Problem 1: Poisson Processes

Let $\{N(t), t \geq 0\}$ be a homogeneous Poisson Process on $(0, \infty)$ with rate λ. Let $\{S_i, i = 1, 2, \cdots\}$ be the points of the Poisson Process, such that $S_1 \leq S_2 \leq S_3 \leq \cdots$.

a) For $0 < s < t$, compute $\mathbb{E}[N(t)]$, $Var[N(t)]$ and $Cov[N(s), N(t)]$. (4p)

Solution: Since $N(t)$ is Poisson(λt), we have $\mathbb{E}[N(t)] = \lambda t$, $Var[N(t)] = \lambda t$ and

$$\begin{aligned} &Cov[N(s), N(t)] = Cov[N(s), N(s) + (N(t) - N(s))] \\ &\quad = Cov[N(s), N(s)] + Cov[N(s), N(t) - N(s)] = Var[N(s)] + 0 = \lambda s. \end{aligned}$$

b) Assume that for $n > 1$ and $t > 0$, $S_n = t$. Provide the distribution of S_{n-1}. (5p)

Solution: Use order statistic property. The $n-1$ points in the interval $(0, t)$ are i.i.d. uniformly distributed. Note further that for $s < t$, "$S_{n-1} \leq s$" is equivalent to "all $n-1$ points in $(0, t)$ are in the interval $(0, s)$", which happens by the i.i.d. property with probability $(s/t)^{n-1}$. That is, for $s < t$ we have $\mathbb{P}(S_{n-1} < s | S_n = t) = (s/t)^{n-1}$.

c) Let Y be an exponentially distributed random variable with expectation $1/\mu$. Assume $N(Y) \geq 1$ (i.e. $S_1 \leq Y$). Provide the distribution of S_1, i.e. compute $\mathbb{P}(S_1 \leq t | S_1 \leq Y)$. (3p)

Solution: Either observe that S_1 and Y are independent exponentials and that the minimum of two independent exponential distributed random variables is exponential distributed with rate the sum of the original rates, furthermore this minimum is independent of which of the two random variables take the minimum. Then observe,

$$\begin{aligned} &\mathbb{P}(S_1 \leq t | S_1 \leq Y) = \mathbb{P}(\min(S_1, Y) \leq t | S_1 \leq Y) \\ &= \mathbb{P}(\min(S_1, Y) \leq t | \min(S_1, Y) = S_1) = \mathbb{P}(\min(S_1, Y) \leq t) = 1 - e^{-(\lambda+\mu)t}. \end{aligned}$$

One can also use direct computation, using conditioning on Y:

$$\begin{aligned} \mathbb{P}(S_1 \leq t | S_1 \leq Y) &= \frac{\mathbb{P}(S_1 \leq t, S_1 \leq Y)}{\mathbb{P}(S_1 \leq Y)} = \frac{\int_0^t \mu e^{-\mu y}(1 - e^{-\lambda y}) dy + \int_t^\infty \mu e^{-\mu y}(1 - e^{-\lambda t}) dy}{\int_0^\infty \mu e^{-\mu y}(1 - e^{-\lambda y}) dy} \\ &= \frac{(1 - e^{-\mu t}) - \frac{\mu}{\lambda+\mu}(1 - e^{-(\lambda+\mu)t}) + e^{-\mu t}(1 - e^{-\lambda t})}{\frac{\lambda}{\lambda+\mu}} = (1 - e^{-(\lambda+\mu)t}). \end{aligned}$$

Problem 2: Renewal Theory

Assume reports of some food-borne diseases are made to the national public health office with independent and identically exponential distributed inter-arrival times with expectation $1/\lambda$. The reports come in at the time somebody calls sick from work (assume that healthy people work 24 hours a day, 7 days a week and can call sick every single moment). Patients stay sick until they take medicine. As soon as N reports have come in, the public health authorities take measures and medicine (which cures the patient immediately at the moment of administration) is provided exactly K days after the N-th patient is reported. All people reported sick at the moment the medicine is provided receive the medicine. Here N is a positive integer, and K a positive real number. When all sick people have taken the medicine the process starts anew.

a) What is the expected time between subsequent moments the public health office provides medicine? (4p)

