EXAMENSARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Optimal Cash Management

av

Elin Oxenhielm

2005 - No 7

Optimal Cash Management

Elin Oxenhielm

Examensarbete i matematik 10 poäng

Handledare: Jan-Erik Björk

2005

Abstract

We study the problem of optimising the flow of cash in a large company when the cash reserve is assumed to follow a jump-diffusion process. This means that we have a Brownian motion to model ordinary fluctuations and a Poisson process to model large fluctuations, jumps, that occur due to rare unpredictable events such as wars or natural disasters. We derive the solution to this stochastic optimal control problem for both a pure diffusion model and a jump-diffusion model. The solution will be of "bang-bang" type, i.e. there exists a "switch point" at which it is optimal to change strategy. By numerical studies we analyse the switch point's qualitative properties, as well as the value of all cash that we take out over time as a function of the initial cash level. We also interpret this problem as the problem of deciding how much to supply of a specific commodity (we study the U.S. crude oil price in detail) based on an observed price, maximising the total supplied amount over time constraint to a maximum production capacity.

Contents

1 Introduction

1.1 The Problem

Imagine that we are fishermen; interested in determining how much salmon we can fish from a lake without threatening the survival of them. We can assume that the number of salmons in the lake evolves in time according to a diffusion process. But sometimes, rare events can occur that can kill many salmons at one point in time, e.g. natural disasters or wars. It is suitable to model these events by a Poisson process with a small intensity rate. This is important, as these events definitely have an impact on how many salmons we can fish without threatening their survival.

The problem of finding how many salmons that we can fish is an optimal control problem, in which we want to maximise the total number of salmons that we can fish, and in the same time we must make sure that they survive. This problem can be translated into many other examples, and they are all equivalent in a mathematical setting. We list some examples here, stating the key mathematical properties of each of them:

1. **The number of salmons in a lake**
   Control function: How many we actually fish up.
   Optimisation incentive: Maximise the total amount of salmons fished over time.
   Conflict: If we fish too much, the survivals of salmons in the lake is threatened.
   Constraint: Bound on maximum fishing capacity.

2. **Cash reserve of a large company**
   Control function: Taking cash out for e.g. dividends and investments.
   Optimisation incentive: Maximise the total amount of cash that we take out over time.
   Conflict: Risk for liquidation if cash level becomes too small.
   Constraint: Bound on maximum amount that we can take out at one point in time.

3. **Prices of raw materials**
   Control function: How much to supply.
   Optimisation incentive: Maximise the total supplied amount over time.
   Conflict: If price becomes too low, it is not profitable to produce.
   Constraint: Bound on suppliers' production capacity.

This Thesis will mainly be concerned with examples 2 and 3, but the problem can easily be translated into example 1 as well as other ones. Note that the examples are analysed under further assumptions, which we will present later.

By studying these examples, we see that *the main problem is to find the specific control function that optimises the optimisation incentive and satisfies the constraint, and in the same time not takes the risk outlined in the conflict.*

So what do we expect the solution to this problem to be? If we look at the second example, the optimisation incentive tells us to maximise the total amount of cash that we take out, so the answer should be to take out cash at the maximum possible speed. On the other hand, the conflict tells us to stop if the cash is below some level, so that there will not be risk for liquidation. The main difficulty is thus to find this "switch point" for which it is optimal to stop the payments. We are also interested in determining the value of all dividends paid out over time for a given cash level, which we optimised.

1.2 Objective

Our analysis is being based on the article by Jeanblanc and Shiryaev [2], in which this problem is analysed for a pure diffusion model. But this article is quite theoretic and not so accessible for non-mathematicians. Therefore, we will make parts of these results explicit and also do some numerical studies, which will serve as an important verification of the results and in the same time give us the possibility to visualise the results in various plots. This is possible as one gets an analytic solution that is relatively easy to deal with.

Further, we will look at the problem as described in the article by Belhaj [1], that presents the jump-diffusion model, the optimal control problem and solves it for some examples of the distribution of the jump's size. We will look at the case with an exponential distribution, as we will see that it is the easiest one to deal with. But in [1] no conclusions are drawn concerning qualitative properties of the results. Our objective is to analyse the qualitative properties by conducting some numerical investigations. The difficulty with this model is that it is too difficult to derive an analytic solution in this case, so we must rely on numerical examples.

1.3 Organisation of the Thesis

- **SECTION 2:** In section 2 we will provide the prerequisites necessary to understand the mathematical formulation of the problem, i.e. basic definitions from stochastic calculus and some information on the Laplace transform, that we will use to solve the optimisation problem. We will also set up the general optimal control problem in both the deterministic and stochastic case.

- **SECTION 3:** In Section 3 we will present the jump-diffusion model we will analyse. We present methods to obtain the switch point and the value function. We also determine the Laplace transform for the general solution explicitly.

- **SECTION 4:** Section 4 will be concerned with analysing the pure diffusion model. In this case it is quite easy to determine the switch point and the value function. We show that the value function is always concave in this case, which is not generally true when we work with the jump-diffusion model. We also study the nature of the solution by numerical studies.

- **SECTION 5:** In Section 5 we will analyse the jump-diffusion model when the size of the jumps is exponentially distributed. We derive a general equation for the switch point. Then we look at the crude oil price and estimates parameters based on this data. After that we do some numerical studies to analyse what happens when we allow one or more jumps to occur.

2 Preliminaries

2.1 Stochastic Calculus

2.1.1 DEFINITION: A *σ-algebra* F on Ω is a collection of subsets on Ω such that

- It is not empty; $\varnothing \in F$ and $\Omega \in F$;
- If $A \in F$ then $A^c \in F$; and
- If $A_1,\ A_2,\ \ldots\ \in F$ then $\bigcup_{i=1}^{\infty} A_i \in F$ and $\bigcap_{i=1}^{\infty} A_i \in F$.

2.1.2 DEFINITION: (Ω, F, P) is a *probability space* if Ω is the set of all possible outcomes, F is a σ-algebra and P denotes the probability function.

2.1.3 DEFINITION: A collection $\{F_t\}_{t>0}$ of σ-algebras on Ω is a *filtration* if

$F_s \subset F_t \quad \forall\ 0 \le s \le t$.

2.1.4 DEFINITION: A *stochastic process* x is a collection of random variables $\{x_t,\ \ t \in T\} = \{x_t(\alpha),\ \ t \in T,\ \ \alpha \in \Omega\}$ defined on a probability space (Ω, F, P).

2.1.5 DEFINITION: A stochastic process $\{w_t\}_{t>0}$ is a *standard Wiener process (or a Brownian motion)* if

- $w_0 = 0$;
- $\{w_t\}_{t>0}$ has stationary independent increments, i.e. $w_t - w_s$ has the same distribution as $w_{t+h} - w_{s+h}\ \forall\ t, s \in T$ such that $t+h, s+h \in T$ and $\forall\ t_i \in T$ such that $t_1 < \ldots < t_n$ and $n \ge 1$ we have that $w_{t_2} - w_{t_1},\ \ldots\ ,\ \ w_{t_n} - w_{t_{n-1}}$ are independent random variables;
- $\forall\ t > 0,\ \ w_t \sim N(0, t)$; and
- $\{w_t\}_{t>0}$ has continuous sample paths.

2.1.6 REMARK: We have that $dw_t \sim N(0,\sqrt{dt})$ as $dw_t = \varepsilon\sqrt{dt}$ where ε is a random drawing from a standardised normal distribution.

2.1.7 DEFINITION: A stochastic process $\{N_t\}_{t>0}$ is a *Poisson process* if it satisfies the first and second properties of the standard Wiener process, and

- $\forall\ t>0,\ \ N_t \sim Poi(\lambda t)$, where *Poi* denotes the Poisson distribution. For example, it has the following properties (called the *Poisson Postulates*):
  - The number of events occurring in nonoverlapping time intervals are independent;
  - The probability structure is time-invariant;
  - The probability of exactly one event in an infinitesimal interval is approximately proportional to its length; and
  - The probability of finding more than one event in an infinitesimal interval is smaller than the probability of finding one.

2.2 The Laplace Transform

2.2.1 DEFINITION: The function $y = y(t)$ defined on $t \in [0,\infty)$ belongs to $L^2[0,\infty)$ if

$$\int_0^\infty |\ y(t)\ |^2\ dt < \infty$$

2.2.2 DEFINITION: The Laplace transform of a function $y(t) \in L^2[0,\infty)$ is a function $\mathscr{L}_y(\xi)$ defined as

$$\mathscr{L}_y(\xi) = \int_0^\infty e^{-\xi\cdot t}\, y(t)dt$$

The Laplace transform satisfies several nice properties. For example, $\mathscr{L}_{y'}(\xi) = \xi\cdot\mathscr{L}_y(\xi) - y(0)$, so one can use Laplace transforms to transform differential equations into algebraic equations.

2.3 Deterministic Optimal Control

2.3.1 DEFINITION: $u(x)$ is called a *utility function* if $u(0)=0$ and if u is a strictly concave continuously differentiable function, such that the derivative $x \mapsto u'(x)$ decreases strictly from $+\infty$ to 0.

2.3.2 EXAMPLE: A common example of a utility function is $u(x)=\sqrt{x}$.

2.3.3 THE PROBLEM: The deterministic optimal control problem is set up as follows: Let $u(x)$ be a utility function and $\rho > 0$ a discount factor. The problem is to find the function $u(x)$ that maximises

$$\int_0^\infty e^{-\rho \cdot t} \cdot u(\dot{x}(t))dt$$

Therefore, we let

$$\mathscr{J}(A) = \text{ Max } \int_0^\infty e^{-\rho \cdot t} \cdot u(\dot{x}(t))dt$$

where $\dot{x}=\frac{dx}{dt}$ and we have the condition $\int_0^\infty \dot{x}(t) \cdot dt = A$, which expresses the total disposal of resources.

2.3.4 THE SOLUTION: The solution to this problem is according to Euler given by the differential equation:

$$e^{-\rho \cdot t} \cdot u'(\dot{x}) = C_0$$

where C_0 is a constant.

2.4 Stochastic Optimal Control

2.4.1 THE PROBLEM: The stochastic optimal control problem is set up as follows: We denote by $c(t)$ the consumption and by $\{x_t\}_{t\geq 0}$ the level of capital at time t that is governed by the following process:

$$dx_t = c(t)\cdot dt + \sigma\cdot x_t\cdot dw_t$$

where $\{w_t\}_{t\geq 0}$ is a standard Wiener process and the x_t in front of it means that we have a geometric process. $\sigma > 0$ is the volatility constant of the Wiener process. The problem is to find the function $t \mapsto c(t)$ that maximises the expected value

$$E\int_0^\infty u(c(t))\cdot e^{-\rho\cdot t}dt$$

We also have the condition that the process stops if $x(\tau) = 0$ for some $\tau \geq 0$. We note that the solution to this problem is not given in advance. As a general reference we refer to the account of stochastic optimisation as presented in chapter 20-21 in the book by Kamien-Schwartz.

2.4.2 THE SOLUTION: If the solution is not given in advance, how does one come about a solution to this problem? We start by defining

$$\mathcal{J}(A;\sigma) = \text{Max } E\int_0^\infty u(c(t))\cdot e^{-\rho\cdot t}dt$$

when the initial asset value is A, i.e. $x(0) = A$. To determine an expression of $\mathcal{J}(A;\sigma)$, we consider a small interval in time $[0,\delta t]$ in which $c(s)\equiv c$ for all $0\leq s\leq \delta t$. The value at time δt is approximately equal to $(1-\delta t)\cdot\mathcal{J}(A;\sigma)$ if the value of the asset at time 0 is $\mathcal{J}(A;\sigma)$, since $\delta t\cdot\mathcal{J}(A;\sigma)$ is consumed in this time interval, given that the consumption is maximised. This gives that the profit is equal to

$$u(c)\cdot\delta t - \rho\cdot A\cdot\mathcal{J}(A;\sigma)\cdot\delta t$$

as we get what we consume in the time interval $[0,\delta t]$ and we subtract the discounted value of this. This profit is approximately equal to

$$\mathcal{J}(x_{\delta t};\sigma) = \mathcal{J}(A - c\cdot\delta t - A\cdot\sigma\cdot dw_{\delta t};\sigma)$$

where $w_{\delta t}$ is a random variable such that $E(w_{\delta t}) = 0$ and $Var(w_{\delta t}) = \delta t$.

Using these properties and evaluating in a Taylor expansion yields the expected profit:

$$E[\mathcal{J}(x_{\delta t};\sigma)] = E[\mathcal{J}(A - c\cdot\delta t + A\cdot\sigma\cdot dw_{\delta t};\sigma)] =$$

$$= \mathcal{J}(A;\sigma) - c\cdot\mathcal{J}'(A;\sigma)\cdot\delta t + \frac{A^2\cdot\sigma^2}{2}\cdot\mathcal{J}''(A;\sigma)\cdot\delta t + o(\delta t)^{3/2}$$

so we should choose the function c so that we maximise

$$u(c) - c\cdot\mathcal{J}'(A;\sigma) \quad\Leftrightarrow\quad u'(c) = \mathcal{J}'(A;\sigma)$$

where the fact that u is strictly concave implies that the problem has a unique solution.

3 The Model

In this section, we use the interpretation of the problem of optimising the flow of cash in a large company.

3.1 Definition of the Model

Let m be the cash held at time zero, $\mu \geq 0$ the constant growth rate, $\sigma > 0$ the Brownian volatility rate, $\lambda \geq 0$ the intensity of the Poisson process $\{N_t\}_{t>0}$ and s a random variable denoting the size of the Poisson jump. s can be discrete or continuous.

We make the following assumptions:

- The liquidation value of the firm is zero, i.e. the firm goes into bankruptcy the first time its cash reserves becomes negative;
- The firm has no access to external financing;
- The firm has no possibility to invest its cash reserves in the stock market or in a risk-free asset.

Under the assumptions and definitions above, the firm's cash reserves evolve according to the process:

$$dm_t = \mu \cdot dt + \sigma \cdot dw_t - s \cdot dN_t - dL_t$$

where $\{L_t\}_{t>0}$ denotes the total amount of cash taken out up to time *t*. L_t is assumed to be non-negative and right-continuous. In deciding the strategy L_t at a time t we only have access to the information given at that specific time, i.e. L_t is adapted to a filtration $\{F_t\}_{t>0}$. L_t is also assumed to be bounded by a so-called technological constraint, i.e. $L_t < K < \infty$ for some constant K. This is the bound on the maximum amount of cash that one can take out at one point in time.

3.1.1 REMARK: In reality, the constants μ, σ, λ and the distribution for s in the model above should be estimated from empirical data using statistical estimation techniques. This means that when one inserts values of these parameters into the results derived in this Thesis one gets an inexact model that always involves some kind of error, which in turn can be estimated by observing the signal/noise ratio in the data and analysing the error of the estimation method one uses. This error should always be analysed before one starts using models that are inexact and the exactness of the results only depends on how good the estimations of the parameters are. The truth lies in the empirical data, not in the mathematical models. These are just approximations of the data and should always be handled with care. We should also say that Brownian motion in general is good for prediction, but bad with matching historical data – as it is a stochastic process.

3.2 The Optimal Control Problem

Recall from the introduction that we wanted to maximise the total amount of cash that we take out. The expected value of this is

$$V(m) = E_m \int_0^{\tau} e^{-\rho \cdot t} dL_t$$

where $\tau = \inf\{t : m(t) \le 0\}$ is the moment of bankruptcy and $\rho > 0$ is the discount factor. The expectation is taken conditioned on the initial reserve m. $V(m)$ is called the value function or the optimal return function. By maximising the expected value like this, we expect the optimal policy of taking out cash to be something like this: when the cash level is large enough, it is optimal to take out the amount K. And if the cash level is small, it is optimal to take out nothing. So the strategy itself is not so exciting. The main problem is to find what we call the *switch point*, below which we shall take out nothing. The existence of such a switch point and the solution to this optimal control problem is given by the following proposition:

3.2.1 PROPOSITION: There is a switch point m^* such that

- When $m \le m^*$, we have that $L(m) = 0$ and V and m^* are given by the following equations:

$$\begin{cases} \rho \cdot V(m) = \mu \cdot V'(m) + \frac{1}{2}\sigma^2 \cdot V''(m) + \lambda \cdot E_s[V(m-s) - V(m)] \\ V(m) = 0 \quad \forall \ m \le 0, \ \ V'(m^*) = 1, \ \ V''(m^*) = 0 \\ V'(m) \ge 0 \quad \forall \ m \ge 0 \end{cases}$$

- When $m > m^*$, we have that $L(m) = K$ and $V(m) = m - m^* + V(m^*)$.

We see that when $m = m^*$, the marginal value of taking out cash equal the marginal value of retaining cash. The value function is reflected at this point; the solution is of "bang-bang" type.

3.3 The Switch Point

How does one find the switch point m^*? According to the above, we should proceed as follows:

- First, we find the function f that solves

$$\begin{cases} \frac{1}{2}\sigma^2 \cdot f''(m) + \mu \cdot f'(m) - \rho \cdot f(m) + \lambda \cdot E_s[f(m-s) - f(m)] = 0 \\ f(m) = 0 \quad \forall \ m \le 0 \end{cases}$$

- Then the switch point m^* is the unique point in the set $\{m : f''(m) = 0, \ \ f'''(m) > 0\}$ that minimises $f'(m)$.

3.3.1 REMARK: The equation that we have to solve is a partial integro-differential equation for non-zero λ, involving the two unknown functions f and the frequency function for the jump size. The equation is only possible to solve analytically in a few cases, as we will see later. An interesting observation is to compare this equation with ones obtained when pricing options when the underlying stock follows a jump-diffusion process. If the option can be realised anytime of the holders choice, and if the holder also can choose how much of the invested value that he/she prefer to realise, we see that one is interested in finding the point in time when it is optimal to realise the option and how much it is optimal to realise at that time. The solution will also in this case be of "bang-bang" type, i.e. at a given point in time it is optimal to either realise as much as you can or to realise nothing – depending on the current value of the underlying stock.

3.4 The Value Function

The value function as a function of the initial cash level m is:

$$V(m) = \begin{cases} \dfrac{f(m)}{f'(m^*)}, & m \le m^* \\ m - m^* + V(m^*), & m^* < m \end{cases}$$

3.4.1 REMARK: One observation we can make immediately is that $V''(m) = 0$ for all $m \ge m^*$, i.e. V is linear in this region. So when analysing concavity/convexity one only needs to consider the case when $m < m^*$. We also see that the slope of the value function is always one after the switch point, independent of any of the parameters.

3.5 The Laplace Transform of the General Solution

To solve the problem, we will determine the Laplace transform of the general solution. This is as far as one can say things about the general solution without making any further assumptions on the frequency function for the jump size.

3.5.1 PROPOSITION: If the function f is in $L^2[0,\infty)$ and if f satisfies the equation

$$\begin{cases} \frac{1}{2}\sigma^2 \cdot f''(m) + \mu \cdot f'(m) - \rho \cdot f(m) + \lambda \cdot E_s[f(m-s) - f(m)] = 0 \\ f(m) = 0 \quad \forall \ m \le 0 \end{cases}$$

then f has the Laplace transform

$$\mathscr{L}_f(\xi) = \frac{\frac{1}{2}\sigma^2 \cdot f'(0)}{\frac{1}{2}\sigma^2 \cdot \xi^2 + \mu \cdot \xi - (\rho + \lambda) + \lambda \cdot E_s[e^{-\xi \cdot s}]}$$

where *s* is a random variable.

PROOF: As $f(0) = 0$, we have that $\mathscr{L}_{f'}(\xi) = \xi \cdot \mathscr{L}_f(\xi)$ and $\mathscr{L}_{f''}(\xi) = \xi^2 \cdot \mathscr{L}_f(\xi) - f'(0)$. If we by $s \mapsto \psi(s)$ denote the frequency function for *s*, the equation to solve becomes

$$\begin{cases} \frac{1}{2}\sigma^2 \cdot f''(m) + \mu \cdot f'(m) - (\rho+\lambda)\cdot f(m) + \lambda \cdot \int_0^\infty f(m-s)\cdot \psi(s)ds = 0 \\ f(m) = 0 \ \forall \ m \le 0 \end{cases}$$

where the Laplace transform of the integral is

$$\int_0^\infty e^{-\xi \cdot t} \int_0^\infty f(t-s)\cdot\psi(s)dsdt = \int_0^\infty \psi(s) \int_0^\infty e^{-\xi\cdot t} f(t-s)dtds =$$

$$= \int_0^\infty \psi(s) \cdot \left[\int_0^s e^{-\xi\cdot t} f(t-s)dt + \int_s^\infty e^{-\xi\cdot t} f(t-s)dt \right] ds =$$

$$= \int_0^\infty \psi(s) \int_s^\infty e^{-\xi\cdot t} f(t-s)dtds = \int_0^\infty \psi(s) \int_0^\infty e^{-\xi\cdot(t+s)} f(t)dtds = \int_0^\infty e^{-\xi\cdot s}\psi(s) \int_0^\infty e^{-\xi\cdot t} f(t)dtds =$$

$$= \mathscr{L}_f(\xi) \int_0^\infty e^{-\xi\cdot s}\psi(s)ds = \mathscr{L}_f(\xi)\cdot E_s[e^{-\xi\cdot s}]$$

We now take the Laplace transform of the whole equation, and obtain:

$$\tfrac{1}{2}\sigma^2 \cdot [\xi^2 \cdot \mathscr{L}_f(\xi) - f'(0)] + \mu\cdot\xi\cdot\mathscr{L}_f(\xi) - (\rho+\lambda)\cdot\mathscr{L}_f(\xi) + \lambda\cdot\mathscr{L}_f(\xi)\cdot E_s\left[e^{-\xi\cdot s}\right] = 0 \quad \Rightarrow$$

$$\mathscr{L}_f(\xi)\cdot\left[\tfrac{1}{2}\sigma^2\cdot\xi^2 + \mu\cdot\xi - (\rho+\lambda) + \lambda\cdot E_s[e^{-\xi\cdot s}]\right] - \tfrac{1}{2}\sigma^2 \cdot f'(0) = 0 \quad \Rightarrow$$

$$\mathscr{L}_f(\xi) = \frac{\frac{1}{2}\sigma^2 \cdot f'(0)}{\frac{1}{2}\sigma^2\cdot\xi^2 + \mu\cdot\xi - (\rho+\lambda) + \lambda\cdot E_s[e^{-\xi\cdot s}]}$$ □

3.5.2 REMARK: We see that in general it will be very difficult to find the inverse Laplace transform and thereby an analytical expression for the function f. In fact, the only cases we can analyse analytically are the ones when either $\lambda = 0$ (when we have a pure diffusion model) or when s has an exponential distribution, as these are the only cases that makes both the nominator and the denominator of the Laplace transform algebraic.

