EXAMENSARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Borsuk's conjecture
and
an introduction to combinatorial geometry

av

Pejman Altafi

2005 - No 11

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 10691 STOCKHOLM

Borsuk's conjecture
and
an introduction to combinatorial geometry

Pejman Altafi

Examensarbete i matematik 20 poäng

Handledare: Paul Vaderlind

2005

Abstract

In the year 1933, the Polish mathematician Karol Borsuk conjectured that an n-dimensional body of diameter d can always be partitioned into $n+1$ parts, each having diameter less than d. Over half a century later, this geometrical conjecture was finally proved to be false in general, with the help of combinatorial arguments. However, there are special cases for which the conjecture holds. Borsuk's Conjecture is the main topic of this paper. At the same time, the paper aims at giving an introduction to the field of Combinatorial Geometry.

Contents

1 Introduction

Borsuk's Conjecture was stated by Karol Borsuk more than half a century ago. He conjectured that one can always decompose n-dimensional bodies of a given diameter into $n+1$ pieces, each of smaller diameter. It is easy to confirm $n+1$ as a "lower bound" just by looking at the regular simplex in $\mathbb{R}^n$. It was not until the year 1992 that this widely studied and accepted (geometrical) conjecture was (combinatorially) disproved by Kahn and Kalai. By this time, Borsuk's Conjecture had been verified for several classes, e.g. for all sets in $\mathbb{R}^2$ and $\mathbb{R}^3$, and for all convex bodies with smooth boundaries, as well as for all centrally symmetric bodies. Kahn and Kalai did not only show the falsity of Borsuk's Conjecture, but they also proved that the minimum number of pieces needed for the decomposition of bodies in $\mathbb{R}^n$ grows exponentially with n.

The primary aim of this paper is to discuss Borsuk's Conjecture. We give the disproof for the general case and we also present the assertion of the conjecture for sets in $\mathbb{R}^2$ and $\mathbb{R}^3$, and for n-dimensional convex bodies with smooth boundaries.

In this paper, the reader will encounter the use of *combinatorial* arguments for proving *geometrical* problems. This field of mathematics is called *Combinatorial Geometry* and one of our objectives is to give an introduction to this subject.

Basic Linear Algebra is a prerequisite for this paper, although most of the necessary definitions are included in the text. Apart from that, mathematical maturity is all that is needed.

We start with some basic definitions. Thereafter, we present the tools needed to prove the Kahn-Kalai Theorem. Finally, after disproving Borsuk's Conjecture for general n-dimensional bodies, we give the assertion of the Conjecture for some of the special cases mentioned above.

I would like to thank Paul Vaderlind for presenting this subject to me and for guiding me throughout my work. Also, thanks to Peter Strömbeck for patiently answering all of my questions.

2 Basic Definitions and Results

In this chapter we present some terminology and basic definitions and results in Abstract Algebra, Geometry and Graph Theory, needed for this paper.

2.1 Abstract Algebra and Geometry

Definition 1. *Let n be an integer. Then $[n] = \{1, 2, \ldots, n\}$.*

Definition 2. *Let X be a set. Then $|X|$ denotes the cardinality of the set X, i.e. the number of its elements.*

Definition 3. *A k-*set *is a set of k elements.*

Definition 4. *If X is a set then 2^X denotes the set of all subsets of the set X and $\binom{X}{k}$ denotes the set of all k-subsets of X.*

Definition 5. *Given a set S, S^n denotes the set of ordered n-tuples:*

$$S^n = \{(s_1, s_2, \ldots, s_n) : s_i \in S\} \tag{1}$$

Definition 6. *We call the set $\{0,1\}^n := \{(\epsilon_1, \ldots, \epsilon_n) : \epsilon_i \in \{0,1\}\}$ the n-*cube *or the* unit cube*. It has 2^n elements. These elements are points in $\mathbb{R}^n$, the n-dimensional Euclidean space.*

Definition 7. *The* incidence vector *of the set $C \subseteq [n] = \{1, 2, \ldots, n\}$ is $(\alpha_1, \ldots, \alpha_n) \in \{0,1\}^n$, where*

$$\alpha_i = \begin{cases} 1 & \text{if} \quad i \in C \\ 0 & \text{if} \quad i \notin C \end{cases} \tag{2}$$

Definition 8. *We define the* distance *between two incidence vectors,* $(x_1, \dots, x_n)$ *and* $(y_1, \dots, y_n)$*, to be the number of places* i*, where* $x_i \neq y_i$*. In other words, the distance between the incidence vectors of two sets,* A *and* B*, is the cardinality of their symmetric difference. (The* symmetric difference *of two sets,* A *and* B*, is the set whose members belong to* A *or* B *and not to both.)*

Definition 9. *The* Euclidean distance *between two vectors,* $u = \{u_1, u_2, \dots, u_n\}$ *and* $v = \{v_1, v_2, \dots, v_n\}$*, in* $\mathbb{R}^n$ *is defined by*

$$d(u, v) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_n - v_n)^2} \tag{3}$$

Note that in the case of two incidence vectors, u *and* v*, their distance is equal to* $d(u, v)^2$*. (The Euclidean distance,* $|x - y|$*, between two points,* $x = \{x_1, x_2, \dots, x_n\}$ *and* $y = \{y_1, y_2, \dots, y_n\}$*, is defined similarly.)*

Definition 10. *A* set system *or a* family of sets *is a set of sets. The sets belonging to the family are its* members*. A set system* F *over a set* X *is a family of subsets of* X*. We say that* X *is the* universe *of* F*.*

Definition 11. *A set system* F *is* k-uniform *if its members are* k*-sets.* F *is* uniform *if it is* k*-uniform for some* k*.*

Definition 12. *Let* L *be a set of positive integers. A set system* F *is* L-intersecting *if* $|A \cap B| \in L$ *for any two distinct* $A, B \in F$*.*

Definition 13. *The* function space $\mathbb{F}^X$ *is the set of functions from the set* X *to a field* $\mathbb{F}$*. This is a linear space with* $f + g$ *and* λf *defined by* $(f + g)(x) = f(x) + g(x)$ *and* $(\lambda f)(x) = \lambda f(x)$ *for* $f, g \in \mathbb{F}^X$ *and* $x \in X$*.*

Definition 14. *The functions* $f_1, \ldots, f_m$*, not all of them equal in the function space* $\mathbb{F}^X$*, are said to be* linearly independent *if, for every* λ_i $(i = 1, \ldots, m)$*, the equation*

$$\lambda_1 f_1 + \ldots + \lambda_m f_m = 0 \tag{4}$$

implies $\lambda_1, \ldots, \lambda_m = 0$*. Here, we say that the solution to equation* (4) *is* trivial.

Definition 15. *A* monomial *is a product of variables with a scalar coefficient. The* degree *of a monomial is the sum of its exponents. A* monic *monomial is a monomial with coefficient* 1*. A* (multivariate) polynomial *is a sum of monomials. A* multilinear polynomial *is a sum of monomials for which the degree in each variable is* ≤ 1*.*

Example 1.

$2x_1^2x_3^5$ *is monomial of degree* 7*.*

$x_1^2x_3^5$ *is a monic monomial of degree* 7*.*

$2x_1^2x_3^5 - 8x_5^9 + x_2^2x_4x_3^3 + 3$ *is a (multivariate) polynomial of degree* 9*.*

$2x_1x_3 - 8x_5 + x_2x_4x_3 + 3$ *is a multilinear polynomial of degree* 3*.*

Proposition 16 (Diagonal Criterion)**.** *Let* $\mathbb{F}$ *be a field and* S *an arbitrary set. For* $i = 1,\ldots,m$ *let* $f_i : S \to \mathbb{F}$ *be functions and* $a_i \in S$ *elements such that*

$$f_i(a_j) \begin{cases} \neq 0 & \text{if } \; i = j \\ = 0 & \text{if } \; i \neq j \end{cases} \tag{5}$$

Then, $f_1, \ldots, f_m$ *are linearly independent.*

Proof. Let $\sum_{i=1}^{m} \lambda_i f_i$ be a linear relation between the f_i. Then, in $\sum_{i=1}^{m} \lambda_i f_i(a_j)$, all but the j^{th} term vanish (condition (5)), and what remains is the $\lambda_j f_j(a_j) = 0$. Again, by condition (5), $f_j(a_j) \neq 0$ and therefore $\lambda_j = 0$. Since this must hold for every j, the linear relation under consideration is trivial. That is $\sum_{i=1}^{m} \lambda_i f_i = 0$ implies $\lambda_1, \ldots, \lambda_m = 0$, which, according to definition (14), proves the independence of $f_1, \ldots, f_m$. □

