MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Instructor: Benjamin Nill

Combinatorics II (Online course)
June 3, 2014
9:00 - 14:00

The exams can be picked up from Katarina Ringels (Hus 6, Rum 204).

Complete and clear solutions must be given
except where explicitly stated otherwise.

No calculators allowed.
If you cannot simplify an expression any further, just leave it.

1. (1,5 points) Give the rook polynomial of a rectangular 7×9 'chessboard'.

No justification necessary, no simplification necessary.

2. (3 points)

(a) (2 points) Count the number of functions $f : \{1, \ldots, 30\} \to \{1, \ldots, 70\}$ such that
- the range of f contains the elements 1 and 2, and
- the range of f has at least eleven elements.

(b) (1 point) How many of the functions in (a) are injective?

Your answers must be as explicit as possible.

3. (4 points) Find a closed formula for the sequence a_n that satisfies $a_0 = 17/4$, $a_1 = 31/4$, and for $n \geq 2$ the recursion relation

$$a_n - 9a_{n-2} = (-3)^{n+1} - 8n.$$

Clearly present every step of your computation.

4. (3 points) Consider in this problem only connected graphs with 48 vertices and 47 edges.

Look at the following list of properties:

(1) bipartite
(2) planar
(3) has an Eulerian trail
(4) complete
(5) chromatic number 2
(6) has a Hamiltonian path

Decide for every property whether it

(A) holds for all such graphs
(S) holds for at least one such graph but not for all such graphs
(N) holds for no such graphs

For instance write, 1A or 1S or 1N, and 2A or 2S or 2N, etc.

You MUST shortly justify your answers.

5. (5 points) Let $k \geq 4$ be an integer, and $S = \{1, \ldots, k\}$. Let G_k be the graph

- whose vertices are the subsets of S of size 4,
- and where two vertices are adjacent, if the corresponding subsets of S intersect.

Note that G_k is connected. (You don't have to prove this).

(a) (1,5 points) Give an explicit numerical criterion (in terms of k) for deciding whether G_k has an Eulerian circuit.

(b) (2 points) Show that G_k is not planar for $k \geq 5$.

For full credits you must present TWO proofs that are as different as possible. Make an effort to give clear and readable arguments.

(c) (1,5 points) Let a_k be the number of vertices of G_k. Write the generating function for a_k as a quotient of two polynomials.

6. (2 points) Let a_r be the number of nonnegative integer solutions of the equation

$$x_1 + 2x_2 + 3x_3 + 4x_4 = r,$$

where $x_2 \leq 2$.

(a) (1 point) Mark all correct interpretations of a_{12} in terms of (unordered) partitions:

(A) a_{12} is the number of partitions of 12 into 4 parts,
where the second smallest summand appears at most twice

(B) a_{12} is the number of partitions of 12 into 4 parts,
where the second smallest summand is at most of size 2

(D) a_{12} is the number of partitions of 12 into 4 parts,
where 2 occurs at most twice as a summand

(E) a_{12} is the number of partitions of 12 into at most 4 parts,
where the second smallest summand appears at most twice

(F) a_{12} is the number of partitions of 12 into at most 4 parts,
where the second smallest summand is at most of size 2

(G) a_{12} is the number of partitions of 12 into at most 4 parts,
where 2 occurs at most twice as a summand

(H) a_{12} is the number of partitions of 12 into parts of size at most 4,
where the second smallest summand appears at most twice

(I) a_{12} is the number of partitions of 12 into parts of size at most 4,
where the second smallest summand is at most of size 2

(J) a_{12} is the number of partitions of 12 into parts of size at most 4,
where 2 occurs at most twice as a summand

(K) a_{12} is the number of partitions of 12 with some part of maximal size 4, where the second smallest summand appears at most twice

(L) a_{12} is the number of partitions of 12 with some part of maximal size 4, where the second smallest summand is at most of size 2

(M) a_{12} is the number of partitions of 12 with some part of maximal size 4, where 2 occurs at most twice as a summand

No justifications necessary.

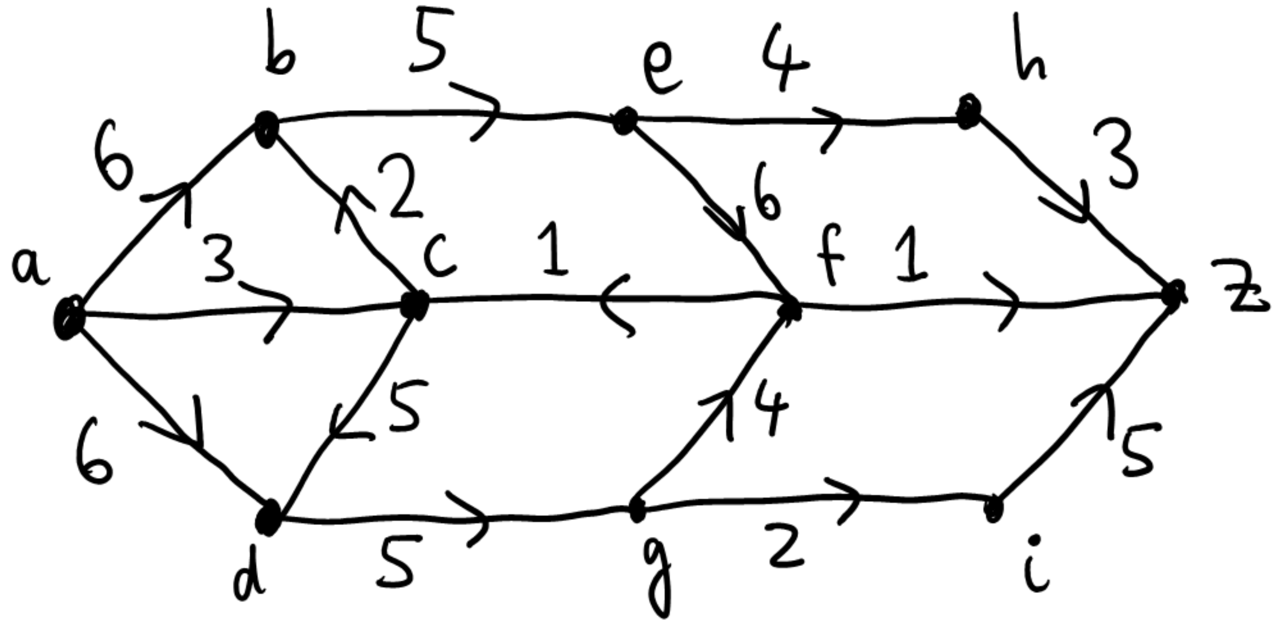
(b) (1 point) Write the generating function of the sequence a_r as a quotient of two polynomials.

