Predicate Logic [DRAFT]

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October 22, 2014

Note on adaptation

These notes are adapted (with permission) from Chapters 9–13 of Jesper Carlström, *Logik*, 2008–2013.

The aim of the adaptation is to give a slightly more modern treatment of predicate logic and its models. The changes to the text are, essentially, the minimum necessary to accomodate this different presentation. The overall organisation is unchanged.

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Contents

Chapter 10

Semantics

In this chapter we will, to a large extent, repeat what we have already done for propositional logic. However, we need to make some modifications to adjust to the more advanced situation we now have.

10.1 Interpretation of terms and formulas

To define an interpretation $\mathcal{A}$ it is not sufficient to choose propositions as interpretations for $P_1, P_2, \ldots$, since these symbols are not necessarily nullary anymore; they could now take arguments. They will instead be interpreted as relations. For instance, P_1 can be interpreted as $\leq$ if it takes two arguments. This relation is viewed as the subset $\{(x, y) \in \mathbb{N}^2 \mid x \leq y\}$.

Precisely, an interpretation or structure $\mathcal{A}$ (for a given arity type) consists of the following:

- A *set* $|\mathcal{A}|$, which is called the *domain* (of individuals); we think about it as the set of the elements about which the language speaks.

Notice that $|\mathcal{A}|$ could be empty!

- For every relation symbol P_j, an r_j-ary relation $P_j^{\mathcal{A}}$ on $|\mathcal{A}|$; that is, a subset of $|\mathcal{A}|^{r_j}$. This means that $\langle b_1, \ldots, b_{r_j} \rangle \in P_j^{\mathcal{A}}$ (or $P_j^{\mathcal{A}}(b_1, \ldots, b_{r_j})$) is a *proposition*, which is true or false for every choice of $b_1, \ldots, b_{r_j} \in |\mathcal{A}|$.
- For every function symbol f_j, an a_j-ary function $f_j^{\mathcal{A}}$ on $|\mathcal{A}|$. This means that $f_j^{\mathcal{A}}(b_1, \ldots, b_{a_j})$ is an *element* of $|\mathcal{A}|$ for every choice of $b_1, \ldots, b_{a_j} \in |\mathcal{A}|$.

10.1.1 Exercise. What special cases do we get when we interpret a nullary relation symbol or function symbol?

10.1.2 Example. Assume that we have a language of arity type $\langle ; 0, 1, 2, 2 \rangle$ and we would like to interpret it involving natural numbers. We take

$$|\mathcal{A}| \stackrel{\text{def}}{=} \mathbb{N} \tag{10.1.3}$$
$$f_1^{\mathcal{A}} \stackrel{\text{def}}{=} 0 \tag{10.1.4}$$
$$f_2^{\mathcal{A}} \stackrel{\text{def}}{=} s \tag{10.1.5}$$
$$f_3^{\mathcal{A}} \stackrel{\text{def}}{=} + \tag{10.1.6}$$
$$f_4^{\mathcal{A}} \stackrel{\text{def}}{=} \cdot. \tag{10.1.7}$$

The above definition of an interpretation is formulated more simply by saying that we interpret in the structure:

$$\langle \mathbb{N}; ; 0, s, +, \cdot \rangle . \tag{10.1.8}$$

A structure is therefore nothing more than a set together with relations and functions. The advantage of the notation (10.1.8) is that one can define the

Notice that in structure, ther more than in t arity type. To first one writes the domain. T corresponding types. To the r writes the rela in the interpre in the last spac the functions.

whole interpretation in one row. The ordering in (10.1.8) is relevant to be able to know what is the interpretation of symbols.

As before, we want now to define an interpretation of the syntax in any structure $\mathcal{A}$.

Consider how a term – for instance, $f_3(x_4, x_{17})$ – should be interpreted. If we are given values for the variables, then we can apply $f_3^{\mathcal{A}}$ to these values to get an element of $\mathcal{A}$. For instance, in $\langle \mathbb{N};;0,s,+,\cdot\rangle$, if we are given the values $x_4 := 5$, $x_{17} = 1$, then $f_3(x_4, x_{17})$ should be given the value $5+1=6$.

Similarly, a formula – for instance, $x_4 \doteq x_{17}$ – can be interpreted as a truth value in Boole, but only once values are given to the variables. For instace, in $\langle \mathbb{N};;0,s,+,\cdot\rangle$, the formula $x_4 \doteq x_{17}$ will be false with the values $x_4 := 5$, $x_{17} := 1$; but for the values $x_4 := 6$, $x_{17} = 6$, the same formula would be true.

We therefore need to consider assignments of values to variables. We formalise these as functions. The assignment $x_4 := 5$, $x_{17} := 1$, for example, is given as the function

$$\begin{aligned}\{4,17\} &\to \mathbb{N}\\ 4 &\mapsto 5\\ 17 &\mapsto 1\end{aligned}$$

Generally, for any finite set $S \subseteq \mathbb{N}$, and any structure $\mathcal{A}$, an assigment of values in $\mathcal{A}$ to the variables indexed by S is represented by a function $f : S \to |\mathcal{A}|$. The value of x_i is given by $f(i)$.

Be careful to note the slightly counter-intuitive direction: X^Y is the set of funtions $Y \to X$!