Proposition 17 (Multilinearization)**.** *Let $\mathbb{F}$ be a field and $\Omega = \{0,1\}^n \subseteq \mathbb{F}^n$. If f is a (multivariate) polynomial of degree $\leq s$ in n variables over $\mathbb{F}$ then there exists a unique multilinear polynomial $\tilde{f}$ of degree $\leq s$ in the same variables such that*

$$f(x) = \tilde{f}(x) \qquad \text{for every } x \in \Omega \tag{6}$$

Proof. $\tilde{f}(x)$ is obtained by the identity $x_i^2 = x_i$, which is clearly valid over Ω (since $0^2 = 0$, $1^2 = 1$). □

Example 2. *If $f(x) = x_1^3 x_2^2 + x_4^3$, we obtain $\tilde{f}(x) = x_1 x_2 + x_4$ and obviously $\tilde{f}(x) = f(x)$ for every $x \in \Omega = \{0,1\}^n$.*

Theorem 18 (Prime Number Theorem)**.** *For every $\epsilon > 0$ and sufficiently large x, the number of primes not greater than x is between the bounds $(1 \pm \epsilon)x/\ln x$. This means that, for sufficiently large x, there are approximately $x/\ln x$ prime numbers $\leq x$.*

Proof. The proof is beyond the scope of this paper. See Widder [ITT] for a proof. □

Definition 19. *Consider a body (a bounded set of points) $C \subset \mathbb{R}^n$. The* diameter *d of C is the greatest (Euclidean) distance between its points, i.e.*

$$d = \sup\{|x - y| : x, y \in C\} \tag{7}$$

In other words, we say that the set C has diameter d if:

(a) *There exists two points, x_0, y_0 in C or its boundary, which are at (Euclidean) distance d.*

(b) For any two points $x, y \in C$, their distance is $\leq d$.

Example 3. *The diameter of a circle is its diameter. The diameter of a square is the length of its diagonal. In fact, the diameter of a polygon (Definition 24) is the maximum distance among its vertices.*

Definition 20. *(Geometric Definition) A set $C \subset \mathbb{R}^n$ is said to be* convex *if, for every $x, y \in C$, the line segment from x to y lies entirely in C (figure 1).*

(Algebraic Definition) In other words, a set $C \in \mathbb{R}^n$ is said to be convex if, for every $x, y \in C$ and $\lambda \in [0, 1]$, we have that

$$(1 - \lambda)x + \lambda y \in C \tag{8}$$

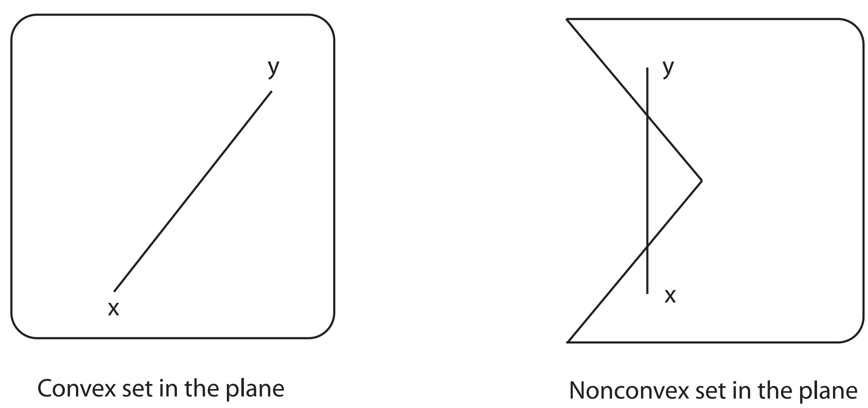

Figure 1

Definition 21. *Let W be a linear space over $\mathbb{R}$. A* convex combination *of the vectors (points) $v_1, \ldots, v_m \in W$ is a linear combination*

$$\sum_{m}^{i=1} \lambda_i v_i \qquad (\lambda_i \in \mathbb{R}^n) \tag{9}$$

where

$$\sum_{i=1}^{m} \lambda_i = 1 \qquad (\lambda_i \geq 0) \tag{10}$$

Definition 22. *The* convex hull *of a subset $S \subseteq W$ is the set of all convex combinations of the finite subsets of S.*

Example 4. *The convex hull of the body in figure* 2(*a*) *is illustrated by figure* 2(*b*). *It is obvious that the convex hull of a convex set S is equal to S.*

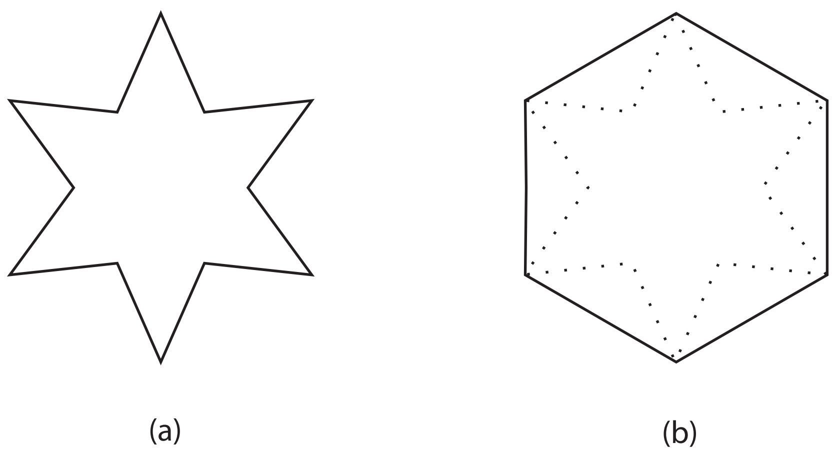

Figure 2

Definition 23. *The set $M \subset \mathbb{R}^n$ is* open *if, for any point $x \in M$, there exists a real number $\epsilon > 0$ such that, given any point $y \in \mathbb{R}^n$ whose Euclidean distance from x is smaller than ϵ, y also belongs to M. The set $M \subset \mathbb{R}^n$ is* closed *if its complement $\mathbb{R}^n \backslash M$ is open.*

Definition 24. *A* polygon *is a body in* $\mathbb{R}^2$ *with* n *sides, where the sides form a closed curve, e.g. a rectangle that has 4 sides. By a* side *we mean a straight line segment including its end points. A* polyhedron *is a body in* $\mathbb{R}^3$*, bounded by a finite number of polygons (faces).*

We call a polyhedron "simple", if it is topologically equivalent to a sphere (i.e. if it was inflated, it would produce a sphere) and its faces are polygons that are topologically equivalent to a disk. In the remainder of this paper we assume that every polyhedron is simple in this sense.

The generalization of the notion of polygon and polyhedron to n *dimensions is a* polytope*. A* spherical *polytope is a polytope bounded by spherical "faces".*

Example 5. *Figure* 3 *shows a spherical tetrahedron (in* $\mathbb{R}^3$*).*

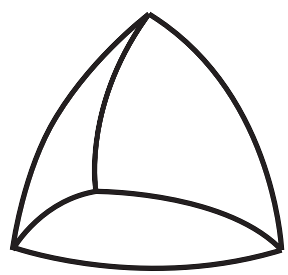

Figure 3

Theorem 25 (Euler's Polyhedral Formula)**.** *For a "simple" polyhedron, let* V*,* E *and* F *denote the number of vertices, edges and faces respectively. Then we have the following relation:*

$$V - E + F = 2 \tag{11}$$

Proof. Imagine a simple hollow polyhedron, i.e. a polyhedron that is topologically equivalent to a sphere, with a surface that can be deformed. Then, if we cut one of the faces of the hollow polyhedron, we can deform the remaining surface until it stretches out flat on a plane as in figure 4(b). The fact that the areas of the faces and the angles between the edges of the polyhedron will change in the process of deformation is of no significance, but what is important is that the network of vertices and edges in the plane contains the same number of vertices and edges as did the original polyhedron, while the number of polygons (faces) will be one less than in the original polyhedron. Our aim is now to prove the theorem by showing that for the plane network $V - E + F = 1$.

Our first step is to "triangulate" the plane network. We do so by drawing diagonals in every polygon of the network that is not already a triangle. The effect of each triangulation is to increase both E and F by 1, thus leaving the value of $V - E + F$ unchanged. Now, the figures consists entirely of triangles (figure 4(c)) and $V - E + F = 1$. We observe that some of the triangles have one or two edges on the boundary of the plane network. Next, we take any boundary triangle and remove that part of it which does not also belong to some other triangle. Thus, from ABC (figure 4(c)) we remove the edge AC and the face, leaving the vertices A, B, C and two edges AB and BC, while from DEF (figure 4(d)) we remove the face, the two edges DF and FE, and the vertex F. The removal of a triangle of the type ABC decreases E by 1 and F by 1, and the removal of a triangle of the type DEF decreases V by 1, E by 2 and F by 1. Thus in both cases the value of $V - E + F$ again remains the same.