7. (2,5 points) In a supermarket one neatly puts 5 different types of apples on two shelfs, on each shelf precisely 20 apples sorted from the left to the right. If every type of apple must be used on each shelf, how many possibilities are there?

As usual it is enough to write down an explicit formula, you don't have to evaluate it.

8. (2,5 points) How many outcomes are possible, if in a lottery 49 identical balls get sorted into 4 numbered urns where the machine is constructed in such a way that the first urn contains an odd number of balls and the last three urns each an even number of balls?

9. (6,5 points) Consider the following network:



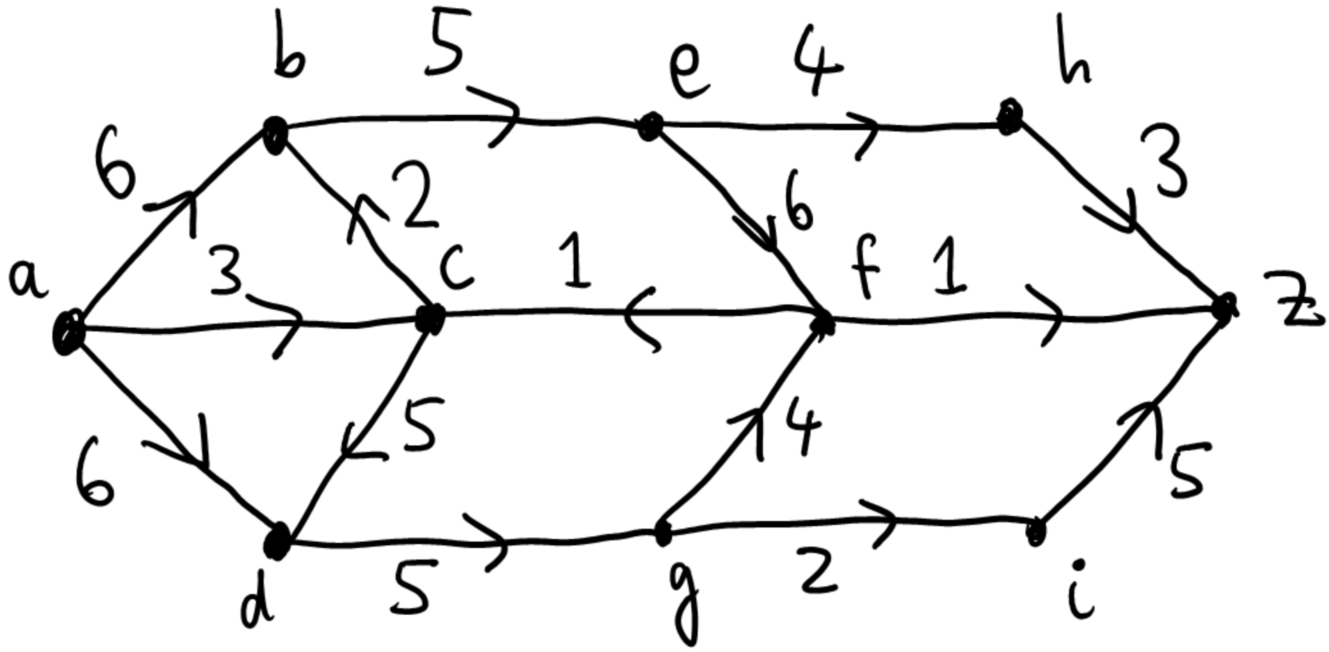
HAND IN THIS PAGE

(a) (2,5 points) Use Dijkstra's algorithm to find for any vertex $v = a, b, c, d, e, f, g, h, i, z$ the distance $d(a, v)$ from a to v.

You MUST USE Dijkstra's algorithm in order to get any credits. Clearly show your table and in which order you choose the vertices (use an extra page, since you will need some space).

(b) (1 points) Give a shortest directed path from a to z.

(c) (2 points) Give a flow with maximal flow value. Enter the maximal flow you found RIGHT HERE into the network (next to the capacities):



Write here the flow value of your flow:

(d) (1 points) Give a cut with the minimal cut capacity RIGHT HERE:

Write here the cut capacity of your cut: