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Legendre Polynomials and their Applications

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LEGENDRE POLYNOMIALS AND THEIR APPLICATIONS

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Contents

1. Special Functions

When I hear the expression "special function", functions like the logarithm, the exponential and the trigonometric come up into my mind. But of course there are many more, like the ones that are described here .

In fact special functions are those functions which have a good relation with physics, and which belong to applied mathematics. These functions have many useful properties that come from their connection with the partial differential equations of mathematical physics.

Orthogonality is one of most important properties that some of these functions have. The orthogonality of $\cos x$ is for example the foundation of the theory of Fourier series, and similar theories may be developed for other special functions.

This branch of mathematic has a history with great names such as Euler, Gauss, Fourier, Legendre and many other.

In this paper we are going to describe Legendre polynomials and their most important properties.

2. Legendre Polynomials

Legendre functions are solutions to Legendre's differential equation:

$$\frac{d}{dx}[(1-x^2)\frac{d}{dx}p(x)] + n(n+1)p(x) = 0 \tag{2.1}$$

They are named after Adrian-Marie Legendre(1752-1833).

This ordinary differential equation is frequently encountered in physics and other technical fields. In particular, it occurs when solving Laplace's equation and related partial differential equations in spherical coordinates (see section 3).

The Legendre differential equation may be solved using the power series method. The solution is finite (i.e the series converges) provided $|x| < 1$. Furthermore, it is finite at $x = \pm 1$ provided n is a non negative integer. In this case the solutions form a polynomial sequence called the Legendre polynomials. Each Legendre polynomial $p_n(x)$ is a nth degree polynomial.

3. Solution Of Legendre's Equation Via Power Series

3.1. **Legendre Functions:** We will use Laplace's equation later, so we will first describe it shortly here.

3.1.1. *Laplace's Equation.* Laplace's equation in cartesian coordinates is

$$\frac{\partial \vartheta^2}{\partial x^2} + \frac{\partial \vartheta^2}{\partial y^2} + \frac{\partial \vartheta^2}{\partial z^2} = 0.$$

Laplace's equation in ordinary spherical coordinates r, θ, Φ is

$$\bigtriangledown^2 \phi = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial \phi}{\partial r}) + \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}(\sin(\theta)\frac{\partial \phi}{\partial \theta}) + \frac{1}{r^2 \sin^2(\theta)}(\frac{\partial^2 \phi}{\partial \Phi^2}) = 0 \tag{3.1}$$

Taking the axis of symmetry over the angle Φ so that $\frac{\partial^2 \phi}{\partial \Phi^2} = 0$, Laplace's equation reduces to

$$\frac{\partial}{\partial r}(r^2 \frac{\partial \phi}{\partial r}) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta \frac{\partial \phi}{\partial \theta}) = 0 \tag{3.2}$$

We solve it by separation of variables, and then the solution will be as follows:
$\phi(r, \theta) = A(r)B(\theta)$.
Differentiating this with respect to r and θ, evaluating the Laplacian of this and dividing through by $A(r)B(\theta)$ gives

$$r^2 \frac{A''(r)}{A(r)} + 2r\frac{A'(r)}{A(r)} + \frac{B''(\theta)}{B(\theta)} + \frac{\cos\theta}{\sin\theta}\frac{B'(a)}{B(\theta)} = 0. \tag{3.3}$$

But r and θ are independent variables therefore this equation is true only if
$r^2 \frac{A''(r)}{A(r)} + 2r\frac{A'(r)}{A(r)}$ and $\frac{B''(\theta)}{B(\theta)} + \frac{\cos\theta}{\sin\theta}\frac{B'(\theta)}{B(\theta)}$
are constants.

Letting $\mu = r^2\frac{A''(r)}{A(r)} + 2r\frac{A'(r)}{A(r)}$ and $-\mu = \frac{B''(\theta)}{B(\theta)} + \frac{\cos\theta}{\sin\theta}\frac{B'(\theta)}{B(\theta)}$, we have the two separate differential equations

$$r^2A''(r) + 2rA'(r) - \mu A(r) = 0 \tag{3.4}$$

$$\sin\theta B''(\theta) + \cos(\theta)B'(\theta) + \mu\sin(\theta)B(\theta) = 0. \tag{3.5}$$

A solution of (3.4) is $A(r) = r^n$ if $\mu = n(n+1)$. A solution of (3.5), if $n = 1$, is $B(\theta) = \cos(\theta)$ with $\mu = n(n+1) = 2$.
The function $\phi(r,\theta) = r\cos(\theta)$ is then a solution of Laplace's equation, corresponding to $n = 1$. We take this as a hint, and consider now the solutions of (3.5)that are functions of the form

$$B(\theta) = P_n(\cos(\theta)) \tag{3.6}$$

Inserting (3.6) in (3.5), and dividing by $\sin\theta$, we get

$$n(n+1)P_n(\cos\theta) + \big(\sin\theta)^2P_n''(\cos\theta)\big) - 2\cos\theta P_n'(\cos\theta) = 0 \tag{3.7}$$

Replacing $\cos(\theta)$ with x, and noting that $(\sin\theta)^2 = 1 - (\cos\theta)^2$, this becomes

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0. \tag{3.8}$$

Replacing $P_n(x)$ with y and $\lambda = n(n+1)$ we get

$$(1-x^2)y'' - 2xy' + \lambda y = 0 \tag{3.9}$$

This equation is called Legendre's equation, where $x = \cos\theta$ ranges from -1 to +1, and we will be interested in the solutions y(x) on $-1 \le x \le 1$. However, there might be some other context which could produce this ODE, and then one will need to also consider possible use of solutions for $|x| > 1$, but assume for now that we only require y(x) for $|x| \le 1$.

Note that the usual boundary conditions are that y(x) should remain finite at the endpoints $x = 1$ and $x = -1$ (corresponding to $\theta = 0$ and $\theta = \pi$).
Since the equation is analytic around x= 0, we can use the standard power series method to determine solutions for this ODE, as follows:

$$y(x) = \sum_{j=0}^{\infty} a_j x^j$$

$$y'(x) = \sum_{j=1}^{\infty} j a_j x^{j-1}$$

$$y''(x) = \sum_{j=2}^{\infty} j(j-1) a_j x^{j-2}$$

and upon substitution of these series into the original equation, we get

$$\sum_{j=2}^{\infty} j(j-1)a_j x^{j-2} - \sum_{j=0}^{\infty}[j(j+1)-\lambda]a_j x^j = 0$$

or

$$\sum_{j=0}^{\infty}[(j+2)(j+1)a_{j+2} - (j(j+1)-\lambda)a_j]x^j = 0$$

The last equation produces the recurrence relation:

$$a_{j+2} = \frac{j(j+1)-\lambda}{(j+2)(j+1)}a_j \tag{3.10}$$

for $j = 0$ we have:

$$a_2 = \frac{-\lambda}{2}a_0 \tag{3.11}$$

for $j = 1$ we have

$$a_3 = \frac{2-\lambda}{3\times 2}a_1 = \frac{2-\lambda}{6}a_1 \tag{3.12}$$

for $j = 2$ we have

$$a_4 = \frac{(2\times 3)-\lambda}{4\times 3}a_2 = \frac{6-\lambda}{12}.\frac{-\lambda}{2}a_0 \tag{3.13}$$

for $j = 3$ we have

$$a_5 = \frac{(3\times 4)-\lambda}{5\times 4}a_3 = \frac{12-\lambda}{20}.\frac{2-\lambda}{6}a_1 \tag{3.14}$$

and so on...
It is clear that every even subscripted coefficient is a multiple of a_0 and every odd coefficient is multiple of a_1.
I mean that a_0 determines a_2 which determines a_4, a_6... and a_1 determines a_3 which determines a_5, a_7...
Now if we let $a_0 = 1, a_1 = 0$ we get the solution:

$$y_1(x) = 1 + \sum_{k=1}^{\infty} a_{2k}x^{2k} \tag{3.15}$$

If we let $a_0 = 0, a_1 = 1$ we get the solution:

$$y_2(x) = 1 + \sum_{k=1}^{\infty} a_{2k+1}x^{2k+1} \tag{3.16}$$

We see that $y_1(x)$ and $y_2(x)$ are linearly independent so the general solution for Legendre Equation can be written as

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

where $y_1(x)$ contains only even powers of x, while $y_2(x)$ contains only odd powers of x.
We now use the ratio test to check that Legendre series converge.