We continue with these operations until finally only one triangle remains, with its 3 edges, 3 vertices and 1 face (figure 4(f)). For this simple network,

$$V - E + F = 3 - 3 + 1 = 1 \tag{12}$$

But we have seen that, by constantly erasing triangles, the value of $V - E + F$ was not changed. Therefore, $V - E + F$ equals 1 for the polyhedron with one face missing. Adding the face that we removed from the original polygon, we see that $V - E + F = 2$ for the complete polyhedron. □

Definition 26. *The points $n_1, n_2, \ldots, n_k \in \mathbb{R}^n$ are* affinely independent *if, for every $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$, where $\sum_{i=1}^{k} \lambda_i = 0$, the equation*

$$\lambda_1 n_1 + \lambda_2 n_2 + \ldots + \lambda_k n_k = 0 \tag{13}$$

is trivial, i.e. every $\lambda_i = 0$.

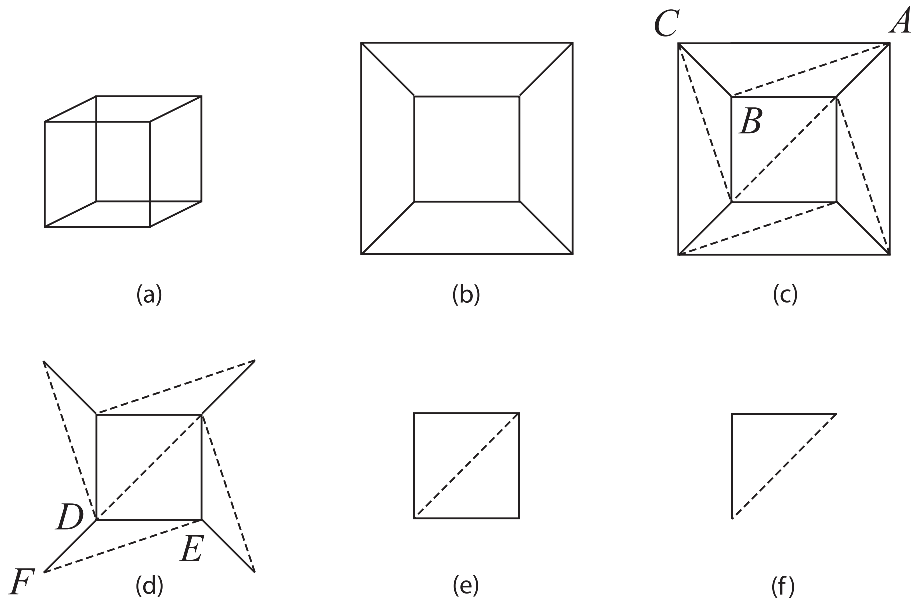

Figure 4

Definition 27. *A* simplex *in $\mathbb{R}^n$ is the convex hull of a set of $n+1$ affinely independent points $n_1, n_2, \dots, n_k$. A* regular simplex *is a simplex, where every pair of the points $n_1, n_2, \dots, n_k$ are at the same distance. In $\mathbb{R}^3$, for example, this means the regular tetrahedron.*

Definition 28. *For a set $L \subset \mathbb{Z}$ ($\mathbb{Z}$ is the set of all integers) and integers r and t we say that*

$$t \in L \pmod{r} \tag{14}$$

if $t \equiv \ell \pmod{r}$ for some $\ell \in L$. A set system F is L-intersecting $\pmod{r}$ if

$$|A \cap B| \in L \pmod{r} \tag{15}$$

for any two distinct sets $A, B \in F$.

Definition 29. *A hyperplane in $\mathbb{R}^n$ is a plane that divides $\mathbb{R}^n$ into two parts. In other words, a hyperplane in $\mathbb{R}^n$ is any affine subspace $S \subset \mathbb{R}^n$ of dimension $n-1$. For example, in $\mathbb{R}^2$ a hyperplane is a straight line.*

Definition 30. *A* tangential hyperplane *in a point p of the boundary of a set $S \subset \mathbb{R}^n$ is a hyperplane in $\mathbb{R}^n$ that intersects with S in p.*

Definition 31. *A body in $\mathbb{R}^n$ is said to have a* smooth boundary *if, for every point in its boundary, there exists a unique tangential hyperplane. For example, the boundary of a cube in $\mathbb{R}^3$ is not smooth since there are more than one (infinitely many) hyperplanes in each vertex.*

2.2 Graph Theory

The purpose of this paper is to illustrate the connection between Geometry and Combinatorics and to give a disproof to Borsuk's Conjecture. The following results from Graph Theory is needed for us to complete our task. For more on Graph Theory, see Biggs [DM].

Definition 32. *A* simple graph *G consists of a finite set V, whose members are called* vertices*, and a set system E of 2-subsets of V, whose members are called* edges*. We write $G = (V, E)$.*

Two vertices, u and v, are said to be adjacent *in the simple graph G if the pair $\{u, v\}$ is one of the edges of G, i.e., $\{u, v\} \in E$. Here we say that u and v are* incident *to the edge $\{u, v\}$. The* degree *of a vertex is the number of its incident edges. In this paper we will only deal with simple graphs.*

Example 6. *The graph $G = (V, E)$, where $V = \{A, B, C, D, E, F, G\}$ and $E = \{\{A, B\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}, \{F, G\}\}$ is illustrated in figure 5.*

Definition 33. *The* complement *of the graph $G(V, E)$ is another graph $\bar{G}$ with the same vertex set, having complementary edge set, i.e., $\bar{G} = (V, \bar{E})$ where $\bar{E} = \binom{V}{2} \setminus E$. ($\binom{V}{2}$ are all the 2-subsets of the set V)*

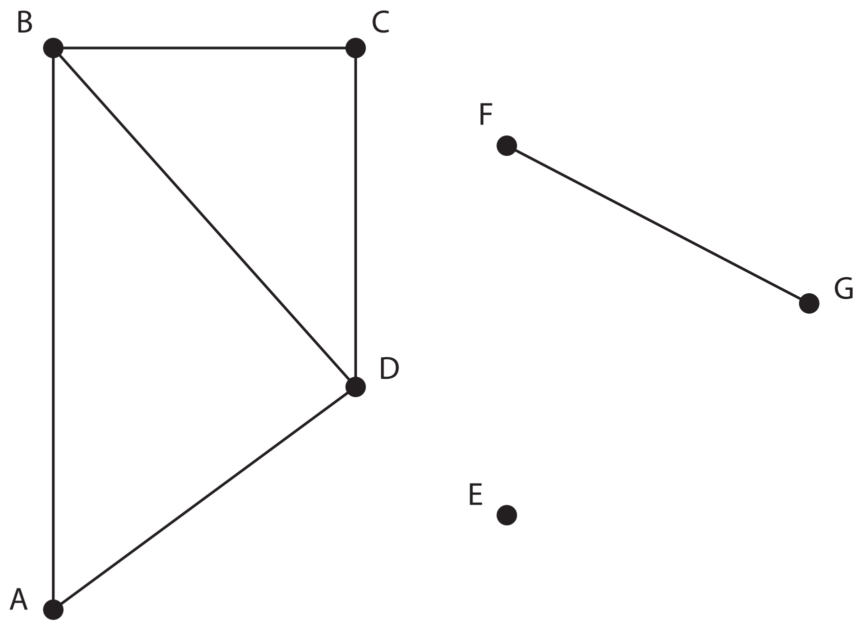

Figure 5

Definition 34. *A* subgraph *is obtained by deleting edges and vertices.* (*Obviously, when we delete a vertex, its incident edges are also deleted.*)

An induced subgraph *is obtained by deleting vertices and only those edges which are incident to them. So an induced subgraph is determined by its set of vertices.*

Definition 35. *An* independent set *in G is a subset $W \subseteq V$, which induces an empty subgraph,* $(W, \emptyset)$ (*an* empty graph *is a graph with no edges*)*. The size of the largest independent set in G is denoted by $\alpha(G)$.*

Definition 36. *A* legal coloring *of G is an assignment of "colors" to each vertex such that adjacent vertices receive different colors. In other words, it is a partition of the vertex set into independent sets, where the members of each independent set have the same color. The minimum number of colors required for a legal coloring is the* chromatic number *of G, denoted by $\chi(G)$.*

Remark 1. *It is, in general, hard to find* $\chi(G)$*, but there are some techniques to find a good lower bound for it. For example, if a graph contains a complete subgraph* K_n*, then* $\chi(G) \geq n$ *(a* complete graph $K_n = (V, E)$ *is a graph, with* n *vertices, where* $E = \binom{V}{2}$*).*

