}3^{50} + \binom{4}{2}2^{50} - \binom{4}{3}1^{50} = 4^{50} - 4\cdot 3^{50} + 6\cdot 2^{50} - 4.$$

b) This equals the previous expression divided by 4! (namely, the Stirling number of the second kind $S(50,4)$).

7. (2,5 points) In how many ways can one distribute 50 identical balls among 4 numbered boxes such that each box contains at least one ball and the fourth box contains at most three balls?

Solution: This equals the number of integer solutions of

$$x_1 + x_2 + x_3 + x_4 = 50, \text{ for } 1 \le x_1,\ 1 \le x_2,\ 1 \le x_3,\ 1 \le x_4 \le 3.$$

After putting a ball in each box, this equals the number of integer solutions of

$$x_1 + x_2 + x_3 + x_4 = 46, \text{ for } 0 \le x_1,\ 0 \le x_2,\ 0 \le x_3,\ 0 \le x_4 \le 2.$$

This equals the coefficient of x^{46} in the generating function

$$\frac{1}{(1-x)^3}(1+x+x^2) = \left(\sum_{k=0}^{\infty}\binom{3+k-1}{k}x^k\right)(1+x+x^2) = \left(\sum_{k=0}^{\infty}\binom{k+2}{2}x^k\right)(1+x+x^2)$$

Therefore, the anwer is $\binom{46+2}{2} + \binom{45+2}{2} + \binom{44+2}{2} = 3244$.

This can also be done by distinguishing the cases $x_4 = 0,1,2$ and counting the number of possibilities to put the remaining balls into the first three boxes.

Or still alternatively, considering the original equation above one gets the generating function

$$(x+x^2+\cdots)^3(x+x^2+x^3) = \frac{x^3}{(1-x)^3}x(1+x+x^2) = \frac{x^3}{(1-x)^3}x\frac{1-x^3}{1-x} = \frac{x^4-x^7}{(1-x)^4},$$

which again gives the answer $\binom{49}{3} - \binom{46}{3} = 3244$.

8. (2 points) Let a_n be the number of (unordered) partitions of n into precisely 4 parts. Write the generating function of the sequence a_n as a quotient of two polynomials.

Solution: By the Ferrers graph argument, the number b_n of (unordered) partitions of n into at most 4 parts equals the number of (unordered) partitions of n into parts ≤ 4. Therefore, as learned in the lecture

$$\sum_{n=0}^{\infty} b_n = \frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^3}\frac{1}{1-x^4}.$$

In the same way, we have for the number c_n of (unordered) partitions of n into at most 3 parts

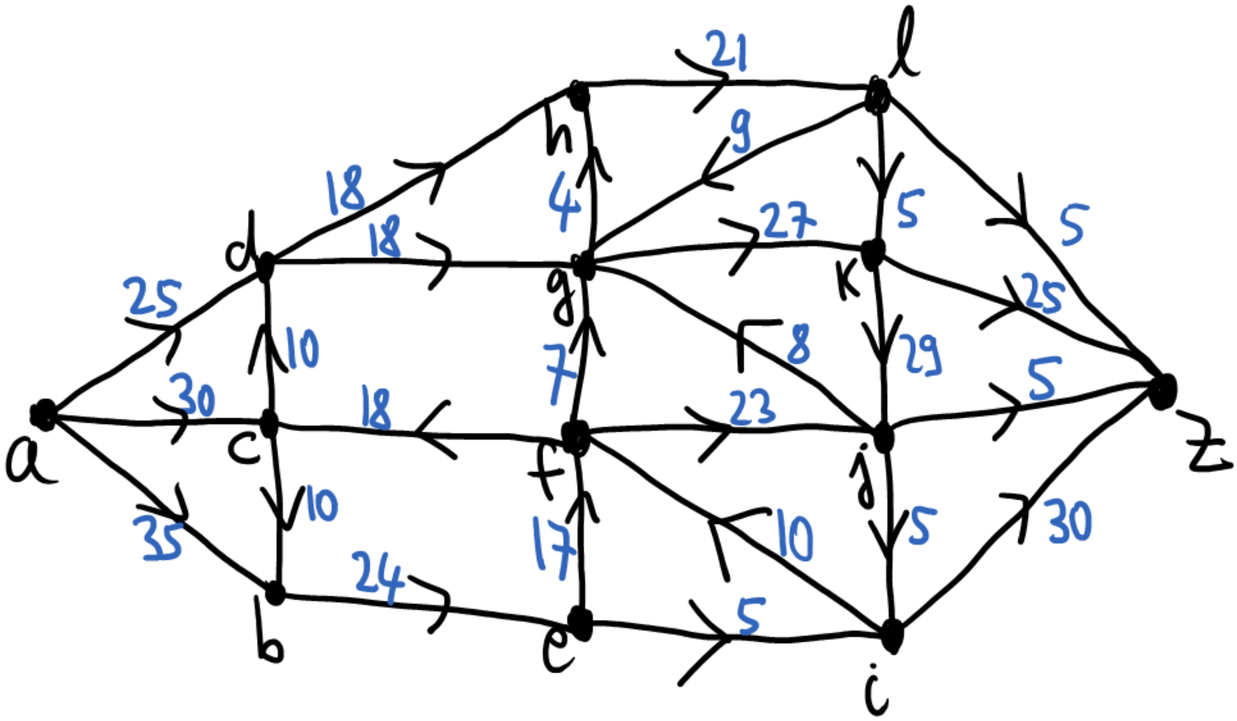
$$\sum_{n=0}^{\infty} c_n = \frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^3}.$$