3.1.2. *Ratio Test.* We examine the limits of the ratio of two successive terms. We find that:

$$\lim_{k\to\infty} |\frac{a_{k+2}x^{k+2}}{a_k x^k}| = \lim_{k\to\infty} |\frac{a_{k+2}}{a_k}|x^2$$

and from equation (3.10) we have

$$\frac{a_{k+2}}{a_k} = \frac{k(k+1)-\lambda}{(k+2)(k+1)} = \frac{k^2+k-\lambda}{k^2+3k+2} \longrightarrow 1$$

when $k \to \infty$.
And thus we see that

$$lim_{k\to\infty}|\frac{a_{k+2}}{a_k}|x^2 = x^2$$

and so the series converges when $|x| < 1$.

3.2. **Polynomial Solutions.** From the recurrence relation it is clear that the separation parameter λ affects the value of all but the first coefficients in $y_1(x)$ and $y_2(x)$. The infinite series may diverge at $|x| = 1$, but we can get convergent series solutions if we allow the separation parameter λ, to take on special values such that the series terminate.

A terminating series will be obtained if we take $\lambda = n(n+1)$ where n is any positive integer.

Then we have the Legendre equation:

$$(1-x^2)y''(x) - 2xy' + n(n+1)y = 0 \tag{3.17}$$

and the recurrence relation:

$$a_{(j+2)} = \frac{j(j+1)-n(n+1)}{(j+2)(j+1)}a_j. \tag{3.18}$$

If n is any even integer, the solution $y_1(x)$ becomes a polynomial of degree n since $a_{n+2} = 0$ and consequently $a_{n+4} = a_{n+6} = ... = 0$.

Thus we have:

$$y_1(x) = a_0 + a_1x^2 + a_2x^4 + ... + a_nx^{2n}$$

and $y_2(x)$ remains an infinite series.

But for an odd integer n, $y_2(x)$ becomes a polynomial:

$$y_2(x) = a_1x + a_2x^3 + a_3x^5 + ... + a_nx^{2n-1}$$

while $y_1(x)$ remains an infinite series.

3.3. **Legendre Polynomial.** We finish determining the series solution for the Legendre equation, in the case where $\lambda = n(n+1)$ with an integer n, by describing more explicitly the general term.

3.3.1. *Even Series.* We have for an even integer n the solution $y_1(x)$:

$$y_1(x) = \sum_{k=0}^{\infty} a_k x^{2k} \tag{3.19}$$

and the recurrence formula:

$$a_{k+1} = \frac{2k(2k+1) - n(n+1)}{(2k+1)(2k+2)} a_k. \tag{3.20}$$

We know that

$x(x+1)-y(y+1) = x^2-y^2+x-y = (x+y)(x-y)+(x-y) = (x-y)(x+y+1))$

and using this on the numerator,we can write the recurrence relation as:

$$a_{k+1} = \frac{(2k-n)(2k+n+1)}{(2k+1)(2k+2)} a_k = -\frac{(n-2k)(n+2k+1)}{(2k+1)(2k+2)} a_k$$

and continue in this way

$$a_k = -\frac{(n-2k+2)(n+2k-1)}{(2k-1)(2k)} a_{k-1}$$

$$= \Big[-\frac{(n-2k+2)(n+2k-1)}{(2k-1)(2k)}\Big]\Big[-\frac{(n-2k+4)(n+2k-3)}{(2k-3)(2k-2)}\Big]\dots$$

$$\dots\Big[-\frac{n(n+1)}{(1)(2)} a_0\Big]$$

$$= (-1)^k \frac{1}{(2k)!}\Big[(n-2k+2)(n-2k+4)\cdots(n-2)n(n+1)(n+3)\dots$$

$$\dots(n+2k-3)(n+2k-1)\Big]a_0$$

But we know that the series will terminate at the mth term for $n = 2m$ (from the recurrence formula (3.20) if $n = 2m$ then $a_{(m+1)} = 0$), so we have the following description of the general term:

$$a_k = (-1)^k \frac{1}{(2k)!}.(2m+2k-1)(2m+2k-3)\dots(2m+3)(2m+1)2m(2m-2)\dots$$

$$\dots(2m-2k+2)a_0$$

$$= \frac{(-1)^k}{(2k)!}.\frac{(2m+2k-1)!!}{(2m-1)!!}.\frac{(2m)!!}{(2m-2k)!!} a_0$$

and so

$$y_1(x) = \frac{(2m)!!}{(2m-1)!!} a_0 \sum_{k=0}^{m} \frac{(-1)^k (2m+2k-1)!!}{(2k)!(2m-2k)!!} x^{2k}$$

(here we use $m!! = m(m-2)(m-4)....$).
To get the standard normalization choose a_0 so that

$$\frac{(2m)!!}{(2m-1)!!} a_0 = (-1)^m$$

Now we can reverse the order of terms in the sum. Letting $k = m - l$ yields

$$y_1(x) = (-1)^m \sum_{l=0}^{m} \frac{(-1)^{m-l}(4m-2l-1)!!}{(2m-2l)!(2l)!!} x^{2m-2l}$$
$$= \sum_{l=0}^{m} \frac{(-1)^l}{2^l l!(2m-2l)!} \cdot \frac{(4m-2l)!}{(4m-2l)!!} x^{2m-2l}$$
$$= \sum_{l=0}^{m} \frac{(-1)^l}{2^l l!(2m-2l)!} \cdot \frac{(4m-2l)!}{2^{2m-l}(2m-l)!} x^{2m-2l}$$

The last formula gives the $2mth$ Legendre polynomial:

$$p_{2m}(x) = \sum_{l=0}^{m} \frac{(-1)^l(4m-2l)!}{2^{2m} l!(2m-2l)!(2m-l)!} x^{2m-2l}$$