Example 7. *Here, we present an algorithm that often leads to a good enough lower bound for* $\chi(G)$ *(*G *is the graph of Example* 6*). The result is illustrated in figure* 6*. We start with one vertex (here* A*) and color it red. Then we continue with* B *and since we cannot use the color red, we color it blue. The vertex* C *cannot be colored blue, but we can color it red. Now, for vertex* D*, we can neither use red nor blue, so we have to introduce the new color green. Proceeding in this manner we see that the number of colors needed is* 3*. We also observe the complete subgraph* K_3*, with vertices* A, B, D*. This means that* $\chi(G)$ *is at least* 3*. And since we have found a legal coloring with* 3 *colors,* $\chi(G) = 3$*. So, generally, in each step we try to pick one of the previously used colors. If this is not possible, we assign a new color to the vertex.*

Proposition 37. *Let* G *be a graph with* n *vertices. The following relation between the chromatic number* $\chi(G)$ *and the independence number* $\alpha(G)$ *holds:*

$$\chi(G) \geq n/\alpha(G) \tag{16}$$

Proof. According to Definition 36, every color class in a legal coloring is independent. So there are $\chi(G)$ independent sets, each of size $\leq \alpha(G)$. That is, there are $\chi(G)$ color classes containing $n_1, n_2, \ldots, n_{\chi(G)}$ vertices with each $n_1, n_2, \ldots, n_{\chi(G)}$ less than or equal to $\alpha(G)$. Then

$$n = n_1 + n_2 + \ldots + n_{\chi(G)} \leq \chi(G) \cdot \alpha(G) \tag{17}$$

□

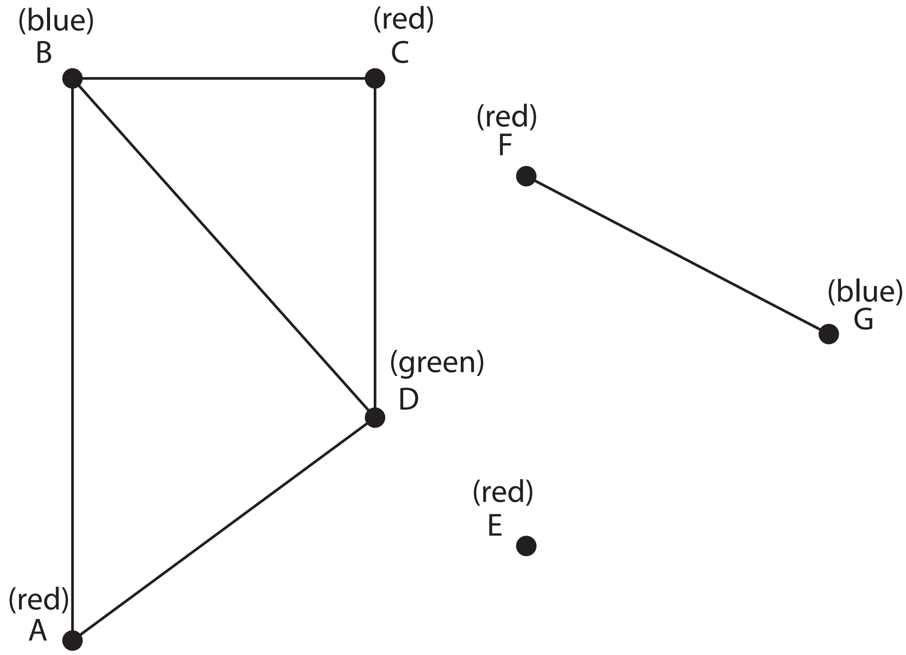

Figure 6

3 More Advanced Tools

In this chapter, some more advanced results are presented. Here we obtain the final tools needed to achieve our main goal; disproving Borsuk's Conjecture.

3.1 The Nonuniform Modular RW Theorem

Let L be a set of s integers and F an L-intersecting k-uniform family of m subsets of a set of n elements. Ray-Chaudhuri and Wilson proved the RW Theorem, which states that

$$m \le \binom{n}{s} \tag{18}$$

when s is small compared to n $\left(n \geq 2s, \binom{n}{s} = \frac{n!}{(n-s)!s!}\right)$.

The RW Theorem has been extended in several ways. One extension is for the set L to consist of s residue classes (mod p) where p is a prime and k, the size of the members of the family, does not belong to these residue classes. We will prove the RW Theorem considering this extension but, for a slightly weaker bound of the form

$$m \leq \binom{n}{s} + \binom{n}{s-1} + \ldots + \binom{n}{0} \tag{19}$$

Theorem 38 (Nonuniform Modular RW Theorem, Alon-Babai-Suzuki, 1991). *Let p be a prime number and L a set of s integers. Assume that $E = \{A_1, \ldots, A_m\}$ is a family of subsets of a set of n elements such that*

$$\begin{array}{llll} (a) & |A_i| \notin L & (\text{mod } p) & (1 \leq i \leq m) \\ (b) & |A_i \cap A_j| \in L & (\text{mod } p) & (1 \leq j < i \leq m) \end{array} \tag{20}$$

Then inequality (19) *holds.*

Proof. We begin by introducing the polynomial

$$F(x, y) = \prod_{\ell \in L} (x \cdot y - \ell) \tag{21}$$

in $2n$ variables where $x, y \in \mathbb{F}_p^n$ (the n-dimensional vector space over the field with p elements) and $x \cdot y = \sum_{i=1}^{n} x_i y_i$ is the standard inner product in $\mathbb{F}_p^n$. Thus,

$$F(x, y) = \prod_{\ell \in L} (x_1 y_1 + x_2 y_2 + \ldots x_n y_n - \ell) \tag{22}$$

Now consider the n-variable polynomials $f_i(x) = F(x, v_i)$, where $v_i \in \mathbb{F}_p^n$ is the incidence vector of the set A_i $(i = 1, \ldots, m)$. It is clear from condition (20) and equation (21) that for $1 \leq i, j \leq m$

$$f_i(v_j) = F(v_j, v_i) = \begin{cases} \neq 0 & \text{if } \ i = j \\ = 0 & \text{if } \ i \neq j \end{cases} \tag{23}$$

By Proposition 17, these equations remain valid if we replace f_i by the corresponding multilinear polynomial $\tilde{f}_i$ (note that every $\tilde{f}_i$ is unique). Clearly, each one of the polynomials $\tilde{f}_1, \ldots, \tilde{f}_m$ corresponds to exactly one of the sets $A_1, \ldots, A_m$. We conclude by the Diagonal Criterion (Proposition 16) that $\tilde{f}_1, \ldots, \tilde{f}_m$ are linearly independent over $\mathbb{F}_p^n$.

On the other hand, all the $\tilde{f}_i$ are multilinear polynomials of degree $\leq s$. Therefore, they can be represented as a linear combination of the following polynomials:

$$\begin{array}{l} 1 \\ x_{i_1} \\ x_{i_1}x_{i_2} \\ x_{i_1}x_{i_2}x_{i_3} \\ \vdots \quad \vdots \\ x_{i_1}x_{i_2}\cdots x_{i_s} \end{array} \tag{24}$$

where the range of $i_1, i_2, \ldots, i_s$ are from 1 to n. The number of polynomials listed above is

$$1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{s} \tag{25}$$

and therefore $\tilde{f}_1, \ldots, \tilde{f}_m$ belong to a space of dimension $\sum_{k=0}^{s} \binom{n}{k}$. $\square$

To show that we lose little by using the upper bound (19), let us examine the contribution of the tail of the sum in (19).

Proposition 39. *For $n \geq 2s$ we have*

$$\binom{n}{s} + \binom{n}{s-1} + \binom{n}{s-2} + \cdots + \binom{n}{0} < \binom{n}{s} \cdot \left(1 + \frac{s}{n-2s+1}\right) \tag{26}$$

Proof. First, we make the observation that for $1 \le k \le s \le n$

$$\frac{\binom{n}{k-1}}{\binom{n}{k}} = \frac{k}{n-k+1} \le \frac{s}{n-s+1} \tag{27}$$

Setting $\alpha = s/(n-s+1)$, it follows that for $s \le n/2$ we have

$$\sum_{k=0}^{s} \binom{n}{k} =$$

$$= \binom{n}{s} + \binom{n}{s} \cdot \frac{s}{n-s+1} + \binom{n}{s} \cdot \frac{s(s-1)}{(n-s+2)(n-s+1)} + \cdots + \binom{n}{s} \cdot \frac{1}{\binom{n}{s}} \le$$

$$\le \binom{n}{s} \cdot 1 + \binom{n}{s} \cdot \alpha + \binom{n}{s} \cdot \alpha^2 + \cdots =$$

$$= \binom{n}{s} \cdot (\sum_{i=0}^{\infty} \alpha^i) = \binom{n}{s} / (1-\alpha) \tag{28}$$

The last equality holds since

$$\sum_{i=0}^{\infty} \alpha^i = \lim_{n\to\infty} \frac{1-\alpha^n}{1-\alpha} = \lim_{n\to\infty} \frac{1}{1-\alpha} \tag{29}$$

(Note that $\alpha < 1$.) To obtain inequality (26) we substitute the value of α. ☐

Remark 2. *Observe that when s is substantially smaller than $n/2$, the term $\binom{n}{s}$ determines the order of the magnitude, since α becomes very small.*

3.2 A Connection Between Geometry and Combinatorics

In 1944 Hadwiger formulated the (geometrical) problem,

> *What is the minimum number $c(n)$ with the property that $\mathbb{R}^n$ can be divided into $c(n)$ subsets $\mathbb{R}^n = S_1 \cup \ldots \cup S_{c(n)}$ such that no pair of points within the same S_i, $i = 1, \ldots, c(n)$ is at unit distance?*

Before treating this problem, we need to make the following definition:

Definition 40. *The* distance-δ graph *in $\mathbb{R}^n$ has the set $\mathbb{R}^n$ as its vertex set and two points are adjacent if their Euclidean distance is δ. The* unit distance graph *corresponds to $\delta = 1$.*

Remark 3. *The unit distance graph and the distance-δ graph on $\mathbb{R}^n$ are isomorphic for any $\delta > 0$. The isomorphism from the unit distance graph to the distance-δ graph is given by $\Phi(p) = \delta p$ for every point $p \in \mathbb{R}^n$.*

With the above definition in mind we see that Hadwiger asks for the chromatic number of the unit distance graph. This problem hasn't even been solved in the plane (all we know is that $3 < c(2) \leq 7$). Frankl and Wilson showed that, for large n, the chromatic number of the unit distance graph on $\mathbb{R}^n$ grows exponentially with n. Before presenting this, we need the following theorem.

Theorem 41 (Omitted Intersection Theorem)**.** *Let p be a prime number and suppose that F is a $(2p-1)$-uniform family of subsets of a set of $4p-1$ elements. If no two members of F intersect in precisely $p-1$ elements, then*

$$|F| \leq 2 \cdot \binom{4p-1}{p-1} \tag{30}$$

Proof. Set $L = 0, \dots, p-2$. Then we have that

$$|F| \le \binom{4p-1}{p-1} + \binom{4p-1}{p-2} + \dots + \binom{4p-1}{0} <$$

$$< \binom{4p-1}{p-1} \cdot (1 + \frac{p-1}{3p-1}) < 2 \cdot \binom{4p-1}{p-1} \tag{31}$$

The first inequality follows from Theorem 38 (F obviously satisfies the assumptions of Theorem 38) and the second inequality follows from Proposition 39. The third inequality is trivial. □

Theorem 42 (Frankl-Wilson, 1981)**.** *For large n, the chromatic number of the unit distance graph on $\mathbb{R}^n$ is greater than 1.139^n.*

Proof. Since the unit distance graph and the distance-δ graph on $\mathbb{R}^n$ are isomorphic for any $\delta > 0$, they both have the same chromatic number $c(n)$.

To prove the theorem, we show that the distance-δ graph of some subset S of the unit cube $\Omega = \{0,1\}^n$ has exponentially large chromatic number for some $\delta > 0$. This δ will depend on n. Then, obviously, the unit distance graph on $\mathbb{R}^n$ will have exponentially large chromatic number (since $S \subseteq \Omega \subset \mathbb{R}^n$).

Each subset $S \subseteq \Omega$ corresponds to a set system $H \subseteq 2^{[n]}$ (e.g. for $n = 3$, we have that $\{(1,1,0),(0,1,1)\} = S \subseteq \{0,1\}^3$ corresponds to $H = \{\{1,2\},\{2,3\}\}$). We see that S consists of the incidence vectors of the members of H.

Let $d(A,B)$ denote the Euclidean distance of the incidence vectors of the sets $A, B \in H$. From Definitions 8 and 9 we conclude that $d(A,B)^2$ is the size of the symmetric difference of A and B. Now if we assume that H is k-uniform, then

$$d(A,B)^2 = 2(k - |A \cap B|) \tag{32}$$

This means that the distances between the elements of S are determined by the intersection sizes of the corresponding sets of H. Thus we have shown the connection between Geometry and Combinatorics.

Now, let us assume that $n = 4p-1$ for some prime p, and that $k = 2p-1$. We say that two sets $A, B \in H$ are adjacent if $|A \cap B| = p-1$. This corresponds to distance $\delta = \sqrt{2p}$, since

$$\delta = d(A, B) = \sqrt{2(k - |A \cap B|)} \tag{33}$$

With the above assumptions in mind, we have defined a graph $G_p = (H_p, E_p)$ with vertex set $H_p = \binom{[4p-1]}{2p-1}$ (i.e. all $(2p-1)$-subsets of the set $[4p-1]$).

Now, our goal is to deduce an exponential lower bound for the chromatic number $\chi(G_p)$. Instead, we will find an upper bound for $\alpha(G_p)$, the size of the largest independent set. By our definition of adjacency for G_p, no two members of an independent set of size $\alpha(G_p)$ intersect in precisely $p-1$ elements. Therefore, applying the "Omitted Intersection Theorem" (Theorem 41), we have that

$$\alpha(G_p) \leq 2 \cdot \binom{4p-1}{p-1} \tag{34}$$

To show the exponential lower bound for $\chi(G_p)$, we use Proposition 37:

$$\chi(G_p) \geq |H_p|/\alpha(G_p) \geq \frac{\binom{4p-1}{2p-1}}{2 \cdot \binom{4p-1}{p-1}} > 1.1397^{4p-1} \tag{35}$$

We have thus shown that the chromatic number of the distance-$\sqrt{2p}$ graph of the set $S(H_p) \subseteq \Omega = \{0,1\}^{(4p-1)} \subset \mathbb{R}^{4p-1}$ is greater than 1.1397^n for all sufficiently large n of the form $n = 4p-1$ ($S(H_p)$ denotes the incidence set $S \subseteq \Omega$ of the set system H_p).

Now, in order to show that exponential growth holds for all sufficiently large n, let p be the largest prime such that $n > 4p-1$. By the "Prime Number Theorem" (Theorem 18), for every $\epsilon > 0$ and every sufficiently large x there exists a prime number p between $(1-\epsilon)x$ and x. Applying this to $x = n/4$, we find a prime number p such that $(1-\epsilon)n < 4p < n$. Therefore,

$$c(n) > c(4p-1) > 1.1397^{4p-1} > 1.1397^{(1-\epsilon)n} > 1.139^n \tag{36}$$

when ϵ is chosen small enough. □

The chromatic number of the unit distance graph on $\mathbb{R}^n$ has been proven to be greater than 1.203^n. But the above verification of the weaker lower bound suffices for the purpose of this paper. For the proof of the stronger lower bound, see Babai and Frankl [LAMC].

4 The Disproof of Borsuk's Conjecture

Borsuk conjectured that every body in $\mathbb{R}^n$ of diameter d can be partitioned into $n+1$ parts each of diameter less than d. (Actually, Borsuk was careful not to state this as a conjecture but as a question, but with time it got known as Borsuk's Conjecture.) The fact that we need, at least, $n+1$ parts is easily observed by looking at the regular simplex in each dimension.

Borsuk's partition problem refers to all subsets of $\mathbb{R}^n$. Obvious arguments show that one may consider, without loss of generality, only sets which are closed and of diameter 1.

After decades of efforts by various mathematicians, Kahn and Kalai managed, in 1993, to give a disproof to Borsuk's, by then, widely accepted Conjecture. They did so by proving that the number of parts in such a partition grows exponentially with n. The simplicity and elegance of this relatively short proof is an excellent example of the value of "Combinatorial Geometry".

Theorem 43 (Kahn-Kalai, 1993)**.** *Let $f(n)$ denote the minimum number such that every set of diameter 1 in $\mathbb{R}^n$ can be partitioned into $f(n)$ pieces of smaller diameter. Then $f(n) > 1.2^{\sqrt{n}}$.*

Proof. The proof is similar to the one given for Theorem 42. We want to show that the number of partitions of some subset of $\mathbb{R}^n$ into parts of smaller diameter is exponential with respect to n. We consider a subset S of the unit cube $\Omega = \{0,1\}^n \subset \mathbb{R}^n$. As we know, such a subset corresponds to a set system $F \subseteq 2^{[n]}$, and we write $S = S(F)$ to denote the set of incidence vectors of the members of F. We assume that F is ℓ-uniform. By the same argument as before, we obtain

$$d(A,B)^2 = 2(\ell - |A \cap B|) \tag{37}$$

for any $A, B \in F$ ($d(A,B)$ denotes the distance of the incidence vectors of the sets A and B). Thus the maximum distance between two members of S occurs when, for their corresponding sets $A, B \in F$, $|A \cap B|$ is minimal. Now, let

$$\mu(F) = \min\{|A \cap B| : A, B \in F\} \tag{38}$$

We observe that a partition of $S(F)$ into sets of smaller diameters means a partition of F as $F = F_1 \cup \ldots \cup F_t$ such that $\mu(F_j) > \mu(F)$ for $j = 1, \ldots, t$. Let $g(F)$ denote the smallest t for which this is possible. Then clearly $f(n) \geq g(F)$ for any uniform set system $F \subset 2^{[n]}$. Thus, we have turned the geometrical problem into a combinatorial one.

Our next step is to associate a graph G with the set system F, where the vertex set consists of the members of F and two sets (vertices) $A, B \in F$ are adjacent if $|A \cap B| = \mu(F)$ (as in Theorem 42 the adjacency is connected to the intersection sizes). Then, we observe that $g(F)$ is the chromatic number $\chi(G)$.

We want to construct a set system F such that the graph G it represents according to the "minimum intersection size adjacency rule" (equation (38)) will be isomorphic to the graph G_p of Theorem 42. (Actually, they are two different representations of the same graph.) This will make the strong lower bound on the chromatic number of G_p (inequality (35)) valid for our graph G.

Assume that n is of the form $(2p-1)(4p-1)$ for some prime number p. We set $m = 4p - 1$, $k = 2p - 1$, and $H_p = \binom{[m]}{k}$, as in Theorem 42.

Moreover, let $X = \binom{[m]}{2}$; $|X| = \binom{m}{2} = n$. We define the set system F over the universe X, i.e. $F \subset 2^X$.

We shall associate a set $\Phi(A) \subset 2^X$ with each $A \in H_p$. The set system to beat "Borsuk's Conjecture" will be

$$F = \{\Phi(A) : A \in H_p\} \tag{39}$$

Remember, our goal is to make a correspondence $A \mapsto \Phi(A)$ such that

$$|A \cap B| = p - 1 \qquad \text{if and only if} \qquad |\Phi(A) \cap \Phi(B)| = \mu(F) \tag{40}$$

for all $A, B \in H_p$. This will establish that Φ is an isomorphism between G_p and G, as desired.

Here is the simple construction: $\Phi(A)$ will be the set of those pairs of elements $[m]$ which are split by A. Formally,

$$\Phi(A) = \{\{x, y\} : x \in A, y \in [m] \backslash A\} \tag{41}$$

Clearly $\Phi(A) \subset X$ and the set system defined by equation 39 is ℓ-uniform with $\ell = k(m-k)$. The correspondence $A \mapsto \Phi(A)$ being one-to-one, all we need to verify is that it preserves adjacency (equation 40).

To this end, assume $|A \cap B| = r$ $(A, B \in H_p)$. It is easy to see that

$$\begin{aligned} |\Phi(A) \cap \Phi(B)| &= \\ &= r(m-2k+r) + (k-r)^2 = \\ &= 2(r-(k-(m/4)))^2 - 2(2k-(m/2))^2 + k^2 \end{aligned} \tag{42}$$

The minimum of this expression is attained when r is as close to $k-(m/4) = p-(3/4)$ as possible, i.e., when $r = p-1$. This completes the proof of the $G \cong G_p$ isomorphism.

By inequality (35) we conclude that

$$g(F) = \chi(G) = \chi(G_p) \geq \frac{\binom{4p-1}{2p-1}}{2 \cdot \binom{4p-1}{p-1}} > 1.1397^{4p-1} = 1.1397^m \tag{43}$$

(The last inequality holds when p is sufficiently large.) Since $m > \sqrt{2n}$, we obtain (for sufficiently large n)

$$f(n) \geq g(F) > 1.1397^{\sqrt{2n}} > 1.203^{\sqrt{n}} \tag{44}$$

completing the proof for all dimensions n of the form

$$n = \binom{4p-1}{2} = (4p-1)(2p-1) \tag{45}$$

where p is a prime. The extension to all dimensions, using the Prime Number Theorem, is analogous to the corresponding argument at the end of the proof of Theorem 42. $\square$

Remark 4. *Concerning an upper bound for the partition of a set of points into parts of smaller diameter, it has been shown* (*Schramm,* 1988) *that, for every* $\epsilon > 0$ *and sufficiently large* d*,*

$$f(d) < (\sqrt{3/2} + \epsilon)^d \tag{46}$$

During the time it took to disprove "Borsuk's Conjecture', several interesting results concerning the conjecture have been obtained. In the following chapters we will give the assertion of "Borsuk's Conjecture" for some special cases.

5 Borsuk's Conjecture in the Plane and in the Room

The highest dimension for which Borsuk's Conjecture (with no restrictions) has been verified, is 3, i.e., the room. In this chapter we will present a geometrical proof for the case $\mathbb{R}^2$ and a combinatorial one for $\mathbb{R}^3$ (it is obvious that Borsuk's Conjecture holds for $\mathbb{R}^1$).

5.1 The Assertion of Borsuk's Conjecture in $\mathbb{R}^2$

In this section we will present the validity of Borsuk's Conjecture for planar bodies. The proof was given by Borsuk himself in 1933. For this purpose he used a lemma, proved by Pal in 1920, stating that every body in $\mathbb{R}^2$ with diameter d can be surrounded by a regular hexagon whose opposite sides are at distance d. Then he showed that it is possible to partition such a hexagon into three parts, each of diameter less than d. We begin by proving this lemma:

Lemma 44 (Pal, 1920)**.** *Every plane set F of diameter d can be enclosed in a regular hexagon whose opposite sides are at distance d (figure 7).*

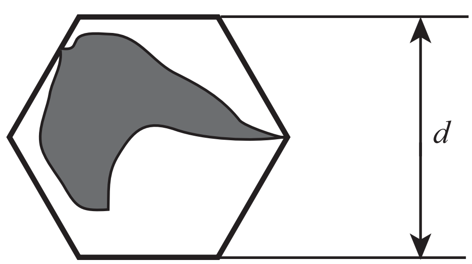

Figure 7

Proof. We may without loss of generality assume that F is topologically closed, i.e. F contains its boundary. Take an arbitrary line l that does not intersect the set F and move it closer to F while keeping it parallel to its original direction, until it touches F (figure 8). The resulting line l_1 has at least one point in common with F and the whole set F lies on one side of l_1. Such a line is called the *support line* of F. Let us draw a second support line l_2, parallel to l_1 (figure 8). Clearly, the whole set F will lie in the strip between the lines l_1 and l_2, and the distance between the lines is at most d, since the diameter of F is d.

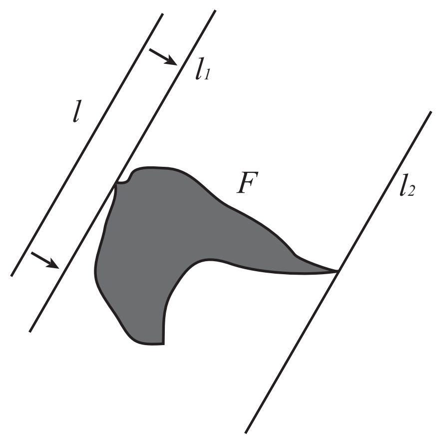

Figure 8

Now, draw two support lines, m_1 and m_2, at an angle of 60° to l_1 (figure 9). The lines l_1, l_2, m_1 and m_2 form a parallelogram $ABCD$ with angle 60° and heights at most d, surrounding the set F.

Next, draw two support lines, p_1 and p_2, at an angle of 120° to l_1, and denote by M and N the bases of the perpendiculars dropped on these lines from the ends of the diagonal AC (figure 9). We shall show that the direction of l_1 can be chosen so that $AM = CN$. Indeed, suppose $AM \neq CN$, say $AM < CN$. Then the value $y = AM - CN$ is negative.

Now, we rotate l_1 through 180° (the set F is kept fixed). The remaining lines l_2, m_1, m_2, p_1 and p_2 will also change their positions (since their positions depend on the choice of l_1). Therefore, as l_1 rotates, the points A, C, M and N will continuously move and continuously vary the value of $y = AM - CN$. But when the line l_1 has rotated through 180°, it will lie in the position formerly occupied by l_2. Hence, we shall obtain the same parallelogram as in figure 9 with the point A and C, and also M and N, reversed. Consequently, y will be positive. If we now plot the graph of the rotation of l_1 from 0° to 180° (figure 10), we see that y is 0 for some position of l_1, i.e. $AM = CN$

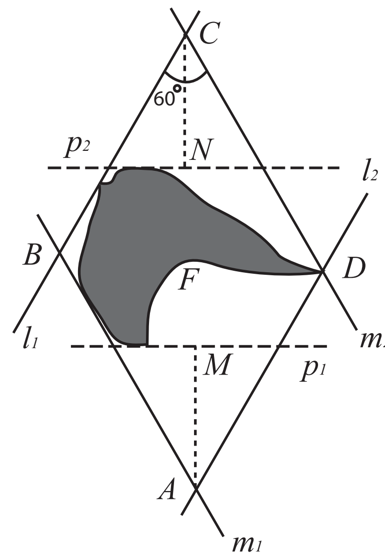

Figure 9

(since as y continuously changes from negative to positive, it must at same point be 0). We shall examine the positions of all our lines when $y = 0$

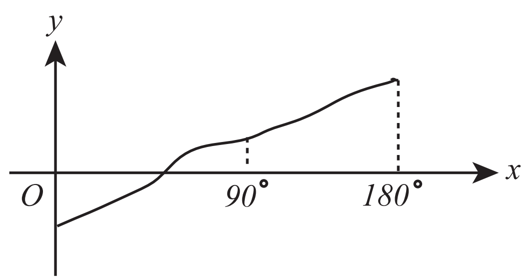

Figure 10

(figure 11). The equality $AM = CN$ implies that the hexagon formed by the lines l_1, l_2, m_1, m_2, p_1 and p_2 is centrally symmetric. Each angle of this hexagon is 120°, and the distance between opposite sides is at most d. If the distance between the lines p_1 and p_2 is less than d, we shall move them apart (moving each the same distance) until the distance equals d. We then move the lines l_1, l_2 and m_1, m_2 in exactly the same way. We thereby obtain a centrally symmetric hexagon (with angles 120°) with opposite sides at distance d from each other (the dotted hexagon in figure 11). From the above it is clear that all the sides of the hexagon are equal, i.e., the hexagon is regular

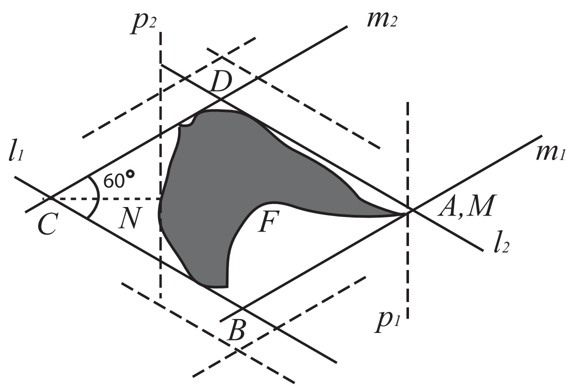

Figure 11

with set F lying inside it. $\square$

Theorem 45 (Borsuk, 1933). *Let F be a body (set of points) in $\mathbb{R}^2$ with diameter d. Then F can always be partitioned into 3 parts, each of diameter less than d.*

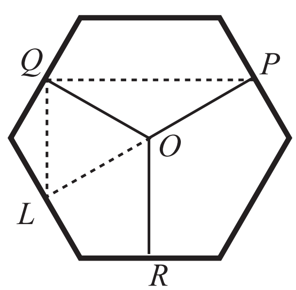

Figure 12

Proof. Since every body $F \subset \mathbb{R}^2$ with diameter d can be surrounded by a regular hexagon, whose opposite sides are at distance d, (Lemma 44) it suffices to show that such a hexagon can be partitioned into three parts each of diameter less than d. The required partition is shown in figure 12 (the points P, Q and R are the centres of the sides and O is the centre of the hexagon). The diameter of each part is less than d since, e.g., in the triangle PQL, the angle Q is a right angle. Therefore, $PQ < PL = d$. Because, the distance PQ is equal to the diameter of all three parts, this proves the theorem. $\square$

A similar proof for bodies in $\mathbb{R}^3$ was given by Grünbaum in 1957. According to a lemma proved by the American mathematician Gale in 1953, every 3-dimensional body of diameter d can be surrounded by a octahedron whose opposite faces are at distance d. Grünbaum then proved that every such octahedron can be partitioned into 4 parts, each of diameter less than d ($\approx 0.9888d$). For the complete proof, see Boltjansky and Gohberg [RPCM].

5.2 The Assertion of Borsuk's Conjecture in $\mathbb{R}^3$

In this section we will prove that a *finite* set of points P in $\mathbb{R}^3$ can be partitioned into at most 4 parts of less diameter. We begin with a definition:

Definition 46. *Let A be a finite set of points in $\mathbb{R}^3$. The* maximum distance *between the points in A is defined to be*

$$\max\{d(p,q) : p, q \in A\} \tag{47}$$

where $d(p,q)$ is the Euclidean distance between the points p and q.

Before proceeding with the proof we need the following theorem, conjectured by Vzsonyi and proved independently by Grünbaum, Heppes and Sterazevicz.

Theorem 47 (Grünbaum, 1956; Heppes, 1956; Sterazevicz, 1957)**.** *Let $f_3^{max}(n)$ denote the maximum number of times the maximum distance can occur among n points in $\mathbb{R}^3$. Then*

$$f_3^{max}(n) = 2n - 2 \tag{48}$$

for every $n \geq 4$.

Proof. We prove this by induction. Equality (48) holds for $n = 4$, where the points are the vertices of the regular tetrahedron. First, we prove that

$$f_3^{max} \leq 2n - 2 \tag{49}$$

Assume that it has already been proven for every integer smaller than n. Fix a set P of n points in $\mathbb{R}^3$ whose diameter $d = 1$. Connect a pair of elements of P by a line segment if their distance is 1. This means that two points are adjacent if they are at distance 1. By $f_P(n)$ we mean the number of times distance 1 occurs in the set P of n elements. Then obviously we want to show that

$$f_P(n) \leq 2n - 2 \qquad \text{for every} \quad P \subset \mathbb{R}^3 \tag{50}$$

We consider two cases:

First, let us assume that there is a point $p \in P$ which is connected to at most two other elements of P. For the set $P - \{p\}$ (consisting of $n - 1$ elements), we have

$$f_{P \setminus \{p\}} \leq f_3^{max}(n-1) \leq 2(n-1) - 2 = 2n - 4 \tag{51}$$

Since p is connected to no more than two elements

$$f_P \leq f_{P \setminus \{p\}} + 2 \leq 2n - 4 + 2 = 2n - 2 \tag{52}$$

Now, let us assume that every element of P is adjacent to at least three points. Considering the correspondence between adjacency and maximum distance, it is easy to see that every point of P is a vertex of the convex hull of P. Draw a unit ball $B(p)$ around each point $p \in P$, and set

$$C = \bigcap_{p \in P} B(p) \tag{53}$$

Obviously, C is a convex set (a spherical polytope) bounded by spherical "faces" and circular arcs ("edges") separating them. Let V, F and E denote the number of vertices, faces and edges of C, respectively. Notice that $F = n$, because each $B(p)$ contributes exactly one face to the boundary of C. On the other hand, $V \geq n$, where strict inequality holds if C has a vertex not belonging to P.

Every vertex x of C is incident with at least three edges of C. Furthermore, if $x \in P$, then the number of edges of C incident with x is equal to the number of points $q \in P$ with $|x-q| = 1$. For example consider that the points q_1, q_2, q_3, q_4 are at distance 1 from the point x. The unit balls surrounding q_1, q_2, q_3, q_4 define four sides s_1, s_2, s_3, s_4, respectively and they intersect in the point x. These sides intersect pairwise in four circular "edges" (figure 13).

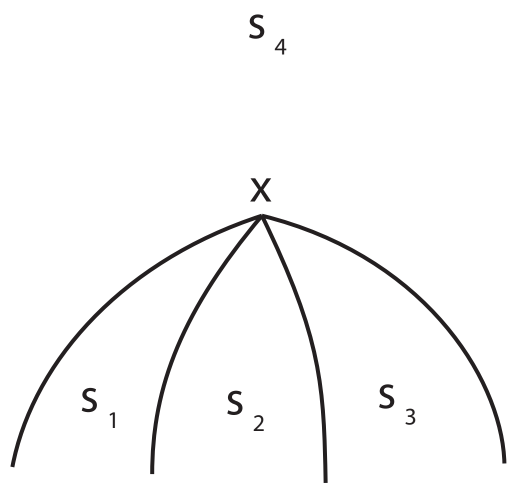

Figure 13

Double-counting the edges of C, we obtain

$$2f_P + 3(V - n) \leq 2E \tag{54}$$

where $3(V - n)$ denotes the number of edges determined by the vertices of C which are not in P.

