MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Instructor: Benjamin Nill

Combinatorics II (Omtentamen)
February 15, 2014
9:00 - 14:00

SOLUTIONS

1. (2,5 points) At a wedding party a kid gets bored. It picks up the 50 nametags of the wedding guests (no two of the wedding guests have the same name) and puts them into 8 empty glasses (some of them can still be empty afterwards). In how many ways can this be done, if

(a) (1 point) the glasses look all different?
(b) (1,5 point) the glasses look the same?

Solution:

(a) Since the nametags are labeled and the glasses are distinct, this corresponds to counting functions $\{1, \ldots, 50\} \to \{1, \ldots 8\}$. There are 8^{50} many.

(b) Since the glasses look the same, this corresponds to distributing 50 distinct objects into 8 identical containers, where some of the containers may be empty. Therefore, the answer is $S(50, 1) + \cdots + S(50, 8)$, see Table 5.13 in the book. (Here, $S(50, k)$ counts the number of ways to distribute 50 distinct objects into k identical containers, where none of the containers may be empty).

2. (3 points) Let r be a positive integer. We define a_r as the number of (unordered) partitions of r into positive summands, where

- no summand is larger than at most 6, and
- each summand appears an even number of times, and
- the numbers 1 and 3 must appear as summands, and
- 2 appears at most two times as a summand.

Write the generating function of a_r as a quotient of two polynomials.

Hint: Make sure to read the problem very carefully.

Solution:

a_r is the number of nonnegative, integer solutions to the following equation:

$$y_1 + 2y_2 + 3y_3 + 4y_4 + 5y_5 + 6y_6 = r,$$

where $y_1 \geq 1, y_2 \leq 2, y_3 \geq 1$, and all y_i's are even. Therefore,

$$\sum_{r=0}^{\infty} a_r = (x^2 + x^4 + \cdots)(1 + (x^2)^2)((x^3)^2 + (x^3)^4 + \cdots) \prod_{i=4}^{6}(1 + (x^i)^2 + (x^i)^4 + \cdots)$$

$$= \frac{x^2}{1 - x^2}(1 + x^4)\frac{x^6}{1 - x^6}\frac{1}{1 - x^8}\frac{1}{1 - x^{10}}\frac{1}{1 - x^{12}}$$

$$= \frac{x^8(1 + x^4)}{(1 - x^2)(1 - x^6)(1 - x^8)(1 - x^{10})(1 - x^{12})}.$$

3. (4 points) Let a_n be the sequence that satisfies $a_0 = 1$, $a_1 = 17$, and for $n \geq 2$ the recursion relation

$$a_n - 6a_{n-1} + 9a_{n-2} = n3^n.$$

Give a closed formula for a_n.

Clearly present every step of your computation.

Solution:

The characteristic equation has only one root 3. Therefore, $a_n^{(h)} = c_1 3^n + c_2 n 3^n$. From the right side we come up with $(An + B)3^n$, and hence multiplying by n^2 yields

$$a_n^{(p)} = n^2(An + B)3^n.$$

Putting this into the recursion equation yields

$$(54n - 54)A + 18B = 9n.$$

Therefore, $A = 1/6, B = 1/2$. This yields for the general solution $a_n = a^{(h)} + a^{(p)}$

$$a_n = c_1 3^n + c_2 n 3^n + n^2(n/6 + 1/2)3^n.$$

Putting in $n = 0$ and $n = 1$ gives $c_1 = 1$ and $c_2 = 4$, thus

$$a_n = (1 + 4n + \frac{n^2}{2} + \frac{n^3}{6})\ 3^n.$$

4. (3 points) Find the number of permutations of $\{1, \ldots, 10\}$

(a) (1,5 points) with at least one fixpoint.
(b) (1,5 points) with precisely one fixpoint.

Remember that a fixpoint of a permutation is an element that is not changed by the permutation.

You have to show your reasoning.

Solution:

Let c_i be the condition that the i-th number is not changed. Then by Theorems 8.1 and 8.2 in the book:

(a) $N - \bar{N} = S_1 - S_2 + S_3 - S_4 + S_5 - S_6 + S_7 - S_8 + S_9 - S_{10}$.