3.3.2. *Odd Series.* The above formula is for even series but in similar way, for an odd integer $n = 2m+1$, the odd series will terminate with $b_{2m+2} = 0$. And we have for the other Legendre solution namely $y_2(x)$:

$$y_2(x) = \sum_{k=0}^{\infty} b_k x^{2k+1},$$

$$b_{k+1} = \frac{(2k+1)(2k+2) - n(n+1)}{(2k+2)(2k+3)} b_k$$

$$b_{k+1} = \frac{(2k+1-n)(2k+1+n+1)}{(2k+2)(2k+3)} b_k$$
$$= \frac{(2k-2m)(2k+2m+3)}{(2k+2)(2k+3)} b_k$$

or:

$$b_k = \frac{(2k-2m-2)(2k+2m+1)}{(2k)(2k+1)} b_{k-1} = -\frac{(2m-2k+2)(2m+2k+1)}{(2k)(2k+1)} b_{k-1}$$
$$= \Big[-\frac{(2m-2k+2)(2m+2k+1)}{2k(2k+1)}\Big]\Big[-\frac{(2m-2k+4)(2m+2k-1)}{(2k-2)(2k-1)}\Big]\dots$$
$$\dots\Big[-\frac{2m(2m+3)}{(2)(3)}\Big] b_0$$

we can write this more compactly as:

$$b_k = \frac{(-1)^k}{(2k+1)!}(2m+2k+1)(2m+2k-1)\dots(2m+3)(2m)(2m-2)\dots$$
$$\dots(2m-2k+4)(2m-2k+2)b_0 = \frac{(-1)^k}{(2k+1)!} \frac{(2m+2k+1)!!}{(2m+1)!!} \frac{(2m)!!}{(2m-2k)!!} b_0$$

This term gives the odd solution, and we can write $y_2(x)$ as:

$$y_2(x) = \sum_{k=0}^{m} b_k x^{2k+1} = \frac{(2m)!!}{(2m+1)!!} b_0 \sum_{k=0}^{m} \frac{(-1)^k(2m+2k+1)!!}{(2k+1)!(2m-2k)!!} x^{2k+1}$$

To get the standard normalization we choose b_0 so that $\frac{(2m)!!}{(2m+1)!!}b_0 = (-1)^m$.
We replace k with $m-l$, which yields:

$$y_2(x) = (-1)^m \sum_{l=0}^{m} \frac{(-1)^{m-l}(4m-2l+1)!!}{(2m-2l+1)!(2l)!!} x^{2m-2l+1}$$

$$= \sum_{l=0}^{m} \frac{(-1)^l}{(2m-2l+1)!2^l l!} . \frac{(4m-2l+2)!}{(4m-2l+2)!!} x^{2m-2l+1}$$

$$= \sum_{l=0}^{m} \frac{(-1)^l}{(2m-2l+1)!2^l l!} . \frac{(4m-2l+2)!}{2^{2m-l+1}(2m-l+1)!} x^{2m-2l+1}.$$

This is the $2(m+1)th$ Legendre polynomial $P_{2m+1}(x)$:

$$P_{2m+1} = \sum_{l=0}^{m} \frac{(-1)^l(4m-2l+2)}{2^{2m+1}l!(2m+1-2l)!(2m+1-l)!} x^{2m+1-2l}.$$

We compare this relation with previous result for $P_{2m}(x)$, and we see that for any odd or even integer n, by replacing m with $n/2$ for the even series and $2m+1$ with n for the odd series, the nth Legendre polynomial can be written as:

$$P_n(x) = \frac{1}{2^n} \sum_{l=0}^{[n/2]} \frac{(-1)^l.(2n-2l)!}{l!(n-2l)!(n-l)!} x^{n-2l}$$

where:
$[n/2] = \frac{n}{2}$ if n is even,
and $[n/2] = \frac{(n-1)}{2}$ if n is odd.

We see that the nth Legendre polynomial P_n is a polynomial of degree n. If n is an even integer, the above derivation clearly shows that the polynomial is an even function of x, while if n is an odd integer, P_n is an odd function. Thus $P_n(-x) = (-1)^n P_n$. For the values of the parameter $\lambda = n(n+1)$ the Legendre polynomial is a solution of Legendre equation which is finite for all x.

The first few polynomials are:

$$P_0(x) = 1$$

$$P_1(x) = \frac{1 \times 2!}{2 \times 1 \times 1} x = x$$

$$P_2(x) = \frac{1 \times 4!}{4 \times 1 \times 2! \times 2!} x^2 + \frac{(-1)2!}{4 \times 1 \times 1 \times 1} x^0 = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

4. Generating Function

Let $f_n(x)$ be a sequence of functions. A function $F(x,t)$ is said to be a generating function of $f_n(x)$ if $F(x,t) = \sum_{n=0}^{\infty} f_n(x)t^n$.
The idea of generating functions is that these functions contain all the sequence, and so these functions can be used in a comprehensive way to find solutions for in example combinatoric problems or differential equations. The generating function for the sequence of Legendre polynomials $P_n(x)$ is given in the following:

Theorem 1. *(Generating Function)*

$$(1-2xt+t^2)^{\frac{-1}{2}} = \sum_{n=0}^{\infty} P_n(x)t^n \tag{4.1}$$

Proof. We will prove that $|2xt-t^2| < 1$ so that we can expand the right hand side using the binomial theorem. If $|x| \leq r$ where r is arbitrary, and $|t| < (1+r^2)^{\frac{-1}{2}} - r$ then it follows that

$$|2xt-t^2| \leq 2|x||t|+|t^2| < 2r(1+r^2)^{\frac{-1}{2}} - 2r^2+1+r^2+r^2-2r(1+r^2)^{\frac{1}{2}} = 1$$

and hence we can expand $(1-2xt+t^2)^{\frac{-1}{2}}$ binomially as follows.
We know that the binomial formula is given by:

$$(1+x)^r = \sum_{k=0}^{\infty} \binom{r}{k} 1^k x^{r-k} \ where\ |x| < 1$$

Here when r is not an integer, then the coefficients are given by:

$$\binom{r}{k} = \frac{1}{k!}\prod_{r=0}^{k-1}(r-n) = \frac{r(r-1)(r-2)\dots(r-k+1)}{k!} \tag{4.2}$$

Our function $\left(1+(t^2-2xt)\right)^{\frac{-1}{2}}$ has $r = -\frac{1}{2}$ which is not an integer and thus the coefficients will be given by:

$$\binom{-\frac{1}{2}}{n} = \frac{1}{n!}\prod_{k=0}^{n-1}(-\frac{1}{2}-k) =$$
$$\frac{-\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2)\dots(-\frac{1}{2}-n+1)}{n!} =$$
$$\frac{-\frac{1}{2}.(-\frac{1}{2}.3)(-\frac{1}{2}.5)\dots(-\frac{1}{2}(2n-1))}{n!} =$$
$$\frac{(-\frac{1}{2})^n(2n-1)!!}{n!} = \frac{(-1)^n(2n-1)!!}{2^n n!}$$

We have by using (4.2)

$$(1+(t^2-2xt))^{\frac{-1}{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n(2n-1)!!}{2^n n!}(t^2-2tx)^n$$

We expand$(t^2-2tx)^n$ binomially again:

$$(t^2-2tx)^n=\sum_{k=0}^{n}\binom{n}{k}(-2tx)^{n-k}t^{2k}=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(-2tx)^{n-k}t^{2k}$$

or

$$(1-2tx+t^2)^{\frac{-1}{2}}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\frac{(2n)!}{2^n n!2^n n!}\frac{n!}{k!(n-k)!}(2x)^{n-k}t^{n+k}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\frac{(2n)!}{2^{2n}n!k!(n-k)!}(2x)^{n-k}t^{n+k} \tag{4.3}$$

Let us relate this to generating functions. The generating function of the Legendre polynomials $P_n(x)$ is

$$g(t,x)=\sum_{n=0}^{\infty}P_n(x)t^n$$

As we see above that we have the double sum $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_{n,k}$, where

$$a_{n,k}=(-1)^k\frac{(2n)!}{2^{2n}n!k!(n-k)!}(2x)^{n-k}t^{n+k} \tag{4.4}$$

are terms that each is correspond with t^{n+k}.