3.5.3 REMARK: The function $E_s[e^{-\xi\cdot s}]$ appears as a familiar and very useful tool in mathematical statistics.

3.5.4 EXAMPLE: Consider the case when s is a discrete stochastic variable, denoting jumps that occur due to interest rate decisions. To be precise, we let $s = 1$ with probability ½ and $s = 2$ with probability ½. Then we have that $E_s[e^{-\xi\cdot s}] = \frac{1}{2}(e^{-\xi} + e^{-2\xi})$ and the Laplace transform for the function f becomes

$$\mathscr{L}_f(\xi) = \frac{\frac{1}{2}\sigma^2 \cdot f'(0)}{\frac{1}{2}\sigma^2 \cdot \xi^2 + \mu \cdot \xi - (\rho + \lambda) + \frac{1}{2}\lambda \cdot (e^{-\xi} + e^{-2\xi})}$$

We see that in this case, the inverse Laplace transform is impossible to find without the use of advanced residue techniques.

4 Analysis of the Pure Diffusion Model

Throughout this section, we assume that there are no Poisson jump present in the model, i.e. $\lambda = 0$. This case was studied by Jeanblanc and Shiryaev in 1995 [2]. Below, we shall make some of their results explicit and do some numerical investigations.

4.1 The Switch Point

4.1.1 PROPOSITION: The function f is

$$f(m) = \frac{f'(0)}{b} \cdot e^{-a \cdot m} \cdot \sinh b \cdot m$$

where $a = \dfrac{\mu}{\sigma^2}$ and $b = \dfrac{\sqrt{\mu^2 + 2\rho \cdot \sigma^2}}{\sigma^2}$

PROOF: The Laplace transform of the function f becomes

$$\mathscr{L}_f(\xi) = \frac{\frac{1}{2}\sigma^2 \cdot f'(0)}{\frac{1}{2}\sigma^2 \cdot \xi^2 + \mu \cdot \xi - \rho} = \frac{f'(0)}{2b} \cdot \left(\frac{1}{\xi + a - b} - \frac{1}{\xi + a + b} \right)$$

so we have that

$$f(m) = \frac{f'(0)}{2b} \cdot (e^{-a \cdot m + b \cdot m} - e^{-a \cdot m - b \cdot m}) = \frac{f'(0)}{b} \cdot e^{-a \cdot m} \cdot \sinh b \cdot m$$

if we let $a = \dfrac{\mu}{\sigma^2}$ and $b = \dfrac{\sqrt{\mu^2 + 2\rho \cdot \sigma^2}}{\sigma^2}$. □

4.1.2 PROPOSITION: Assuming that $f'(0) > 0$, we have that the switch point is

$$m^* = \frac{1}{b} \cdot \ln\left(\frac{b + a}{b - a} \right)$$

where $a = \dfrac{\mu}{\sigma^2}$ and $b = \dfrac{\sqrt{\mu^2 + 2\rho \cdot \sigma^2}}{\sigma^2}$

PROOF: By taking derivatives we obtain:

$$f'(m) = \frac{f'(0)}{b} \cdot e^{-a \cdot m} \cdot [-a \cdot \sinh b \cdot m + b \cdot \cosh b \cdot m]$$

$$f''(m) = \frac{f'(0)}{b} \cdot e^{-a \cdot m} \cdot [(a^2 + b^2) \cdot \sinh b \cdot m - 2a \cdot b \cdot \cosh b \cdot m]$$

$$f'''(m) = \frac{f'(0)}{b} \cdot e^{-a \cdot m} \cdot [-a \cdot (a^2 + 3b^2) \cdot \sinh b \cdot m + b \cdot (3a^2 + b^2) \cdot \cosh b \cdot m]$$

Letting $f''(m^*) = 0$ gives that

$$(a^2 + b^2) \cdot \sinh b \cdot m^* - 2a \cdot b \cdot \cosh b \cdot m^* = 0 \quad \Leftrightarrow$$

$$(a^2 + b^2) \cdot [e^{b \cdot m^*} - e^{-b \cdot m^*}] - 2a \cdot b \cdot [e^{b \cdot m^*} + e^{-b \cdot m^*}] = 0 \quad \Leftrightarrow$$

$$(a^2 + b^2) \cdot [e^{2b \cdot m^*} - 1] - 2a \cdot b \cdot [e^{2b \cdot m^*} + 1] = 0 \quad \Leftrightarrow \quad e^{2b \cdot m^*} \cdot [a^2 + b^2 - 2a \cdot b] - [a^2 + b^2 + 2a \cdot b] = 0$$

$$\Leftrightarrow \quad e^{2b \cdot m^*} \cdot (a - b)^2 = (a + b)^2 \quad \Leftrightarrow \quad e^{2b \cdot m^*} = \left(\frac{a + b}{a - b} \right)^2 \quad \Leftrightarrow \quad 2b \cdot m^* = \ln \left(\frac{a + b}{a - b} \right)^2 \quad \Leftrightarrow$$

$$m^* = \frac{1}{b} \cdot \ln \left(\frac{b + a}{b - a} \right)$$

as we have that $a < b$.

We have to analyse when $f'''(m^*) > 0$, which is when $f'(0) > 0$ as seen by rewriting f''' as

$$f'''(m) = \frac{f'(0)}{2b} \cdot e^{-m \cdot (a+b)} \cdot [e^{2b \cdot m} \cdot (-a^3 - 3a \cdot b^2 + 3a^2 \cdot b + b^3) + a^3 + 3a \cdot b^2 + 3a^2 \cdot b + b^3] =$$

$$= \frac{f'(0)}{2b} \cdot e^{-m \cdot (a+b)} \cdot [e^{2b \cdot m} \cdot (b - a)^3 + (b + a)^3] \qquad \square$$

4.2 The Value Function

4.2.1 PROPOSITION: The value function is given by

$$V(m) = \begin{cases} \frac{1}{b} \cdot \left(\frac{b + a}{b - a} \right)^{\frac{a}{b}} \cdot e^{-a \cdot m} \cdot \sinh b \cdot m, & m \leq \frac{1}{b} \cdot \ln \left(\frac{b + a}{b - a} \right) \\ m - \frac{1}{b} \cdot \ln \left(\frac{b + a}{b - a} \right) + \frac{2a}{b^2 - a^2}, & \frac{1}{b} \cdot \ln \left(\frac{b + a}{b - a} \right) < m \end{cases}$$

where $a = \dfrac{\mu}{\sigma^2}$ and $b = \dfrac{\sqrt{\mu^2 + 2\rho \cdot \sigma^2}}{\sigma^2}$

PROOF: Inserting the switch point m^* into f' we obtain

$$f'(m^*) = \frac{f'(0)}{b} \cdot e^{-a\cdot m^*} \cdot [-a \cdot \sinh b \cdot m^* + b \cdot \cosh b \cdot m^*] =$$

$$= \frac{f'(0)}{2b} \cdot e^{-a\cdot m^*} \cdot [-a \cdot (e^{b\cdot m^*} - e^{-b\cdot m^*}) + b \cdot (e^{b\cdot m^*} + e^{-b\cdot m^*})] =$$

$$= \frac{f'(0)}{2b} \cdot e^{-m^*\cdot(a+b)} \cdot [-a \cdot (e^{2b\cdot m^*} - 1) + b \cdot (e^{2b\cdot m^*} + 1)] = \frac{f'(0)}{2b} \cdot e^{-m^*\cdot(a+b)} \cdot [e^{2b\cdot m^*} \cdot (b-a) + b + a] =$$

$$\frac{f'(0)}{2b} \cdot \left(\frac{b+a}{b-a}\right)^{-\frac{a}{b}-1} \cdot \left[\left(\frac{b+a}{b-a}\right)^2 \cdot (b-a) + b + a\right] = \frac{f'(0)}{2b} \cdot \left(\frac{b+a}{b-a}\right)^{-\frac{a}{b}-1} \cdot \left[\frac{b^2 + 2a\cdot b + a^2}{b-a} + b + a\right]$$

$$= \frac{f'(0)}{2b} \cdot \left(\frac{b+a}{b-a}\right)^{-\frac{a}{b}-1} \cdot \left[\frac{2b^2 + 2a\cdot b}{b-a}\right] = f'(0) \cdot \left(\frac{b+a}{b-a}\right)^{-\frac{a}{b}} = f'(0) \cdot \left(\frac{b-a}{b+a}\right)^{\frac{a}{b}}$$

so we have that

$$\frac{f(m)}{f'(m^*)} = \frac{1}{b} \cdot \left(\frac{b+a}{b-a}\right)^{\frac{a}{b}} \cdot e^{-a\cdot m} \cdot \sinh b \cdot m$$

and

$$\frac{f(m^*)}{f'(m^*)} = \frac{1}{b} \cdot \left(\frac{b+a}{b-a}\right)^{\frac{a}{b}} \cdot e^{-a\cdot m^*} \cdot \sinh b \cdot m^* =$$

$$\frac{1}{2b} \cdot \left(\frac{b+a}{b-a}\right)^{\frac{a}{b}} \cdot e^{-a\cdot m^*} \cdot [e^{b\cdot m^*} - e^{-b\cdot m^*}] = \frac{1}{2b} \cdot \left(\frac{b+a}{b-a}\right)^{\frac{a}{b}} \cdot [e^{-m^*\cdot(a-b)} - e^{-m^*\cdot(a+b)}] =$$

$$\frac{1}{2b} \cdot \left(\frac{b+a}{b-a}\right)^{\frac{a}{b}} \cdot \left[\left(\frac{b+a}{b-a}\right)^{-\frac{a}{b}+1} - \left(\frac{b+a}{b-a}\right)^{-\frac{a}{b}-1}\right] = \frac{1}{2b} \cdot \left[\frac{b+a}{b-a} - \frac{b-a}{b+a}\right] =$$

$$\frac{1}{2b} \cdot \left[\frac{(b+a)^2 - (b-a)^2}{b^2 - a^2}\right] = \frac{1}{2b} \cdot \left[\frac{4a\cdot b}{b^2 - a^2}\right] = \frac{2a}{b^2 - a^2} \qquad \square$$

4.3 Concavity of the Value Function

We recall from earlier observations that $V''(m) = 0$ for all $m \geq m^*$. The following proposition provides a robust result that not holds in the case with a non-zero λ, as we will see in the next section.

4.3.1 PROPOSITION: $V''(m) < 0$ for all $m < m^*$, i.e. the function V is strictly concave in this interval.

PROOF: $V''(m) = \dfrac{f''(m)}{f'(m^*)} = \dfrac{1}{b} \cdot \dfrac{f'(0)}{f'(m^*)} \cdot e^{-a\cdot m} \cdot [(a^2 + b^2) \cdot \sinh b \cdot m - 2a \cdot b \cdot \cosh b \cdot m] =$

$$= \frac{1}{2b} \cdot \left(\frac{b+a}{b-a} \right)^{\frac{a}{b}} \cdot e^{-a\cdot m} \cdot [(a^2 + b^2) \cdot (e^{b\cdot m} - e^{-b\cdot m}) - 2a \cdot b \cdot (e^{b\cdot m} + e^{-b\cdot m})] =$$

$$= \frac{1}{2b} \cdot \left(\frac{b+a}{b-a} \right)^{\frac{a}{b}} \cdot e^{-(a+b)\cdot m} \cdot [(a^2 + b^2) \cdot (e^{2b\cdot m} - 1) - 2a \cdot b \cdot (e^{2b\cdot m} + 1)] < 0$$

if

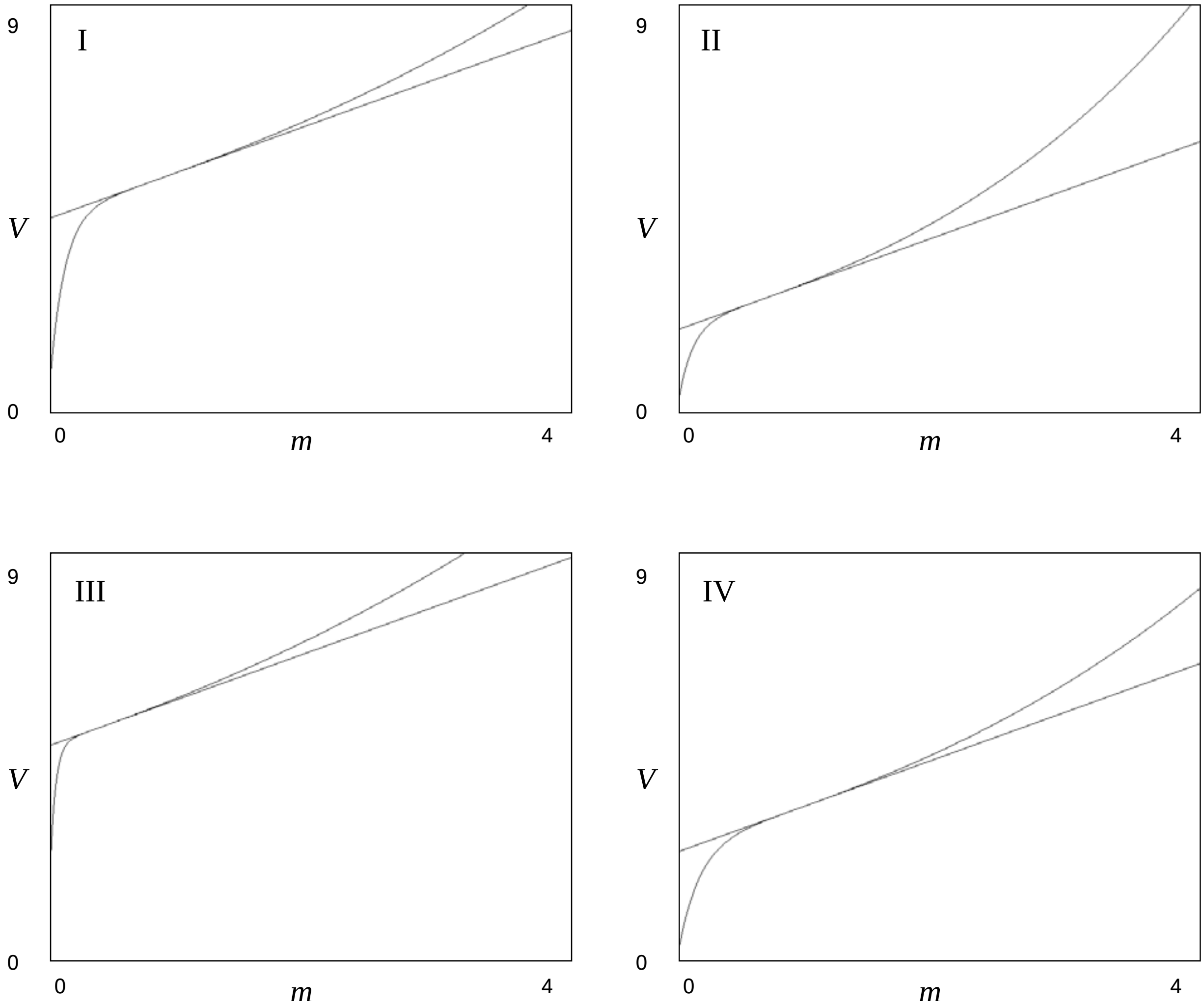
$$e^{2b\cdot m} \cdot (a-b)^2 - (a+b)^2 < 0 \quad \Leftrightarrow \quad e^{2b\cdot m} < \left(\frac{a+b}{a-b} \right)^2 \quad \Leftrightarrow \quad m < m^*$$

which was assumed. □

4.4 Numerical Study

The objective with this section is to analyse the results obtained in the pure diffusion model for various values of the parameters. We start by observing the components of the value function and we expect it to be concave up to the switch point and linear after that. Then we will study the switch point and how it depends on the parameters. We expect it to be increasing with σ and decreasing with ρ. But how does it depend on μ? Our calculations will show that this dependence looks like the beta distribution, an interesting fact that may have economic implications. Finally we will study when the firm is sensitive towards the Brownian risk induced by a non-zero σ. We will show that when σ is small the firm is sensitive up to the switch point, and when σ increases above some switch point, the firm is sensitive up to a point below the switch point that will converge quite fast to zero.

4.4.1 The Value Function



4.4.1.1 FIGURE: In this figure, the two components of the value function $m \mapsto V(m)$ are plotted. The value function follows the function that starts at zero, and when it reaches the switch point it switches to follow the linear line. We see that the value function is indeed concave up to the switch point, and after that it becomes linear (i.e. either concave or convex). The parameter values are as follows:

I: $\mu = 1/2$, $\sigma = 1/3$, $\rho = 1/10$

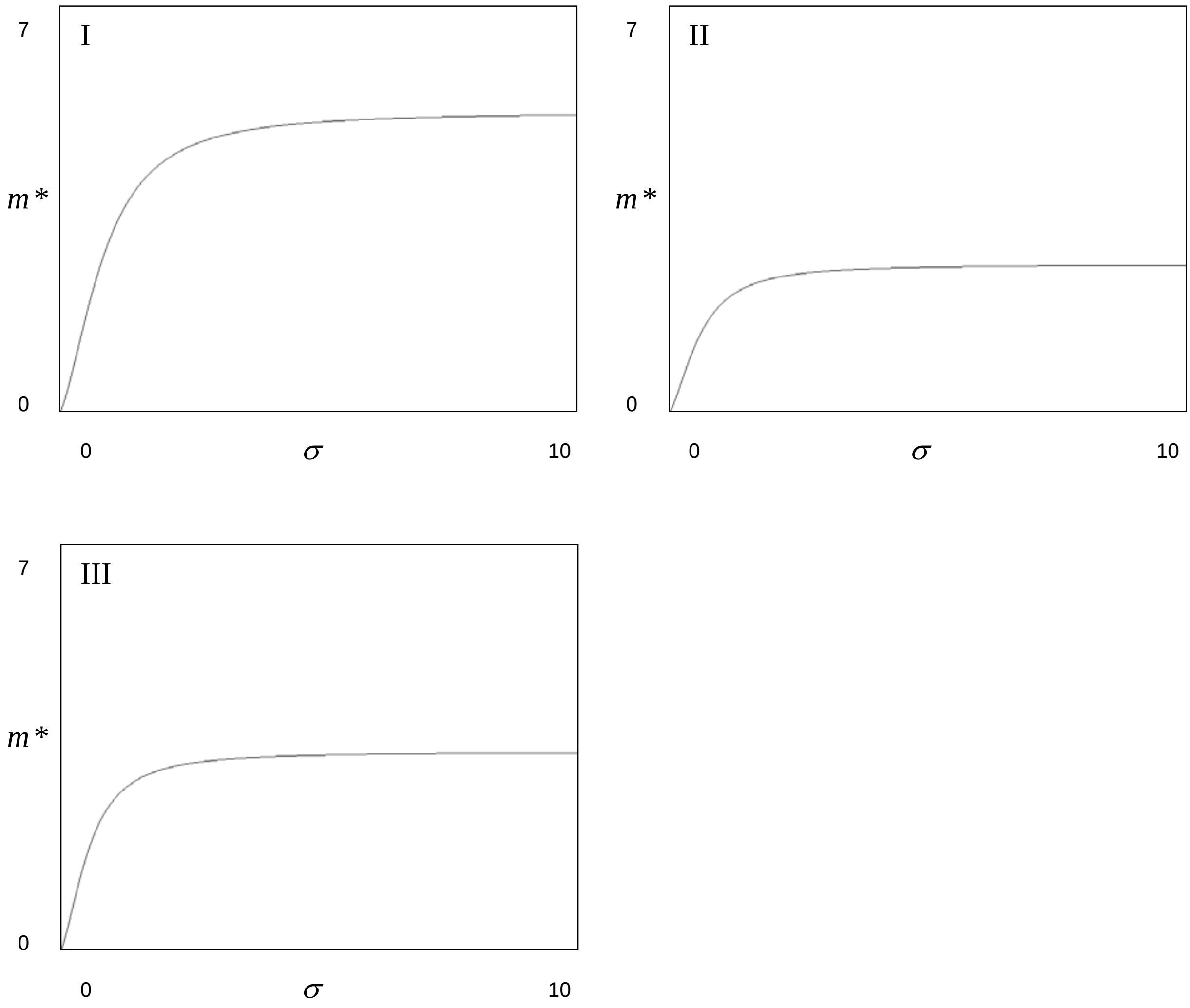
II: $\mu = 1/2$, $\sigma = 1/3$, $\rho = 1/5$

III: $\mu = 1/2$, $\sigma = 1/5$, $\rho = 1/10$

IV: $\mu = 1/3$, $\sigma = 1/3$, $\rho = 1/10$

4.4.2 The Switch Point

We will next study what happens when we let one of the parameters be free, and we fix the rest of them at the same values as above.

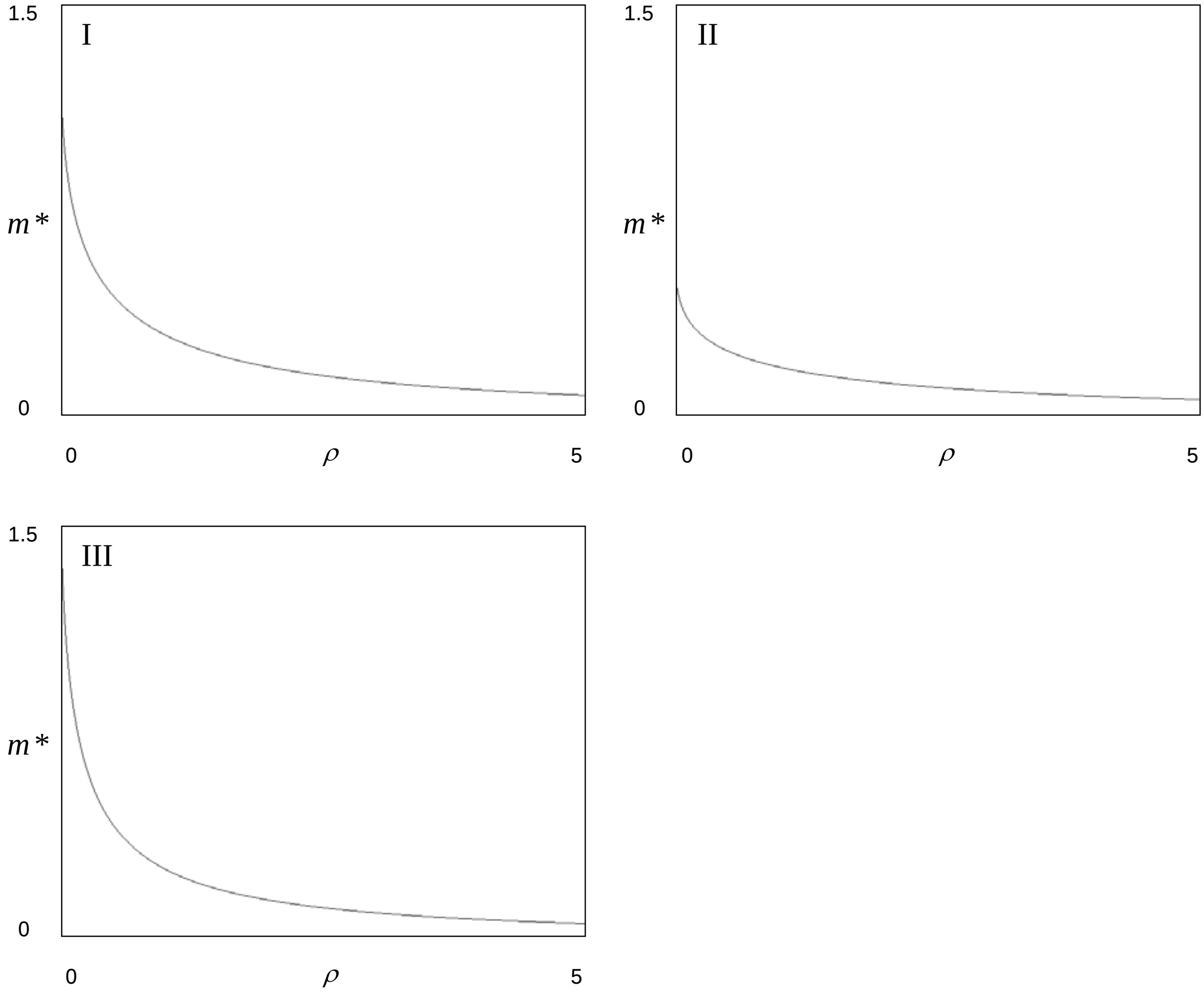


4.4.2.1 FIGURE: The function $\sigma \mapsto m^*(\sigma)$. Of course we were expecting the switch point to be an increasing function of σ, as a larger σ means a greater risk and therefore the firm needs to retain more cash against possible liquidation. On the other hand, if σ converges towards zero, there is no need to retain any cash as the model gets deterministic, i.e. the σ-risk is eliminated. This is kind of a strange solution, as it is optimal to take out everything, wait until the cash grows with μ and then take everything out once again. The parameter values are as follows:

I: $\mu = 1/2$, $\rho = 1/10$

II: $\mu = 1/2$, $\rho = 1/5$

III: $\mu = 1/3$, $\rho = 1/10$

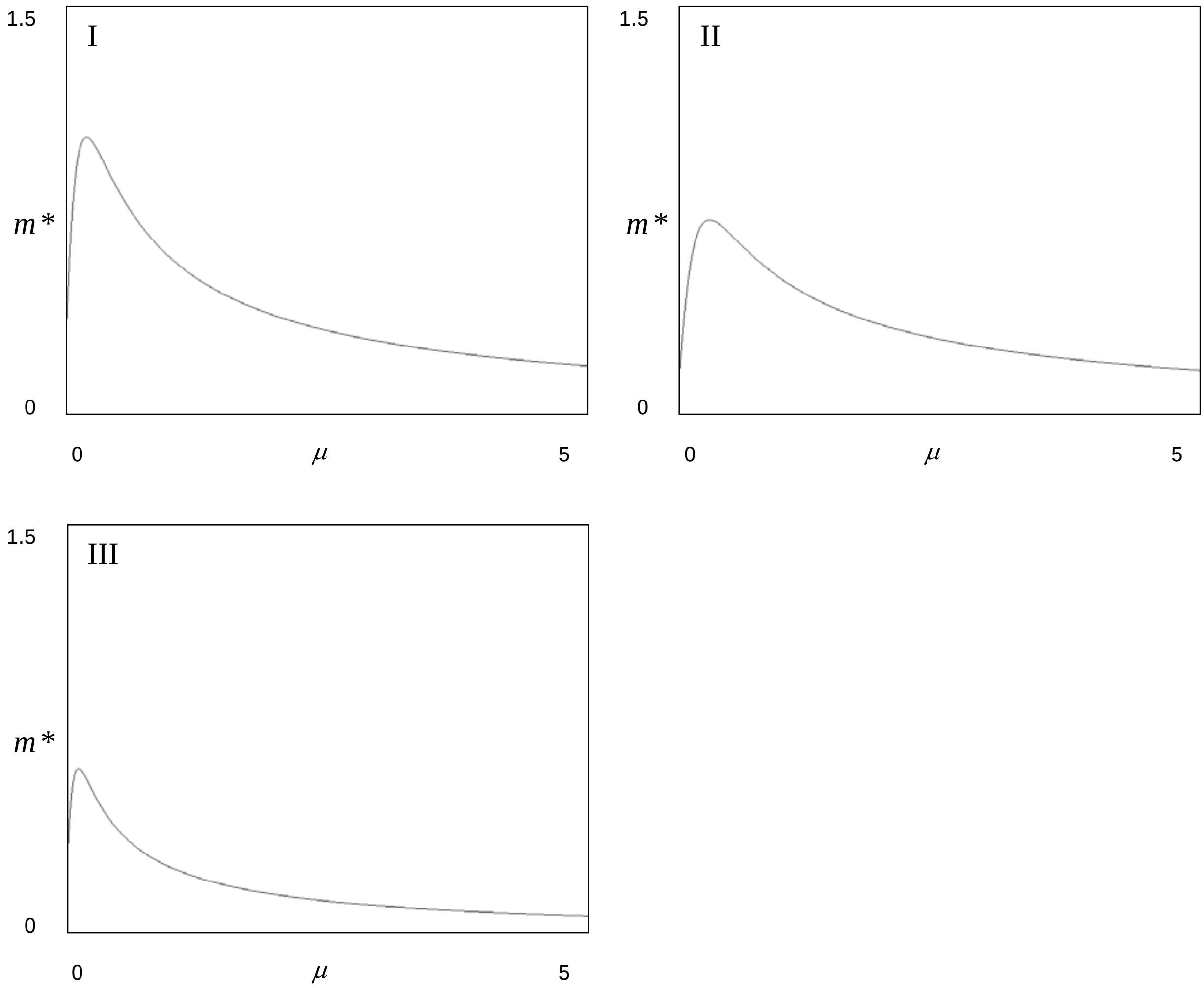


4.4.2.2 FIGURE: The function $\rho \mapsto m^*(\rho)$. We see that the switch point starts declining rapidly when ρ increases if we start at zero, and we see that ρ must be held sufficiently small for the switch point to not converge to zero. So the switch point is very sensitive to changes in the discount factor! The special thing with the discount factor is that it is an external parameter compared to the other ones that are given from data generated internally within the firm. The discount factor "decides" how profitable it is to retain cash compared to consuming and this figure illustrates how the relationship with discount factor looks like. The parameter values are as follows:

I: $\mu = 1/2$, $\sigma = 1/3$

II: $\mu = 1/2$, $\sigma = 1/5$

III: $\mu = 1/3$, $\sigma = 1/3$



4.4.2.3 FIGURE: The function $\mu \mapsto m^*(\mu)$. We see that this function has a shape similar to a beta distribution; it starts at zero, it is concave up to a maximum point and then convex after that. This is less intuitive than the other plots. The existence of a maximum point can be explained by the following conflict: The fact that the firm goes into ruin quite fast because of the $\sigma > 0$ gives that a larger value of μ is desirable. On the other hand, if μ is too large then the firm does not need to retain any cash as the large μ provides a guarantee that cash will flow in fast. We see that when μ converges towards zero; the switch point converges towards zero. It is thus optimal to not retain any cash when μ is zero as the σ-risk can bring it down. The parameter values are as follows:

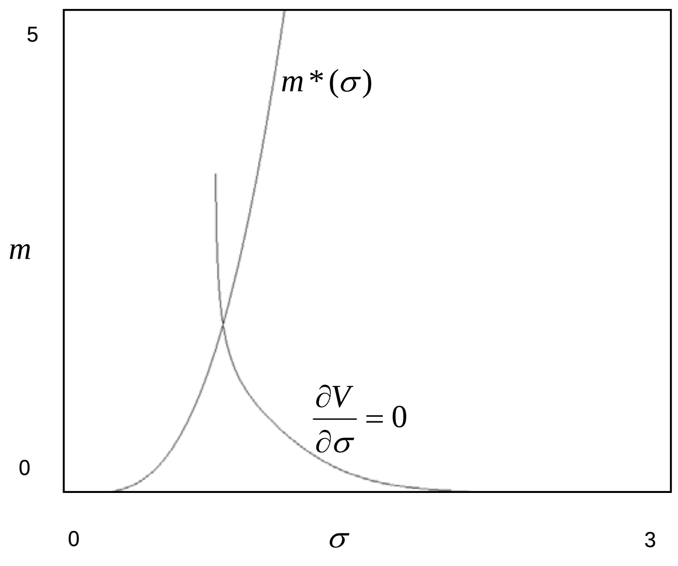
I: $\sigma = 1/3$, $\rho = 1/10$

II: $\sigma = 1/3$, $\rho = 1/5$

III: $\sigma = 1/5$, $\rho = 1/10$

4.4.3 Sensitivity Against the Brownian Risk

Finally, we discuss a variation, which perhaps is a bit tentative. The sensitivity is studied by observing the sign of the derivative of the value function with respect to σ. When $m > m^*$, this derivative is independent of m and thus it has the same sign for all m (it is positive). Then the firm is not sensitive to the Brownian risk. When $m \le m^*$, the derivative of the value function with respect to σ is a function of m, so for a fixed value of σ there is in the pure diffusion model one point for which the sign of the derivative changes. This means that for m smaller than this point the derivative is negative and the firm is sensitive to the Brownian risk – and for m larger than this point the firm is not sensitive. We note that there is just one such point, as indicated by the following figure:



4.4.3.1 FIGURE: The function that starts in 0 is the switch point as function of σ. This is plotted as this is the upper bound on m for which the plot is valid. The other function is obtained by letting the derivative of the value function with respect to σ be equal to zero and taking out m as a function of σ, to see at which values of m that the sign of this derivative changes. We see that for σ smaller than the intersection between these two functions, we have that the sign changes at the switch point m^*; the firm is thus sensitive up to the switch point. In this figure the intersection point between the two functions is approx. $m = 1.35$ and $\sigma = 0.8$. For σ larger than that, the figure indicates that there is a point less than m^* for which the sign changes. This means that even if the cash goes below the switch point the firm does not need to be sensitive for the Browninan risk. The parameter values are: $\mu = 1/2$, $\rho = 1/10$.

5 Analysis of the Jump-Diffusion Model

Throughout this section we analyse the jump-diffusion model as outlined in section 3. We assume that s is exponentially distributed with parameter Δ, which means that the frequency function for s is

$$\psi(s) = \Delta \cdot e^{-\Delta \cdot s} \text{ such that } \int_0^\infty \Delta \cdot e^{-\Delta \cdot s} ds = 1 \text{ and } \Delta \geq 0$$

The expected size of the jump is $E(s) = 1/\Delta$. This model is analysed to some extent in [1]. We will first derive an equation that gives us a unique switch point. Then we will study a numerical example, derive parameters from given data, solve the switch point equation and determine the value function. We then analyse some qualitative properties of the switch point and the value function that we get.