We write $|\mathcal{A}|^S$ for the set of all such functions. The reason for this notation is that if S is has n elements, $\{i_1, \ldots, i_n\}$, then $|\mathcal{A}|^S \simeq |A|^n$. For instance, functions $\{4,17\} \to |\mathcal{A}|$ correspond to pairs of elements $(b_4, b_{17}) \in |\mathcal{A}|^2$. We will often think of such functions as tuples $\vec{b}$ in this way, and write their values as b_i rather than $\vec{b}(i)$.

It would perhaps be more natural to call these (a_4, a_{17}), but we are using $\vec{a}$ already for arity types.

Alternatively, to emphasise the view of them as valuations of variables, we will write them sometimes as e.g. $\langle x_4 := 5, x_{17} := 1\rangle$.

We can now extend an interpretation $\mathcal{A}$ to terms precisely as we did with formulas in propositional logic. Given any term $t \in \mathrm{Term}(S)$, and values $\vec{b} \in |\mathcal{A}|^S$ for its variables, we obtain a value in $[\![t]\!](\vec{b}) \in |\mathcal{A}|$ In other words, the interpretation of t is a function $[\![t]\!]^{\mathcal{A}} : |A|^S \to |\mathcal{A}|$. This is defined recursively, as follows:

10.1.9 Definition.

$$\begin{aligned}[\![x_i]\!](\vec{b}) &\stackrel{\text{def}}{=} b_i\\ [\![f_i(t_1,\ldots,t_{a_i})]\!](\vec{b}) &\stackrel{\text{def}}{=} f_i^{\mathcal{A}}([\![t_1]\!],\ldots,[\![t_{a_i}]\!])\,.\end{aligned}$$

Note that as usual, when the interpretation $\mathcal{A}$ is clear from context, we write just $[\![t]\!]$ rather than $[\![t]\!]^{\mathcal{A}}$.

10.1.10 Example. Interpret the language of arity type $\langle;0,1,2,2\rangle$ in the structure

$$\langle \mathbb{N};;0,s,+,\cdot\rangle \tag{10.1.11}$$

Compute the expression $[\![f_3(f_4(f_2(f_1),x_0),x_1)]\!](\vec{b})$ as far as possible, for $\vec{b} \in \mathbb{N}^{\{0,1\}}$

Solution.

$$\begin{aligned}[\![f_3(f_4(f_2(f_1),x_0),x_1)]\!](\vec{b}) &\stackrel{\text{def}}{=} f_3^{\mathcal{A}}(f_4^{\mathcal{A}}(f_2^{\mathcal{A}}(f_1^{\mathcal{A}}),[\![x_0]\!](\vec{b})),[\![x_1]\!](\vec{b}))\\ &\stackrel{\text{def}}{=} s(0)\cdot b_0 + b_1\\ &= 1\cdot(b_0+b_1) = b_0+b_1.\end{aligned}$$

The answer is thus a function of b_0 and b_1, the values assigned by $\vec{b}$ to x_0 and x_1. We cannot calculate any further if we do not have specific values for x_0 and x_1. □

Next, we will assign interpretations to formulas.

As with terms, a formula $\varphi \in \mathrm{Form}(S)$ will have a defined truth value only once values are chosen for its variables: it may be true for some values of the variables, false for others.

This suggests interpreting a formula as a function $[\![\varphi]\!] : |\mathcal{A}|^S \to \mathrm{Boole}$. It will often be slightly more convenient to view them slightly differently: as subsets of $|\mathcal{A}|^S$.

Generally, the power set $\mathcal{P}(X)$ of a set may always be identified with the function set Boole^X. A subset $R \subseteq X$ corresponds to a function $X \to \mathrm{Boole}$ by taking $R(x)$ to be 1 when $x \in R$, and 0 when $x \notin R$. Conversely, a function $R : X \to \mathrm{Boole}$ determines the subset $\{x \in X \mid R(x) = 1\}$.

We will mostly treat them as subsets; but we could equally well have used functions into Boole, and it is useful to keep both viewpoints in mind.

To define the interpretation of quantifiers, we need one more technical detail. We will have to say things like "the assignment of values $\vec{b}$, but with x_3 given the value 7". We denote $\vec{b}[3 \mapsto 7]$, or $\vec{b}[x_3 := 7]$. Precisely:

▶ **10.1.12 Definition** (updating)**.** Let X be any set, and suppose $S \subseteq \mathbb{N}$, and $\vec{b} \in X^S$, and $a \in X$. Define $\vec{b}[i \mapsto a] \in X^{S \cup \{i\}}$ by:

$$\vec{b}[i \mapsto a](j) \stackrel{\text{def}}{=} \begin{cases} a & \text{if } j = i \\ b_j & \text{otherwise} \end{cases}$$

This will either extend $\vec{b}$ with a new value, if i was not in S, or change the value of b_i if i was already in S. Generally we refer to $\vec{b}[i \mapsto a]$ as *updating* $\vec{b}$.

We do not have to know anything about v to compute this.

10.1.13 Example. If $a \in |\mathcal{A}|$, and $\vec{b} \in |\mathcal{A}|^S$ then

$$[\![x_0]\!]^{\mathcal{A}}(\vec{b}[0 \mapsto a]) = \vec{b}[0 \mapsto a](0) = a.$$

10.1.14 Exercise. Simplify

a) $\vec{b}[i \mapsto a][i \mapsto b]$

b) $\vec{b}[i \mapsto [\![i]\!]^{\mathcal{A}}(\vec{b})]$

c) $\vec{b}[i \mapsto [\![i]\!]^{\mathcal{A}}(\vec{b}[i \mapsto a])]$

10.1.15 Exercise. Show that if $i \neq j$, then

$$\vec{b}[i \mapsto a][j \mapsto b] = \vec{b}[j \mapsto b][i \mapsto a],$$

but that this is not necessarily true if $i = j$. Show that in that case, both sides of the equation can be simplified.

