Logic – Homework 1 – Sample solutions 24.ix.2014

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1. (a) $\models (P_1 \to \neg P_1)$ does not hold:

take the interpretation A with $A(P_1) = 1$.

Then $$[\![P_1 \to \neg P_1]\!]^A = [\![P_1]\!]^A \to (\neg [\![P_1]\!]^A)$$
$$= 1 \to 0 = 0$$

so $P_1 \to \neg P_1$ fails in this interpretation.

(b). Natural deduction shows that

$$P_1 \wedge (P_2 \vee P_3) \vdash (P_1 \wedge P_2) \vee (P_1 \wedge P_3) \;;$$

it follows by soundness that

$$P_1 \wedge (P_2 \vee P_3) \models (P_1 \wedge P_2) \vee (P_1 \wedge P_3) .$$

$$\dfrac{\dfrac{P_1 \wedge (P_2 \vee P_3)}{P_2 \vee P_3}\,{\scriptstyle \wedge E} \qquad \dfrac{\dfrac{\dfrac{P_1 \wedge (P_2 \vee P_3)}{P_1}\,{\scriptstyle \wedge E} \quad [P_2]^1}{P_1 \wedge P_2}\,{\scriptstyle \wedge I}}{(P_1 \wedge P_2) \vee (P_1 \wedge P_3)}\,{\scriptstyle \vee I} \qquad \dfrac{\dfrac{\dfrac{P_1 \wedge (P_2 \vee P_3)}{P_1}\,{\scriptstyle \wedge E} \quad [P_3]^1}{P_1 \wedge P_3}\,{\scriptstyle \wedge I}}{(P_1 \wedge P_2) \vee (P_1 \wedge P_3)}\,{\scriptstyle \vee I}}{(P_1 \wedge P_2) \vee (P_1 \wedge P_3)}\,{\scriptstyle \vee E\text{-}1}$$

1(b): Again, by soundness, it suffices to show that

$$P_1 \vee (P_2 \wedge P_3) \models (P_1 \vee P_2) \wedge (P_1 \vee P_3):$$

$$\dfrac{\dfrac{P_1 \vee (P_2 \wedge P_3) \qquad \dfrac{[P_1]^1}{P_1 \vee P_2}\,\vee I \qquad \dfrac{\dfrac{[P_2 \wedge P_3]^1}{P_2}\,\wedge E}{P_1 \vee P_2}\,\vee I}{P_1 \vee P_2}\,\vee E\text{-}1 \qquad \dfrac{P_1 \vee (P_2 \wedge P_3) \qquad \dfrac{[P_1]^2}{P_1 \vee P_3}\,\vee I \qquad \dfrac{\dfrac{[P_2 \wedge P_3]^2}{P_3}\,\wedge E}{P_1 \vee P_3}\,\vee I}{P_1 \vee P_3}\,\vee E\text{-}2}{(P_1 \vee P_2) \wedge (P_1 \vee P_3)}\,\wedge I$$

2(a) $(P_1 \vee P_2),\ \neg P_2 \vdash P_1$:

$$\dfrac{P_1 \vee P_2 \qquad [P_1]^1 \qquad \dfrac{\dfrac{\neg P_2 \qquad [P_2]^1}{\bot}\,\to E}{P_1}\,\bot E}{P_1}\,\vee E\text{-}1$$

2(b) $\varphi \vdash \top$:

$$\dfrac{}{\top}\,\top\text{-}I$$

2 (c): $\vdash ((P_1 \to P_2) \to P_1) \to P_1$

$$
\dfrac{\dfrac{\dfrac{[\neg P_1]^2 \qquad \dfrac{[((P_1 \to P_2) \to P_1]^1 \qquad \dfrac{\dfrac{\dfrac{[\neg P_1]^2 \quad [P_1]^3}{\bot}\,{\to}E}{P_2}\,\bot E}{(P_1 \to P_2)}\,{\to}I\text{-}3}{P_1}\,{\to}E}{\bot}}{P_1}\,RAA\text{-}2}{((P_1 \to P_2) \to P_1) \to P_1}\,{\to}I\text{-}1
$$

3. (a) $\vee$-introduction 1 is $\dfrac{\varphi_1}{\varphi_1 \vee \varphi_2}$.

So: need to show: if $\mathcal{D} = \dfrac{\begin{matrix}\mathcal{D}_1 \\ \vdots\end{matrix}}{\varphi_1 \vee \varphi_2}$, where $\mathcal{D}_1$ ends in φ_1, and if soundness holds for $\mathcal{D}_1$, then soundness holds for $\mathcal{D}$.

The undischarged assumptions of $\mathcal{D}$ are precisely the same as those of $\mathcal{D}_1$, since $\vee$-I does not discharge any assumptions. So if these all hold, in some interpretation A, then by soundness for $\mathcal{D}_1$, φ_1 holds

(3(a) cont'd) in A, and so $[\![\varphi_1 \vee \varphi_2]\!]^A = 1 \vee [\![\varphi_2]\!]^A = 1$, so $\varphi_1 \vee \varphi_2$ holds in A, as desired.

3(b) $\perp$-elimination: $\dfrac{\perp}{\varphi}$. So: if $\mathcal{D} = \dfrac{\mathcal{D}_1}{\varphi}$, where $\mathcal{D}_1$ ends in $\perp$, and if soundness holds for $\mathcal{D}_1$, NTS soundness for $\mathcal{D}$.

Again, this rule discharges no assumptions; so their undischarged assumptions are the same.

By soundness for $\mathcal{D}_1$, if these assumptions ~~all~~ hold in some interp'n A, then so does $\perp$. But $[\![\perp]\!] = 0$ in any model so this is impossible! So there is no such model; so, vacuously, in any such model, φ holds, as required.

4(a). Suppose $\Gamma \models \psi_1 \wedge \psi_2$. Then in any model A of Γ, $[\![\psi_1 \wedge \psi_2]\!]^A = 1$, so $[\![\psi_1]\!]^A = [\![\psi_2]\!]^A = 1$. So $\Gamma \models \psi_1$, and $\Gamma \models \psi_2$.

4(b). This claim does not hold; e.g. take $\varphi_1 = P_1$, $\varphi_2 = P_2$, $\Gamma = \{ P_1 \vee P_2 \}$. Then trivially $\Gamma \models P_1 \vee P_2$; but it is not the case that either $\Gamma \models P_1$ or $\Gamma \models P_2$.

(A countermodel to ~~$P_1 \vee P_2$~~ $P_1 \vee P_2 \models P_1$ is given by $A(P_1)=0$, $A(P_2)=1$; conversely, $A(P_1)=1$, $A(P_2)=0$ shows that $P_1 \vee P_2 \not\models P_2$.)