The inner sum in the double sum is represented by the vertical lines of the array:

$$\mathbf{a_{n,k}}=\begin{pmatrix} a_{00} & a_{10} & a_{20} & a_{30} & a_{40} & \cdots \\ & a_{11} & a_{21} & a_{31} & a_{41} & \cdots \\ & & a_{22} & a_{32} & a_{42} & \cdots \\ & & & a_{33} & a_{43} & \cdots \\ & & & & a_{44} & \cdots \\ \vdots & \vdots & \ddots & & & \end{pmatrix}$$

We can rearrange the sum of the $a_{n,k}$ in other ways, for instance by grouping the terms which have the same powers of t together, that is $n+k$ is constant.

Thus we can group the terms which have the same values (n+k), and get the double sum as a_{nk}:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}a_{n,k}=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}a_{n-k,k}$$

By using this sum, we write the generating function as:

$$\begin{aligned}
&[1-2xt+t^2]^{-1/2} = \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}(-1)^k \frac{(2n-2k)!(2x)^{n-2k}}{2^{2n-2k}(n-k)!k!(n-2k)!}t^n \\
&= \sum_{n=0}^{\infty}\left[\frac{1}{2^n}\sum_{k=0}^{[n/2]}(-1)^k \frac{(2n-2k)!x^{n-2k}}{(n-k)!k!(n-2k)!}\right]t^n
\end{aligned}$$

We see that the term in brackets is just the same as our representation for the Legendre polynomial $P_n(x)$.
Thus we have proved that :

$$(1-2xt+t^2)^{\frac{-1}{2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

where $g(t,x) = (1-2xt+t^2)^{\frac{-1}{2}}$ is the generating function for the Legendre polynomials. □

5. Recurrence Relations

The generating function can be used to devise various relationships among the set of the polynomials $P_n(x)$.

Consider differentiating with respective to t:

$$\frac{\partial g(t,x)}{\partial t} = \frac{x-t}{\left(1-2xt+t^2\right)^{3/2}} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1}$$

By using formula (4.1) this can be written as:

$$\frac{x-t}{(1-2xt+t^2)}\sum_{n=0}^{\infty} P_n(x)t^n = \sum_{n=0}^{\infty} nP_n(x)t^{n-1}$$

or:

$$\sum_{n=0}^{\infty} P_n(x)(t-x)t^n + \sum_{n=0}^{\infty} nP_n(x)(1-2xt+t^2)t^{n-1} = 0$$

which can be reformulated as:

$$\sum_{n=0}^{\infty} P_n(x)t^{n+1} - x\sum_{n=0}^{\infty} P_n(x)t^n + \sum_{n=0}^{\infty} nP_n(x)t^{n-1}$$

$$-2x\sum_{n=0}^{\infty} nP_n(x)t^n + \sum_{n=0}^{\infty} nP_n(x)t^{n+1} = 0$$

If we shift all indexes so that all powers of t have the same degree, we find:

$$\sum_{n=0}^{\infty} P_{n-1}(x)t^n - x\sum_{n=0}^{\infty}(2n+1)P_n(x)t^n + \sum_{n=0}^{\infty}(n+1)P_{n+1}(x)t^n +$$
$$+\sum_{n=0}^{\infty}(n-1)P_{n-1}(x)t^n = 0$$

which means that:

$$-x(2n+1)P_n(x) + nP_{n-1}(x) + (n+1)P_{n+1}(x) = 0 \tag{5.1}$$

Now we will rewrite the last equation as the following theorem:

Theorem 2. *The following relation:*

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \tag{5.2}$$

is true and gives the Legendre polynomials, by starting with $P_0(x) = 1$, $P_1(x) = x$.

We see that this recurrence relation connects three Legendre polynomials with consecutive indices.By using this theorem we can calculate the Legendre polynomials step by step, starting from
$P_0(x) = 1, P_1(x) = x$.

Example:

We use the recurrence relation to obtain the first four Legendre polynomials using that $P_0(x) = 1, P_1(x) = x$

Put n=0,1,2,3,4 in the recurrence relation to obtain

$$\begin{aligned}
P_1 - xP_0 &= 0 \Rightarrow P_1(x) = x \\
2P_2 - 3xP_1 + P_0 &= 0 \Rightarrow P_2(x) = \frac{1}{2}(3x^2-1) \\
3P_3 - 5xP_2 + 2P_1 &= 0 \Rightarrow P_3(x) = \frac{1}{3}[-2x + \frac{5}{2}x(3x^2-1)] = \frac{1}{2}(5x^3-3x) \\
4P_4 - 7xP_3 + 3P_2 &= 0 \Rightarrow P_4(x) = \frac{1}{4}[-\frac{3}{2}(3x^2-1) + \frac{7}{2}(5x^3-3x)] =
\end{aligned}$$
$$= \frac{1}{8}(35x^4 - 30x^2 + 3)$$

One can devise another recursion relation, by differentiating $g(t,x)$ with respect to x

$$\frac{\partial g(t,x)}{\partial x} = \frac{t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} P_n'(x)t^n$$

which can be reformulated as:

$$\frac{t}{(1-2xt+t^2)}\sum_{n=0}^{\infty} P_n(x)t^n = \sum_{n=0}^{\infty} P_n'(x)t^n$$

$$-\sum_{n=0}^{\infty} P_n(x)t^{n+1} + \sum_{n=0}^{\infty} P_n'(x)(1-2xt+t^2)t^n = 0$$

or

$$-\sum_{n=1}^{\infty} P_{n-1}(x)t^n + \sum_{n=0}^{\infty} P_n'(x)t^n - 2x\sum_{n=1}^{\infty} P_{n-1}'(x)t^n + \sum_{n=2}^{\infty} P_{n-2}'(x)t^n = 0$$

We can write the last equation in this way:

$$P_{n-1}(x) = P_n'(x) - 2xP_{n-1}'(x) + P_{n-2}'(x)$$

for $n \geq 2$

or:

$$P_n(x) = P_{n+1}'(x) - 2xP_n'(x) + P_{n-1}'(x)$$

for $n \geq 1$

This equation can be combined with the first recursion relation to give another beautiful form:

Theorem 3. $P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n'(x)$

and also others, such as:

Theorem 4. $(2n+1)(1-x^2)P_n'(x) = n(n+1)\big(P_{n-1}(x) - P_{n+1}(x)\big)$

6. Special Values Of $P_n(x)$

Theorem 5. (i) $P_n(1) = 1$

(ii) $P_n(-1) = (-1)^n$

(iii)

$$P_n(0) = \begin{cases} \frac{(2n)!}{2^{2n}(n!)^2} & \textit{if } n \textit{ is even} \\ 0 & \textit{if } n \textit{ is odd} \end{cases}$$

Proof. (i) By setting $x = 1$ in equation $(1-2xt+t^2)^{\frac{-1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x)$,

$$\sum_{n=0}^{\infty} t^n P_n(1) = (1-2t+t^2)^{\frac{-1}{2}} = (1-t)^{-1} = \sum_{n=0}^{\infty} t^n$$

so that, by equating the coefficients of t^n, $P_n(1) = 1$.

(ii) Similarly,

$$\sum_{n=0}^{\infty} t^n P_n(-1) = (1+2t+t^2)^{\frac{-1}{2}}$$

$$= (1+t)^{-1} = \sum_{n=0}^{\infty} (-1)^n t^n$$

leading to $P_n(-1) = (-1)^n$.