We can now recursively define the interpretations of formulas.

▶ **10.1.16 Definition.** Let $\mathcal{A}$ be an interpretation. For each formula $\varphi \in \mathrm{Form}(S)$, we recursively define $[\![S \mid \varphi]\!]^{\mathcal{A}} \subseteq |A|^S$ as follows.

$$\begin{aligned}
[\![S \mid t_1 \doteq t_2]\!] &\stackrel{\text{def}}{=} \{\vec{b} \in |\mathcal{A}|^S \mid [\![t_1]\!](\vec{b}) = [\![t_2]\!](\vec{b})\} \\
[\![S \mid P_i(t_1, \ldots, t_{r_i})]\!] &\stackrel{\text{def}}{=} \{\vec{b} \in |\mathcal{A}|^S \mid ([\![t_1]\!](\vec{b}), \ldots, [\![t_{r_i}]\!](\vec{b})) \in P_i^{\mathcal{A}}\} \\
[\![S \mid \top]\!] &\stackrel{\text{def}}{=} |\mathcal{A}|^S \\
[\![S \mid \bot]\!] &\stackrel{\text{def}}{=} \emptyset \\
[\![S \mid \varphi \wedge \psi]\!] &\stackrel{\text{def}}{=} [\![S \mid \varphi]\!] \cap [\![S \mid \psi]\!] \\
[\![S \mid \varphi \wedge \psi]\!] &\stackrel{\text{def}}{=} [\![S \mid \varphi]\!] \cup [\![S \mid \psi]\!] \\
[\![S \mid \varphi \to \psi]\!] &\stackrel{\text{def}}{=} \{\vec{b} \in |\mathcal{A}|^S \mid \vec{b} \notin [\![S \mid \varphi]\!] \vee \vec{b} \in [\![S \mid \psi]\!]\} \\
[\![S \mid \forall x_i \varphi]\!] &\stackrel{\text{def}}{=} \{\vec{b} \in |\mathcal{A}|^S \mid \vec{b}[i \mapsto a] \in [\![S \cup \{i\} \mid \varphi]\!] \text{ for all } a \in |A|\} \\
[\![S \mid \exists x_i \varphi]\!] &\stackrel{\text{def}}{=} \{\vec{b} \in |\mathcal{A}|^S \mid \vec{b}[i \mapsto a] \in [\![S \cup \{i\} \mid \varphi]\!] \text{ for some } a \in |A|\}
\end{aligned}$$

Chapter 12

Natural deductions

12.1 New rules

Natural deduction in predicate logic is almost precisely the same as for propositional logic, but with even more rules. These are collected in Figure 12.1 (page 6). The rule "refl" is called *reflexivity* and the rule "subst" is called the *substitution rule*.

Besides these, the one other change is that we need to keep track of variables in formulas. Just as we defined a logical consequence relation $\psi_1, \ldots, \psi_n \vDash_S \varphi$ for each set of variables S, similarly we need to have deductions of formulas with free variables, not just closed formulas. So at each step of the derivation, instead of just a formula, we now have a set of variables S besides a formula $\varphi \in \text{Form}(S)$. We write this as $(S)\ \varphi$, or $(\vec{x})\ \varphi$.

Just as rules discharged *hypotheses* before, now they may also discharge *variables* in undischarged hypotheses. If i is discharged in a hypothesis $(S \cup \{i\})\ \varphi$, this means that after discharging it is considered just as $(S)\ \varphi$. To remember which rules discharge variables, note that it happens exactly as it is required in order to keep the variables of all undischarged assumptions $(S)\ \psi$ the same as the variables of the conclusion.

Note the various restrictions appearing in the rules. To be able to use some of them, it is required that some terms are free for some variables, while for some other rules it is required that variables do not occur freely in certain formulas. These restriction are important – disregarding them can lead to deriving false formulas. To remember these restrictions, notice how they go together with variable discharging. If a rule discharges (or could discharge) i in assumptions, then x_i must not occur free in these assumptions, since otherwise one could end up with assumptions $(S)\ \varphi$, with $\varphi \notin \text{Form}(S)$.

The principles for derivations are otherwise the same as in propositional natural deduction. This chapter, therefore, does not contain any theory; only examples and exercises. We just have to modify some definitions.

▶ **12.1.1 Definition.** By $\varphi_1, \ldots, \varphi_n \vdash_S \varphi$ we mean that there is a derivation of $(S)\ \varphi$, with only the rules of figure 5.1 and 12.1 and without any undischarged assumptions, except possibly $(S)\ \varphi_1, \ldots, (S)\ \varphi_n$. (Compare Definition 5.5.1.) When S is empty, we write just $\vdash$ instead of $\vdash_\emptyset$. Again, we will oftern write $\vdash_{\vec{x}}$ rather than $\vdash_S$.