5.1 A General Equation for the Switch Point

This section culminates in Theorem 5.1.4 that gives a general equation for the switch point. But the theorem needs some preparation. Using the frequency function above, we have that

$$E_s[e^{-\xi \cdot s}] = \int_0^\infty e^{-s \cdot \xi} \cdot \psi(s) ds = \int_0^\infty e^{-s \cdot \xi} \cdot \Delta \cdot e^{-\Delta \cdot s} ds = \Delta \cdot \int_0^\infty e^{-s \cdot (\xi + \Delta)} ds = \Delta \cdot \left[\frac{e^{-s \cdot (\xi + \Delta)}}{-(\xi + \Delta)} \right]_0^\infty = \frac{\Delta}{\xi + \Delta}$$

So the Laplace transform of the function f becomes

$$\mathscr{L}_f(\xi) = \frac{\frac{1}{2}\sigma^2 \cdot f'(0) \cdot (\xi + \Delta)}{[\frac{1}{2}\sigma^2 \cdot \xi^2 + \mu \cdot \xi - (\rho + \lambda)] \cdot (\xi + \Delta) + \lambda \cdot \Delta} =$$

$$= \frac{\frac{1}{2}\sigma^2 \cdot f'(0) \cdot (\xi + \Delta)}{[\frac{1}{2}\sigma^2 \cdot \xi^2 + \mu \cdot \xi - \rho] \cdot (\xi + \Delta) - \lambda \cdot \xi} =$$

$$= \frac{f'(0) \cdot (\xi + \Delta)}{\left[\xi^2 + \frac{2\mu \cdot \xi}{\sigma^2} - \frac{2\rho}{\sigma^2} \right] \cdot (\xi + \Delta) - \frac{2\lambda \cdot \xi}{\sigma^2}}$$

The polynomial

$$\left[\xi^2+\frac{2\mu\cdot\xi}{\sigma^2}-\frac{2\rho}{\sigma^2}\right]\cdot(\xi+\Delta)-\frac{2\lambda\cdot\xi}{\sigma^2}$$

has three distinct real roots for the parameter values that we are interested in, which can be checked by algebraic computations (the discriminant is positive). We do it numerically for all set of parameters that we will use. We have

$$\left[\xi^2+\frac{2\mu\cdot\xi}{\sigma^2}-\frac{2\rho}{\sigma^2}\right]\cdot(\xi+\Delta)-\frac{2\lambda\cdot\xi}{\sigma^2}=0$$

$$\Leftrightarrow\quad \xi^3+\left(\Delta+\frac{2\mu}{\sigma^2}\right)\cdot\xi^2+\frac{2}{\sigma^2}(\mu\cdot\Delta-\rho-\lambda)\cdot\xi-\frac{2\rho\cdot\Delta}{\sigma^2}=0$$

and by using the standard technique for finding roots to cubic polynomials, we have proved the following proposition:

5.1.1 PROPOSITION: The roots to the polynomial equation

$$\xi^3+\left(\Delta+\frac{2\mu}{\sigma^2}\right)\cdot\xi^2+\frac{2}{\sigma^2}(\mu\cdot\Delta-\rho-\lambda)\cdot\xi-\frac{2\rho\cdot\Delta}{\sigma^2}=0$$

are

$$\xi_k=-\frac{b}{3}+2\sqrt{-\frac{p}{3}}\cos\left(\theta+\frac{k\cdot\pi}{3}\right)$$

for $k\in\{0,1,2\}$

where

$$b=\Delta+\frac{2\mu}{\sigma^2},\ c=\frac{2}{\sigma^2}\cdot(\mu\cdot\Delta-\rho-\lambda),\ d=-\frac{2\rho\cdot\Delta}{\sigma^2},\ a=-\frac{b}{3},\ p=c-\frac{b^2}{3},$$

$$q=d-\frac{b\cdot c}{3}+\frac{2b^3}{27},\ \theta=\frac{1}{3}\arccos\left(-\frac{q}{2}\cdot\left(-\frac{3}{p}\right)^{3/2}\right)$$

We have that the Laplace transform becomes

$$\mathscr{L}_f(\xi)=\frac{\frac{1}{2}\sigma^2\cdot f'(0)\cdot(\xi+\Delta)}{(\xi-\xi_0)\cdot(\xi-\xi_1)\cdot(\xi-\xi_2)}=\tfrac{1}{2}\sigma^2\cdot f'(0)\cdot\left(\frac{K}{\xi-\xi_0}+\frac{F}{\xi-\xi_1}+\frac{D}{\xi-\xi_2}\right)$$

where *K*, *F* and *D* will be determined by partial decomposition.

DETERMINING *K, F* AND *D*: The equality above is equivalent to

$$\xi+\Delta=K\cdot(\xi-\xi_1)\cdot(\xi-\xi_2)+F\cdot(\xi-\xi_0)\cdot(\xi-\xi_2)+D\cdot(\xi-\xi_0)\cdot(\xi-\xi_1)$$

$$=K\cdot(\xi^2-\xi\cdot(\xi_1+\xi_2)+\xi_1\cdot\xi_2)+F\cdot(\xi^2-\xi\cdot(\xi_0+\xi_2)+\xi_0\cdot\xi_2)+D\cdot(\xi^2-\xi\cdot(\xi_1+\xi_0)+\xi_1\cdot\xi_0)$$

$$= \xi^2 \cdot (K+F+D) - \xi \cdot (K \cdot (\xi_1 + \xi_2) + F \cdot (\xi_0 + \xi_2) + D \cdot (\xi_1 + \xi_0)) + K \cdot \xi_1 \cdot \xi_2 + F \cdot \xi_0 \cdot \xi_2 + D \cdot \xi_1 \cdot \xi_0$$

$$\Leftrightarrow \begin{cases} K + F + D = 0 \\ K \cdot (\xi_1 + \xi_2) + F \cdot (\xi_0 + \xi_2) + D \cdot (\xi_1 + \xi_0) = -1 \\ K \cdot \xi_1 \cdot \xi_2 + F \cdot \xi_0 \cdot \xi_2 + D \cdot \xi_1 \cdot \xi_0 = \Delta \end{cases}$$

$$\Leftrightarrow \; K = -F - D \text{ and}$$

$$\begin{cases} (-F-D) \cdot (\xi_1 + \xi_2) + F \cdot (\xi_0 + \xi_2) + D \cdot (\xi_1 + \xi_0) = -1 \\ (-F-D) \cdot \xi_1 \cdot \xi_2 + F \cdot \xi_0 \cdot \xi_2 + D \cdot \xi_1 \cdot \xi_0 = \Delta \end{cases}$$

$$\Leftrightarrow \begin{cases} F \cdot (\xi_0 - \xi_1) + D \cdot (\xi_0 - \xi_2) = -1 \\ F \cdot \xi_2 \cdot (\xi_0 - \xi_1) + D \cdot \xi_1 \cdot (\xi_0 - \xi_2) = \Delta \end{cases}$$

$$\Leftrightarrow \; F = \frac{-1 - D \cdot (\xi_0 - \xi_2)}{\xi_0 - \xi_1} \text{ and}$$

$$\xi_2 \cdot (-1 - D \cdot (\xi_0 - \xi_2)) + D \cdot \xi_1 \cdot (\xi_0 - \xi_2) = \Delta \;\Leftrightarrow -\xi_2 + D \cdot [\xi_1 \cdot (\xi_0 - \xi_2) - \xi_2 \cdot (\xi_0 - \xi_2)] = \Delta \;\Leftrightarrow$$

$$D = \frac{\Delta + \xi_2}{\xi_1 \cdot (\xi_0 - \xi_2) - \xi_2 \cdot (\xi_0 - \xi_2)}$$

Summing up, we have that

5.1.2 PROPOSITION: Partial decomposition yields that

$$K = -F - D$$

$$F = \frac{-1 - D \cdot (\xi_0 - \xi_2)}{\xi_0 - \xi_1}$$

$$D = \frac{\Delta + \xi_2}{\xi_1 \cdot (\xi_0 - \xi_2) - \xi_2 \cdot (\xi_0 - \xi_2)}$$

Finally, taking the inverse Laplace transform yields

$$f(m) = \tfrac{1}{2}\sigma^2 \cdot f'(0) \cdot (K \cdot e^{m \cdot \xi_0} + F \cdot e^{m \cdot \xi_1} + D \cdot e^{m \cdot \xi_2}) =$$

$$= \tfrac{1}{2}\sigma^2 \cdot f'(0) \cdot ((-F-D) \cdot e^{m \cdot \xi_0} + F \cdot e^{m \cdot \xi_1} + D \cdot e^{m \cdot \xi_2}) =$$

$$= \tfrac{1}{2}\sigma^2 \cdot f'(0) \cdot (F \cdot (e^{m \cdot \xi_1} - e^{m \cdot \xi_0}) + D \cdot (e^{m \cdot \xi_2} - e^{m \cdot \xi_0}))$$

and we have proved the following proposition:

5.1.3 PROPOSITION: Assume that s is exponentially distributed with parameter Δ. Let ξ_0, ξ_1 and ξ_2 be the roots of Proposition 5.1.1 and let F and D be given by Proposition 5.1.2. Then

$$f(m)=\tfrac{1}{2}\sigma^2\cdot f'(0)\cdot(F\cdot(e^{m\cdot\xi_1}-e^{m\cdot\xi_0})+D\cdot(e^{m\cdot\xi_2}-e^{m\cdot\xi_0}))$$

Differentiating the function of Proposition 5.1.3 twice and letting the second derivative equal to zero yields the following theorem:

5.1.4 THEOREM: Assume that s is exponentially distributed with parameter Δ. Let ξ_0, ξ_1 and ξ_2 be the roots of Proposition 5.1.1 and let *F* and *D* be given by Proposition 5.1.2. Then the unique point $m*$ that minimises $f'(m)$ is given by

$$F\cdot\xi_1^2 e^{m*\cdot\xi_1}+D\cdot\xi_2^2 e^{m*\cdot\xi_2}-e^{m*\cdot\xi_0}\cdot\xi_0^2\cdot(D+F)=0$$

provided $f'(0)\neq 0$ .We call this the *switch point equation*.

5.2 Numerical Study: The Price of Crude Oil

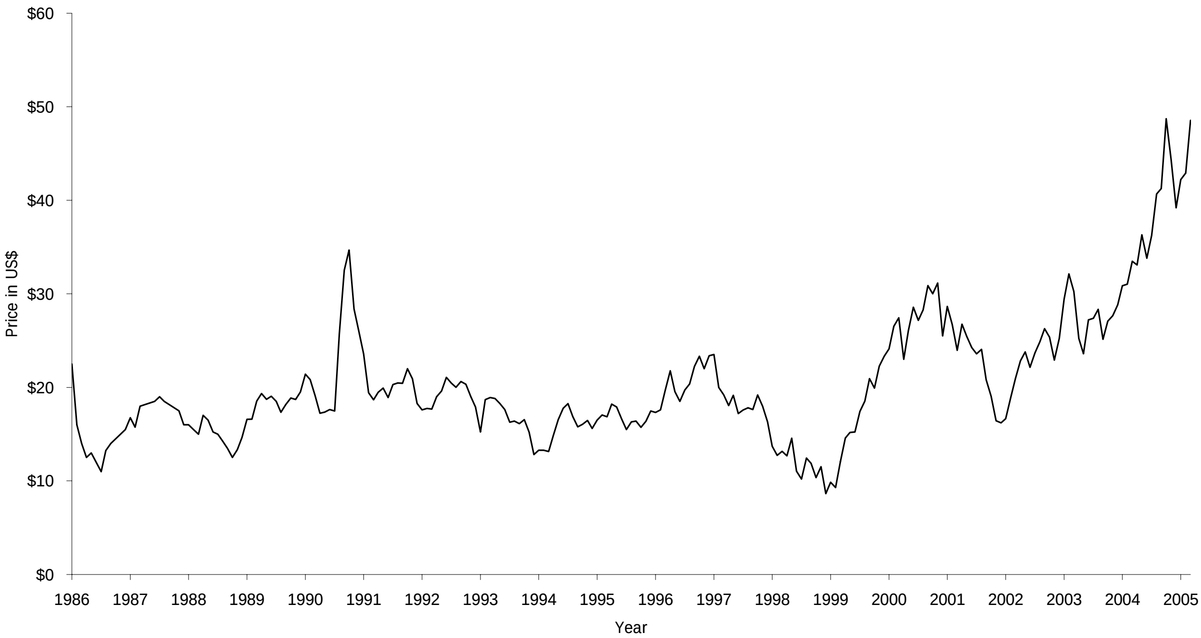
We will analyse the solution in detail for the U.S. crude oil price. The model we use suits the price quite well, especially as there are big jumps present in the price data, which can be modelled by a Poisson process. We will see that the small fluctuations look very much like a Brownian sample path, which also shows that this model is very suitable to use. But one must have access to a large amount of data; otherwise the model will not fit. Lots of data also ensure better parameter estimations. Note that this approach suits prices of raw materials in general, not only crude oil.

Before the subsequent analysis, we insert an introductory discussion.