(iii) Finally,

$$\sum_{n=0}^{\infty} t^n P_n(0) = (1+t^2)^{\frac{-1}{2}} = \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n} n! n!} t^{2n}$$

giving for even integers ($n = 2m$)

$$P_{2m}(0) = \frac{(2m)!}{2^{2m}(m!)^2}$$

and for odd integers ($n = 2m+1$)

$$P_{2m+1}(0) = 0.$$

□

7. Orthogonality Of the Legendre Polynomials

Definition 1. A system of real functions $f_n(x)$ (n=0,1,2,3,....) is said to be orthogonal with weight $f(x)$ on the interval $[a, b]$ if :

$$\int_a^b f_m(x) f_n(x) f(x) dx = 0$$

for every $n \neq m$ and $f(x)$ is a fixed nonnegative function which does not depend on n or m.

Example:
The function $\cos nx$ are orthogonal with weight 1 on $[0, \pi]$ because: $\int_0^\pi \cos mx \cos nx dx = 0$ $(n \neq m)$.
The orthogonality property is important because the functions with this property can often used to expand arbitrary functions in an infinite series expansion.An example is the Fourier series expansion of a function$f(x) = \sum_{n=1}^{\infty} a_n \sin nx$ where a_n is the expansion coefficients.
A similar property holds for the Legendre polynomials. One of the most important properties of Legendre polynomials is their orthogonality on $[-1, 1]$.

Theorem 6. (i) *If $m \neq n$, then $\int_{-1}^1 P_m(x) P_n(x) dx = 0$*
(ii) *For each n, we have $\int_{-1}^1 P_n^2(x) dx = \frac{2}{2n+1}$*

Proof. (i) Let $P_m(x)$ and $P_n(x)$ be two polynomials which satisfy Legendre's equation.
We have for $m \neq n$:

$$(1-x^2)P_m'' - 2xP_m' + m(m+1)P_m = 0 \quad (7.1)$$
$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0 \quad (7.2)$$

Multiply (7.1) by $P_n(x)$ and (7.2) by $P_m(x)$ and subtract the resultant expression giving

$$(1-x^2)(P_nP''_m - P_mP''_n) - 2x(P_nP'_m - P_mP'_n) +$$
$$+[m(m+1) - n(n+1)]P_mP_n = 0 \tag{7.3}$$

But

$$\frac{d}{dx}\big(P_nP'_m - P_mP'_n\big) = P_nP''_m + P'_nP'_m - P'_mP'_n - P_mP''_n = P_nP''_m - P_mP''_n$$

therefore, (7.3) reduces to:

$$(1-x^2)\frac{d}{dx}\big(P_nP'_m - P_mP'_n\big) - 2x\big(P_nP'_m - P_mP'_n\big) = [n(n+1) - m(m+1)]P_mP_n.$$

We can continue by writing the left hand side as:

$$\frac{d}{dx}\big[(1-x^2)(P_nP'_m - P_mP'_n)\big],$$

and therefore:

$$\frac{d}{dx}\big[(1-x^2)(P_nP'_m - P_mP'_n)\big] = [n(n+1) - m(m+1)]P_mP_n \tag{7.4}$$

Finally we integrate this equality over $[-1,1]$:

$$(1-x^2)(P_nP'_m - P_mP'_n)\mid_{-1}^{1} = 0 =$$
$$= \big[n(n+1) - m(m+1)\big]\int_{-1}^{1} P_m(x)P_n(x)dx \tag{7.5}$$

since $n \neq m$ (7.5) reduces to:

$$\int_{-1}^{1} P_m(x)P_n(x)dx = 0 \tag{7.6}$$

and this is the statement in the theorem on the orthogonality of Legendre polynomials for $n \neq m$ with weight $f(x) = 1$ on [-1,1].

(ii) This time we will use the generating function, square it:

$$g^2 = (1-2xt+t^2)^{-1} = \left(\sum_{k=0}^{\infty} P_n(x)t^n\right)^2$$

We integrate from -1 to 1 with respect to x:

$$\int_{-1}^{1}\frac{dx}{1-2xt+t^2} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\int_{-1}^{1} P_k(x)P_{n-k}(x)dxt^n$$

each integral on the right vanishes except when $k = n-k$ (the other terms are zero due to orthogonality), therefore the only nonzero terms in the series are those for which n is even, $n = 2k$. Hence:

$$\begin{aligned}
\int_{-1}^{1}\frac{dx}{1-2xt+t^2} &= \int_{-1}^{1}dx\sum_{n=0}^{\infty}P_n^2(x)t^{2n}\\
\Big[-\frac{1}{2t}ln(1-2xt+t^2)\Big]_{-1}^{1} &= \sum_{n=0}^{\infty}\int_{-1}^{1}P_n^2(x)dxt^{2n}\\
-\frac{1}{2t}\Big[ln(1-t)^2-ln(1+t)^2\Big] &= \sum_{n=0}^{\infty}\int_{-1}^{1}P_n^2(x)dxt^{2n}\\
or: -\frac{1}{t}\Big[ln(1-t)-ln(1+t)\Big] &= \sum_{n=0}^{\infty}\int_{-1}^{1}P_n^2(x)dxt^{2n}.
\end{aligned}$$

Expand the left hand side as a power series in t. Since

$$ln(1+t) = -\sum_{p=1}^{\infty}\frac{(-t)^p}{p},$$

we have that:

$$\sum_{n=0}^{\infty}\int_{-1}^{1}P_n^2(x)dx.t^{2n} = \frac{1}{t}(-\sum_{p=1}^{\infty}\frac{-t^p}{p}+\sum_{p=1}^{\infty}\frac{(-1)^p.t^p}{p})$$

$$=\frac{1}{t}\sum_{p=1}^{\infty}\frac{t^p}{p}\big(1-(-1)^p\big)$$

The terms in the sum vanish for even powers and only odd powers of t survive in the sum.

Put $p=2n+1$, then

$$\sum_{n=0}^{\infty}\int_{-1}^{1}P_n^2(x)dxt^{2n} = \frac{1}{t}\sum_{n=0}^{\infty}\frac{t^{2n+1}}{2n+1}.2 = \sum_{n=0}^{\infty}\frac{2}{2n+1}t^{2n} \tag{7.7}$$

from (7.7) we see that we must have:

$$\int_{-1}^{1}P_n^2(x)dx = \frac{2}{2n+1} \tag{7.8}$$

□

8. Rodrigues Formula

If one looks at the series expansion for the Legendre polynomials, one finds it can be rewritten somewhat:

$$P_n(x) = \frac{1}{2^n}\sum_{l=0}^{[n/2]}\frac{(-1)^l(2n-2l)!}{l!(n-2l)!(n-l)!}x^{n-2l} \tag{8.1}$$

Let $D = \frac{d}{dx}$, we know that $D^s x^m = \frac{m!x^{m-s}}{(m-s)!}$.
Take $s = n, m = n - 2l$, then we rewrite the formula as:

$$D^n x^{2n-2l} = \frac{(2n-2l)!x^{2n-2l-n}}{(2n-2l-n)!} = \frac{(2n-2l)!x^{n-2l}}{(n-2l)!} \tag{8.2}$$

Setting this equality into (8.1), we have

$$P_n(x) = \frac{1}{2^n} \sum_{l=0}^{[n/2]} \frac{(-1)^l D^n x^{2n-2l}}{l!(n-l)!} \tag{8.3}$$

