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Lilavati in the history of mathematics

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Abstract

Lilavati
In
The History of Mathematics.

The main objective of this paper is to provide a review of ***Lilavati***, a work written in the 12^{th} century by Bhaskara II, also known as Bhaskaracharyya. In his work, the author presents mathematical problems in a poetic form and most of these are to be regarded as recreational. Generally, and somewhat surprisingly, little concern is paid to the theoretical background of formulas anywhere in this work, the author instead concentrating on the mechanical application of the methods being described. Nevertheless, there are a number of problems from the epoch in which ***Lilavati*** was composed that may be solved by the application of modern algebra, especially indeterminate equations. In addition to an analysis of the mathematical problems presented in ***Lilavati***, the present paper also provides an outline of the importance of ***Lilavati****,* and other work by Bhaskaracharyya, in the context of a number of significant events in the general history of mathematics.

The second edition of the translation of ***Lilavati*** by Henry Thomas Colebrooke, with notes by Haran Chandra Banerji, comprising 13 chapters and an appendix, preserved in the original Sanskrit, has been used for the purposes of this paper. This text consists of 278 verses and deals with various subjects: tables, the number system, arithmetic operations, fractions, zero, rule of three, compound rule of three, mixture, interest, progressions, plane geometry and the measurement of geometric quantities, stacks, saw, etc.

The perspective adopted in this paper is to focus in particular on the number zero and its function and Bhaskaracharyya's method of squaring a number, extraction of the square root by hand, the cube of a number, the cube root of a number, completing and forming perfect squares and dealing with problems in proportionality, principal and interest on money, permutations and combinations, arithmetical progression, geometrical progression, Pythagoras theorem, an invariant (*lamba*) perpendicular in geometry and pulverizer. Comparisons are drawn with modern mathematical methods and some general conclusions are drawn from these with regard to the contemporary relevance of the work of Bhaskaracharyya.

Contents

1. INTRODUCTION

The present paper provides a review of Lilavati**,** Bhaskara's most famous book, which was subsequently translated by the English astronomer Henry Thomas Colebrooke around 1817. Bhaskara was born in 1114 in Vijayapura, India and died in 1185 in Ujjain, India. He is also known as Bhaskaracharyya and sometimes as Bhaskara II. Apparently, Bhaskara dedicated the book to his beautiful daughter, who was soon to enter into marriage, and the book is named after her. This story, as it appears in a Persian manuscript, goes as follows: *Lilavati was the name of Bhaskaracharya's daughter. From casting her horoscope, he discovered that the auspicious time for her wedding would be a particular hour on a certain day. He placed a cup with a small hole at the bottom of a vessel filled with water, arranged so that the cup would sink at the beginning of the propitious hour. When everything was ready and the cup was placed in the vessel, Lilavati suddenly out of curiosity bent over the vessel and a pearl from her dress fell into the cup and blocked the hole in it. The lucky hour passed without the cup sinking. Bhaskaracharya believed that the only way to console his dejected daughter, who now would never get married, was to write her a manual of mathematics!*

The following equation is one of the most puzzling mathematical operations described in Lilavati:

$\dfrac{0\times(x+\frac{x}{2})\times 3}{0}=63\Rightarrow x=14.$, as we know that division by zero in not defined.

During this medieval epoch, the common mathematics in use in India had been passed down through the well-known Vedas (Vedic Scriptures, Samhita)[1], Ved Vyas (pre-1000 BC), Sulvasutras (800 BC), Apasthmaba (600 BC), the Jaina (500 BC) and the Bakhshali Indian mathematicians. In addition, as described by Victor J. Katz in chapter six of his "A History of Mathematics," Aryabhatta I (476 - 550 CE) and Brahmagupta (598 – 670 CE) both made remarkable contributions to Indian mathematics during this time. Aryabhata I initiated the *kuttaka,* and Brahmagupta adopted zero as a genuine number in his mathematics. It has also been suggested that Pythagoras, the Greek mathematician and philosopher, who lived in the 6th century BC, was familiar with the Hindu Upanishads[2] and learnt his basic geometry from the Sulvasutras.[3] Furthermore, there are indications that the following statement found in Baudhayana's geometrical Sutra suggested what later became universally known as Pythagoras theorem:

The chord which is stretched across the diagonal of a square produces an area of double the size.

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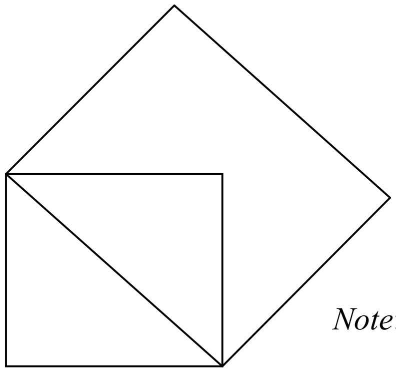
[1] The entire body of sacred writings, chief among which are four books, the Rig-Veda, the Sama-Veda, the Atharva-Veda and the Yajur-Veda.

[2] A class of speculative prose treatise with the principal message: the unity of Brahman and Atman. Brahman is ''the Creator'', the first member of Trimurti, with Vishnu the Preserver and Shiva the Destroyer.
Atman is 'The World Soul,' from which all individual souls derive, and to which they return as the supreme goal of existence.

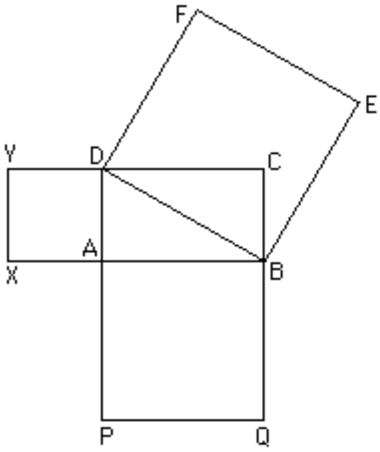
[3] A terse saying embodying a general truth or astute observation. A definition.



Note: This sutra is valid for squares only!

The Katyayana Sulvasutra however, gives a more general version of the sutra:

The rope which is stretched along the length of the diagonal of a rectangle produces an area which the vertical and horizontal sides make together.



The Katyayana form of Pythagoras's theorem

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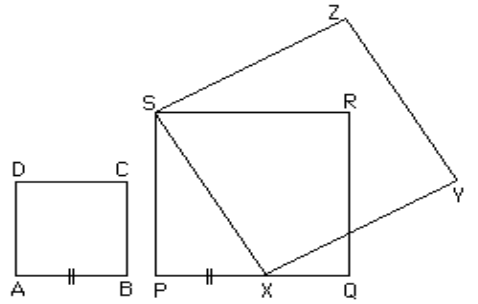
Below is the construction, based on Pythagoras's theorem, for making a square equal in area to two given unequal squares.

ABCD and *PQRS* are the two given squares. Mark a point *X* on *PQ* so that *PX* is equal to *AB*. Then the square on *SX* has area equal to the sum of the areas of the squares *ABCD* and *PQRS*. This follows from Pythagoras's theorem since $SX^2 = PX^2 + PS^2$.

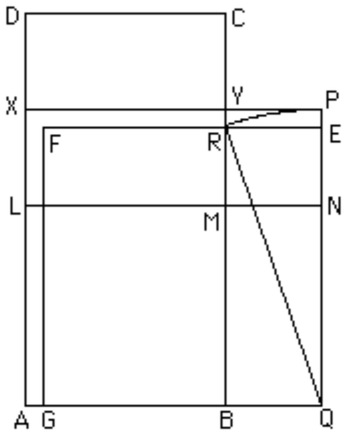


To find a square equal to the sum of two given squares - given in all Sulbasutras

The next construction which we examine is that to find a square equal in area to a given rectangle. Here is the version as it appears in the Baudhayana Sulvasutra.

Consider the diagram on the right!

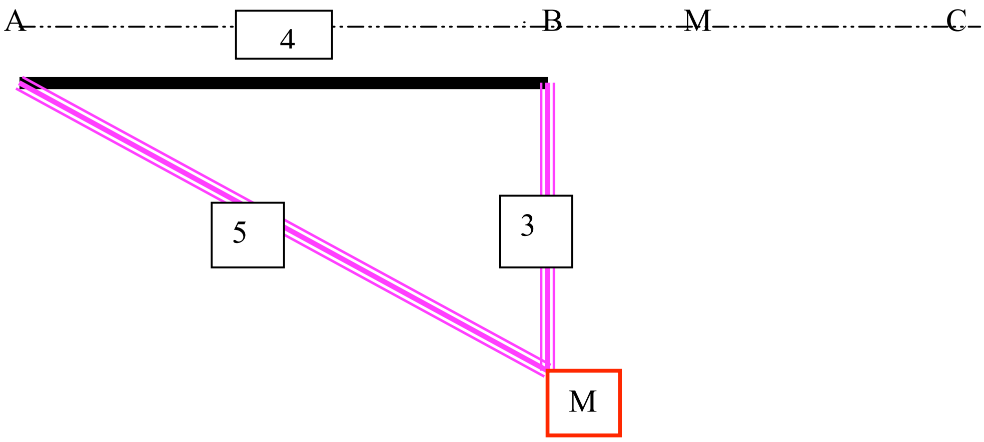
$EQ^2 = QR^2 - RE^2$
$= QP^2 - YP^2$
$= ABYX + BQNM = ABYX + XYCD$
$= ABCD.$



Finding a square equal to a given rectangle

The rectangle *ABCD* is given. Let *L* be marked on *AD* so that *AL* = *AB*. Then complete the square *ABML*. Now bisect *LD* at *X* and divide the rectangle *LMCD* into two equal rectangles with the line *XY*. Now move the rectangle *XYCD* to the position *MBQN*. Complete the square *AQPX*. Now the square we have just constructed is not the one we require and a little more work is needed to complete the work. Rotate *PQ* about *Q* so that it touches *BY* at *R*. Then *QP* = *QR* and we see that this is an ideal "rope" construction. Now draw *RE* parallel to *YP* and complete the square *QEFG*. This is the required square equal to the given rectangle *ABCD*.

In addition, the Pythagorean integer triples [4] 3, 4, 5 and 8, 15, 17 also 12, 35, 37 used in the construction of right angles, are contained in the Sulvasutra; for example, take a stretch **AB** four units long. Then double it so that **BC = AB.** This implies that we have a stretch 8 units long. Now take a white chord of length **AC** (**i.e.** eight units long), nail one end of it to the point **A** and lay it along the stretch **AC.** Mark a point **M** in ink on the chord so that BM=1 unit! If we now take the other end of the chord and nail it to the point **B,** then draw **t**he chord down vertically by pinching it at the point **M**, where we can now see that we have constructed a right-angled triangle of units 3, 4 and 5 by taking **BM** perpendicular to **AB**.