(b) $S_1 - 2S_2 + 3S_3 - 4S_4 + 5S_5 - 6S_6 + 7S_7 - 8S_8 + 9S_9 - 10S_{10}$.

Here, $S_i = \binom{10}{i}(10 - i)!$.

If you have a hard time understanding these results, try it with 3 instead of 10 and draw some Venn-diagrams.

An easier solution can also be found by thinking about derangements ...

5. (3 points) Let G be a (loop-free, undirected, **connected**) graph with 24 vertices and 45 edges. Prove that if G is bipartite, then G cannot be planar.

Hint: How 'small' can a region of a bipartite planar graph be?

(I forgot to add the assumption 'connected' on the exam.)

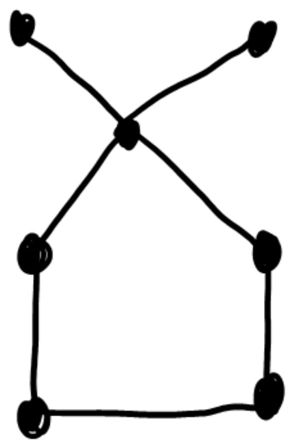
Solution:

Assume G is bipartite and planar. A planar graph with 24 vertices and 45 edges has by Euler's formula 23 regions. Since a cycle in a bipartite graph must have *even* number of vertices, any such cycle has at least four vertices. Since the boundary of a region in a planar graph contains a cycle, the degree of every region is at least 4. (Note that this also holds for the unbounded region - except possibly if G has no cycles, in which case G is a tree, but this is not possible because of $45 \neq 24 - 1$). Again, since G is planar, we get for the number e of edges of G

$$2e = \sum_R \deg(R) \geq 23 \cdot 4 = 92.$$

Hence, $e \geq 46$, a contradiction.

6. (3 points) Find the chromatic number *and* the chromatic polynomial of the following graph:



Solution:

Clearly, it is not possible to color the vertices with 1 or 2 colors so that no edge has vertices of the same color. (The reason is the 4-cycle!) So, the chromatic number is $\chi(G) = 3$.

In order to compute the chromatic polynomial one applies the Decomposition Theorem (Theorem 11.10 in the book) several times just as in Example 11.36 or Example 11.37. Collecting all the terms gives

$$P(G, \lambda) = \lambda(\lambda - 1)^3(\lambda - 2)(\lambda^2 - 2\lambda + 2).$$

7. (2,5 points) At the international arctic research station it has become a ritual that every newcomer has to write down a new word into the station's logbook using only the letters in the word ARCTICA. For instance, RTAA is an allowed word, but not CAAAR. After how many visitors is it impossible to find a new word not yet written down?

As usual it is enough to write down an explicit formula, you don't have to evaluate it.

Solution:

The word has 2 A's, 2 C's, 1 R, 1 S, and 1 T. The exponential generating function for the number of words using these letters is

$$\left(1 + x + \frac{x^2}{2}\right)^2 (1+x)^3 = 1 + 5x + 11x^2 + 14x^3 + \frac{45}{4}x^4 + \frac{23}{4}x^5 + \frac{7}{4}x^6 + \frac{1}{4}x^7.$$

Therefore, to get the number of all words of lengths 1 up to 7 we have to sum up the coefficients of $\frac{x}{1!}, \frac{x^2}{2!}, \ldots, \frac{x^7}{7!}$:

$$5 \cdot 1! + 11 \cdot 2! + 14 \cdot 3! + \frac{45}{4} \cdot 4! + \frac{23}{4} 5! + \frac{7}{4} 6! + \frac{1}{4} 7! = 3591.$$

8. (2,5 points) In how many ways can one put 13 identical balls in 20 numbered boxes such that each of the first 5 boxes contain at least one ball, and each of the last 15 boxes contain at most 3 balls?