Assume that the suppliers on the market that we are considering only supply one particular commodity (which we below will assume to be crude oil), that the market sets the price of this commodity and that we can observe how this price evolves in time. Then the suppliers will go out of business if they do not make a profit. This means that the price for the commodity will never go below some positive non-zero threshold, below which it is not profitable to supply this commodity. The suppliers may be companies of different size that have different investment capacity, so the threshold is some average level for all companies contributing to the price index for this commodity. When the price reaches the threshold, the profit of supplying the commodity is equal to zero. This threshold is the switch point, as the suppliers should stop their production if the price goes below that point. We see that the control function is the amount of the commodity supplied, and the optimisation incentive is to maximise the total amount supplied over time. Of course, this is a simplification of reality, but I think that it is more close to reality than to just assume that the log-price follows e.g. a jump-diffusion process when modelling a price index.

5.2.1 Estimation of the Parameters

The following figure is a visualisation of the data set that we will use:

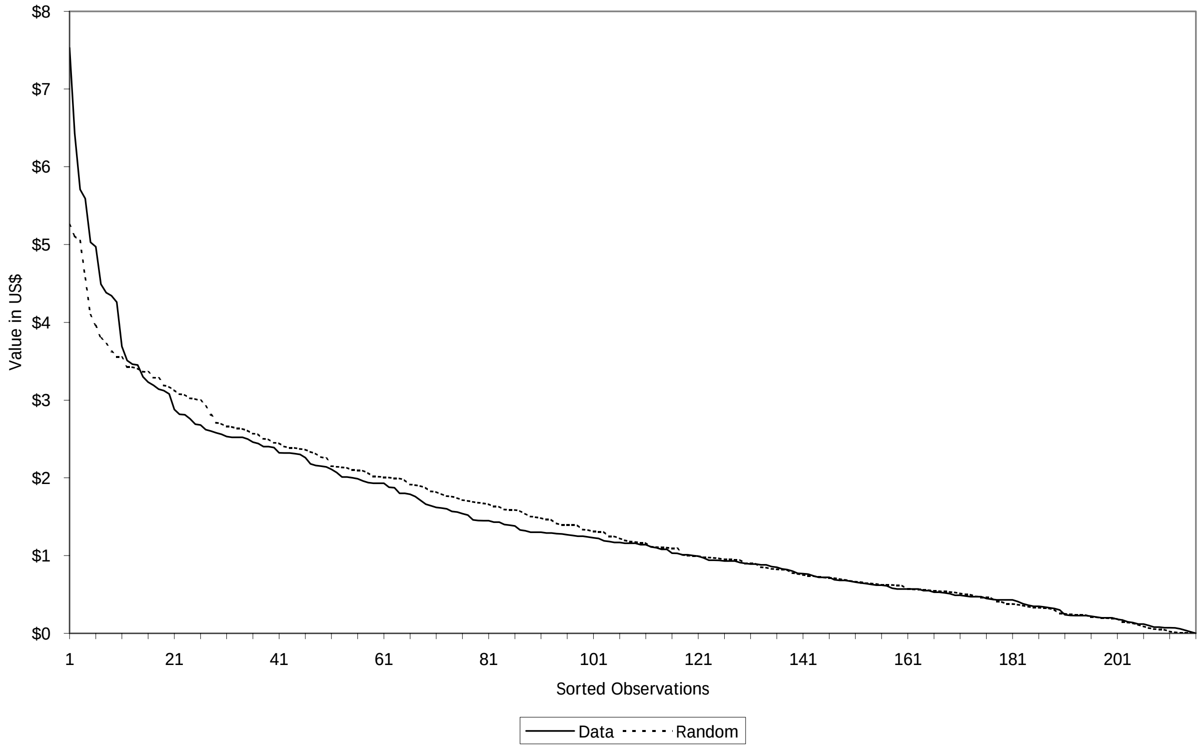


5.2.1.1 FIGURE: The U.S. crude oil price between January 1986 and March 2005.

We see that it is suitable to assume that the price follows a jump-diffusion process

$$dm_t = \mu \cdot dt + \sigma \cdot dw_t - s \cdot dN_t - dL_t$$

The parameters that we will work with are the following (note that these are approximate values): $\lambda = 2/224 = 1/112$ (two Poisson events occur in the lapse of 224 months), $\rho = 0.03$, $\mu = 0.07$ (obtained by regression on the data set after taking out the Poisson jumps). After normalising the initial value to 1, we see that $\Delta = 1/2$. In the original units, $\sigma = 1.9$, which is the deviation from a simulated normal distribution, and we choose the value that fits our modified data best (obtained by experimenting with different σ's and plotting against data), which is approx. 1.9. Below, we plot the deviations from the data set together with simulated random numbers that have a normal distribution with expectation zero and standard deviation 1.9:

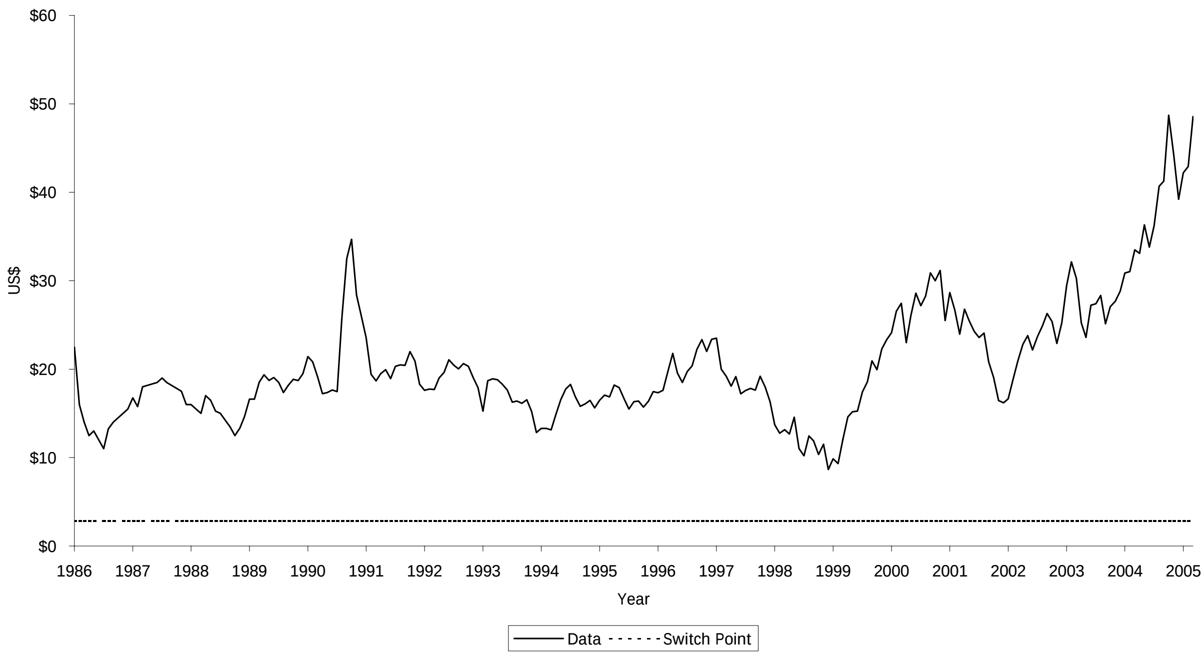


5.2.1.2 FIGURE: The deviations in the data set and random deviations from a normal distribution with expectation zero and deviation 1.9. This shows in some sense how well the model is suited for the data. We see that the data has a larger probability for large values than the normal distribution has, but the similarity is quite good! Here, one has the possibility to choose a σ that is good with small jumps, or good with large jumps. We have chosen to take a σ that goes as good as possible with both small and large jumps.

After normalisation, $\sigma = 1.9/22.5 \approx 0.0844$.

5.2.2 The Switch Point

First, we calculate the switch point with all the parameters as estimated to be 0.126277 (normalised) and 2.8412325 (in the original units \$). This is visualised together with the data below.



5.2.2.1 FIGURE: The original data and the calculated switch point. We see that historically, the price has never fallen below the switch point.

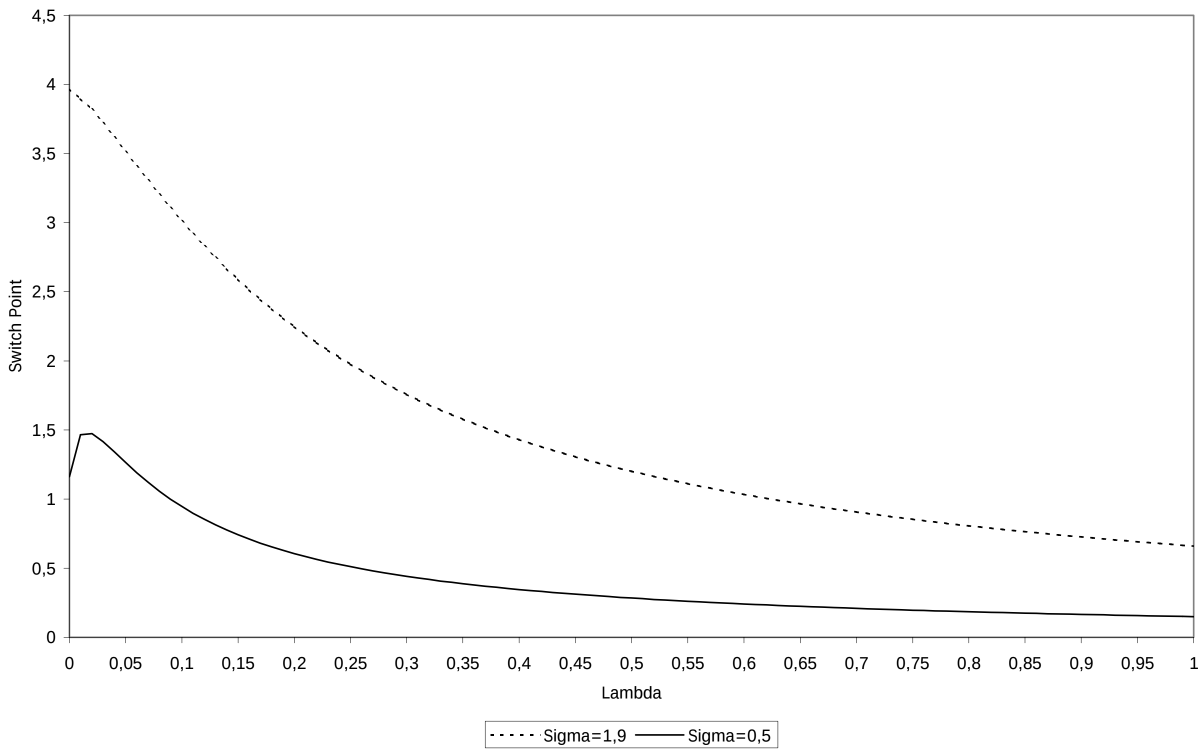
Next, our interest lies in analysing the qualitative properties of the function $(\sigma,\lambda) \mapsto m^*(\sigma,\lambda)$.

When $\lambda = 0$, the probability of a large upward jump is zero. If λ is controllable, the controller may want to increase λ a little, to get a positive probability of a large upward jump that may bring a large increase in value to the controller. It is like winning on a lottery because of the small probability that the event occurs but the great value that an upward jump may bring. But to be able to play in this lottery, the controller must increase λ from zero and thereby also increase the probability of a downward jump. So this is not a lottery where you only can win something – you can also lose a large amount. This conflict means that for small λ, the function $\lambda \mapsto m^*(\lambda)$ can be increasing or decreasing. We will see that the other risk parameter, σ, decides whether the function $\lambda \mapsto m^*(\lambda)$ is increasing or decreasing for small

λ. We will see that when σ is sufficiently small, the expected profit of an upward jump is greater than the expected loss of a downward jump for small λ and thus the function $\lambda \mapsto m^*(\lambda)$ will be increasing for small λ. And when σ grows, there exists a threshold for which the opposite becomes true, i.e. the function $\lambda \mapsto m^*(\lambda)$ will be decreasing for small λ.