Instructions on how to numerically calculate the diagonal of a square with a side of 1 unit are provided in another stanza of Sulvasutra:

"extend the measure (unit) with a 1/3 and then with a ¼ of the latter and lessen by 1/34 of the last declared quantity ¼×1/3."

Translating this stanza into the decimal system, we get a value for the diagonal which is compatible with our modern calculation of $\sqrt{2}$.

$\sqrt{2} \approx 1+1/3+1/(4\times 3)-(1/34)\times(1/(4\times 3))$. $\Leftrightarrow \sqrt{2} \approx 1.4142156862744$.

$\sqrt{2} \approx 1.4142135623731$, Calculated by TI-92 (Texas Instruments).

These scriptures do not, however, give any reasons as to why these operations should be carried out in this way. These explanations have been left entirely to be the result of our modern educated reasoning. It is also known that during this medieval epoch, attempts were made to divide a segment into seven equal parts, to find a solution

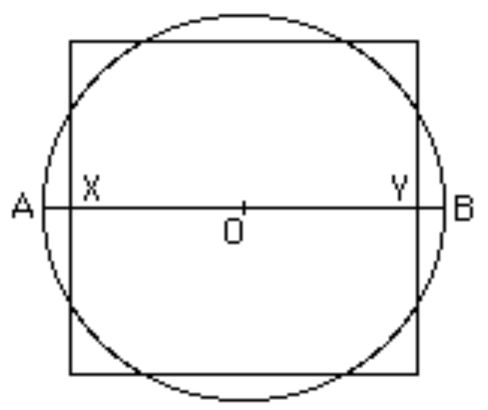
[4] These triples are derived from the formulae:
$a = m^2 - n^2,\ b = 2mn\ \ and\ c = m^2 + n^2, m > n;\ \ m > 0, n > 0, \quad a^2 + b^2 = c^2$. As we can see the Pythagorean triples can also be derived from the following formulas:

$$m^2 + \left\{\frac{m^2-1}{2}\right\}^2 = \left\{\frac{m^2+1}{2}\right\}^2, m = 2\text{k+1; where k=0,1,2,3,.....}$$

$$n^2 + \left(\left(\frac{n}{2}\right)^2 - 1\right)^2 = \left(\left(\frac{n}{2}\right)^2 + 1\right)^2, n = 2\text{k; where k=1,2,3,.....}$$

to the general linear equation and to square a circle and conversely find a circle equal in area to a given square. The circumference of a circle was evaluated to $\pi \times$ the diameter of the circle, where $\pi \approx \frac{22}{7}$ ($\approx$3.14285) is a sacred number from the Vedic times.

It is said that all the Sulvasutras contain a method to square the circle. It is an approximate method based on constructing a square of side $^{13}/_{15}$ times the diameter of the given circle as in the diagram below. This corresponds to taking $\pi = 4 \times (^{13}/_{15})^2 = {}^{676}/_{225} = 3.00444$, so it is not a very good approximation.

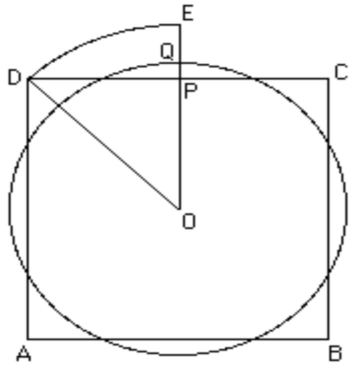


The Sulbasutras 13/15 method of "circling the square"

Note:

Sulvasutra is also called Sulbasutra in various books!

The Sulvasutras also examine the converse problem of finding a circle equal in area to a given square. The following construction appears. Given a square *ABCD* find the centre *O*. Rotate *OD* to position *OE* where *E* is the midpoint of the side of the square *DC*. Let *Q* be the point on *PE* such that *PQ* is one third of *PE*. The required circle has centre *O* and radius R (R= *OP*+ *PE*). Again it is worth calculating what value of π this implies to get a feel for how accurate the construction is. Now if the square has side 2*a* then the radius of the circle is *r* where *r = OE –EQ (*$\Leftrightarrow$ *r=OP+PQ)*.



The Sulbasutras method of "circling the square"

Since *OE* $= 2\sqrt{a}$ and *EQ*=$\frac{2}{3}(a\sqrt{2}-a)$, $r = a(\frac{\sqrt{2}}{3}+\frac{2}{3})$.

Then $2a\times 2a = \approx \pi \times r^2 = \pi\left(\left(a(\frac{\sqrt{2}}{3}+\frac{2}{3})\right)\right)^2$, which gives $\pi = \frac{36}{(\sqrt{2}+2)^2} \approx 3.088$.

Furthermore, geometric descriptions of ellipses also seem to have appeared during the age of Vedas.

The **Vedas** are books of knowledge containing hymns and offering verses in Vedic Sanskrit c 1500 – 1200 BC. Hymns were composed to the deities: Indra, Mitra, Varuna, Visnu and to Agni (fire), Sarya or Savitar (Sun), Usas (dawn), and Maruts (Storm).
The **Rig-Veda** is the oldest and most important, comprising more than a thousand hymns.
The **Yajur-Veda** is comprised of liturgical and ritualistic formulae in verse and prose.
The **Sama-Veda** consists of hymns, many of which occur in the Rig-Veda, for which musical notation is added or indicated.
The **Atharva-Veda** in verse and prose, comprising charm, prayer, curses, spells, etc. as well as some theosophical and cosmogony hymns, and written in a cruder and more popular style than the preceding.
Veda in Sanskrit means knowledge, sacred lore or a sacred book. It also means: I know.

The other events worth mentioning are the following ones:

Jainism and infinity

According to the information available, a new religion and philosophy, Jainism arose in India around the 6th century BC. Jainism to a certain extent replaced the Vedic religion and gave birth to Jaina mathematics. The most important idea of Jaina mathematics was the *infinite***.** In Jaina cosmology time was thought of as eternal and without form, the world, infinite; it was never created and has always existed. Space pervades everything and is without form. All the objects of the universe exist in space, which is divided into the space of the universe and the space of non-universe. There is a central region of the universe in which all-living beings, including men, animals, gods and devils, live. Above this central region is the upper world, which is divided into two parts. Below the central region is the lower world, which is divided into seven tiers. These cosmological concepts, as it appears, have been a motivating factor in the development of mathematical idea of the infinite. Still surprising, the Jaina cosmology contained a period of 2^{588} ($\approx 1.0130653244347 \times 10^{177}$) years for an unknown, not yet deciphered, phenomenon. The other remarkable numerical speculation, for the number of human beings that could have ever existed on Earth, is 2^{96} ($\approx 7.92282162514266 \times 10^{28}$) comes from *Anuyoga Dwara Sutra.*

As we can see, there was a great fascination with large numbers in Indian thought over a long period of time and this again almost required them to consider infinitely large measures. The first point worth making is that they had different infinite measures which they did not define in a rigorous mathematical fashion, but nevertheless are quite sophisticated. The way that the first unnameable number was effectively produced is by means of a recursive construction. The Jaina construction begins with a cylindrical container of very large radius r^q (taken to be the radius of the earth) and having a fixed height h. The number $n^q = f(r^q)$ is the number of very tiny white mustard seeds that can be placed in this container. Next, $r_1 = g(r^q)$ is defined by a complicated recursive sub procedure, and then as before a new larger number $n_1 = f(r_1)$ is defined. The *Anuyoga Dwara Sutra* then states:

> *Still the highest enumerable number has not been attained.*

The whole procedure is repeated, yielding a truly huge number, which is called *jaghanya- parita-asamkhyata*, meaning "unenumerable of low enhanced order." Continuing the process yields the smallest unenumerable number.

Jaina mathematics recognized five different types of infinity:

> *... infinite in one direction, infinite in two directions, infinite in area, infinite everywhere and perpetually infinite.*

The *Satkhandagama Sutra* permutations and combinations are stated and in the *Bhagabati Sutra* rules are given for the number of permutations of 1 selected from n, 2 from n, and 3 from n. Similarly rules are given for the number of combinations of 1 from n, 2 from n, and 3 from n. Numbers are calculated in the cases where $n = 2$, 3 and 4. These rules can be translated as P_r^n and C_r^n respectively. $P_r^n = n!/(n-r)!$ is the number of permutations of size r from the n elements. (an ordered arrangement of distinct elements). $C_r^n = P_r^n\ /r! = \binom{n}{r} = n!/(n-r)!r!$ is the number of combinations of size r from a set of size *n*, with no reference to order (corresponds to r! permutations of size *r* from the *n* elements). The author then says that one can compute the numbers in the same way for larger *n*. He writes:

> *In this way,* 5, 6, 7,…. 10*, etc. or an enumerable, unenumerable or infinite number of may be specified. Taking one at a time, two at a time, ... , ten at a time, as the number of combinations are formed they must all be worked out.*

Another concept, which the Jainas seem to have gone at least some way towards understanding, was that of the logarithm. They had begun to understand the laws of indices. For example the *Anuyoga Dwara Sutra* states:

The first square root multiplied by the second square root is the cube of the second square root.

The second square root was the fourth root of a number. This therefore is the formula

$(\sqrt{a}) \times (\sqrt{\sqrt{a}}) = (\sqrt{\sqrt{a}})^3$. $\quad [(\sqrt{\sqrt{a}}) = a^{1/4}$, the fourth root of a number]

As an illustration take a = 16, which yields $a^{1/4} = 2$, for
$(\sqrt{16}) \times (\sqrt{\sqrt{16}}) = (\sqrt{\sqrt{16}})^3$.

Again the *Anuyoga Dwara Sutra* states:

> *... the second square root multiplied by the third square root is the cube of the third square root.*

The third square root was the eighth root of a number. This therefore is the formula

$$(\sqrt{\sqrt{a}}) \times (\sqrt{\sqrt{\sqrt{a}}}) = (\sqrt{\sqrt{\sqrt{a}}})^3.$$

Some historians studying these works believe that they see evidence for the Jainas having developed logarithms to base 2.

Considering the expressions of the recursive and the square root formulas given above, one could indeed recognize that the Jaina mathematicians were definitely well equipped with the concept of exponential functions, which in fact can be classified as inverse functions related to logarithmic functions.

Although the Jainas were well ahead of their contemporaries in some aspects of mathematics, they were not, however, so competent with regard to discovering the real value of *pi*. The approximation $pi = \sqrt{10}$ seems to be one which was frequently used by them.

In the years of our C.E., Aryabhata (476 AD), besides providing a systematic treatment of the position of the planets in space, carried out calculations for π and discovered that π ≈ 62832/20000 = 3.1416, which is a surprisingly accurate approximation for π. In fact, we have π ≈ 3.14159265 correct to eight places. The following is the rule given by Aryabhata for the calculation of π:

Add four to one hundred, multiply by eight and then add sixty-two thousand. The result is approximately the circumference of a circle of diameter twenty thousand. By this rule the relation of the circumference to diameter is given. (100+4) × 8 + 62000 = 62832).

He calculated the circumference of the Earth to be 5000 *yojanas* (36000 km), one *yojana* being 7.2 kms. The actual figure, according to Harris Benson, in "University Physics," is 40023.9 km. The mean radius of the Earth is estimated to be 6.37×10^{6} m.

Aryabatha's value for the length of the year is 365 days 6 hours 12 minutes 30 seconds, which is an overestimate since the true value is less than 365 days 6 hours, where, according to Victor Katz, in 'A History of Mathematics', the true solar year is $11\frac{1}{4}$ minutes less than $365\frac{1}{4}$ days.(Incidentally, Aryabatha used $\pi = \sqrt{10}$ in practice, and the highest number he used in calculations is of ten digits, say 9000000000 i.e. nine *padmas*).

Aryabhata, besides presenting the sine of an angle in his work *Aryabhatia*, introduced the versine (versineθ=1-cosθ) into trigonometry.

Yativrsabha (6th C) produced Tiloyapannattti, a work in applied mathematics: trigonometric tables, various units for measuring distances and time and also description of the system of infinite time measures. Varahamira compiled texts written previously on astronomy and made important additions to Aryabhatta's trigonometric formulas. His works on permutations and combinations complemented what had been previously achieved by Jain mathematicians and provided a method of calculation of C_r^n that closely resembles the much more recent *Pascal's Triangle.*

In the 7th C, Bhaskara I included in his treatise, the *Mahabhaskariya,* three verses, which give an approximation of the trigonometric sine function by means of a rational fraction. The formula, proposed by Bhaskara is amazingly accurate. The use of the formula leads to a maximum error of less than one percent. Bhaskara's formula is

$$\sin x = \frac{16x\times(\pi - x)}{5\pi^2 - 4x(\pi - x)}.$$

Bhaskara I attributes this work to Aryabhata I. Here are the values given by the formula compared with the correct value for sin *x,* for *x:* from 0 to $\pi/_{2;}$ in steps of $\pi/_{20}$.

$x = 0$	formula = 0.00000	$\sin x = 0.00000$	error = 0.00000
$x = \pi/20$	formula = 0.15800	$\sin x = 0.15643$	error = 0.00157
$x = \pi/10$	formula = 0.31034	$\sin x = 0.30903$	error = 0.00131

$x = 3\pi/20$	formula = 0.45434	$\sin x = 0.45399$	error = 0.00035
$x = \pi/5$	formula = 0.58716	$\sin x = 0.58778$	error = -0.00062
$x = \pi/4$	formula = 0.70588	$\sin x = 0.70710$	error = -0.00122
$x = \pi/10$	formula = 0.80769	$\sin x = 0.80903$	error = -0.00134
$x = 7\pi/20$	formula = 0.88998	$\sin x = 0.89103$	error = -0.00105
$x = 2\pi/5$	formula = 0.95050	$\sin x = 0.95105$	error = -0.00055
$x = 9\pi/20$	formula = 0.98753	$\sin x = 0.98769$	error = -0.00016
$x = \pi/2$	formula = 1.00000	$\sin x = 1.00000$	error = 0.00000

Brahmagupta, a contemporary of Bhaskara I, wrote important works on mathematics and astronomy. In particular he wrote *Brahmasphutasiddhanta* (The Opening of the Universe), in 628 C.E. The work was written in 25 chapters. Brahmagupta's understanding of the number systems went far beyond that of others of his period. In the *Brahmasphutasiddhanta* he defined zero as the result of subtracting a number from itself. He gave some properties as follows:

> *When zero is added to a number or subtracted from a number, the number remains unchanged; and a number multiplied by zero becomes zero.*

He also gives arithmetical rules in terms of fortunes (positive numbers) and debts (negative numbers):

> *A debt minus zero is a debt.*
> *A fortune minus zero is a fortune.*
> *Zero minus zero is a zero.*
> *A debt subtracted from zero is a fortune.*
> *A fortune subtracted from zero is a debt.*
> *The product of zero multiplied by a debt or fortune is zero.*
> *The product of zero multiplied by zero is zero.*
> *The product or quotient of two fortunes is one fortune.*
> *The product or quotient of two debts is one fortune.*
> *The product or quotient of a debt and a fortune is a debt.*
> *The product or quotient of a fortune and a debt is a debt.*

So the *Bindu* (zero), although used as a placeholder in the place-value numeral system of earlier centuries in India, was given an algebraic definition and its mathematical relation definitely established in the treatises of Brahmagupta in the 7th century AD.

In addition to the *Brahmasphutasiddhanta,* Brahmagupta wrote a second work on mathematics and astronomy which is the *Khandakhadyaka,* written in 665 when he was 67 years old.

In the 9th century, Mahaviracharya wrote *Ganit Saar Sangraha,* where he describes the then currently-used method of calculating the Least Common Multiple

(LCM) of given numbers. He also derived formulae to calculate the area of the ellipse and a quadrilateral inscribed within a circle.

Sridhara, in his book *Patiganita,* provided mathematical formulae for a variety of practical problems involving ratios, barter, simple interest, mixtures, purchase and sale, rates of travel, wages, and filling cisterns. Sections of the book are also devoted to arithmetic and geometric progressions, including progressions with fractional numbers or terms, and formulas for the sum of certain finite series are provided.

The 10th century prominent mathematicians were Vijayanandi of Benares who wrote *Karanatilaka* (translated into Arabic by Al-Beruni), and Sripati of Maharashtra.

It is between the 7th and 11th century that the Indian numerals developed into their modern form, and along with the symbols denoting various mathematical functions (such as plus, minus, square root, etc) that eventually became the foundation stone of modern mathematical notation.

Coming to the 12th century, we have Bhaskara II, or Bhaskaracharya. His father, Mahesvara, was a Brahmin[5] and was famed as an astrologer. Bhaskaracharya became head of the astronomical observatory at Ujjain, the leading mathematical center in India at that time. Outstanding mathematicians such as Varahamihira and Brahmagupta had worked there and built up a strong school of mathematical astronomy.

Given that he was building on the knowledge and understanding of Brahmagupta it is not surprising that Bhaskaracharyya understood zero and negative numbers. However, his understanding went further even than that of Brahmagupta. To give some examples before we examine his work in a little more detail below, we can mention here that he knew that $x^2 = 9$ had two solutions. He also gave the following useful identity:

$$\sqrt{\frac{1}{2}\times\left(a+\sqrt{a^2-b}\right)}\pm\sqrt{\frac{1}{2}\times\left(a-\sqrt{a^2-b}\right)}$$

Although Lilavati contains a vast number of interesting problems and examples in arithmetic, geometry, combinatorial maths and *kuttaka* (pulverizer, i.e. a quantity such that a given number being multiplied by it and the product added to a given quantity, may be divisible by a given divisor without a remainder), the scope of this paper is limited to the study of zero and its function and Bhaskaracharyya's method of

squaring a number
extraction of the square root by hand
the cube of a number
the cube root of a number
completing and forming perfect squares
proportionality
principal and interest on money
permutations and combinations
arithmetical progression
geometrical progression

[5] Belonging to Brahman, the highest caste of Indian society. The other castes are: Kshatriya, Vaisia and Sudra. There are also casteless/outcasts in the Indian society and, naturally, all non-Indians are casteless!

Pythagoras theorem
an invariant (*lamba*) perpendicular in geometry
pulverizer.

In addition, an outline of the importance of Lilavati, together with work by Bhaskara besides Lilavati, and its relation to other historical events.

2. ZERO AND ITS FUNCTION

As it is now established, the concept of zero originated in ancient India. This numeral represented by a dot was termed Pujyam. Param-Pujya is a prefix used in reverence to elders in written communication. In this case it means respected or esteemed. Pujyam also means holy. In India the more current term for zero is Shunyam, meaning a blank. The reason why the term Pujya came to be sanctified can only be guessed. As it is widely known, Indian philosophy has glorified concepts like the material world being an illusion (Maya), the act of renouncing the material world (Tyaga) and the goal of merging into the void eternity (Nirvana). Herein could lay the reason why the mathematical concept of zero got a philosophical connotation of reverence. The concept of zero or Shunya is derived from the concept of a void. The concept of void existed in Brahmin philosophy and so possibly the derivation of a symbol for it. The concept of Shunyata, influenced South-east Asian cultures through the Buddhist concept of Nibbana 'attaining salvation by merging into the void of eternity'. In ancient India the terms used to describe zero included Pujyam, Shunyam, and Bindu. The concept of a void or blank was termed as Shukla and Shubra.

So as we see, the zero had been a mystical symbol until it was fully recognized as a number by the Italian mathematician Leonardo de Pisa (c.1175-1250). He opened the European World to the Hindu-Arabic notation for numerals and algorithms for arithmetic. In ancient times the zero was used as a place holder in many countries of the East. Its substantial value, as can be seen, was propagated by the Indian mathematicians of the 7^{th} century. Consequently the negative numbers could henceforth join number theory. Later the zero evolved into a more complex status, such that Bhaskara perceived its role as being the same as that played by the other numbers of the decimal system. Incidentally, the word 'zero' is derived from *zephyrum* in Arabic, which in the Italian language became *zenero*, then *zepiro*. Later it contracted to zero.

When zeros can be separated as factors forming the indeterminate expression 0/0 from the other quantities in account, Bhaskara interprets the expression 0/0 as equal to 1, although for $a \neq 0$ he considers *a/0* as infinity. Otherwise the zero is equal to the conventional null value. In the formulation and solution of the following problem:

$$\frac{0 \times (x + \frac{x}{2}) \times 3}{0}$$

equated to 63 yields $x=14$

We can see that, on the one hand this operation is not valid at all in modern mathematics because of the undefined expression, $\frac{0}{0}$. But on the other hand if we adopt

Bhaskara's hint we can see that when 0 is replaced by x_0, a value next to 0, and letting $x_0 = \frac{1}{10^n}$, (where $n \in Z^+$.); we can judiciously simplify $\frac{x_0}{x_0}$ to a unit. At any rate, as far as the conventional mathematical rules are concerned Bhaskaracharyya is completely wrong about the solution of his problems involving the zero.