12.1.2 Example. Show that $\vdash_{x_0} (x_0 \doteq x_0)$.

Solution. Since x_0 is a term, we can use the rule for reflexivity.

$$\frac{}{(x_0)\ x_0 \doteq x_0}\ \text{refl}$$

□

12.1.3 Example (Symmetry)**.** Show that $t \doteq s \vdash_S s \doteq t$, for any $s, t \in \text{Term}(S)$.

φ, σ denote formulas in appropriate variables
t, s denote terms in appropriate variables
x_i denotes an arbitrary variable

In any substitution $t[s/x_i]$, it is assumed that s is free for x_i in t.

$$\frac{(S)\ \varphi \quad (S)\ \psi}{(S)\ \varphi \wedge \psi}\ \wedge I$$

and all other propositional rules with variable annotations added similarly;

$$\frac{}{(S)\ t \doteq t}\ \text{refl} \qquad \frac{(S)\ \varphi[t/x_i] \qquad (S)\ t \doteq s}{(S)\ \varphi[s/x_i]}\ \text{subst}$$

$$\frac{\begin{matrix}[i] \\ \vdots \\ (S \cup \{i\})\ \varphi\end{matrix}}{(S)\ \forall x_i \varphi}\ \forall I \qquad \frac{(S)\ \forall x_i \varphi}{(S)\ \varphi[t/x_i]}\ \forall E$$

x_i must not occur freely in any undischarged assumption. If $i \notin S$, then this rule discharges it in those assumptions.

$$\frac{(S)\ \varphi[t/x_i]}{(S)\ \exists x_i \varphi}\ \exists I \qquad \frac{(S)\ \exists x_i \varphi \qquad \begin{matrix}[i] \quad [(S \cup \{i\})\ \varphi] \\ \vdots \\ (S \cup \{i\})\ \sigma\end{matrix}}{(S)\ \sigma}\ \exists E$$

x_i must not occur freely in σ, or in any undischarged assumption in the right subtree, except possibly φ; if $i \notin S$, this rule discharges it in all such assumptions.

Figure 12.1: Additional rules for natural deduction in predicate logic

Solution. If we let $\varphi = (x_i \doteq t)$, where we choose i so that x_i does not occur in t, we get

$$\varphi[t/x_i] = (x_i[t/x_i] \doteq t[t/x_i]) = (t \doteq t)\,, \tag{12.1.4}$$
$$\varphi[s/x_i] = (x_i[s/x_i] \doteq t[s/x_i]) = (s \doteq t)\,, \tag{12.1.5}$$

and we can use the rule for substitution.

$$\dfrac{\dfrac{}{(S)\ t \doteq t}\,\text{refl} \qquad (S)\ t \doteq s}{(S)\ s \doteq t}\,\text{subst}$$

□

Note that the formula φ does not itself occur in the derivation. In fact, looking just at the derivation, we cannot see which variable x_i has been chosen, and therefore we cannot tell which formula φ was in the substitution rule. However, we can always decide whether an application of the substitution rule is correct by noting how the formulas above and under the line differ. With that information, one can see whether there exists a formula φ which can be used in the rule, but the choice may not be unique.

12.1.6 Exercise (transitivity)**.** Show that $u \doteq t, t \doteq s \vdash_S u \doteq s$.

> *Hint.* Find the formula φ which could be used together with the substitution rule.

12.1.7 Example. Construct a derivation of $\forall x_0 \forall x_1 (x_0 \doteq x_1 \to x_1 \doteq x_0)$.

Solution.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(x_0,x_1)\ x_0 \doteq x_0}\,\text{refl} \qquad [(x_0,x_1)\ x_0 \doteq x_1]^1}{(x_0,x_1)\ x_1 \doteq x_0}\,\text{subst}}{(x_0,x_1)\ x_0 \doteq x_1 \to x_1 \doteq x_0}\,{\to}I_1}{(x_0)\ \forall x_1 (x_0 \doteq x_1 \to x_1 \doteq x_0)}\,\forall I}{(\emptyset)\ \forall x_0 \forall x_1 (x_0 \doteq x_1 \to x_1 \doteq x_0)}\,\forall I$$

Here, before the uses of $\forall I$, we have an implication introduction which discharges the assumption $(x_0, x_1)\ x_0 \doteq x_1$. This is why the $\forall$-introductions are allowed. □

Do you see that both uses of $\forall I$ would be forbidden if the assumption $x_0 \doteq x_1$ was undischarged? Otherwise, check the rules in Figure 12.1.

12.1.8 Exercise. Construct a derivation of

$$\forall x_0 \forall x_1 \forall x_2 ((x_0 \doteq x_1) \wedge (x_1 \doteq x_2) \to (x_0 \doteq x_2))\,.$$

A special case of $\forall E$ is

$$\dfrac{(S)\ \forall x_i \varphi}{(S)\ \varphi}\,\forall E \tag{12.1.9}$$

which one gets by taking $t = x_i$, since $\varphi[x_i/x_i] = \varphi$ (Exercise 9.2.21). It occurs, for example, in the solution of the following problem.

12.1.10 Example. Show that $\forall x_i \neg\varphi \vdash_S \neg\exists x_i \varphi$ (for any $\varphi \in \mathrm{Form}(S)$).

Solution.

$$\dfrac{\dfrac{[(S)\ \exists x_i \varphi]^2 \qquad \dfrac{\dfrac{(S)\ \forall x_i \neg\varphi}{(S)\ \neg\varphi}\,\forall E \qquad [(S)\ \varphi]^1}{(S)\ \bot}\,{\to}E}{(S)\ \bot}\,\exists E_1}{(S)\ \neg\exists x_i \varphi}\,{\to}I_2$$

We must check that the variable restrictions are satisfied. The only rule we use which has such restrictions is $\exists E$. It requires, in the above case, that x_i does not occur freely in $\bot$ (which is not the case), and that x_i does not occur freely in some undischarged assumption in the derivation of $\bot$, except possibly in φ. In our case, $\forall x_i \neg\varphi$ is the only undischarged assumption, except from φ, and x_i does not occur freely in it. (It occurs bound, but that is not a problem.)