But what happens when $\lambda \to \infty$? Then the risk of a large downward jump becomes so big so that it is dangerous to keep anything and thus the function $\lambda \mapsto m^*(\lambda)$ must converge to zero, and the convergence speed must be decreasing in σ. So we see that if the function $\lambda \mapsto m^*(\lambda)$ is increasing for small λ, there exists a λ for which the function attains a global maximum. And if the function $\lambda \mapsto m^*(\lambda)$ is decreasing for small λ then it is decreasing for all λ.

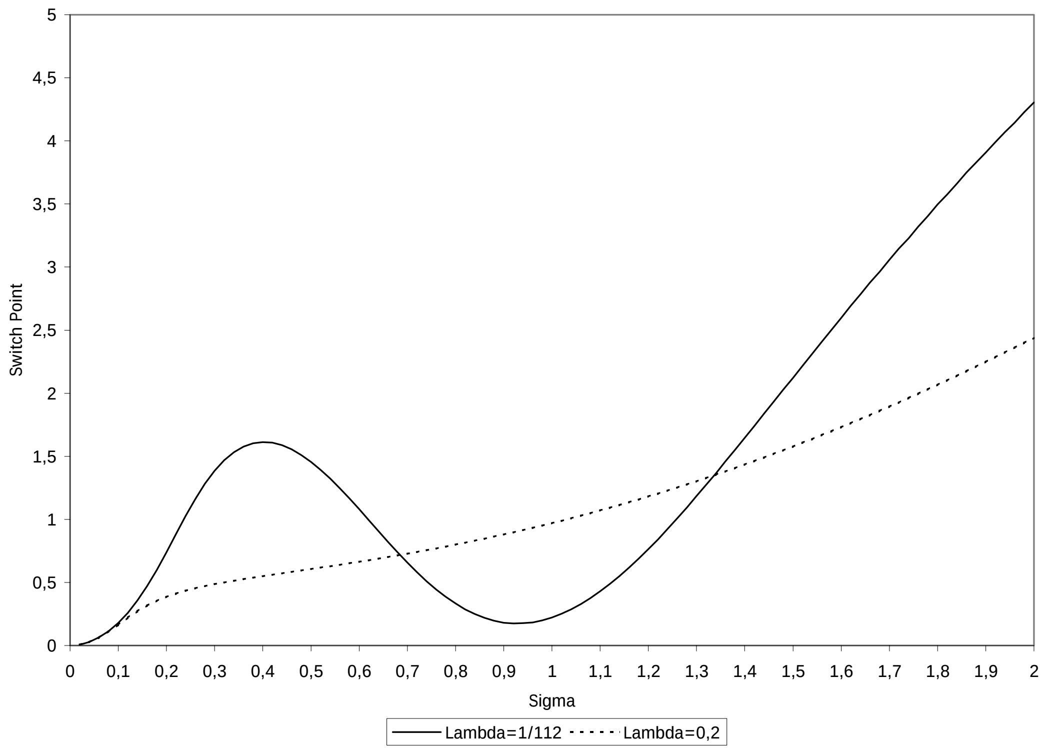
All these properties of the function $\lambda \mapsto m^*(\lambda)$ are illustrated by the figure below:



5.2.2.1 FIGURE: The function $\lambda \mapsto m^*(\lambda)$ as a numerical solution to the switch point equation with normalised parameters as estimated in section 5.2.1, except for λ that is free, and σ that we prescribe two values according to the figure.

It is also an interesting case to let σ be free and look at the function $\sigma \mapsto m^*(\sigma)$ for some fixed values of λ. We will see that when λ is small, there is a great uncertainty in the model whether the λ-event will occur or not. This will give the function $\sigma \mapsto m^*(\sigma)$ a special shape; it will fluctuate widely up and down. We will also see that for non-zero λ, the function $\sigma \mapsto m^*(\sigma)$ need not be monotone anymore. For larger λ, the result will be more robust as one knows that the λ-event probably will occur, and can be able to act thereafter. The figure below indicates that the function $\sigma \mapsto m^*(\sigma)$ actually seems to be monotone for sufficiently large λ.

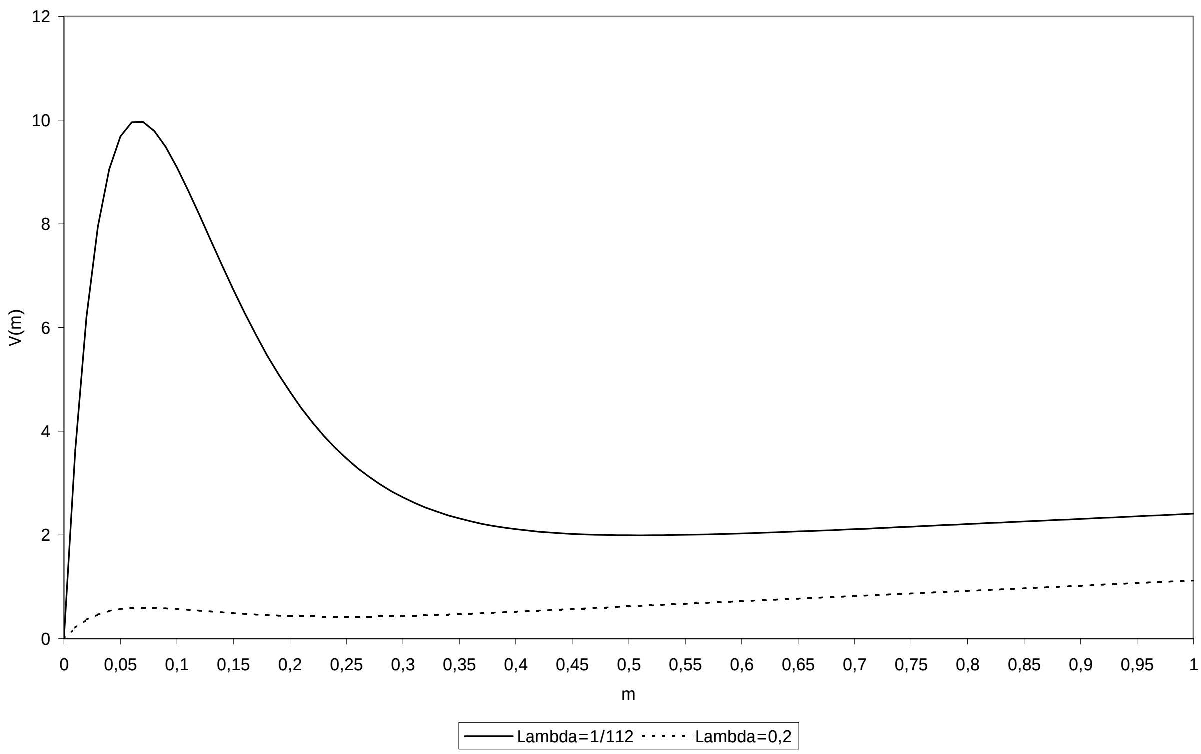
Another question to pose is: Does the limit $\lim_{\sigma \to 0} m^*(\sigma, \lambda)$ exist? It seems to be that way, as indicated by the following figure:



5.2.2.2 FIGURE: The function $\sigma \mapsto m^*(\sigma)$ for two values of λ, as a numerical solution to the switch point equation with normalised parameters as estimated in section 5.2.1. This figure visualises the uncertainty induced by a small λ, which causes large fluctuations in the function $\sigma \mapsto m^*(\sigma)$.

We see in the figure above that the limit $\lim_{\sigma \to 0} m^*(\sigma, \lambda)$ seems to exist and be equal to a very small positive amount. And this is quite natural, as the suppliers should always, if not the price is at a very small positive level, produce at their maximum capacity when $\sigma \to 0$; if λ is small then the event is not likely to occur, so it is better to supply down to a very small price and earn the money even if the price gets down. And if λ is large, the risk induces that it also is better to supply down to a very small price, as the highly probable event can bring the price down as well.

5.2.3 The Value Function



5.2.3.1 FIGURE: The function $m \mapsto V(m)$ for the normalised parameters of section 5.2.1 for two prescribed values of λ.

Below, we analyse the value function qualitatively.

We see that when the initial cash level m is small, we are very sensitive to changes in λ. For a small λ, the possibility of a large upward jump assigns a large value when m is small, reflected in the peak in the figure. For small m, the profit of the event of winning in the lottery has a great positive impact on the value. But when m increases, we are less vulnerable to the effect of a large jump so the impact of a large jump decreases and so the value function decreases. And then the switch point comes and after that the value function is linear. We see that the smaller λ, the greater is the peak. We were expecting that the larger λ, the more dangerous the lottery becomes as then we can lose a large amount. Of course, when λ grows, we get more and more afraid of a large downward jump and this is reflected in the value function, which decreases when λ increases.

A non-zero λ seems to provide the existence of such a peak, i.e. the existence of a point m such that $V'(m) = 0$. This is consistent with the result that we obtained in the previous section; that the value function always is concave when $\lambda = 0$. Then, it is obvious that no point m with $V'(m) = 0$ exists. When λ is non-zero, the property $V'(m) = 0$ can be used to find the m for which the value function attains this local maximum. When λ was zero, it was a bad thing to have an initial cash level below the switch point, as the value function was concave. But in the case of a non-zero and quite small λ, we see that *having an initial cash level that is below the switch point but not too close to zero is a good thing*. We see that an "optimal initial cash level" is the one that maximises the value function and not its derivative.

5.3 Conclusions

We will here summarise the results that we found by the numerical investigation above.

We showed numerically that when the Poisson intensity rate is small, we are blinded by the profit that a large upward jump may bring and does not care about the equally large probability of a downward jump. But when the Poisson intensity rate increased, the focus turned towards the risk of a downward jump and the equally large probability of an upward jump did not matter anymore.

We also found the following:

THE SWITCH POINT

- Small Brownian volatility rate: the switch point as a function of the Poisson intensity rate is increasing, attains a global maximum and then decreases towards zero. Large Brownian volatility rate: It is decreasing all the time towards zero.
- When Poisson intensity rate is zero, the switch point as a function of the Brownian volatility rate is monotone increasing. For non-zero Poisson intensity rate, the switch point as a function of the Brownian volatility rate need not be monotone anymore. Especially in the case of a small Poisson intensity rate, when there is a great uncertainty whether the Poisson event will occur or not in a given time interval. In this case it fluctuates widely up and down. It becomes monotone when the Poisson intensity rate is sufficiently large.

THE VALUE FUNCTION

- When Poisson intensity rate is zero, we saw that it was a bad thing to have an initial cash level below the switch point, as the value function is concave. But in the case of a non-zero and quite small Poisson intensity rate, we showed that having an initial cash level below the switch point is a good thing, as the possibility of an upward jump gives a large value being in this interval if not initial cash level is very close to zero. We saw that an "optimal initial cash level" is the one at which the first derivative of the value function is equal to zero.

By interpreting the problem as a problem of deciding how much to supply of a specific commodity based on an observed price, we found when studying the crude oil price that the price had historically never been below the switch point that we obtained by calculations.

References

[1] Belhaj, M. (2004), Optimizing Dividend Payments when Cash Reserves Follow a Compound Process, Stochastic Finance 2004.

[2] Jeanblanc, P. and A. N. Shiryaev (1995), Optimization of the flow of dividends, Russian Mathematical Surveys, 50, 257-277.

[3] Mikosch, T. (1999), Elementary Stochastic Calculus with Finance in View, World Scientific.

[4] Cont, R. and Tankov, P. (2004), Financial Modelling With Jump Processes, Chapman & Hall

[5] Dahlgren, M. (2004), Some Applications of Variational Methods in Mathematical Finance and Numerics, Lund Institute of Technology