3a. SQUARING A NUMBER

Squaring as we know is a mathematical operation, which consists of multiplying a number by itself. It has a multiple use, especially in finding an area of the square of a known side. Suppose we are to find the square of 297.Then according to Bhaskara's method or rule: *The multiplication of two like numbers together is the square. The square of the last digit is to be placed over it; and the rest of the digits, doubled and multiplied by the that last, to be placed above them respectively; then repeating the number, except the last digit again (perform the like operation). Or twice the product of two parts, added to the sum of the squares of the parts, is the square (of the whole number). Or the product of the sum and difference of the number and an assumed quantity is the square.*

We have

i)

$$
\begin{array}{rr}
7^2 = & 49 \\
7 \times 2 \times 29 = & 406 \\
9^2 = & 81 \\
9 \times 2 \times 2 = & 36 \\
2^2 = & 4 \\
 & \ldots\ldots\ldots\ldots\ldots\ldots \\
(297)^2 = & 88209
\end{array}
$$

or beginning from the left side of the number 297

ii)

$$
\begin{array}{rr}
(200)^2 = & 40000 \\
2 \times 200(90+7) = & 38800 \\
(90)^2 = & 8100 \\
2 \times 90 \times 7 = & 1260 \\
7^2 = & 49 \\
 & \ldots\ldots\ldots \\
(297)^2 = & 88209
\end{array}
$$

The calculation as we can see is based upon the identities:
$(a+b)^2$, $\left(a+(b+c)\right)^2$ and $(a+b+c)^2 = a^2 + 2a(b+c) + b^2 + 2bc + c^2$.
The other identities for squaring purposes are given as *(a+b) (a-b)* + $b^2 = a^2$, and *ab+ a(a-b)=*a^2 *;* *a* being the proposed (given) and *b,* the assumed (arbitrary) quantity.

Obviously the method of squaring which Bhaskara uses is easy to follow and is quite compatible with our modern version.

3b. EXTRACTION OF SQUARE ROOT BY HAND

Having carried out the squaring of a number it is quite natural to seek a process by which one could find the square root of a given number. It is an operation inverse to squaring. When we are given a number, we know that its square root will be some number, such that when we multiply the number (root) by itself we obtain the given number.

Bhaskara gives the following rule for extracting square root: *Having deducted from the last of the odd digits the square number, double its root; and by that dividing the subsequent even digit, and subtracting the square of the quotient from the next uneven place, note in a line (with the preceding double number) the double of the of the quotient. Divide by the (number as noted in a) line the next even place, and deduct the square of the quotient from the following uneven one, and note the double of the quotient in the line. Repeat the process (until the digits be exhausted). Half the (number noted in the) line is the root.*

Since this rule is so ambiguous and difficult to execute straightforwardly, Colebrooke therefore interprets it into the following method of extracting the square root of a number: for example, to find the square root of 88209, observe first that the root we are looking for cannot exceed three (integer) digits. Counting from the right we mark off two places (8/82/09) and we work on the figures to left of the slash. In this example we start working from left on 8.

As we see that $2 < \sqrt{8} < 3$, the square of 3 is 9>8, so we cannot take 3 in our calculation. The only choice is 2, our first digit in $\sqrt{8\ 82\ 09}$. Next, we subtract 4 (the square of 2) from 8 and obtain 4 as a remainder to which we add the pair 82 of the original given number. The subsequent result we reduce to a single digit by dividing it by a suitable divisor (by taking 2 times the first digit of $\sqrt{8\ 82\ 09}$ and annexing to it a right number) which will give us the second digit in the root of our number. By trial we see that 49 is the right candidate for the division which yields 9 as the quotient and, continuing in the same manner we get 7, the third digit in the square root, 297 of our number 88209.

$$
\begin{array}{rl}
 & 8\ 82\ 09\ (297 \\
 & 4 \\
 & \text{-----------} \\
49) & 482 \\
 & 441 \\
 & \ldots\ldots\ldots\ldots\ldots \\
587) & 4109 \\
 & 4109 \\
\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots \\
 & 0000
\end{array}
$$

Moreover there is another method of calculation of the root, which was in use long before Bhaskara came up with his idea. It was given by Aryabhata I.(5^{th} C.E.):

*One should always divide the avarga (*an even place*) by twice the (square) root of the (preceding) varga (*an odd place*). Counting from right to left, the odd places are* *called varga and the even places are called avarga. After subtracting the square (of the quotient) from the varga the quotient will be the square root to the next place.*

The nearest square root to the number in the last odd place on the left is set down in a place apart, and after this are set down the successive quotients of the division performed. The number subtracted is the square of that figure in the root represented by the quotient of the preceding division. The divisor is the square of that part of the root which has already been found. If the last subtraction leaves no remainder the square root is exact.

	88209 (2=root 1
Subtract the square of the quotient, *root1*	4

Twice the root 1, 2×2	4) 48 (9=next digit of root
	36

	122
Square of the new quotient, 9^2	81

Two times the present part of the root, 2×29	58) 410 (7=next digit of root
	406

	49
Square of the latest quotient	49

	00

Perhaps Aryabhata's method is more appealing for this type of operation!

3c. THE CUBE OF A NUMBER

Like the square, a cube is a quantity $c^3 = c \times c \times c$.

This review would be incomplete if one ignored the idea of finding the cube of a number. Hence Bhaskaracharyya bases his calculations on the following formulae:

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$
$$(a+b)^3 = a^3 + b^3 + 3ab(a+b).$$
$$\Big(\sqrt{a^2})^3\Big) = (a^2)^3.$$

The cube of 125 is $125 \times 125 \times 125 = 1953125$. Or
$(125)^3 = (120+5)^3 = 120^3 + 5^3 + 3 \times 600(120+5) = 1953125.$

As we see this method is the same as the one we use in our modern times.

3d. THE CUBE ROOT OF A NUMBER

To complement the idea of cube Bhaskaracharyya proposes a method of finding the cube root of a number, an inverse operation of cubing: *The first digit is cube's place; and the two next, uncubic; and again, the rest in like manner. From the last cubic place take the (nearest) cube, and set down its root apart. By thrice the square of that root divide the next (or uncubic) place of figures, and note the quotient in a line (with the quantity before found). Deduct its square taken into thrice the last (term), from the next (digit); and its cube from the succeeding one. Thus the line (in which the result is reserved) is the root of the cube. The operation is repeated (as necessary).*

Here again we have a difficult method to follow, but it can be visualized the following way: To extract the cube root of 1953125, we take the cube root of a million.

Then 1953125(100

subtract from the dividend the cube of the first quotient $(100)^3 = 1000000$

Division by 3 times the quotient squared, $3\times(100)^2$) 953125(20

(**)Note, we are unable to use 30 in the quotient here!

600000

353125

3 times the new quotient squared times the previous quotient $3\times(20)^2\times100 =$ 120000

233125

cube of the new quotient to be subtracted $(20)^3 = 8000$

$3\times$the sum of the quotients squared $3\times(120)^2 = 43200)$ 225125(5

216000

9125

$3\times$the latest quotient squared$\times$the sum of the former quotients, $3\times5^2\times120 = 9000$

125

subtract the cube of the latest quotient $5^3 = 125$

000

The cube root is the summation of 100+20+5=125.

(**) $30\times3\times100^2 = 900000$ deducted from 953125 will give us 53125, out of which the subtraction of $3\times(20)^2\times100 =$ 120000 would give no positive remainder!

According to Aryabhata I, the Extraction of the Cube Root of a number may be obtained as follows:

One should divide the second aghana by three times the square of the (cube) root of the (preceding) ghana. The square (of the quotient) multiplied by three times the *purva (that part of the cube root already found) is to be subtracted from the first aghana, and the cube (of the above division) is to be subtracted from the ghana.*

Counting from right to left, the first, fourth, etc., places are named ghana (cubic); the second, fifth, etc., places are called the first aghana (non-cubic) places; and the third, sixth, etc., places are called second aghana (non-cubic) places. The nearest cube root to the number in (or up to and including) the last Ghana place on the left is the first figure of the cube root. After it are placed the quotients of the successive divisions. If the last subtraction leaves no remainder the cube root is exact.

Example: To find a cube root of 1860867 (Note: the cube root of 1 is 1)

```
                                                   1860867 (root=1
  Cube of root is subtracted from the dividend     1
                                                   --------------
The remainder divided by 3 times the square of 1, 3 )08(2=quotient (next digit of root)
                                                    6
                                                   -----
(That part of the cube root already found is a purva)  26
    Square of quotient multiplied                       12
    By 3 times the purva (2^2 × 3 × 1)          ------------------------
                                                        140
    Subtracting the cube of quotient, 2^3                 8
                                                   ---------------
    3 times square of root; 3×(12)^2          432) 1328(3=next digit of root
                                                    1296
                                                   -----------
                                                       326
    Square of quotient multiplied by 3 times the purva 324
                                                   ---------------
                                                          27
    Cube of quotient                                      27
                                                   --------------
                                                          00
```

The cube root of 1860867 is 123

Although there are several different methods of extracting roots, the method above might be one of the more straightforward and fast.

4. COMPLETING AND FORMING PERFECT SQUARES

Completing a square is a process by which we convert an expression of type $At^2 + 2Bt + C$ into a compact form: A perfect square is a quantity which is the exact square of another quantity, where by quantity Bhaskara means rational numbers. 9 is perfect square and in Bhaskara's meaning even $\frac{9}{4}$ is perfect

square. So is a quadratic expression that factors into two identical binomials
$a^2+b^2+2ab=(a+b)(a+b)=(a+b)^2$, a perfect square.
Or $a^2+b^2+c^2+2ab+2ac+2bc=(a+b+c)^2$.

In this section Bhaskara proposes a rule whereby we are able to compose a number and then turn it into a square.

The square of an arbitrary number, multiplied by eight and lessened by one, then halved and divided by the assumed number, is one quantity; its square, halved and added to one, is the other. Or unity, divided by double an assumed number and added to that number, is a first quantity; and unity is the other. These give pairs of quantities, the sum and difference of whose squares, lessened by one, are squares.