Solution: We wait for N independent interarrival times all with expectation $1/\lambda$ and K additional days. So the expected time between provisions of medicine are $N/\lambda + K$

b) What is the expected total time of sick leave in the country because of the food borne disease in one cycle (the period between moments of administration of medicine)? (4p)

Solution: For the i-th patient, where $i \leq N$, the expected duration of sick leave is $(N-i)/\lambda + K$, where $(N-i)/\lambda$ is the expected time until the N-th patient calls sick. In the final K time units of a cycle, people call sick according to a Poisson Process with intensity λ, and the number of people getting sick (say X_K) is Poisson distributed with expectation λK. Conditioned on $X_K = k$ the times those k people get sick are distributed as k i.i.d. uniforms on this interval by the order statistic property. So the total expected sick leave of those k patients is $kK/2$. So the total expected sick leave is $\sum_{i=1}^{N}((N-i)/\lambda + K) + \mathbb{E}[X_k K/2] = \frac{N(N-1)}{2\lambda} + KN + \lambda K^2/2$.

c) What is in the long run the average total time of sick leave per week in the country because of the food borne disease? (4p)

By renewal theorems we know that this long run average converges to

$$\frac{\mathbb{E}[\text{sickleave per cycle}]}{\mathbb{E}[\text{duration of cycle}]} = \frac{\frac{N(N-1)}{2\lambda} + KN + \lambda K^2/2}{N/\lambda + K} = \frac{N(N-1) + 2\lambda KN + (\lambda K)^2}{2N + 2\lambda K}$$

Problem 3: Queueing Theory

Consider an $M/M/1$ queue in which customers arrive at rate λ and workloads are exponentially distributed with expectation $1/\mu$. Assume that the queue has a maximal capacity K, which means that if there are K customers in the queue, newly arriving customers will leave immediately.

a) What is the asymptotic distribution of the number of customers in the system? (4p)

Solution: Let ρ_i be the asymptotic probability of having system size i. Note that the maximal system size is $K+1$ (K in the queue and 1 in service.) Use balance equations for $i = 0, 1, \cdots K$: $\lambda\rho_i = \mu\rho_{i+1}$. This implies that for $i = 0, 1, \cdots K+1$, $\rho_i = (\lambda/\mu)^i \rho_0$ and in order to have

$$1 = \sum_{i=0}^{K+1} \rho_i = \rho_0 \frac{1 - (\lambda/\mu)^{K+2}}{1 - \lambda/\mu}, \text{ we need } \rho_0 = \frac{1 - \lambda/\mu}{1 - (\lambda/\mu)^{K+2}}.$$

b) What proportion (in the long run) of arriving potential customers will depart immediately without being served? (4p)

Solution: In the long run, the time the process is at its maximal capacity is given by $\rho_{K+1} = \frac{(1-\lambda/\mu)(\lambda/\mu)^{K+1}}{1-(\lambda/\mu)^{K+2}}$. This is also the fraction of potential customers which have to leave without being served (by PASTA).

c) Now assume the server in the queue is replaced by two servers, which work at half speed. i.e. the system becomes an $M/M/2$ queue with parameters λ and $\mu/2$. What proportion (in the long run) of arriving potential customers will then depart immediately without being served? (4p)

Solution: Note that the maximum system size is now $K+2$. The balance equations become $\lambda\rho_0 = (\mu/2)\rho_1$ and $\lambda\rho_i = \mu\rho_{i+1}$ for $i = 1, \cdots K+1$. This implies that for $i = 1, \cdots K+2$ we have, $\rho_i = (\lambda/\mu)^{i-1}\rho_1$ and $\rho_0 = (\mu/(2\lambda))\rho_1$.

$$1 = \sum_{i=0}^{K+2} \rho_i = \rho_1 \left((\mu/(2\lambda)) + \sum_{i=1}^{K+2} (\lambda/\mu)^{i-1} \right) = \rho_1 \left((\mu/(2\lambda)) + \frac{1 - (\lambda/\mu)^{K+2}}{1 - \lambda/\mu} \right),$$

So, $\rho_1 = \left((\mu/(2\lambda)) + \frac{1-(\lambda/\mu)^{K+2}}{1-\lambda/\mu} \right)^{-1}$ and therefore,

$$\rho_{K+2} = \left((\mu/(2\lambda)) + \frac{1 - (\lambda/\mu)^{K+2}}{1 - \lambda/\mu} \right)^{-1} (\lambda/\mu)^{K+1} = \frac{(1 - \lambda/\mu)(\lambda/\mu)^{K+1}}{\frac{\mu+\lambda}{2\lambda} - (\lambda/\mu)^{K+2}}.$$