Solution:

We want the coefficient of x^{13} in

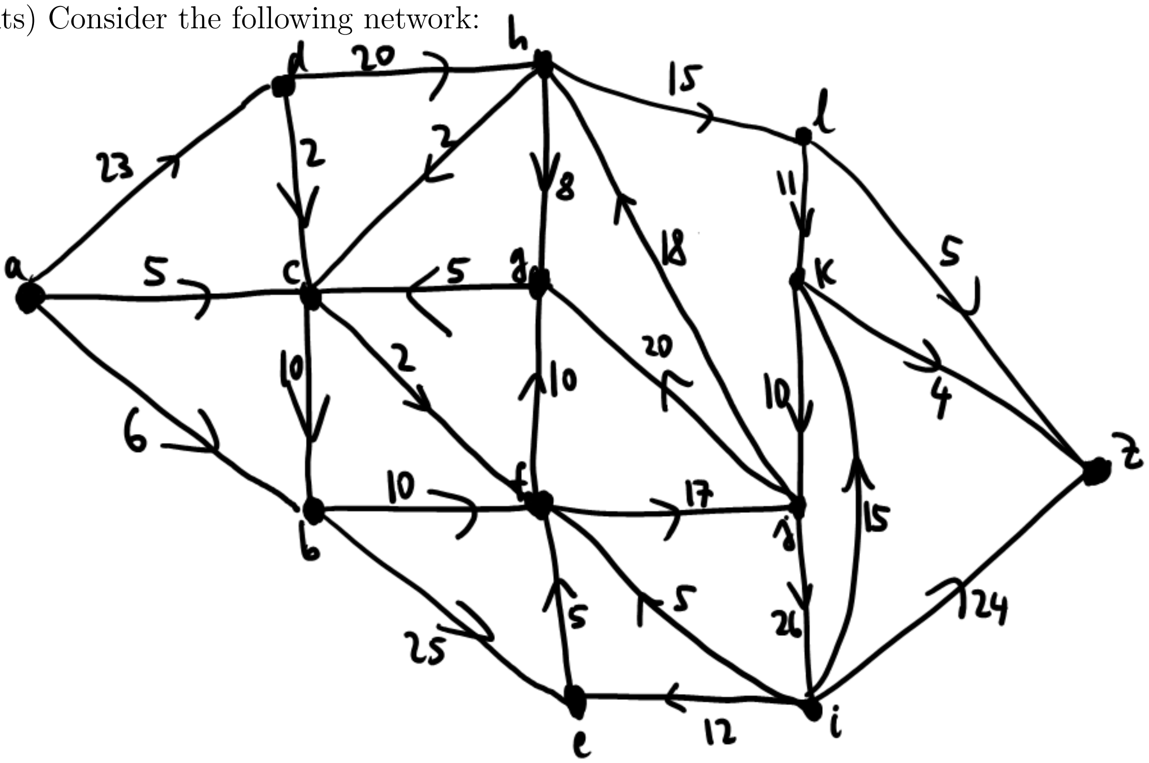
$$(x + x^2 + \cdots)^5 (1 + x + x^2 + x^3)^{15} = \frac{x^5}{(1-x)^5} \frac{(1-x^4)^{15}}{(1-x)^{15}} = \frac{x^5 (1-x^4)^{15}}{(1-x)^{20}}$$

$$= x^5 \left(\sum_{i=0}^{15} \binom{15}{i} (-1)^i (x^4)^i \right) \left(\sum_{r=0}^{\infty} \binom{19+r}{r} x^r \right).$$

Therefore, we look for the coefficient of x^8 in the product of the two sums. This yields

$$\binom{19+8}{8} + \binom{15}{1}(-1)\binom{19+4}{4} + \binom{15}{2} = 2087355.$$

9. (6,5 points) Consider the following network:



(a) (2 points) Give the shortest directed path from a to l. (I repeat: to l, not to z). Show your reasoning.

(b) (3 points) Give a flow with maximal flow value **ON THE NEXT PAGE**.