If we let n be any number, then we will have, for $n\in \mathrm{Z}^+$

$$\frac{1}{2n}(8n^2-1) \qquad \text{and} \qquad \frac{1}{2}\left\{\frac{1}{2n}(8n^2-1)\right\}^2+1$$

According to the given rule, the sum of the squares of the above numbers (see * below) minus 1 is

$$\frac{1}{4}\left(4n-\frac{1}{2n}\right)^4+2\left(4n-\frac{1}{2n}\right)^2=\left(4n-\frac{1}{2n}\right)^2\left\{\frac{1}{4}\left(4n-\frac{1}{2n}\right)^2+2\right\}=$$
$$=\left(4n-\frac{1}{2n}\right)^2\left(2n+\frac{1}{4n}\right)^2.$$

Similarly, the difference of the squares of the below given numbers lessened by 1 is a square, according to the description of the second part of the rule: The numbers are $\frac{1}{2n}+n$ and 1; $n\in \mathrm{Z}^+$

$$\left\{\left(\frac{1}{2n}+n\right)^2\pm(1)^2\right\}-1=\left(\frac{1}{2n}\pm n\right)^2.$$

*The detailed operation is as follows:

$$\frac{1}{2n}(8n^2-1)=4n-\frac{1}{2n};$$
then

$$\left\{\frac{1}{2n}(8n^2-1)\right\}^2+\left(\frac{1}{2}\left\{\frac{1}{2n}(8n^2-1)\right\}^2+1\right)^2-1=\left(4n-\frac{1}{2n}\right)^2+\left\{\frac{1}{2}\left(4n-\frac{1}{2n}\right)^2+1\right\}^2-1$$
$$=\left(4n-\frac{1}{2n}\right)^2+\frac{1}{4}\left(4n-\frac{1}{2n}\right)^4+2\times\frac{1}{2}\left(4n-\frac{1}{2n}\right)^2+1^2-1=\frac{1}{4}\left(4n-\frac{1}{2n}\right)^4+2\left(4n-\frac{1}{2n}\right)^2$$
$$=\left(4n-\frac{1}{2n}\right)^2\left\{\frac{1}{4}\left(4n-\frac{1}{2n}\right)^2+2\right\};$$

$$\left(4n-\frac{1}{2n}\right)^2\left\{\frac{1}{4}\left(16n^2-\frac{8n}{2n}+\frac{1}{4n^2}\right)+2\right\}=\left(4n-\frac{1}{2n}\right)^2\left\{4n^2-1+\frac{1}{16n^2}+2\right\}$$
$$=\left(4n-\frac{1}{2n}\right)^2\left(2n+\frac{1}{4n}\right)^2.$$

Following Bhaskara, Colebrooke interprets the above acquired expressions as perfect squares in contrast to our definition of the term. No doubt, the operations used in the process of forming squares are rather skilful, but the above results are just numerical squares and cannot be considered as perfect squares.

5. PROPORTIONALITY

Two quantities are said to be proportional if they have the same or a constant ratio or relation: The quantities y and x are proportional if $\frac{y}{x}=k$, where k is constant of proportionality.

The operations on proportional quantities in Lilavati do not differ from our modern ones for cross ratios; such as

$$\frac{5}{2}:x=\frac{3}{7}:\frac{9}{1}$$
$$\because x\times\frac{3}{7}=\frac{5}{2}\times 9\Leftrightarrow x=52\frac{1}{2}$$

This setting and the solution of the problem comes from: *If two and a half palas* (measure) *of saffron be obtained for three-sevenths of a nishka* (money), *say, how much is gotten for nine nishkas.*

Operations on proportionality executed by Bhaskara seem to be the same as we use these days.

6. PRINCIPAL (initial capital) AND INTEREST ON MONEY

Although lending and borrowing money at the time of Lilavati was by private citizens, the financial problems in composition of Principal and Interest do not differ from those of our times and are based on the following formulae:

The interest you get on your initial capital, after a period of time is I;

$$I = \frac{P \times r \times 12 \times t}{100}.$$

$$\because A = P\left\{1 + \frac{1}{100} \times r \times 12 \times t\right\}$$

$$\because P = \frac{A \times 100}{100 + r \times 12 \times t}, \quad and\ I = \frac{A \times r \times 12 \times t}{100 + r \times 12 \times t}.$$

Where A =Amount in balance, r=rate per month, I=interest, t= years and P=principal.

In addition, an extension of these formulas is succeeded by a rule where the Principal is known: *The arguments* (the principals) *taken into their respective times are divided by the fruit* (interest) *taken into elapsed* (passed) *times; the several* (distinct) *quotients, divided by their sum and multiplied by the mixed quantity, are the parts as severally* (distinctly) *lent.*

If we let x, y , z be the portions of money lent at r_1, r_2, r_3 per cent, per month and let I=common interest in t_1, t_2, t_3 months respectively. Then $x+y+z=a$ and

$$I = \frac{x \times r_1 \times t_1}{100} = \frac{y \times r_2 \times t_2}{100} = \frac{z \times r_3 \times t_3}{100};$$

$$\because x : y : z = \frac{100 \times 1}{r_1 \times t_1} : \frac{100 \times 1}{r_2 \times t_2} : \frac{100 \times 1}{r_3 \times t_3};$$

$$\because x = \frac{100 \times 1}{r_1 \times t_1} \times \frac{a}{\frac{100 \times 1}{r_1 \times t_1} + \frac{100 \times 1}{r_2 \times t_2} + \frac{100 \times 1}{r_3 \times t_3}};$$

with similar values for y and z.

In modern times we might apply advanced methods using the exponential function, $y = y_0 e^{ax}$, where y denotes the amount in balance with compound interest, y_0 is the initial amount, a denotes the rate of interest and x expresses the time period. Or we might take the problem into a recurrence relation, $p_n = p_0(1+r)^n$, $p_{n+1} = p_n + rp_n$, where p_n is the amount in balance after the basic period n, r is the rate of interest and p_{n+1} is the amount with compound interest accumulated after the desired periods of investment of the capital p_0.

Thus we see that although our calculations regarding the simple (non-compound) rate of interest are the same as Bhaskara's, the modern method by which we do our calculations is of course much more advanced than Bhaskara's regarding the computation of compound interest.

7. PERMUTATIONS AND COMBINATIONS

Permutation is an application of the rule of product: If a procedure can be broken down into first and second phases (or more), and if there are *m* possible alternatives for the first phase and if, for each of these alternatives, there are *n* possible alternatives for the second phase, then the total procedure can be carried out, in the designated order, *mn* ways.

Suppose a menu at a restaurant consists of 3 appetizers, 4 main dishes and 3 desserts. In how many different ways can we compose a dinner consisting of an appetizer, a main dish and a dessert? – We can choose an appetizer three different ways. After that we can select a main dish 4 different ways and lastly a dessert in 3 different ways. Altogether we have $3\times4\times3=36$ different ways of composing our dinner. If, in addition, we would take either coffee or tea for digestion, how many different ways then could we compose our meal?

If in general we have a set of *n* elements: $a_1, a_2, ..., a_n$, we can write these elements in $n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = n!$ different ways in a definite order (each element appears only once in the set so formed).

$$\begin{array}{lll} a_1 & .... & a_n \\ a_2 & .... & a_{n-1} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ a_n & .... & a_1 \end{array}$$

Such an order of a set is a permutation. If we take only *k* elements and place them in a certain order, we get arranged subsets with *k* elements. According to the rule of product we have $n(n-1)(n-2)\cdots(n-k+1)$ arranged subsets with *k* elements. The number of permutations of size *k* for the *n* elements is equivalent to

$$P_k^n = n(n-1)(n-2)\cdots(n-k+1)\times\frac{(n-k)(n-k-1)\cdots 3\cdot 2\cdot 1}{(n-k)(n-k-1)\cdots 3\cdot 2\cdot 1} = \frac{n!}{(n-k)!}.$$

A combination (a selection) of *k* elements out of *n* distinct elements of a given set, with no reference to order corresponds to *k*! permutations of size *k* from the *n* elements. Thus the number of combinations of size *k* from a set of size *n* is

$$\frac{P_k^n}{k!} = \frac{n!}{k!(n-k)!}.$$

The permutations and combinations dealt in Lilavati are the ones that are carried out by the formulas P_r^n and C_r^n.

$P_r^n = \dfrac{n!}{(n-r)!}$ implying $C_r^n = \dfrac{P_r^n}{r!}$

An illustration follows:

In a pleasant edifice, with eight doors, constructed by a skilful architect, as a palace for the lord of the land, tell me the permutations of entrance doors taken one, two, three, and so on. Say mathematician, how many are the combinations in one composition, with ingredients of six different tastes, sweet, pungent, astringent, sour, salt, and bitter, taking them by ones, twos, threes and so on.

The answer is given by

$$C_r^n = \frac{P_r^n}{r!} = \binom{n}{r} = \frac{n!}{(n-r)!r!}$$

So the changes on the entrance doors of the palace amount to $(2^8 - 1) = 255$.
In case, all the entrance doors being shut taken into account; $(2^8 - 1) + 1 = 255 + 1$.
Likewise for the second part of the problem the answer will be $(2^6 - 1) = 63$.

In this section Bhaskaracharyya gives the results correctly out, but has no reference to the general formulas for the permutations or for the combinations.

8a. ARITHMETICAL PROGRESSION

The arithmetical Progression consists mainly of the series which are still used in our modern times:

1+2+3+4+5+6+…………*+n* $= \frac{n}{2}(n+1)$. (a sum corresponding to a triangular number).

It follows that the sum of *n* terms of the series whose *n*th term is $\frac{n}{2}(n+1)$ is the sum of

n triangular numbers$= \frac{1}{6}n(n+1)(n+2) = \frac{n}{2}(n+1)\frac{(n+2)}{3}$.

The sum of the square numbers beginning from 1 and the sum of the cubic numbers is given below:

$$1^2+2^2+..........+n^2=\frac{n}{6}(n+1)(2n+1)=\frac{n}{2}(n+1)(\frac{2n+1}{3}).$$

$$1^3+2^3+..........+n^3=\left\{\frac{n}{2}(n+1)\right\}^2.$$

This is how Bhaskara II sees the arithmetical progression in his rule: *Half the period multiplied by the period added to unity, is the sum of the arithmeticals one and so on, and is named their addition. This being multiplied by the period added to two, and being divided by three, is the aggregate of the additions.*

Twice the period added to one and divided by three, being multiplied by the sum (of arithmeticals), is the sum of the squares. The sum of the cubes of the numbers one, and so on, is pronounced by the ancients equal to the square of the addition.