Since $a_n = b_n - c_n$, we get

$$\sum_{n=0}^{\infty} a_n = \frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^3}\left(\frac{1}{1-x^4} - 1\right) = \frac{x^4}{(1-x)(1-x^2)(1-x^3)(1-x^4)}.$$

As usual, there are also other ways to solve this problem.

9. (6,5 points) Consider the following network:



(a) (2 points) Give the shortest directed path from a to j. Show your reasoning.

(b) (3 points) Give a flow with maximal flow value **ON THE NEXT PAGE**.

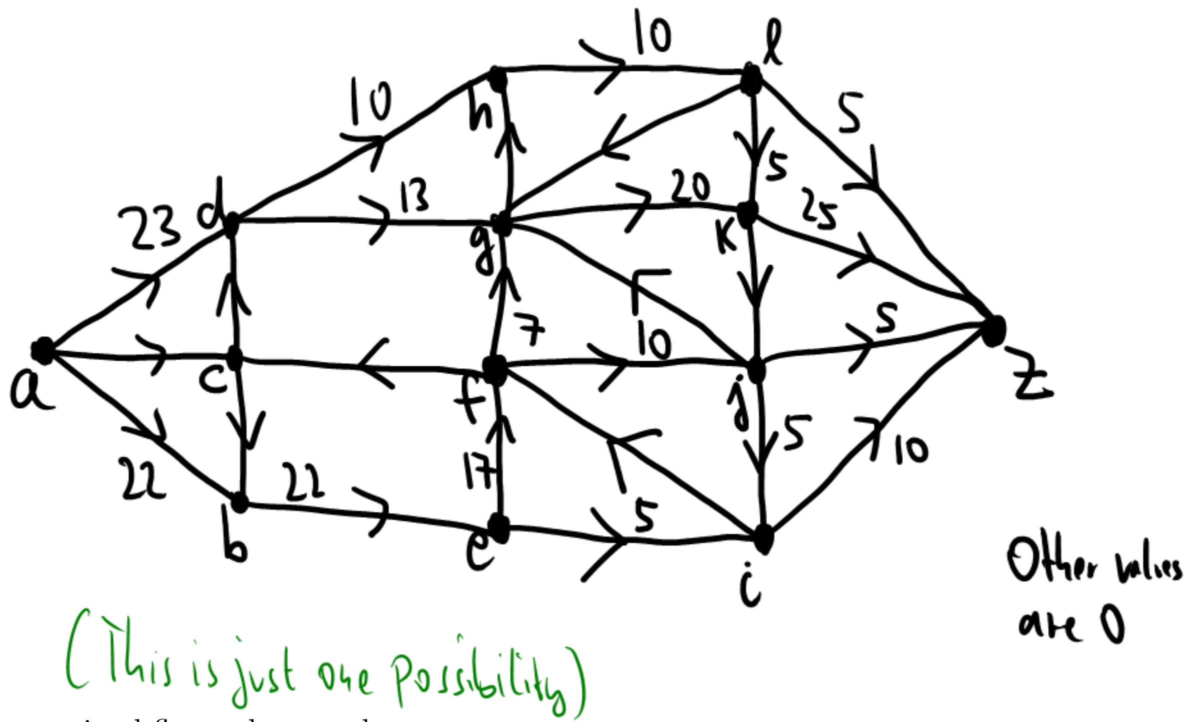
(c) (1,5 points) Give a cut with the minimal cut capacity **ON THE NEXT PAGE**.

Solution:
a) The shortest directed path from a to j is a-b-e-i-f-j of length 97. This can be done via Dijkstra's algorithm. It is also possible to do an exhaustive search by comparing all possible direct paths, however, in this case one can easily overlook some possibility.

b)+c) see next page.

SOLUTION FOR THIS PAGE

Enter the max flow you found right here into the network:



The maximal flow value equals:

45

A cut set with minimal cut capacity is given as:

$P = \{a, b, c, d, e, f, g, h, j, k, l\}$, $\bar{P} = \{i, z\}$.

The minimal cut capacity equals:

45