Problem 4: Brownian Motion and Stationary Processes

Let $\{X(t), t \geq 0\}$ be a standard Brownian motion.

a) Provide the definition of a standard Brownian Motion. (4p)

Solution: A *standard Brownian motion* $\{X(t); t \geq 0\}$ is a stochastic process in continuous time and continuous state space $\mathbb{R}$ which satisfies
1) $X(0) = 0$
2) X has independent increments, i.e. $X(t_3) - X(t_2)$ and $X(t_1) - X(t_0)$ are independent if $0 \leq t_0 < t_1 \leq t_2 < t_3$
3) for $0 \leq s < t$, $X(t) - X(s) \sim \mathcal{N}(0, t-s)$
4) $\mathbb{P}(X \in \mathcal{C}[0,\infty)) = 1$, where $\mathcal{C}[0,\infty)$ is the space of continuous functions $f : [0,\infty) \to \mathbb{R}$. That is, X is almost surely continuous

b) Let $\mu > 0$. What is the probability that $\frac{X(t)-1}{t} < \mu$ for all $t > 0$? That is, compute $\mathbb{P}[\sup_{0<t<\infty} \frac{X(t)-1}{t} < \mu]$. (4p)

Solution: Observe that $Y(t) = X(t) - \mu t$ is a Brownian motion with drift $-\mu$ and that $\mathbb{P}[\sup_{0<t<\infty} \frac{X(t)-1}{t} < \mu] = \mathbb{P}[\sup_{0<t<\infty} Y(t) < 1]$. We know from the last line of the cheat-sheet with μ replaced by $-\mu$, $y = 1$ and $t \to \infty$ that this probability is given by $e^{-2\mu}$.

c) Let $M(t) = \sup_{0<s<t} X(s)$. Compute $\mathbb{P}[M(1) = M(2)]$. (4p)

Hint: Derive the joint density of $X(1)$ and $M(1)$. Furthermore, note that

$$\mathbb{P}[M(1) = M(2)] = \mathbb{P}[\sup_{1<s<2} X(s) \leq \sup_{0<s<1} X(s)] = \mathbb{P}[\sup_{1<s<2} (X(s) - X(1)) \leq \sup_{0<s<1} (X(s) - X(1))].$$

Solution: Note that for $m, x > 0$ we have by the reflection principle that

$$\mathbb{P}(M(1) > m, X(1) < m - x) = \mathbb{P}(X(1) > m + x) = \int_{m+x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy.$$

By setting $z = m - x$, this implies that for $m > 0$ and $m > z$

$$\mathbb{P}(M(1) > m, X(1) < z) = \mathbb{P}(X(1) > 2m - z) = \int_{2m-z}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy.$$

So, the joint density of $M(1)$ and $X(1)$ is given by

$$f_{M(1),X(1)}(m,z) = \frac{d}{dz}\frac{-d}{dm}\int_{2m-z}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy = \frac{d}{dz}\sqrt{\frac{2}{\pi}}e^{-(2m-z)^2/2}$$
$$= \sqrt{\frac{2}{\pi}}(2m-z)e^{-(2m-z)^2/2}.$$

And the joint density of $M(1)$ and $M(1)-X(1)$ is given by

$$f_{M(1),M(1)-X(1)}(m,x) = \sqrt{\frac{2}{\pi}}(m+x)e^{-(m+x)^2/2},$$

which in turn gives that

$$f_{M(1)-X(1)}(x) = \int_0^{\infty}\sqrt{\frac{2}{\pi}}(m+x)e^{-(m+x)^2/2}dm = \sqrt{\frac{2}{\pi}}e^{-x^2/2}.$$

In particular, $M(1)-X(1)$ is distributed as $M(1)$.

Note that $\sup_{1<s<2} X(s)-X(1)$ is also distributed as $M(1)$ and is independent of $M(1)$. So $\mathbb{P}[\sup_{1<s<2}(X(s)-X(1)) \le \sup_{0<s<1}(X(s)-X(1))]$ is the probability that the first of two i.i.d. continuous random variables is less than the second, which is 1/2.