Somewhat surprisingly, here in this item of Lilavati, Bhaskaracharyya evokes the proper flavour of the general formula concerning the above operations.

8b. GEOMETRICAL PROGRESSION

A geometric progression is an infinite sequence of numbers, such as 5, 10, 20, 40, … where the division of each term by its immediate predecessor is a constant, the common ratio.

Referring to a geometrical progression Bhaskara extends the idea using a pedagogically devised problem leading to a geometric series:

The initial quantity being two, my friend; the daily augmentation, a threefold increase; and the period, seven; say what the sum in this case is.

To find a solution to this problem, we let *a* be the first term of the geometric series and *r* be the common ratio. Then,

$$a+ar+......+ar^{n-1}=\frac{a(r^n-1)}{r-1}.$$

The first term: *2=a*.
Increasing multiplier: *3=r*
Period: *7=n*
The sum=*2186.*

This problem is based on the interpretation of Bhaskara's rule: *The period being an uneven number, subtract one, and note 'multiplicator'; being an even one, halve it, and note 'square,' until the period be exhausted. Then the produce arising from multiplication and squaring (of the common multiplier) in the inverse order from the last, being lessened by one, the remainder divided by the common multiplier less one, and multiplied by the initial quantity, will be the sum of a progression increasing by a common multiplier.*

Here again we see Bhaskara leading us to a general formula, stated above. Hence no time gap is felt between his epoch and ours in this matter, because we use the same formula in modern times.

9. PYTHAGORAS THEOREM

Apart from the most famous theorem

$$a^2+b^2=d^2,$$
$$2ab+(a-b)^2=a^2+b^2,$$
$$(a+b)(a-b)=a^2-b^2,$$

d represents the hypotenuse of a right-angled triangle with sides a and b.

Bhaskaracharyya proposes a rule, imposing algebraic terms for the construction of a right-angled triangle:
A side (horizontal) *is put. From the multiplied by twice some assumed number and divided by one less than the square of the assumed number an upright* (vertical) *is obtained. This being set apart is multiplied by the arbitrary number, and the side as put is subtracted; the remainder will be the hypotenuse. Such a triangle is termed right-angled.*

Let a denote the given side, and n the assumed number.
Then proceeding by the rule, we get $\frac{2an}{n^2-1}$ for the vertical, and

$$\frac{2an}{n^2-1}\times n-a=a\times\frac{n^2+1}{n^2-1} \text{ for hypotenuse.}$$

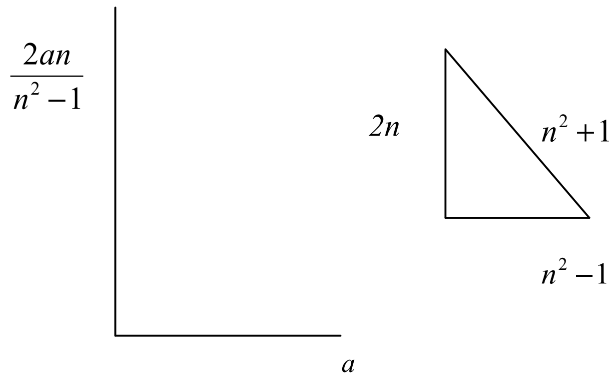
Verification:

$$a^2+\{\frac{2an}{n^2-1}\}^2=a^2+\frac{4a^2n^2}{(n^2-1)^2}=\frac{(n^2-1)^2a^2+a^2 4n^2}{(n^2-1)^2}=$$
$$=a^2\left[\frac{(n^2-1)^2+4n^2}{(n^2-1)^2}\right]=a^2\,\frac{n^4-2n^2+1+4n^2}{(n^2-1)^2}=a^2\,\frac{(n^2+1)^2}{(n^2-1)^2}=\left\{a\frac{n^2+1}{n^2-1}\right\}^2.$$

How these expressions for the vertical and hypotenuse are arrived at, could be visualized by the quantities $2n, (n^2-1), (n^2+1)$ taken to represent the vertical, side (horizontal) and hypotenuse of a right-angled triangle, because

$$(2n)^2+(n^2-1)^2=(n^2+1)^2.$$

Now consider another right-angled triangle similar to the above, the side being a Then, since the sides of the two triangles are proportional, the vertical of the second triangle will obviously be $\frac{2an}{n^2-1}$. Again, as $n\times$vertical of the first triangle = its side + its hypotenuse, so $n\times$vertical of the second triangle = its side + its hypotenuse.



Thus $n\times\frac{2an}{n^2-1} = a$ + hypotenuse, whence hypotenuse = $\{n\times\frac{2an}{n^2-1}\}-a$.

An assumed number *n*, being the difference between the hypotenuse and vertical is shown by the following rule*: A side (*horizontal*) is put. Its square, divided by an arbitrary number, is set down in two places: and the arbitrary number being added and subtracted, and the sum and difference halved, the results are the hypotenuse and upright. Or, in like manner, the side and hypotenuse may be deduced from the upright. Both results are rational quantities.*

Illustration: Let *a* denote the given side, and *n* the assumed number. Then by the rule we have $\frac{1}{2}\left\{\frac{a^2}{n}+n\right\}$ for hypotenuse, and $\frac{1}{2}\left\{\frac{a^2}{n}-n\right\}$ for vertical.

Verification:

$$\mathrm{a}^2+\frac{1}{4}\left\{\frac{(\mathrm{a}^2-n^2)^2}{n^2}\right\}=\left\{\frac{4a^2n^2+(\mathrm{a}^2-n^2)^2}{4n^2}\right\}=\frac{(\mathrm{a}^2+n^2)^2}{4n^2}=\left\{\frac{1}{2}\left(\frac{a^2}{n}+n\right)\right\}^2.$$

There is also an extension of the above rule: *Let twice the product of two assumed numbers be the upright* (vertical)*; and the difference of their squares, the side (*horizontal*): the sum of their squares will be the hypotenuse, and a rational number.*

This rule can be interpreted as follows: Let *a* and *b* be the assumed numbers. Then *2ab* is the upright and (a^2-b^2) the side. The hypotenuse is $\sqrt{(2ab)^2+(\mathrm{a}^2-b^2)^2}=(\mathrm{a}^2+b^2)$.

Proceeding further we come across another, rather commonly applicable, rule: *The square of the ground intercepted between the root and tip is divided by the (length of the) bamboo and the quotient severally added to, and subtracted from, the bamboo: the* *moieties (of the sum and difference) will be the two portions of it representing hypotenuse and upright.*

As a reference we shall use the following example:
Let a denote the height of the bamboo, b: the distance between root and tip; and x the height at which the bamboo is broken.
Then,

$$b^2 = (a-x)^2 - x^2, \quad \because \frac{b^2}{(a-x)+x} = (a-x)-x; \quad \Rightarrow \frac{b^2}{a} = (a-x)-x, \because a = (a-x)+x \quad \textit{identically}.$$

$$\text{Hence } \frac{1}{2}\left(a+\frac{b^2}{a}\right) = a-x \textit{ or hypotenuse}, \quad \textit{and} \quad \frac{1}{2}\left(a-\frac{b^2}{a}\right) = x \textit{ or upright}.$$

Another practical rule: *The quotient of the square of the side divided by the difference between the hypotenuse and upright is twice set down; and the difference is subtracted from the quotient (in one place) and added to it (in the other). The moieties (of the remainder and sum) are in their order the upright and hypotenuse.*

The interpretation follows:
Let a denote the difference between hypotenuse and vertical, b the side, and x the vertical. Then,

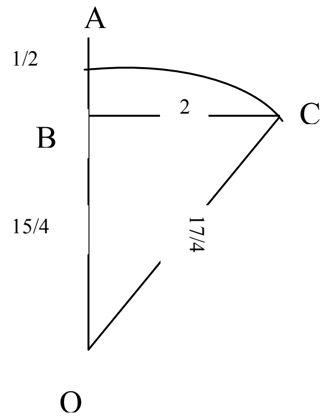
$$b^2 = (a+x)^2 - x^2;$$

$$x = \{(b^2/a) - a\} \div 2, \quad \textit{and } a+x = \{(b^2/a)+a\} \div 2$$

An example: *Friend, the space between the lotus (as it stood) and the spot where it submerged, is the side. The lotus as seen (above water) is the difference between the hypotenuse and upright. The stalk is the upright, for the depth of water is measured by it. Say what the depth of the water is.*

In a certain lake swarming with ruddy geese and cranes, the tip of a bud of lotus was seen a span above the surface of the water. Forced by the wind it gradually advanced, and was submerged at the distance of two cubits. Compute quickly, mathematician, the depth of the water.

Let O be the root of the lotus, A its tip; and C the point on the surface of the water is submerged. Then, while it advances by the force of the wind, O remains fixed, and the lotus describes an arc of a circle, of which O is the centre, and OA the radius. Hence OC = OA.



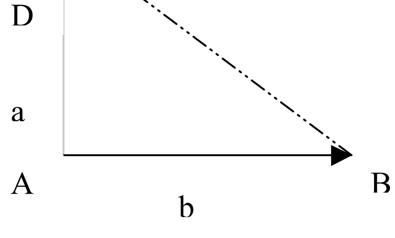
Difference of hypotenuse and vertical, a = 1/ 2 cubit. Side = 2.