By Euler's polyhedral formula (Theorem 25)

$$V - E + F = 2 \tag{55}$$

Thus,

$$\begin{aligned} 2f_P &\leq 2E - 3(V-n) = 2(V + F - 2) - 3(V - n) = \\ &= 2(n + F - 2) - (V - n) \leq 2(n + n - 2) - 0 = 2(2n - 2) \end{aligned} \tag{56}$$

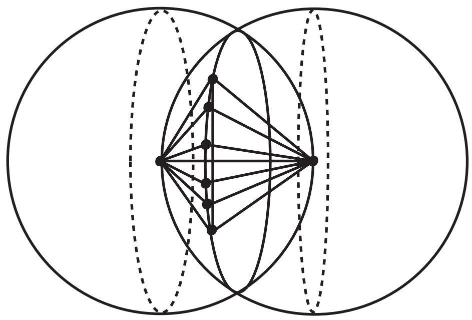

Figure 14

as required. The last inequality follows by the fact that $V \geq n$.

Thus from equations (52) and (56) we obtain

$$f_3^{max}(n) = \max f_P(n) \leq 2n - 2 \tag{57}$$

Figure 14 shows that

$$f_3^{max}(n) \geq 2n - 2 \tag{58}$$

for $n = 8$. It is easy to see that inequality (58) holds for every n. Equations (57) and (58) gives us the desired result

$$f_3^{max} = 2n - 2 \tag{59}$$

□

Theorem 48. *Let P be a finite set of points in $\mathbb{R}^3$. Then P can be partitioned into at most 4 subsets of smaller diameter.*

Proof. We prove this by induction. Let $|P| = n$. For $n \leq 4$, the assertion is obvious. Let $n > 4$ and assume that the assertion is true for every set of size at most $n - 1$.

Now, think of the points of P as the vertices of a graph. Also, let two points be adjacent if and only if their distance is equal to the diameter of P. It follows from Theorem 47 that the resulting graph has a vertex p of degree at most 3. By the induction hypothesis, we can partition $P - \{p\}$ into 4 parts P_1, P_2, P_3, P_4, where diam $P_i <$ diam P $(1 \leq i \leq 4)$. So, there exists an index i such that p is not adjacent to any element of P_i. Hence, diam $(P_i \cup \{p\}) <$ diam P. Put $i = 1$ and the partition of P into 4 parts of less diameter is given by

$$(P_1 \cup \{p\}) \cup P_2 \cup P_3 \cup P_3 \tag{60}$$

$\square$

Once again, we have observed the beauty of solving a geometrical problem combinatorially. For a similar proof for bodies in $\mathbb{R}^2$, see Pach and Agarwal [CM].

6 Borsuk's Conjecture for Bodies in $\mathbb{R}^n$

As we saw in Section 4, Borsuk's Conjecture is generally not true. It has not even been proven for bodies in $\mathbb{R}^4$. However, in some special cases, the conjecture is true in $\mathbb{R}^n$. In this chapter we will present Hadwiger's proof of the assertion of Borsuk's Conjecture for convex bodies with smooth boundaries in $\mathbb{R}^n$. But first we need the following theorem:

Definition 49. *An n-dimensional* sphere *is a set of points in $\mathbb{R}^n$ each having a fixed distance r to a point C called the* center *of the sphere. A* ball *is a sphere together with all of its interior points.*

Theorem 50. *Every n-dimensional ball E with diameter d can be partitioned into $n+1$ pieces, each of diameter less than d.*

Proof. For the proof, see Boltjansky and Gohberg [RPCM]. $\square$

Theorem 51 (Hadwiger, 1946)**.** *Every n-dimensional convex body with smooth boundary and diameter d can be partitioned into $n+1$ parts of diameter less than d.*

Proof. Let F be any n-dimensional convex body with smooth boundary having diameter d. Consider also an n-dimensional ball E having the same diameter d, and construct some partition of this ball E into $n+1$ parts $M_0, M_1, \ldots, M_n$, each of diameter less than d. We now construct a partition of the boundary G of the body F into $n+1$ sets $N_0, N_1, \ldots, N_n$.

Let A be an arbitrary boundary point of F. Draw the tangential hyperplane of F passing through A (this is, by Definition 31, unique), and draw parallel to it the tangential hyperplane of the ball E, so that the body F and the ball E lie on the same side of these hyperplanes (figure 15). Denote by $f(A)$ the point at which the second hyperplane touches the ball E. We shall consider the point A belonging to the set N_i if the corresponding point $f(A)$ belongs to the set M_i ($i = 0, 1, \ldots, n$). Consequently, the whole boundary G of the body F is partitioned into $n+1$ sets $N_0, N_1, \ldots, N_n$.

We shall prove, by contradiction, that each of the sets $N_0, N_1, \ldots, N_n$ has

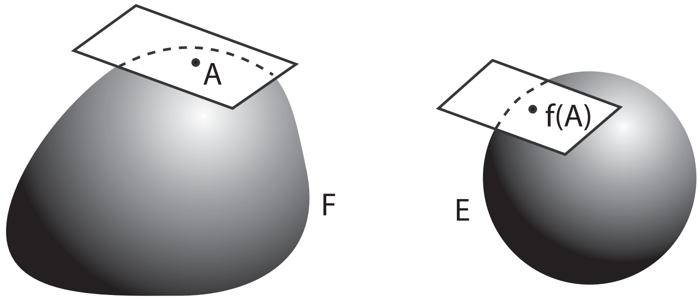

Figure 15

diameter less than d. Assume that a certain set N_i has diameter d and let A and B be two points of the set N_i at distance d from each other. Construct two hyperplanes Γ_A, Γ_B passing through the points A and B perpendicular to the segment AB. Clearly, F lies in the region between these hyperplanes (otherwise the diameter of F would be greater than d), i.e., Γ_A and Γ_B are tangential hyperplanes of F, passing through A and B. These tangential planes being parallel implies that the corresponding points $f(A)$ and $f(B)$ are at distance d. On the other hand, as A and B belong to the set N_i, the points $f(A)$ and $f(B)$ also belong to M_i and, therefore, the distance between $f(A)$ and $f(B)$ is less than d. This contradiction shows that none of the sets $N_0, N_1, \ldots, N_n$ has diameter d.

Now let O be an arbitrary interior point of F. For any $i = 0, 1, \ldots, n$, we shall denote by P_i the "cone" with apex O and curvilinear base the set N_i. Clearly, the constructed "cones" $P_0, P_1, \ldots, P_n$ fill the whole body F, i.e., we have obtained a partition of F into $n+1$ parts. Furthermore, it is clear that each of the sets P_i has diameter less than d (because the diameter of the "base" N_i is less than d). Hence, the constructed partition divides the body F into $n+1$ parts of diameter less than d, proving the theorem. □

References

[LAMC] Babai, L. and Frankl, P. (1992), *Linear Algebra Methods in Combinatorics; with Applications to Geometry and Computer Science, Preliminary*, Version 2. Department of Computer Science, University of Chicago.

[DM] Biggs, N. L. (1989), *Discrete Mathematics*. Oxford Science Publications.

[DFSP] Boltjansky, V. G. and Gohberg, I. (1980), *The Decomposition of Figures into Smaller Parts*. The University of Chicago Press.

[RPCM] Boltjansky, V. G. and Gohberg, I. (1985), *Results and Problems in Combinatorial Geometry*. Cambridge University Press.

[WM] Courant, R. and Robbins, H. (1996), *What Is Mathematics*? Oxford University Press.

[PSPM] Grünbaum, B. (1963), "Borsuk's Problem and Related Questions". *Proceedings of Symposia in Pure Mathematics*, Vol. 7. American Mathematical Society, Rhode Island.

[CBC] Kahn, J. and Kalai, G. (1993), "A Counterexample to Borsuk's Conjecture". *Bulletin of the American Mathematical Society*, Vol. 29, No. 1, pp. 60-62.

[CM] Pach, J. and Agarwal, K. (1995), *Combinatorial Geometry*. New York: Wiley.

[ITT] Widder, D. V. (1971), *An Introduction to Transform Theory*. Academic Press.

Internet Resources

- www.mathworld.wolfram.com
- www.wikipedia.org
- www.planetmath.org
- www.math.su.se/~yishao/kurser/opt04ht/k1.pdf