In what follows, the fact that the variable restrictions need to be satisfied will not be explicitly checked, but these checkings must always be done before one can assert that the derivation is correct. □

On the exam you do not need to justify why the variable restrictions are satisfied if you are not explicitly instructed to do so. Otherwise, a derivation is considered to be *wrong* if the restrictions are not satisfied.

In the next few exercises, assume that $\varphi, \psi \in \mathrm{Form}(S \cup \{i\})$.

12.1.11 Example. Show that $\neg\exists x_i\varphi \vdash_S \forall x_i\neg\varphi$.

Solution.

$$\dfrac{\dfrac{\dfrac{(S\cup\{[i]\})\ \neg\exists x_i\varphi \qquad \dfrac{[(S\cup\{i\})\ \varphi]^1}{(S\cup\{i\})\ \exists x_i\varphi}\,\exists I}{(S\cup\{i\})\ \bot}\,{\to}E}{(S\cup\{i\})\ \neg\varphi}\,{\to}I_1}{(S)\ \forall x_i\neg\varphi}\,\forall I$$

□

12.1.12 Example. Show that $\exists x_i\neg\varphi \vdash_S \neg\forall x_i\varphi$.

Solution.

$$\dfrac{\dfrac{(S)\ \exists x_i\neg\varphi \qquad \dfrac{[(S\cup\{i\})\ \neg\varphi]^1 \qquad \dfrac{[(S\cup\{[i]^1\})\ \forall x_i\varphi]^2}{(S\cup\{i\})\ \varphi}\,\forall E}{(S\cup\{i\})\ \bot}\,{\to}E}{(S)\ \bot}\,\exists E_1}{(S)\ \neg\forall x_i\varphi}\,{\to}I_2$$

□

12.1.13 Example. Show that $\neg\forall x_i\varphi \vdash_S \exists x_i\neg\varphi$.

Solution. Here we must use RAA twice.

$$\dfrac{\dfrac{(S)\ \neg\forall x_i\varphi \qquad \dfrac{\dfrac{\dfrac{[(S\cup\{[i]^2\})\ \neg\exists x_i\neg\varphi]^3 \qquad \dfrac{[(S\cup\{i\})\ \neg\varphi]^1}{(S\cup\{i\})\ \exists x_i\neg\varphi}\,\exists I}{(S\cup\{i\})\ \bot}\,{\to}E}{(S\cup\{i\})\ \varphi}\,\mathrm{RAA}_1}{(S)\ \forall x_i\varphi}\,\forall I_2}{(S)\ \bot}\,{\to}E}{(S)\ \exists x_i\neg\varphi}\,\mathrm{RAA}_3$$

□

12.1.14 Example. Show that $\vdash_S \forall x_i(\varphi\wedge\psi) \leftrightarrow \forall x_i\varphi \wedge \forall x_i\psi$.

Solution.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[(S)\ \forall x_i(\varphi\wedge\psi)]^1}{(S)\ \varphi\wedge\psi}\,\forall E}{(S)\ \varphi}\,\wedge E}{(S)\ \forall x_i\varphi}\,\forall I \qquad \dfrac{\dfrac{\dfrac{[(S)\ \forall x_i(\varphi\wedge\psi)]^1}{(S)\ \varphi\wedge\psi}\,\forall E}{(S)\ \psi}\,\wedge E}{(S)\ \forall x_i\psi}\,\forall I}{(S)\ \forall x_i\varphi\wedge\forall x_i\psi}\,\wedge I}{(S)\ \forall x_i(\varphi\wedge\psi)\to\forall x_i\varphi\wedge\forall x_i\psi}\,{\to}I_1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[(S)\ \forall x_i\varphi\wedge\forall x_i\psi]^2}{(S)\ \forall x_i\varphi}\,\wedge E}{(S)\ \varphi}\,\forall E \qquad \dfrac{\dfrac{[(S)\ \forall x_i\varphi\wedge\forall x_i\psi]^2}{(S)\ \forall x_i\psi}\,\wedge E}{(S)\ \psi}\,\forall E}{(S)\ \varphi\wedge\psi}\,\wedge I}{(S)\ \forall x_i(\varphi\wedge\psi)}\,\forall I}{(S)\ \forall x_i\varphi\wedge\forall x_i\psi\to\forall x_i(\varphi\wedge\psi)}\,{\to}I_2}{(S)\ \forall x_i(\varphi\wedge\psi)\leftrightarrow\forall x_i\varphi\wedge\forall x_i\psi}\,\wedge I$$

□

The examples 12.1.14 and 12.1.15 show how the rules for $\forall$ and $\exists$ are used in more complicated cases. Note that, in Example 12.1.15, it is important to use $\exists E$ sufficiently far down in the derivation so that the variable restrictions are satisfied.

12.1.15 Example. Show that $\vdash_S \exists x_i(\varphi\vee\psi) \leftrightarrow \exists x_i\varphi \vee \exists x_i\psi$.

You can try to construct a derivation by yourself.