One more rule: *The height of the tree multiplied by its distance from the pond, is divided by twice the height of the tree added to the space between the tree and the pond: the quotient will be the measure of the leap.*

This rule steers the example that follows: Let D be the top of the tree, and B the position of the pond. The first ape is supposed to descend from D to A, and then go from A to B; while the second ape is supposed to jump vertically upwards from D to C, and then to leap directly from C to B. Now let DA = a, AB = b, and CD = x, which is required. Then by the question, we have

$$x+\sqrt{(a+x)^2+b^2}=a+b;$$
$$\because (a+x)^2+b^2=(a+b)^2-2(a+b)x+x^2,$$
$$x=ab/(2a+b)$$

Example:

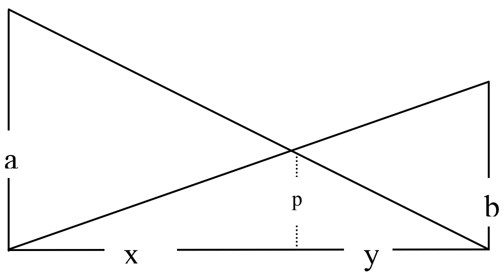


From a tree a hundred cubits high, an ape descended and went to a pond two hundred cubits distant: while another ape, jumping (vaulting) to some height off the tree, proceeded with velocity diagonally to the same spot. If the space travelled by them be equal, tell me quickly, learned man, the height of the leap, if thou have diligently studied calculation.
(Answer: 50 cubits)

10. AN INVARIANT PERPENDICULAR (*lamba)*

To find the perpendicular (*lamba*) *p* in the figure below, we are to follow the following rule: *The product of two erect bamboos being divided by their sum, the quotient is the perpendicular from the junction (intersection) of threads passing reciprocally from the root (of one) to the tip (of the other). The two bamboos, multiplied by an assumed base, and divided by their sum, are the portions of the base on the respective sides of the perpendicular.*

Certainly, similar triangles play a great role in illustrating the solution to this geometrical problem as shown below:



$$\frac{p}{a}=\frac{y}{x+y},\quad \frac{p}{b}=\frac{x}{x+y};\qquad \text{since}\quad \frac{y}{x+y}+\frac{x}{x+y}=\frac{y+x}{x+y}.$$

$$\because\ p\left(\frac{1}{a}+\frac{1}{b}\right)=1.$$

So $p=\dfrac{ab}{a+b}$. Thus p is independent of x and y, provided *a* and *b* are given.
Again, let *(x+ y)* = *k*, be any assumed number.
Then $x=\dfrac{pk}{b}=\dfrac{ak}{a+b};\quad y=\dfrac{pk}{a}=\dfrac{bk}{a+b}.$

And here is a practical example: *Tell the perpendicular drawn from the intersection of strings stretched mutually from the roots to the summits of two bamboos fifteen and ten cubits high, standing upon ground of unknown extent.*
Solution: perpendicular p = 6.

$$\left[15\times\frac{10}{15+10}\right]$$

To find the segments of the base, let the ground be assumed 5; the segments come out 3 and 2. Or putting 10, they are 6 and 4. Or taking 15, they are 9 and 6. In every instance the perpendicular is the same, i.e. 6.

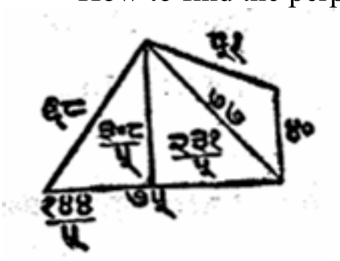
Note: For a, b fixed, the base may vary by assuming a greater or less quantity for it, the perpendicular will always be the same.

Continued rule for finding the perpendicular (the diagonals either given or assumed): *In the triangle within the quadrilateral, the perpendicular is found as before taught; the diagonal and side being sides, and the base, a base.*

This continued rule is based upon the rule given for triangles: *In a triangle, the sum of two sides being multiplied by their difference, is divided by the base; the quotient is subtracted from, and added to, the base which is twice set down: and being halved, the results are segments corresponding to those sides.*

The above rule is easily understood by working with a triangle where the sides and the base are given quantities.

How to find the perpendicular of the below given quadrilateral figure?



In this case a diagonal is drawn from the top extremity (vertice) of the left side to the origin of the right one is given to be 77. By this diagonal, a triangle is constructed within the quadrilateral. In it the diagonal drawn is one side, 77 and the original left side is another, 68; the base, 75. Then by proceeding by the rule, the segments are found 144/5 and 231/5; and the perpendicular, 308/5.

The summary of the operation follows: When in a triangle the two sides *a* and *b* along with the base *c* (composed of two segments *x* and *y*), are given; we are to add the given sides first and then multiply the sum (*a*+*b*) by the difference of the sides (*a*-*b*). This yields $a^2 - b^2$, which is to be divided by the base (*x*+*y*). The result *z* thus obtained is to be subtracted from the same base and then divided by 2 to obtain the values of the segments *x* and *y* respectively.

So we have

$$\frac{(a+b)(a-b)}{(x+y)} = \frac{a^2 - b^2}{(x+y)} = z,$$

$$(x+y) - z = w,$$

$$\frac{w}{2} = x$$

In our case we have to interchange the values for *a* and *b* to get the right result!

Despite the lack of theoretical proofs for the formulas and rules given by Bhaskara, we can see that his contribution in geometry has been truly magnificent in Lilavati!

11. PULVERIZER

Pulverizer is about solving the indeterminate equation: $ax \pm by = c$. It is the most interesting item in Lilavati, based on the Euclidean algorithm and linking up to modern algebra. The pulverizer (*kuttaka)* was actually initiated by Aryabhata in the 5th century C.E. However, Bhaskara contributes with an improved method giving the following rule, concerning the integer solutions to the indeterminate equation: *In the first place, as preparatory to the investigation of a pulverizer, the dividend, divisor and additive quantity are, if practicable, to be reduced by some number. If the number by which the dividend and divisor are both measured, do not also measure the additive quantity, the question is an ill put (or impossible) one.*

The last remainder, when the dividend and divisor are mutually divided, is their common measure. Being divided by that common measure, they are termed reduced quantities. Divide mutually the reduced dividend and divisor, until unity be the remainder in the dividend. Place the quotients one under the other, and the additive quantity beneath them, and cipher (zero) *at the bottom. By the penult multiply the number next above it and add the lowest term. Then reject the last and repeat the operation until a pair of numbers be left. The uppermost of these being abraded* (rubbed off) *by the reduced dividend, the remainder is the quotient. The other (or lowermost) being in like manner abraded by the reduced divisor, the remainder is the multiplier.*

Thus precisely is the operation when the number of quotients is even. But if the number be odd, the numbers as found must be subtracted from their respective abraders, and the residues will be the true quotient and multiplier.

As usual Bhaskara sets an example: *Say quickly, mathematician, what that multiplier is, by which two hundred and twenty-one being multiplied, and sixty-five added to the product, the sum divided by a hundred and ninety-five becomes exhausted.*
The dividend in this problem is 221, the divisor is 195 and the additive is 65. The common divisor of these numbers, as we see, is 13. By this divisor (the common measure) we reduce the given numbers to their least terms 17, 15 and 5 respectively.

The reduced dividend and divisor are to be divided reciprocally,

```
17)15(0
    00
 -------------
    15)17 (1
       15
    ……………
       02)15 (7
           14
    …………..
            1
```

and the quotients put one under the other, the additive under them, and zero (*cipher*) at the bottom. The series which results is:

$$\begin{array}{c}1\\7\\5\\0\end{array}$$

Then multiplying by the penult (second last) the number above it and proceeding as directed, the two quantities are got; 35 and 40. i.e. ($0+7\times 5=35$) and ($5+35\times 1=40$), yielding:

$\dfrac{17\times 35+05}{15}=40$. The other values for the equality Bhaskara gives are (6, 5) and

(23, 20). However, as we note, these are only a few of the solutions available for this type of a linear equation.

In transforming the above problem into our modern algebra we would get it set in the following form:

$$\frac{221x+65}{195}=y \Leftrightarrow 221x-195y=-65.$$

The $GCD(221,195)$ is 13 and $13\,|\,65$.

So we are able to reduce the equation to $15y-17x=5$.
Since 15 and 17 are prime numbers, *GCD* (17,15)=1. This signifies that we can solve an auxiliary equation $15y-17x=1$.

$$\begin{cases}17=1\times 15+2\\15=7\times 2+1\end{cases}$$

Thus we have
$$1=15-7\times 2=15-7(17-15)=15-7\times 17+7\times 15=8\times 15-7\times 17.$$

$$\because\begin{cases}y_0=8\\x_0=7\end{cases}$$

So a solution to the equation $15y-17x=5$ is $\begin{cases}y_1=8\times 5=40\\x_1=7\times 5=35\end{cases}$

The general solution of $15y-17x=5$ is as follows: $y=40+(-17)n;\quad x=35-15n$.
$$y=40+(-17)n;\quad x=35-15n$$

Since $x>0$ and $y>0$ are to be positive integers; *i*) $35-15n>0 \Leftrightarrow 35>15n; n\le 2$
ii) $40-17n>0 \Leftrightarrow 40>17n; n\le 2$

All the positive solutions to our equation are got for n = 2,1,0,-1,-2,….

So that *(x,y)*=(5,6), (20,23), (35,40), (50,57),..

The method used by Bhaskaracharyya in solving this type of a problem is tedious and does not give us any general solution as we can acquire in the present day method of solving the linear equations. However Bhaskara proves his ingenuity in dealing with such mathematical problems in his time.

12. THE IMPORTANCE OF LILAVATI

The importance of Lilavati lies in its challenging recreational mathematical problems set in poetic form, covering the property of zero, calculations including negative numbers, working out square and cube roots. It also deals with the solutions of problems involving money and capital gains, arithmetical and geometrical progressions, plane and solid geometry. In addition, the book takes up combinatorics and indeterminate equations of the first order, completing and forming perfect squares, and combination of digits.

Besides mathematics, the book extends the concept of enumeration by words from *kharva* to *parardha*: *kharva* (ten billion), *nikharva* (hundred billion), *mahapadma* (thousand billion), *sanku* (ten thousand billion), *jaldhi* or *samudra* (hundred thousand billion), *antya* (million billion), *madhya* (ten million billion) and *parardha* (hundred million billion).

Bhaskaracharya's rule (a.0)/0 = a, given in Lilavati, is equivalent to the concept of a non-zero "infinitesimal". Although his claim is not without foundation, perhaps it is seeing in it ideas beyond what Bhaskaracharyya intended.

13. BHASKARA BESIDES LILAVATI

As mentioned before, Lilavati is a part of *Siddhanta Siromani*, with the sequels, *Bijaganita* (algebra), *Goladhyaya* (sphere/celestial globe) and *Grahaganita* (mathematics of the planets), produced by Bhaskaracharyya.
He derived a cyclic method '*Chakrawal*' for solving equations of the form $ax^2+bx+c=y$ and the method for finding the solutions of the problem $Nx^2+1=y^2$, the so called Pell's equation.

Bhaskaracharyya is thought to be the first to show that d(*sin x) = cos x* d*x*.

In Astronomy Bhaskaracharyya postulated *'Objects fall on earth due to a force of attraction by the Earth. Therefore the Earth, the planets, constellations, the moon and the Sun are held in orbit due to this attraction,'* in his work, *Surya Siddhanta*. As we know, Sir Isaac Newton rediscovered this Law of Gravity in 1687. Bhaskara II also calculated the time taken for the Earth to orbit the Sun to 365.258756484 days against 365.2596 days (The modern accepted measurement) and broached the fields of infinitesimal calculation and integration.