Actually we can replace the upper limit on this sum by n because $D^n x^{2n-2l} = 0$for every value of l such that

$$n/2 < l \leq n.$$

This is seen since then

$$n/2 - n < l - n \leq 0$$

or $0 \leq n - l < n/2$ which means that

$$0 \leq 2n - 2l < n$$

and thus $D^n x^{2n-2l} = 0$, Thus we rewrite (8.3) as

$$\begin{aligned} P_n(x) &= \frac{1}{2^n} \sum_{l=0}^{n} \frac{(-1)^l}{l!(n-l)!} . \frac{d^n}{dx^n}\left(x^{2n-2l}\right) \\ &= \frac{1}{2^n} \frac{d^n}{dx^n} \sum_{l=0}^{n} \frac{(-1)^l}{l!(n-l)!} x^{2n-2l} \end{aligned} \tag{8.4}$$

By the binomial theorem we can write:

$$\begin{aligned} P_n(x) &= \frac{1}{2^n.n!} \frac{d^n}{dx^n} \sum_{l=0}^{n} \frac{n!}{l!(n-l)!} (-\frac{1}{x^2})^l x^{2n} \\ &= \frac{1}{2^n.n!} \frac{d^n}{dx^n} \left[(1 - \frac{1}{x^2})^n . x^{2n}\right] \end{aligned}$$

and thus we have proved the following theorem

Theorem 7.

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n. \tag{8.5}$$

This is known as Rodrigues formula for Legendre polynomials.
We will prove that (8.5) satisfies Legendre equation (2.1) directly. This gives another proof of the theorem. By putting (8.5) in (2.1), we get:

$$\frac{d}{dx}\left[(1-x^2)\frac{1}{2^n n!}\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n\right] + n(n+1)\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n = 0$$

$$\frac{1}{2^n n!}\left[-2x\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n + (1-x^2)\frac{d^{n+2}}{dx^{n+2}}(x^2-1)^n + n(n+1)\frac{d^n}{dx^n}(x^2-1)^n\right] = 0$$

multiplying by -1 the above equation becomes

$$\frac{1}{2^n n!}\left[2x\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n+(x^2-1)\frac{d^{n+2}}{dx^{n+2}}(x^2-1)^n-n(n+1)\frac{d^n}{dx^n}(x^2-1)^n\right]=0 \tag{8.6}$$

But we can rewrite the terms as:

$$-n(n+1)\frac{d^n}{dx^n}(x^2-1)^n = n(n+1)\frac{d^n}{dx^n}(x^2-1)^n - 2n(n+1)\frac{d^n}{dx^n}(x^2-1)^n \tag{8.7}$$

and

$$2x\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n = 2(n+1)x\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n - 2nx\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n \tag{8.8}$$

Setting (8.7) and (8.8) into (8.6) gives

$$\begin{aligned}\frac{1}{2^n n!}\Bigg[&2(n+1)x\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n - 2nx\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n + \\ &(x^2-1)\frac{d^{n+2}}{dx^{n+2}}(x^2-1)^n + n(n+1)\frac{d^n}{dx^n}(x^2-1)^n \\ &- 2n(n+1)\frac{d^n}{dx^n}(x^2-1)^n\Bigg] = 0\end{aligned} \tag{8.9}$$

Now rewrite the above equation in the following form:

$$\begin{aligned}\frac{1}{2^n n!}\Bigg[&n(n+1)\frac{d^n}{dx^n}(x^2-1)^n + 2(n+1)x\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n + \\ &(x^2-1)\frac{d^{n+2}}{dx^{n+2}}(x^2-1)^n - 2n(n+1)\frac{d^n}{dx^n}(x^2-1)^n \\ &- 2nx\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n\Bigg] = 0\end{aligned} \tag{8.10}$$

This equation recalls to us Leibniz theorem:

Lemma 1.

$$\begin{aligned}\frac{d^n}{dx^n}(uv) = \frac{d^n u}{dx^n}.v + n\frac{d^{n-1}u}{dx^{n-1}}.v' + \frac{n(n-1)}{1.2}\frac{d^{n-2}u}{dx^{n-2}}v'' + ...+ \\ +\frac{n(n-1)...(n-k+1)}{k!}\frac{d^{n-k}u}{dx^{n-k}}.\frac{d^k}{dx^k}v + \cdots + u\frac{d^n}{dx^n}v\end{aligned}$$

Thus we can write (8.10) in this way:

$$\frac{1}{2^n n!}\frac{d^{n+1}}{dx^{n+1}}\left[(x^2-1)\frac{d}{dx}(x^2-1)^n - 2nx(x^2-1)^n\right] = 0 \tag{8.11}$$

Note that the terms between brackets in(8.11) are actually equal to zero and thus their differential must be equal to zero.We have thus proven that the polynomials given by Rodrigues formula satisfy Legendre's equation.

The great advantage of Rodrigues formula is its form as an *nth* derivative. This means that in an integral, it can be used repeatedly in an integration by parts to evaluate the integral. The orthogonality of the Legendre polynomials, for example, follows very quickly when Rodrigues formula is used. There is a Rodrigues formula for many, but not all, orthogonal polynomials. It can be used to find recurrence relation, the differential equation, and many other properties of them. We will now give an example which shows that this formula is useful for proving various properties of the $P_n(x)$.

Example: We can show that the Legendre polynomials are orthogonal by integrating and using Rodrigues formula.

For an arbitrary function $g(x)$ defined in the interval $-1 \leq x \leq 1$, consider the integral

$$I = \int_{-1}^{1} g(x)P_n(x)dx$$

Using Rodrigues formula, we write the integral as:

$$\int_{-1}^{1} g(x)\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n dx$$

$$= \frac{1}{2^n n!}\int_{-1}^{1} g(x)\frac{d^n}{dx^n}(x^2-1)^n dx$$

$$= \frac{1}{2^n n!}\int_{-1}^{1} g(x)d\big[\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\big] \tag{8.12}$$

Let $u(x) = g(x) \Rightarrow du = g'(x)d(x)$

$$dv(x) = d\big[\frac{d^{n-1}}{dx^{n-1}}(x^2-1)\big] \Rightarrow$$

$$v(x) = \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n$$

Integrate (8.13) by parts

$$I = \frac{1}{2^n n!}\Bigg(\underbrace{\big[g(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\big]_{-1}^{1}}_{=0} - \int_{-1}^{1} g'(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n dx\Bigg)$$

$$= \frac{-1}{2^n n!}\int_{-1}^{1} g'(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n dx \tag{8.13}$$

continue integrating by parts the equality (8.14), by setting

$$dv(x) = \big[\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\big]dx = d\big[\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n\big]$$

$$\Rightarrow v(x) = \frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n$$

$$u(x) = g'(x) \Rightarrow du = g''(x)dx$$

Now we write the integral as

$$I = \frac{-1}{2^n.n!}\left(\underbrace{\left[g(x)\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n\right]_{-1}^{1}}_{=0} - \int_{-1}^{1} g''(x).\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n.dx\right)$$

$$= \frac{1}{2^n.n!}\int_{-1}^{1} g''(x).\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n.dx \tag{8.14}$$

and so on. After n partial integrations this yields the final result:

$$\int_{-1}^{1} g(x)P_n(x)dx = \frac{(-1)^n}{2^n.n!}\int_{-1}^{1}(x^2-1)^n\frac{d^n}{dx^n}g(x)dx \tag{8.15}$$

Now we can use the last equality to obtain orthogonality of Legendre polynomials, by replacing $g(x)$ with $P_m(x)$

$$\int_{-1}^{1} g(x)P_n(x)dx = \frac{(-1)^n}{2^n.n!}\int_{-1}^{1}(x^2-1)^n\frac{d^n}{dx^n}\left[\frac{1}{2^m m!}\frac{d^m}{dx^m}(x^2-1)^m\right]dx \tag{8.16}$$

Case 1)Suppose $m \neq n$

If $m \neq n$, let $m < n$, then the degree of $(x^2-1)^m$ is $2m < m+n$, and so $\frac{d^{m+n}}{dx^{m+n}}(x^2-1)^m = 0$ $(m < n)$. Thus

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{(-1)^n}{2^n.n!}\int_{-1}^{1}(x^2-1)^n.$$
$$\frac{d^{n+m}}{dx^{n+m}}\left[\frac{1}{2^m.m!}(x^2-1)^m\right]dx = 0. \tag{8.17}$$

So for $n < m$ $\int_{-1}^{1} P_m(x)P_n(x)dx = 0$.