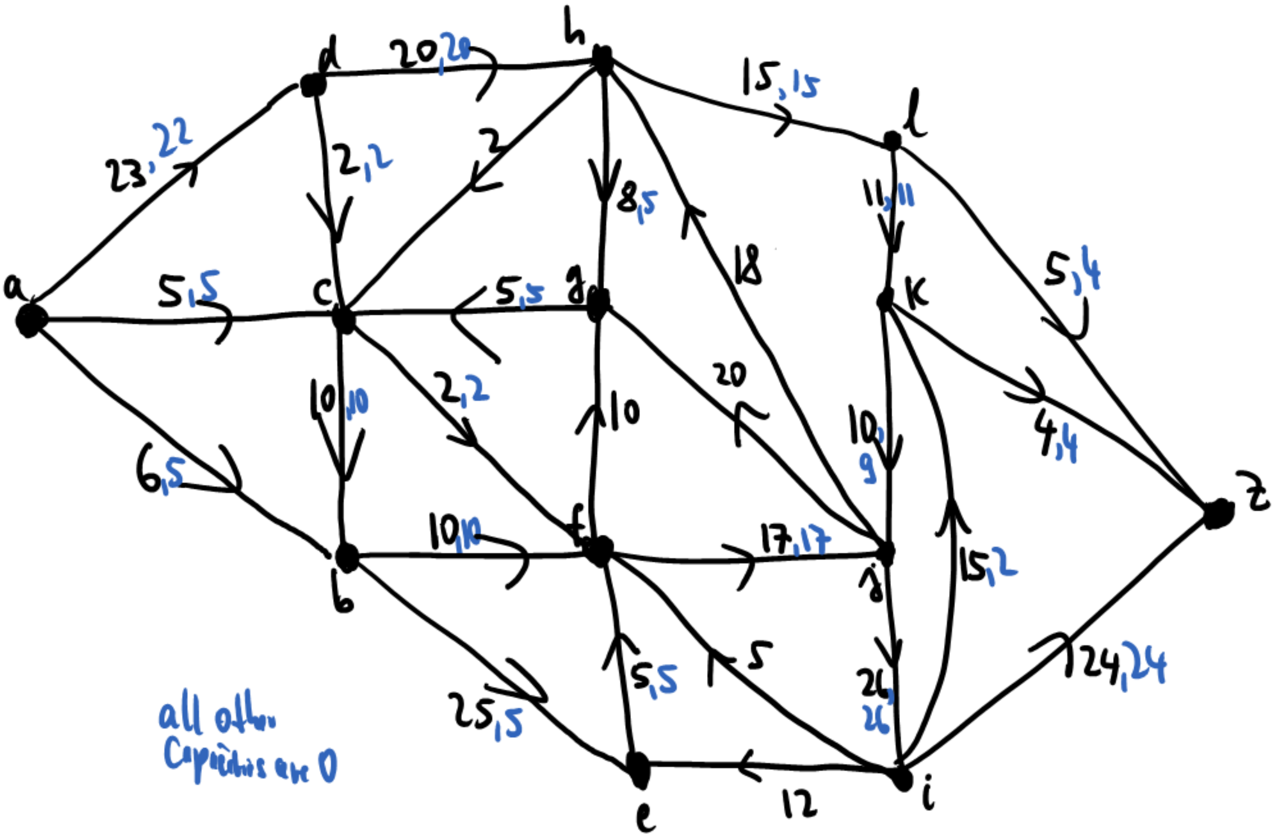
(c) (1,5 points) Give a cut with the minimal cut capacity **ON THE NEXT PAGE**.

Solution to (a):

Dijkstra's algorithm (or close inspection) shows that a-c-f-j-h-l is the unique directed path of minimal length $5 + 2 + 17 + 18 + 15 = 57$. (Note: the more obvious path a-d-h-l has length $23 + 20 + 15 = 58$.)

SOLUTIONS TO 9(b,c)

Here are the capacities of one possible max flow (next to the capacities):



The maximal flow value equals:

32

A cut set with minimal cut capacity is given as:

$P = \{a, b, c, d, e, g, h\}$, $\bar{P} = \{f, i, j, k, l, z\}$.

The minimal cut capacity equals:

32