Alternatively we can compute:

$$\mathbb{P}[M(1) = M(2)] = \mathbb{P}[M(1)-X(1) = M(2)-X(1)]$$
$$= \mathbb{P}[\sup_{1<s<2}(X(s)-X(1)) \le \sup_{0<s<1}(X(s)-X(1))]$$
$$= \int_0^{\infty} f_{M(1)-X(1)}(x)\mathbb{P}[\sup_{1<s<2}(X(s)-X(1)) \le x]dx$$
$$= \int_0^{\infty}\sqrt{\frac{2}{\pi}}e^{-x^2/2}\mathbb{P}[M(1)\le x]dx = 1 - \int_0^{\infty}\sqrt{\frac{2}{\pi}}e^{-x^2/2}\mathbb{P}[M(1) > x]dx$$
$$= 1 - \int_0^{\infty}\sqrt{\frac{2}{\pi}}e^{-x^2/2}\int_x^{\infty}\sqrt{\frac{2}{\pi}}e^{-y^2/2}dydx = 1 - \frac{2}{\pi}\int_0^{\infty}\int_x^{\infty}e^{-(x+y)^2/2}dydx$$
$$= 1 - \frac{1}{\pi}\left(\int_0^{\infty}\int_x^{\infty}e^{-(x+y)^2/2}dydx + \int_0^{\infty}\int_y^{\infty}e^{-(x+y)^2/2}dxdy\right)$$
$$= \frac{1}{\pi}\int_0^{\infty}\int_0^{\infty}e^{-(x+y)^2/2}dydx = 1/2.$$

Where, the last equality can be obtained by transforming to polar coordinates.

Problem 5: Simulation

Let X be exponentially distributed with expectation $1/\lambda$ and let Y be gamma distributed with density $f_Y(x) = (2\lambda)^2 x e^{-2\lambda x}$.

a) Argue that X is distributed as $-\frac{1}{\lambda}\log[U]$, where U is a uniform random variable taking values on $[0,1]$. (4p)

Solution: Use inverse function method:

$$\mathbb{P}[X \le x] = 1 - e^{-\lambda x} = \mathbb{P}(U < 1 - e^{-\lambda x}) = \mathbb{P}(\log[1-U] > -\lambda x) = \mathbb{P}(-\frac{\log[1-U]}{\lambda} < x).$$

The result follows by observing that $1-U$ and U have the same distribution

b) Show that Y can be simulated using only uniform random variables taking values in $[0,1]$ by $-\frac{1}{2\lambda}(\log[U_1] + \log[U_2])$, where U_1 and U_2 are independent uniform random variables taking values in $[0,1]$. (4p)

Solution: This is immediate from observing that Y is gamma distributed with parameters 2 and 2λ, which in turn is distributed as the sum of two i.i.d. exponential random variables with parameter 2λ. The result follows than from a).

c) Let $U_1, U_2, \cdots$ and $V_1, V_2, \cdots$ be independent uniform random variables taking values in $[0,1]$. Show that Y can be simulated using only uniform random variables taking values in $[0,1]$ in the following way:

For $k = 1, 2, \cdots$, define $Y_k = -\frac{1}{\lambda}\log[U_k]$.
If $V_1 < Y_1 \lambda e^{-(\lambda Y_1 - 1)}$, then $Y = Y_1$.
If $V_1 \ge Y_1 \lambda e^{-(\lambda Y_1 - 1)}$ and if $V_2 < Y_2 \lambda e^{-(\lambda Y_2 - 1)}$, then $Y = Y_2$ and so on.
That is,

$$Y = Y_J, \qquad \text{where} \qquad J = \min\{k \ge 1; V_k < Y_k \lambda e^{-(\lambda Y_k - 1)}\}.$$

(4p)

Solution: This is rejection method using $g(x) = \lambda e^{-\lambda x}$. So $f_Y(x)/g(x) = 4\lambda x e^{-\lambda x}$, which is maximal in $x = 1/\lambda$ and therefore has maximal value $c = 4e^{-1}$.

Note that the Y_k's have density $g(x)$. So Y_1 should be accepted with probability $f_Y(Y_1)/cg(Y_1) = \lambda Y_1 e^{-\lambda Y_1 + 1}/4$.