By symmetry: $\int_{-1}^{1} P_m(x)P_n(x)dx = 0 for m > n$

We have proved that

$$\int_{-1}^{1} P_m(x)P_n(x)dx = 0 for (m \neq n) \tag{8.18}$$

Case 2)Suppose n=m

We have

$$\int_{-1}^{1} P_n(x)P_n(x)dx = \frac{1}{(2^n n!)^2}\int_{-1}^{1}(u^{(n)})^2dx$$

From the integration by parts procedure just used we rewrite this as

$$\frac{(-1)^n}{(2^n n!)^2}\int_{-1}^{1} u(x)u^{(2n)}(x)dx = \frac{(-1)^n(2n)!}{(2^n n!)^2}\int_{-1}^{1}(x^2-1)^n dx$$

This last integral is equal to

$$\int_{-1}^{1}(x^2-1)^n dx = \frac{(-1)^n 2^{2n+1}(n!)^2}{(2n+1)!}$$

Inserting this into the integral expression gives

$$\int_{-1}^{1} P_n(x)P_n(x)dx = \frac{(-1)^n(2n)!)}{(2^n n!)^2}\frac{(-1)^n 2^{2n+1}(n!)^2}{(2n+1)!} = \frac{2}{2n+1}$$

So we have

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{m,n}$$

9. Expansion Of Functions In Legendre Series

Definition 2. (Sturm-Liouville Form)
The differential equation:

$$\frac{d}{dx}\big(p(x)\frac{d}{dx}y(x)\big) + q(x)y(x) = \lambda w(x)y(x)$$

is said to be in Sturm Liouville form. The function w is known as the weight function.

The Legendre equation can be put in Sturm-Liouville form, since $\frac{d}{dx}(1-x^2) = -2x$, so that the Legendre equation is equivalent to

$$\big((1-x^2)y'\big)' + n(n+1)y = 0$$

From the Sturm-Liouville theory, the Legendre polynomials form a complete set and therefore functions on the interval $[-1,1]$ can be expanded in term as a basis. To get pointwise convergence certain properties of f e.g. continuity are needed(see the references)

Theorem 8. *For any arbitrary function $f(x)$ on the interval $[-1,1]$:*

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \tag{9.1}$$

is a generalized Fourier Legendre series.

To obtain a_n, we multiply both sides in (9.1) by $P_m(x)$ and integrate:

$$\int_{-1}^{1} f(x)P_m(x)dx = \int_{-1}^{1}\sum_{n=0}^{\infty} a_n P_n(x)P_m(x)dx$$

$$= \sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_n(x)P_m(x)dx \tag{9.2}$$

but by the orthogonality of Legendre polynomials, we know that
$\int_{-1}^{1} P_n(x)P_m(x)dx = 0$ for $n \neq m$ and
$\int_{-1}^{1} P_n(x)P_m(x)dx = \int_{-1}^{1} P_n^2(x) = \frac{2}{2n+1}$ for $n = m$
so(9.2) can be rewritten as

$$\int_{-1}^{1} f(x)P_n(x)dx = a_n\frac{2}{2n+1}$$

so $a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx$

So we can develop an arbitrary function on $[-1, 1]$ in a Fourier-Legendre series where

$$f(x) = \sum_n a_n P_n(x), a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx \tag{9.3}$$

This kind of approach is useful where one might expand an unknown function in Legendre Polynomials and then convert the problem of determining $f(x)$ into the problem of determining the expansion coefficients a_n which may be easier.

Example

Write the function $f(x) = (x+a)(x-a), a > 0$ as a series of Legendre polynomials.

Solution

The given function can be rewritten as $f(x) = x^2 - a^2$ and it is a second degree polynomial which can therefore be trivially expressed in terms of P_0, P_1, P_2 as:

$$f(x) = x^2 - a^2 = \sum_{n=0}^{2} a_n P_n(x)$$

Here, applying the result from (9.3),

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx$$

and so by direct calculation we have

$$\begin{aligned}
a_0 &= \frac{1}{2}\int_{-1}^{1}(x^2 - a^2)dx = \frac{1}{2}[\frac{1}{3}x^3 - a^2x]_{-1}^{1} = \frac{1}{3} - a^2 \\
a_1 &= \frac{3}{2}\int_{-1}^{1}(x^3 - a^2x)dx = \frac{3}{2}[\frac{1}{4}x^4 - \frac{1}{2}a^2x^2]_{-1}^{1} = 0 \\
a_2 &= \frac{5}{2}\int_{-1}^{1}\frac{1}{2}(x^3 - a^2x)(3x^2 - 1)dx = \frac{5}{4}\int_{-1}^{1}(3x^4 - (1+3a^2)x^2 + a^2)dx \\
&= \frac{5}{4}[\frac{3}{5}x^5 - \frac{(1+3a^2)}{3}x^3 + a^2x]_{-1}^{1} = \frac{2}{3} - 2a^2
\end{aligned}$$

Then the required expression is
$f(x) = \left(\frac{1}{3} - a^2\right)P_0(x) + \left(\frac{1}{3} - 2a^2\right)P_2(x)$

10. Associated Legendre Functions

Separation of variables for Laplace's equation leads also to variants of Legendre's equation of the following form: $(1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + [n(n+1) - \frac{m^2}{1-x^2}]\Theta = 0$.

We saw that for $m = 0$ we have Legendre function which we studied before, but now we want to consider the more general case of m as nonzero integer.