Solution. See Figure 12.2 (page 9). □

12.1.16 Example. a) What is wrong with the following derivation?

$$\dfrac{\dfrac{[(x_1)\ \exists x_0(x_0\doteq x_1)]^1 \qquad \dfrac{[(x_0,x_1)\ x_0\doteq x_1]^2}{(x_0,x_1)\ \exists x_1(x_1\doteq x_1)}\,\exists I}{(x_1)\ \exists x_1(x_1\doteq x_1)}\,\exists E_2}{(x_1)\ \exists x_0(x_0\doteq x_1)\to\exists x_1(x_1\doteq x_1)\,.}\,{\to}I_1$$

b) Can one derive $\exists x_0(x_0\doteq x_1)\to\exists x_1(x_1\doteq x_1)$?

$$
\dfrac{
  \dfrac{
    [(S)\ \exists x_i(\varphi\vee\psi)]^3
    \qquad
    \dfrac{
      [(S\cup\{i\})\ \varphi\vee\psi]^2
      \quad
      \dfrac{\dfrac{[(S\cup\{i\})\ \varphi]^1}{(S\cup\{i\})\ \exists x_i\varphi}\,\exists I}{(S\cup\{i\})\ \exists x_i\varphi\vee\exists x_i\psi}\,\vee I
      \quad
      \dfrac{\dfrac{[(S\cup\{i\})\ \psi]^1}{(S\cup\{i\})\ \exists x_i\psi}\,\exists I}{(S\cup\{i\})\ \exists x_i\varphi\vee\exists x_i\psi}\,\vee I
    }{(S\cup\{i\})\ \exists x_i\varphi\vee\exists x_i\psi}\,\vee E_1
  }{
    \dfrac{(S)\ \exists x_i\varphi\vee\exists x_i\psi}{(S)\ \exists x_i(\varphi\vee\psi)\to\exists x_i\varphi\vee\exists x_i\psi}\,{\to}I_3
  }\,\exists E_2
  \qquad
  \dfrac{
    [(S)\ \exists x_i\varphi\vee\exists x_i\psi]^7
    \quad
    \dfrac{[(S)\ \exists x_i\varphi]^6 \quad \dfrac{\dfrac{[(S\cup\{i\})\ \varphi]^4}{(S\cup\{i\})\ \varphi\vee\psi}\,\vee I}{(S\cup\{i\})\ \exists x_i(\varphi\vee\psi)}\,\exists I}{(S)\ \exists x_i(\varphi\vee\psi)}\,\exists E_4
    \quad \cdots
  }{
    \dfrac{(S)\ \exists x_i(\varphi\vee\psi)}{(S)\ \exists x_i\varphi\vee\exists x_i\psi\to\exists x_i(\varphi\vee\psi)}\,{\to}I_7
  }
}{(S)\ \exists x_i(\varphi\vee\psi)\leftrightarrow\exists x_i\varphi\vee\exists x_i\psi}\,\wedge I
$$

Figure 12.2: Solution of Example 12.1.15.

Solution. a) By $\exists I$ the formula under the line is $\exists x_1\varphi$, where $\varphi = (x_1 \doteq x_1)$. The formula above the line should be of the form $\varphi[t/x_1]$, that is, $t \doteq t$ for some term t. But since x_0 and x_1 are different variables, this is not correct.

b) Sure, for instance:

$$\dfrac{\dfrac{\overline{(x_1)\ x_1 \doteq x_1}\ \text{refl}}{(x_1)\ \exists x_1(x_1 \doteq x_1)}\ \exists I}{(x_1)\ \exists x_0(x_0 \doteq x_1) \to \exists x_1(x_1 \doteq x_1)\,.}\ {\to}I$$

Remember that one does not *have to* discharge anything when introducing implication. □

12.2 Misc. exercises

Unless otherwise stated, assume that all formulas' free variables are as required for the stated relations to make sense; e.g. all overall free variables are among the ambient variables S.

When $S = \emptyset$, the empty structure gives a counterexample to Ex. 12.2.1, showing (once we have soundness) that the condition "S is non-empty" is necessary here (and in the next few exercises).

12.2.1 Exercise. Show that $\neg\exists x_i\varphi \vdash_S \neg\forall x_i\varphi$ for any $\varphi \in \mathrm{Form}(S \cup \{i\})$, provided S is non-empty.

12.2.2 Exercise. Show that $\vdash_S (\forall x_i\varphi) \leftrightarrow \varphi$, provided x_i does not occur freely in φ and S is non-empty.

12.2.3 Exercise. Show that $\vdash_S \exists x_i\varphi \leftrightarrow \varphi$, if x_i does not occur freely in φ and S is non-empty.

12.2.4 Exercise. Show that $\vdash_S \forall x_i(\varphi \vee \psi) \leftrightarrow (\forall x_i\varphi) \vee \psi$, if x_i does not occur freely in ψ. Notice in which part of the derivation this assumption is used.

12.2.5 Exercise. Show that $\vdash_S \exists x_i(\varphi \wedge \psi) \leftrightarrow (\exists x_i\varphi) \wedge \psi$, if x_i does not occur freely in ψ. Notice in which part of the derivation this assumption is used.

