Homework 3 of 3

Logic, Stockholm University, Autumn 2014

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http://kurser.math.su.se/course/view.php?id=186

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Due Thursday 23 October, in class (or by email before class). Problems are marked with the per milles they count for on the final grade. This homework contains 4 problems, and is worth 35 ‰ of the final grade.

Make sure to state clearly whenever you are using the Soundness or Completeness theorems.

N.B. There will only be 3 homeworks, not 4 as originally planned.

1. (10 ‰) Give natural deduction proofs showing each of the following (assuming appropriate arity types):

   (a) $\vdash\ \forall x_0\, \exists x_1\, (x_1 \doteq x_0)$

   (b) $\vdash_{x_0,x_1}\ \exists x_2\ (f_1(x_0, x_1) \doteq x_2)$

   (c) $\vdash\ \forall x_0, x_1\ ((x_o \doteq x_1) \to (f_1(x_0) \doteq f_1(x_1)))$

   (d) $\exists x_0\, \forall x_1\ (x_1 \doteq x_0)\ \vdash\ \forall x_0\, \exists x_1\ (f(x_1) = x_0)$

2. (8 ‰) Which of the following entailments hold? (Justify your answers with proofs, countermodels, etc. as appropriate.)

   (a) For all φ, ψ in $\mathrm{Form}(S \cup \{i\})$, $\quad \exists x_i\, (\varphi \vee \psi) \models_S (\exists x_i\, \varphi) \vee (\exists x_i\, \psi)$.

   (b) For all φ, ψ in $\mathrm{Form}(S \cup \{i\})$, $\quad (\exists x_i\, \varphi) \vee (\exists x_i\, \psi) \models_S \exists x_i\, (\varphi \vee \psi)$.

   (c) For all φ, ψ in $\mathrm{Form}(S \cup \{i\})$, $\quad \exists x_i\, (\varphi \wedge \psi) \models_S (\exists x_i\, \varphi) \wedge (\exists x_i\, \psi)$.

   (d) For all φ, ψ in $\mathrm{Form}(S \cup \{i\})$, $\quad (\exists x_i\, \varphi) \wedge (\exists x_i\, \psi) \models_S \exists x_i\, (\varphi \wedge \psi)$.

3. (7 ‰) Show that the following is provable:

$$\vdash \big(\forall x_0, x_1, x_2\ (x_0 \doteq x_1 \lor x_1 \doteq x_2 \lor x_0 \doteq x_2)\big) \\ \lor \big(\forall x_0, x_1\ \exists x_2\ (\neg(x_2 \doteq x_0) \land \neg(x_2 \doteq x_1))\big)$$

4. (10 ‰) Define closed formulas $\varphi_{\text{inj}}, \varphi_{\text{surj}}, \varphi_{\text{invol}} \in \text{Form}(\emptyset)$ as follows:

$$\begin{aligned} \varphi_{\text{inj}} &:= \forall x_0, x_1\ \ (f_1(x_0) \doteq f_1(x_1) \to x_0 \doteq x_1) \\ \varphi_{\text{surj}} &:= \forall x_0\ \exists x_1\ \ (f_1(x_1) \doteq x_0) \\ \varphi_{\text{invol}} &:= \forall x_0\ \ (f_1(f_1(x_0)) \doteq x_0) \end{aligned}$$

Which of the following theories are consistent? (Justify your answers appropriately.)

(a) $\{\varphi_{\text{inj}}, \neg\varphi_{\text{surj}}\}$

(b) $\{\neg\varphi_{\text{inj}}, \varphi_{\text{surj}}\}$

(c) $\{\varphi_{\text{invol}}, \neg\varphi_{\text{surj}}\}$

(d) $\{\varphi_{\text{invol}}, \varphi_{\text{inj}}\}$