MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Alexander Berglund

Tentamensskrivning i
Topologi, MM8002/SF2721
7.5 hp
2016-08-23

(1) Let X and Y be topological spaces. Decide whether the following statements are true or false. If true, give a proof. If false, give a counterexample.
  (a) The equality $\overline{A \times B} = \overline{A} \times \overline{B}$ holds for all subsets $A \subseteq X$, $B \subseteq Y$.
  (b) If $f\colon X \to Y$ is continuous, then $f^{-1}(A^\circ) = f^{-1}(A)^\circ$ for all $A \subseteq Y$.
  (c) Every continuous bijection between homeomorphic spaces is a homeomorphism.
  (d) If X is compact, then the topology on X is induced by a metric. [20 points]

(2) A *partial order* on a set X is a binary relation $\leq$ satisfying

(Reflexivity)	$x \leq x$ for all $x \in X$,
(Transitivity)	if $x \leq y$ and $y \leq z$, then $x \leq z$,
(Antisymmetry)	if $x \leq y$ and $y \leq x$, then $x = y$.

Declare a subset $U \subseteq X$ to be open if $x \leq y$ and $y \in U$ implies $x \in U$.
  (a) Prove that this defines a topology on X.
  (b) Prove that a function $f\colon X \to Y$ between partially ordered sets is continuous if and only if it is order-preserving in the sense that
$$x \leq y \Rightarrow f(x) \leq f(y).$$
  (c) Consider the set of positive integers $\mathbb{Z}_+$ with the topology induced by the partial order given by the divisibility relation; $x \leq y$ if and only if x divides y. Is $\mathbb{Z}_+$ compact in this topology? Connected? Hausdorff? [20 points]

(3) Let $a > 1$ be a real number and consider the map
$$\mathbb{Z} \times \big(\mathbb{R}^n \setminus \{0\}\big) \to \mathbb{R}^n \setminus \{0\},$$
$$(k, x) \mapsto a^k x.$$
  (a) Show that the map defines an action of the discrete group $\mathbb{Z}$ on $\mathbb{R}^n \setminus \{0\}$.
  (b) Show that the orbit space is compact.
  (c) Show that the orbit space is homeomorphic to $S^{n-1} \times S^1$. [15 points]

(4) Suppose that X and Y are homotopy equivalent topological spaces. Show that X is path-connected if and only if Y is. [10 points]

(5) Let X be the subspace of $\mathbb{R}^3$ given by the union of the unit circles in the xy-, xz- and yz-planes, respectively. Calculate the fundamental group, the Euler characteristic and the Betti numbers of X. [15 points]

(6) Let G be a topological group acting on a topological space X. Show that if both G and the orbit space X/G are connected, then so is X. [20 points]

Corrected exams will be returned on Tuesday August 30, 13:00 – 13:15, in room 414. Thereafter exams may be collected in room 204.