12.2.6 Exercise (from the exam on 2005-08-23)**.**

a) Construct a derivation of $\vdash_S ((\exists x_i\varphi) \to \psi) \to ((\forall x_i\varphi) \to \psi)$, for non-empty S.

b) Construct a derivation of $\vdash_S ((\forall x_i\varphi) \to \psi) \to ((\exists x_i\varphi) \to \psi)$, assuming that x_i does not occur freely in φ.

c) An attempt to derive $(\forall x_i\varphi \to \psi) \to (\exists x_i\varphi \to \psi)$ could be the following tree, but if x_i occurs freely in φ or ψ, the derivation is not correct. Explain what the error is and what is wrong. Point out the precise location of errors!

$$\dfrac{\dfrac{\dfrac{[(S)\ \exists x_i\varphi]^3 \qquad \dfrac{\dfrac{[(S \cup \{[i]\})\ \varphi]^2}{(S \cup \{i\})\ \forall x_i\varphi}\ \forall I_1 \qquad [(S \cup \{[i]^2\})\ \forall x_i\varphi \to \psi]^4}{(S \cup \{i\})\ \psi}\ {\to}E}{(S)\ \psi}\ \exists E_2}{(S)\ \exists x_i\varphi \to \psi}\ {\to}I_3}{(S)\ (\forall x_i\varphi \to \psi) \to (\exists x_i\varphi \to \psi)}\ {\to}I_4$$

d) Show that, if $\psi = (x_i \doteq x_i)$, there is a correct derivation of $((\forall x_i\varphi) \to \psi) \to ((\exists x_i\varphi) \to \psi)$.

12.2.7 Exercise (from the exam on 2003-01-09)**.** Derive $(\exists x_i P_1(x_i) \to \forall x_i P_2(x_i)) \leftrightarrow \forall x_j((\exists x_i P_1(x_i)) \to P_2(x_j))$.

12.2.8 Exercise (from the exam on 2004-08-17)**.** Derive $\forall x_i\varphi \vee \exists x_i\neg\varphi$.

Hint. One has to use RAA several times.

12.2.9 Exercise (from the exam on 2004-08-17)**.** Explain why the following derivation is not correct if x_0 occurs freely in φ (specify precisely which step(s) are wrong, and explain why).

$$\dfrac{\dfrac{\dfrac{[(x_0)\ \forall x_0\exists x_1\varphi]^2}{(x_0)\ \exists x_1\varphi}\,\forall E \qquad \dfrac{\dfrac{[(x_0,x_1)\ \varphi]^1}{(x_0)\ \forall x_0\varphi}\,\forall I}{(x_0)\ \exists x_1\forall x_0\varphi}\,\exists I}{(\emptyset)\ \exists x_1\forall x_0\varphi}\,\exists E_1}{(\emptyset)\ \forall x_0\exists x_1\varphi\to\exists x_1\forall x_0\varphi}\,{\to}I_2$$

12.2.10 Exercise (from the exam on 2005-01-07)**.** Derive $\forall x_i(\neg\varphi\vee\neg\psi)\leftrightarrow\neg\exists x_i(\varphi\wedge\psi)$.

12.2.11 Exercise (from the exam on 2004-10-18)**.**

a) Explain why the following is not a correct derivation if $\psi=(x_0\doteq x_0)$.

$$\dfrac{\dfrac{[(S\cup\{0\})\ \exists x_0(\varphi\wedge\psi)]^2 \qquad \dfrac{\dfrac{\dfrac{[(S\cup\{0\})\ \varphi\wedge\psi]^1}{(S\cup\{0\})\ \varphi}\,\wedge E}{(S\cup\{0\})\ \exists x_0\varphi}\,\exists I \qquad \dfrac{[(S\cup\{0\})\ \varphi\wedge\psi]^1}{(S\cup\{0\})\ \psi}\,\wedge E}{(S\cup\{0\})\ (\exists x_0\varphi)\wedge\psi}\,\wedge I}{(S\cup\{0\})\ (\exists x_0\varphi)\wedge\psi}\,\exists E_1}{(S\cup\{0\})\ \exists x_0(\varphi\wedge\psi)\to(\exists x_0\varphi)\wedge\psi}\,{\to}I_2$$

Specify precisely which step is wrong and explain why!

b) Show that there is a correct derivation of $\vdash_{S\cup\{x_0\}}\exists x_0(\varphi\wedge\psi)\to(\exists x_0\varphi)\wedge\psi$, where $\psi=(x_0\doteq x_0)$.

12.2.12 Exercise (from the exam 2002-08-20)**.** Derive $\vdash_S\exists x_i\varphi\vee\psi\leftrightarrow\exists x_i(\varphi\vee\psi)$, where S is non-empty and x_i does not occur freely in ψ. Specify where these assumptions are used.

12.2.13 Exercise (from the exam on 2002-10-21)**.** Derive $((\exists x_iP_1(x_i))\to(\forall x_iP_2(x_i)))\leftrightarrow\forall x_i(P_1(x_i)\to\forall x_iP_2(x_i))$.

12.2.14 Exercise (from the exam on 2003-08-19)**.** Derive $\vdash_S(\psi\to\exists x_i\varphi)\leftrightarrow\exists x_i(\psi\to\varphi)$, where S is non-empty and x_i does not occur freely in ψ.

12.2.15 Exercise (from the exam on 2003-10-20)**.** Derive $((\exists x_i\varphi)\to\psi)\leftrightarrow\forall x_i(\varphi\to\psi)$, where x_i does not occur freely in ψ. Specify in which part of the derivation this assumption is used.