One can also obtain the above associated Legendre equation from the Legendre equation, by differentiating the Legendre equation m times. Start from

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

and apply $\frac{d^m}{dx^m}$.
We will need to make use again of Leibniz lemma for the derivative of a product:

$$\frac{d^m}{dx^m}\Big[A(x)B(x)\Big] = \sum_{s=0}^{m} \frac{m!}{s!(m-s)!}\frac{d^{m-s}}{dx^{m-s}}A(x)\frac{d^s}{dx^s}B(x)$$

Then proceeding to differentiate,

$$\frac{d^m}{dx^m}\left[(1-x^2)y''\right] - 2\frac{d^m}{dx^m}(xy') + n(n+1)y^{(m)} = 0$$

we have

$$\begin{aligned}\frac{d^m}{dx^m}(xy') &= \sum_{s=0}^{m}\left(\begin{array}{c} m \\ s \end{array}\right)\frac{d^s}{dx^s}(x)\frac{d^{m-s}}{dx^{m-s}}(y') \\ &= \left(\begin{array}{c} m \\ 0 \end{array}\right)xy^{(m+1)} + \left(\begin{array}{c} m \\ 1 \end{array}\right).1.y^{(m)} \\ &= xy^{(m+1)} + my^{(m)}\end{aligned}$$

and

$$\frac{d^m}{dx^m}\left[(1-x^2)y''\right] = \sum_{s=0}^{m}\left(\begin{array}{c} m \\ s \end{array}\right)\frac{d^s}{dx^s}(1-x^2)\frac{d^{m-s}}{dx^{m-s}}(y'')$$

$$= \left(\begin{array}{c} m \\ 0 \end{array}\right)(1-x^2)y^{(m+2)} + \left(\begin{array}{c} m \\ 1 \end{array}\right)(-2x)y^{(m+1)} + \left(\begin{array}{c} m \\ 2 \end{array}\right)(-2)y^{(m)}$$

$$= (1-x^2)y^{(m+2)} - 2mxy^{(m+1)} - 2\frac{m(m+1)}{2!}y^{(m)}$$

Thus, the mth derivative of Legendre's equation becomes

$$(1-x^2)y^{(m+2)} - 2mxy^{(m+1)} - m(m-1)y^{(m)} - 2xy^{(m+1)} - 2my^{(m)} + n(n+1)y^{(m)} = 0.$$

We rewrite this as:

$$(1-x^2)y^{(m+2)} - 2(m+1)xy^{(m+1)} + [n(n+1) - m(m+1)]y^{(m)} = 0.$$

Let us define a new variable, $u = y^{(m)}$. The equation becomes

$$(1-x^2)u'' - 2(m+1)xu' + (n-m)(n+m+1)u = 0$$

Note that for $m = 0$ the last equality is exactly Legendre differential equation which has the solution $y = P_n(x)$. Therefore we conclude that there is a relation between the solution of the associated Legendre differential equation and the solution of the Legendre equation given by

$$u(x) = constant. \frac{d^m}{dx^m} P_n(x).$$

If we multiply the last equation by $(1-x^2)^m$, we can reformulate the result as:

$$\frac{d}{dx}\left((1-x^2)^{m+1}\frac{du}{dx}\right) + (n-m)(n+m+1)(1-x^2)^m u(x) = 0$$

We can then do a change of variables, namely: $v(x) = (1-x^2)^{m/2}u(x)$ or $u(x) = (1-x^2)^{-m/2}v(x)$.

Then

$u'(x) = mx(1-x^2)^{-(m+1)/2}v + (1-x^2)^{-m/2}v'$ and

$u''(x) = \left[m(1-x^2)^{-(m+2)/2} + m(m+2)x^2(1-x^2)^{-(m/2+2)}\right]v +$

$+ 2mx(1-x^2)^{-(m+2)/2}v' + (1-x^2)^{-m/2}v''$

The differential equation is

$(1-x^2)u'' - 2(m+1)xu' + (n-m)(n+m+1)u = 0.$

Multiplying this by $(1-x^2)^{m/2}$ we get finally

$$(1-x^2)v'' + 2mxv' + \left[m + \frac{m(m+2)x^2}{1-x^2}\right]v +$$

$$-2(m+1)xv' - \frac{2m(m+1)}{1-x^2}x^2 v + (n-m)(n+m+1)v = 0$$

We sum all the terms of v'', v', and v, we get

$$(1-x^2)v'' + (2mx - 2mx - 2x)v' +$$

$$+\left[m + \frac{m(m+2)x^2}{1-x^2} - \frac{2m(m+1)}{1-x^2}x^2 + n^2 + nm + n - nm - m^2 - m\right]v = 0$$

or

$$(1-x^2)v'' - 2xv' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]v = 0$$

This is **the associated Legendre equation**. It's solution is :

$$v(x) = (1-x^2)^{m/2}u(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)$$

We can write these solutions as:

$$\boxed{P_n^m(x) \equiv (1-x^2)^{m/2}\tfrac{d^m}{dx^m}P_n(x)}.$$

They are called the associated Legendre functions.

These functions are not polynomials.However, since P_n is an nth degree polynomial, we can differentiate it only n times before obtaining zero. thus, for $P_n^m(x)$ to be nonzero, we must have $m \leq n$.

If we insert Rodrigue's formula into the associated Legendre functions, we will have:

$$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}.\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n = \frac{1}{2^n n!}(1-x^2)^{m/2}\frac{d^{n+m}}{dx^{n+m}}(x^2-1)^n$$

This means that we that m can be allowed to take negative values, namely $-n \leq m \leq n$.

11. Dirichlet problem for Laplace's equation

The Dirichlet problem for Laplace's equation consists in finding a solution u on some domain D such that u on the boundary of D is equal to some given function. Since the Laplace operator appears in the heat equation, one physical interpretation of this problem is as follows: fix the temperature on the boundary of the domain and wait until the temperature in the interior doesn't change anymore; the temperature distribution in the interior will then be given by the solution to the corresponding Dirichlet problem. In the other words we will find the function $u = u(r, \theta)$ such that:
1. u is harmonic in the domain $r < a$ and continuous in $r \leq a$.
2. u satisfies the boundary condition $u\mid_{r=a} = f(\theta)$ where $f(\theta)$ is continuous in the interval $0 \leq \theta \leq \pi$.

Assume that the solution is of the form $U = R(r)\Theta(\theta)$. Separation of variables shows that for some constant λ the function $R(r)$ satisfy the radial differential equation:

$$r(rR)'' - \lambda R = 0$$

and for the same constant λ we have

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) + \lambda\Theta = 0 \tag{11.1}$$

where Θ, Θ' and Θ'' are continuous on the interval $0 \leq \theta \leq \pi$.
In fact, for $x = \cos(\theta)$ the equation (11.1)is Legendre equation in the following form:

$$\frac{d}{dx}\big[(1-x^2)\frac{d\Theta}{dx}\big] + \lambda\Theta = 0$$

where $\lambda = n(n+1)$ and $\Theta(\theta) = P_n(\cos\theta)$ where $P_n(x)$ is the Legendre polynomials of degree n. This determines the possible λ.
We write the radial equation as

$$r^2R'' + 2rR' - \lambda R = 0$$

which is a so-called Cauchy-Euler equation and has the general solution:

$$R = C_1 r^n + C_2 r^{-n-1}$$

We choose $C_2 = 0$(because the solution is bounded at the center $r = 0$ of the disk). So we have $R = C_n r^n, n = 0, 1, 2, ...$

The solutions of Laplace's equation of the form $U = R(r)\Theta(\theta)$ are the function $r^n P_n(\cos\theta)$. An arbitrary linear combination is:

$$U(r,\theta) = \sum_{n=0}^{\infty} B_n r^n P(\cos\theta)$$

and under suitable conditions this is a solution of the boundary value problem.
Now suppose given a function f on the interval $-1 < x < 1$ such that $f(x) = F(\arccos x)$ or $F(\theta) = f(\cos\theta)$. We write $f(x)$ in Legendre series:

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

where

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx.$$

We now have the following theorem

Theorem 9. *The equation*

$$U(r,\theta) = \sum_{n=0}^{\infty} a_n r^n P_n(\cos\theta)$$

$$= \sum_{n=0}^{\infty} \frac{2n+1}{2} r^n P_n(\cos\theta)\int_{-1}^{1} f(s)P_n(s)ds$$

solves Dirichlet problem in the unit disk.

12. References

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