By means of his *Chakrawal* method for solving indeterminate equations of degree 2, in Pell's equation $Nx^2 + 1 = y^2$ for $n = 8, 11, 32, 61$ and 67, when $n = 61$ Bhaskaracharyya found the solutions $x = 226153980$, $y = 1776319049$ and when $n = 67$, he found the solutions $x = 5967$, $y = 48842$.

Of particular interest are Bhaskaracharyya's well-known trigonometric equalities:

sin (a+b) = sin a cos b + cos a sin b; sin (a-b) = sin a cos b – cos a sin b.

14. SOME RECREATIONAL EXAMPLES FROM LILAVATI

1. Beautiful and dear Lilavati, whose eyes are like a fawn's! Tell me numbers resulting from one hundred and thirty-five, taken into twelve, if thou be skilled in multiplication by whole or by parts, whether by subdivision of form or separation of digits. Tell me auspicious woman, the quotient of the product divided by the same multiplier.

2. The quarter of a sixteenth of the fifth of three-quarters of two-thirds of a moiety of a *dramma* was given to beggar by a person, from whom he asked alms: tell me how many cowry shells the miser gave, if thou be conversant, in arithmetic, with the reduction termed subdivision of fractions.

3. Pretty girl, with tremulous eyes, if thou know the correct method of inversion, tell me the number, which multiplied by three, and added to three-quarters of the product, and divided by seven, and reduced by subtraction of a third part of the quotient, and then multiplied into itself, and having fifty-two subtracted from the product and the square root of the remainder extracted, and eight added, and the sum divided by ten, yields *two.*

4. The root subtracted, and the difference given. One pair out of a flock of geese remained sporting in the water, and saw seven times the half of the square root of the flock proceeding to the shore tired of diversion. Tell me, dear girl, the number of the flock.

5. Out of a heap of pure lotus flower, a third part, a fifth and a sixth were offered respectively to the gods Siva, Vishnu and the Sun; and a quarter was presented to

Bhavani. The remaining six lotuses were given to the venerable preceptor. Tell quickly the whole number of lotuses.

6. Out of a swarm of bees, one fifth part settled on a blossom of Kadamba, and one third on a flower of silindhri; three times the difference of those numbers flew to the bloom of a Kutaja. One bee which remained hovered and flew about in the air allured at the same moment by the pleasing fragrance of a jasmine and pendants. Tell me charming woman, the number of bees.

7. The root and a fraction both subtracted. Of a flock of geese, ten times the square root of the number departed for the Manasa lake, on the approach of a cloud: and eighth part went to a forest of Sthalapadminis: three couples were seen in sport on the eater abounding with delicate fibres of the lotus. Tell dear girl, the whole number of the flock.

8. The eighth part of a troop of monkeys, squared, was skipping in a grove and delighted with their sport. Twelve remaining were seen on the hill, amused with chattering to each other. How many were they in all?

9. The fifth part of a troop of monkeys less three, being squared, had gone to a cave; and one monkey was in sight, having climbed on a branch. Say how many they were.

10. Enraged in the battle, Partha [Arjuna] shot a host of arrows to kill Karna. With half the arrows he turned aside the host of arrows of that [opponent]; with four times the square-root of the total, he killed his horses; with six arrows, he killed [his charioteer] Salya; then with three arrows he destroyed the umbrella, the banner, and the bow [of his enemy], and with one, he cut off his head. How many arrows did Arjuna shoot? [This is from an episode in the *Mahabharata*.]

11. On an expedition to seize his enemy's elephants, a king marched two yojanas the first day. Say intelligent calculator, with what increasing rate of daily march he proceeded, he reaching his foe's city, a distance of eighty yojanas, in a week.

12. If a bamboo, measuring thirty two cubits and standing upon level ground, be broken in one place by the force of the wind, and the tip of it meet the ground at sixteen cubits: say mathematician, at how many cubits from the root it is broken.

13. A snake's hole is at the foot of a pillar, nine cubits high, and a peacock is perched on its summit. Seeing a snake at the distance of thrice the pillar gliding towards his hole, he pounces obliquely upon him. Say quickly at how many cubits from the snake's hole they meet, both proceeding an equal distance.

14. In a certain lake swarming with ruddy geese and cranes, the tip of a bud of lotus was seen a span above the surface of the water. Forced by the wind it gradually advanced, and was submerged at the distance two cubits. Compute quickly, mathematician, the dept of the water.

15. How many statues of Sambhu [Siva] [can] there be, with [his attributes] the rope, the elephant-hook, the serpent, the drum, the skull, the trident, the corpse-bier, the dagger, the arrow, and the bow held in his different hands? And how many of Hari [Visnu], with the club, the discus, the lotus, and the conch-shell?

16. The [number of] different [numbers which was explained] previously is divided by the [number of] different [numbers that can be produced] in the [number of] places [occupied by] identical digits---[these divisors are] computed separately [when there is more than one set of identical digits]---[and that] is the [number of] different [numbers produced from the specified digits].

17. Tell me quickly, mathematician, how many numbers are [produced] with the numbers two, two, one, and one, and [how much is] their sum, and similarly with the numbers four, eight, five, five, and five, if you are skilled in the procedure for the "net of numbers."

DEFINITIONS OF TECHNICAL TERMS in Lilavati

Money by tale

1. One rupee is equal to one hundred paise in modern India.

2. Twice ten cowry shells are *kákiní*; four of these are a *pana*; sixteen of which must be here considered as a *dramma*; and in like manner, a *nishka*, as consisting of sixteen of these.

Weights

3. A *gunja* (or seed of *Abrus*) is reckoned equal to two barley-corns; a *valla*, to three *gunjas*; and eight of these are a *dharana*; two of which make a *gadyánaka*; In a like manner, one *dhataka* is composed of fourteen *vallas*.

4. Half ten *gunjas* are called a *másha*, by such as are conversant with the use of the blance; a *karsha* contains sixteen of what are termed *máshas*; a *pala*, four *karshas*. A *karsha* of gold is named *suvarna*.

Measures

5. Eight breadths of a barley-corn are here a finger; four times six fingers, a cubit; four cubits, a staff; and a *krosa* contains two thousand of these; and a *yojana*, four *krosas.*

6. So a bamboo pole consists of ten cubits; and a field (or a plane figure) bounded by four sides, measuring twenty bamboo poles, is a *nivartana*.

7. A cube, which in length, breadth and thickness measures a cubit, is termed a solid cubit: and, in the meting of corn and the like, a measure, which contains a solid cubit, is a *khári* of *Magadha* as it is denominated in science.

8. A *drona* is the sixteenth part of a *khári*; an *ádhaka* is a quarter of *dorna*; a *prastha* is a fourth part of an *ádhaka*; and a *kudaba* is by the ancients termed a quarter of a *prastha*.

NAMING THE NUMBERS BY WORDS

Numeration by words expressed in regularly increasing decuple multiples found in Lilavati ($10^n, n = 0,...,17.$):

Eka= unit
D*asa*= ten
S*ata*= hundred
S*ahasra*= thousand
A*yuta*= ten thousand
L*aksha*= myriad (hundred thousand)
P*rayuta* = million
K*oti* = ten million
A*rbuda*= hundred million
A*bja or padma*= thousand million (billion)
K*harva*= ten billion
N*ikharva*= hundred billion
M*ahapadma*= thousand billion (trillion)
S*anku*= ten trillion
J*aldhi* or *samudra*= hundred trillion

A*ntya*= quadrillion
M*adhya*= ten quadrillion
P*arardha*= hundred quadrillion

The new words in enumeration entered in the 12^{th} C: (*kharva…… parardha*)

15. CONCLUSION

Although Bhaskaracharyya understood the importance of the powerful zero as a number, he did not veritably interpret its role in the denominator of fractions.

With regard to the conventional mathematical rules, Bhaskaracharyya is completely wrong about the solution of his problem involving the zero as a divisor.

The method of squaring which Bhaskara uses is obviously easy to follow and is quite compatible with our modern version.

In extracting the square root by hand, Bhaskara provides a good, but not easy, method to find roots of numbers when we are not equipped with mechanical gadgets.

Perhaps Aryabhata's method is more appealing for this type of an operation!
In finding a cube of a number, the method used is the same as the one we use in modern times. Regarding the extraction of cube roots, despite the several different methods at our disposal, the Bhaskaracharyya's, as well as Aryabhata's, methods might be the more straightforward and fast.

Completing and forming perfect squares produced by Bhaskara II is clearly a skilled operation applicable in all ages and in every part of the world!

Operations on proportionality executed by Bhaskaracharayya seem to be the same as those we use these days.

On the topic of principal and interest on money, we see that although our calculations regarding the simple (non-compound) rate of interest are the same as Bhaskara's, the computation regarding compound interest which we perform is, of course, much more advanced than Bhaskara's.

In the section of permutations and combinations, Bhaskaracharyya arrives at the correct results, but has no reference to the general formulas for permutations or for combinations.

Somewhat surprisingly, here in arithmetical progression, Bhaskaracharyya evokes the proper flavour of the general formulas concerning the operations conducted in the text. He does likewise in the geometrical progression leading us to a general formula, stated in the text. Hence no time gap is felt in this matter, because we use the same formula in modern times.

Pythagoras theorem, being part of Bhaskara's Geometry, is propounded through the same ideas as used today in our schools.

An invariant (*lamba*) perpendicular in geometry, based on the perspective of similar triangles, is undoubtedly well applicable in the calculations carried out in modern times.

In solving the indeterminate equation of the pulverizer, the method used by Bhaskaracharyya is rather tedious and does not give us any general solution such as we can acquire by the present day method of solving the indeterminate linear equations. However Bhaskara does exhibit his ingenuity in dealing with such mathematical problems of the epoch.

Most of the problems treated in Lilavati are solved by mechanical appliance of rules stated by Bhaskaracharyya without any theoretical proofs. Nevertheless, it can be perceived that he had great mathematical intuition.

References

http://www-groups.dcs.st-andrews.ac.uk/~history/HistTopics/Jaina_mathematics.html
Jan Thompson, Historiens matematik. ISBN 91-44-31011-0. Studentlitteratur, Lund 1991
Victor J. Katz, A History of Mathematics. ISBN 0-321-01618-1
Morris Kline, Mathematics for the Nonmathematician. ISBN 0-486-24823-2. Dover Publ.
Walter E. Clark, The Aryabhatiya of Aryabhata. University of Chicago, 1930
Lilavati, Henry Thomas Colebrooke's Translation
Webster's Encyclopaedic Unabridged Dictionary of the English Language, G. Books
Encyclopaedia Britannica, ISBN 0-85229-173-6

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