12.3 Summary

We have extended the formal system with new rules to cover the new ingredients in the language. The rules from propositional logic still hold. The most important thing to remember from here is the ability to construct derivations by using both the old and the new rules. You should also be able to decide if yours or someone elses's derivation is correct, for which you need to know both the rules and the limitations that there are for the variables. For instance, one rule (which?) is only allowed to be used when a certain variable does not occur freely in any undischarged assumption, and another (which?) has a more complicated set of limitations. Remember also that every rule that contains a substitution in its formulation requires that the term is free for the variable in the formula.

Solutions to the exercises

10.1.1 If P_j is nullary, then its interpretation will be a proposition which is either true or false. If f_j is nullary, its interpretation will be an element of the domain (a constant).

10.1.14 a) $\mathcal{A}[i \mapsto a][i \mapsto b] = \mathcal{A}[i \mapsto b]$

b) $\mathcal{A}[i \mapsto [\![i]\!]^{\mathcal{A}}] = \mathcal{A}$

c) $\mathcal{A}[i \mapsto [\![i]\!]^{\mathcal{A}[i\mapsto b]}] = \mathcal{A}[i \mapsto b]]$

10.1.15 Assume that $i \neq j$. It is sufficient to check that

$$\vec{b}[i \mapsto a][j \mapsto b](k) = \vec{b}[j \mapsto b][i \mapsto a](k)$$

holds for all $k \in S \cup \{i, j\}$. There are three different cases: $k = i$, $k = j$, and all other values of k.

If $k = i$, we get $\vec{b}[i \mapsto a][j \mapsto b](k) = \vec{b}[i \mapsto a](i) = a$ and $\vec{b}[j \mapsto b][i \mapsto a](k) = a$. If $k = j$ we get $\vec{b}[i \mapsto a][j \mapsto b](k) = b$ and $\vec{b}[j \mapsto b][i \mapsto a](k) = \vec{b}[j \mapsto b](j) = b$. For every other k we get $\vec{b}[i \mapsto a][j \mapsto b](k) = b_k$, and $\vec{b}[j \mapsto b][i \mapsto a](k) = b_k$.

If, on the other hand, $i = j$, then $\vec{b}[i \mapsto a][j \mapsto b](i) = b$, but $\vec{b}[j \mapsto b][i \mapsto a](i) = a$. The left-hand and right-hand valuations are therefore not equal (unless $a = b$), and in fact can be simplified, according to previous exercise, to $\mathcal{A}[j \mapsto b]$ and $\mathcal{A}[i \mapsto a]$ respectively.

12.1.6 Let $\varphi = (u \doteq x_i)$, where x_i is a variable which does not occur in u. Then $\varphi[t/x_i] = (u \doteq t)$ and $\varphi[s/x_i] = (u \doteq s)$. We can, hence, use the substitution rule.

12.1.8 End the derivation by using $\rightarrow I$ and three $\forall I$. To derive $x_0 \doteq x_2$, the substitution rule is used, and above it $\wedge E$.

12.2.1 End with $\rightarrow I$. To derive $\bot$, we use that from $\forall x_i \varphi$ we can derive φ by $\forall E$, using some variable from S as the term; then we can derive $\exists x_i \varphi$ by $\exists I$. This gives a contradiction.

12.2.2 End with $\wedge I$, and above it use $\rightarrow I$. For $\forall x_i \varphi \vdash \varphi$, only one instance of $\forall E$ is needed. For $\varphi \vdash \forall x_i \varphi$, use $\forall I$, which is possible because x_i does not occur freely in φ.

12.2.3 Conclude as in the previous exercise. For $\exists x_i \varphi \vdash \varphi$, one instance of $\exists E$ is used, which is allowed because x_i does not occur freely in φ. For the other direction we use $\exists I$, relying on non-emptiness of S.

12.2.4 End with $\wedge I$ and thereafter $\rightarrow I$. For $\forall x_i(\varphi \vee \psi) \vdash_S \forall x_i \varphi \vee \psi$ RAA is used as the last step. To get to $\bot$ we use $\rightarrow E$ with $\neg(\forall x_i \varphi \vee \psi)$ as the main premise. The side premise is derived from φ, by $\forall I$ (this step requires that x_i does not occur freely in any undischarged assumption, but this condition is satisfied, since x_i does not occur freely in ψ) followed by $\vee I$. Finally, the formula φ is derived from $\forall x_i(\varphi \vee \psi)$ and $\neg(\forall x_i \varphi \vee \psi)$.

For $\forall x_i \varphi \vee \psi \vdash_S \forall x_i(\varphi \vee \psi)$ one discharges $\forall I$ (which requires that x_i does not occur freely in ψ). Thereafter, an instance of $\vee E$.

12.2.5 For $\exists x_i(\varphi \wedge \psi) \vdash \exists x_i \varphi \wedge \psi$ one discharges by $\exists E$, which is possible since x_i does not occur freely in ψ. The side derivation ends with $\wedge I$.

To derive $\exists x_i(\varphi \wedge \psi)$ from $\exists x_i \varphi \wedge \psi$ one ends with $\exists E$ applied to $\exists x_i \varphi$, which is possible since x_i does not occurs freely in ψ. The formula $\exists x_i \varphi$ is derived, in turn, using $\wedge E$ from $\exists x_i \varphi \wedge \psi$. To derive $\exists x_i(\varphi \wedge \psi)$ from φ and $\exists x_i \varphi \wedge \psi$ one ends with $\exists I$ and above it $\wedge I$. The formula ψ is derived through $\wedge E$ from $\exists x_i \varphi \wedge \psi$.