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MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Hall's Marriage Theorem

av

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HALL'S MARRIAGE THEOREM

Contents

1. Graph Theory

A graph is a mathematical object that catches the notation of connection. Most people are familiar with the children's puzzle of trying to connect three utilities (water, telephone and electricity) to three houses without having any of the "wires" cross. The widely thought about how the graph theory originated is found in a puzzle that was posed by the two towns folk of Kningsberg, Prussia in the early 1700's. Köningsberg was built largely on the pergel river, this island sits near where tow branches of the river join, and the borders of the town spread over to the banks of the river as well as a nearby promontory. Between these four land masses, seven bridge had been erected.

Somebody asked "is it possible to take a walk through town, crossing each of the seven bridges just once, and ending up wherever you started?"
The famous mathematician Leonard Euler heard of the problem, solved it (it is not possible) and in the process invented Graph Theory.
In this paper I am going to describe Hall's Marriage Theorem which belongs to this branch i mathematic, and prove it in several ways. To understand the different proofs of this theorem I have to introduce some different definitions, conceptions and theorems.

2. Definitions

A graph G consists of a finite set V of vertices (points, nods) and a finite set E of edges, where each edge is an unordered pair of vertices. We write $G = G(V, E)$.

We say that two vertices a and b are adjacent if there is an edge $e = (a, b)$ in E, while the edge $e \in E(G)$ is said to be incident to a and b.

Two edges e and f intersect if they share a common endpoint. If they do not intersect we say that they are disjoint.

A walk is a sequence of vertices $v_1, ..., v_n$ where $(v_i, v_{i+1}) \in E(G)$ and $i = 1, 2, ..., n-1$. A path is a walk without repeated vertex in the sequence.
A set of vertices $X \subseteq V$ is a covering of $G = (V, E)$ if every edge of E contains at least one endpoint in X.

3. Matchings

Definition 1. A matching M of a graph $G = (V, E)$ is a set of edges, no two of which are incident to a common vertex.

Definition 2. A matching M in a graph G is maximum if there is no matching M' in G so that $|M| < |M'|$, where $|M|$ is the size of the set M.

Definition 3. Let $G = (V, E)$ be a graph, and M a matching in G.
1. If v is an end vertex of some edge in M, we say that M saturates v. Otherwise, we say that v is unsaturated by M.
2. An $M-$alternating path is a path whose edges alternate between edges in M and edges not in M.
3. An augmenting path is an $M-$alternating path whose ends are both not saturated by M.

Theorem 1. *(Berge,1957) M is a maximum matching in G if G has no $M-$augmenting path.*

Before proving this theorem we will need some more definitions.

1. We define an edge to be weak with respect to a matching M if it is not in the matching. Vertex is weak if it incident to a weak edges.
2. The symmetric difference between two sets S and T as:
$S \oplus T = (S \setminus T) \cup (T \setminus U)$.
3. A covering of a graph G is a subset K av V such that every edge of G has at least one end in K. A covering K is minimum covering if G has no covering K' med $|K'| < |K|$.
If K is a covering of G, and M is a matching of G, then K contains at least one end of each of the edges in M. Thus for any matching M and any covering K, $|M| \leq |K|$.

Observation. An augmenting path P has an odd number of edges, with one more edge that is not covered by the matching. Note that we can always increase a matching by removing the even edges of P from M and adding the odd edges of P to M increases the size of the matching by one.
Now to Berge's proof.

Proof. We prove first that if there is an augmenting path so the matching is not maximum.
Let M be a maximum matching in G and suppose that G contains an M-augmenting path

$$P : v_0, v_1, ..., v_k,$$

where k is clearly odd (by the definition of augmenting path). If N is defined to be

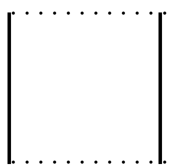
$$N = (M - \{v_1v_2, v_3v_4, ..., v_{k-2}v_{k-1}\}) \cup \{v_0v_1, v_2v_3, ..., v_{k-1}v_k\}$$

then N is a matching in G, and it contains one more edge than M; thus, M is not a maximum matching.
To prove the other direction, we will show that if the matching is not maximum then there is some augmenting path. Suppose that M is not a maximum matching and there does not exist an M-augmenting path and let N be a maximum matching in G. Now, consider the symmetric difference $M \oplus N$. Since no two edges in a matching meet at a common vertex(matching defintion), every vertex is incident with at most one vertex in M and one vertex in N. We represent N by dotted edges and M by solid edges, each connected component of $M \oplus N$ is one of the following types:
1. A path that begins and ends with a solid edges:

2. A path that begins with a solid edge and ends with a dotted edge:

3. A path that begins and ends with a dotted edge:

4. An alternating cycle such as:



Since N is a maximum matching then$|N| > |M|$, and there must be a component with more dotted edges than solid ones. Note that any alternating cycle must have an even length because every vertex is incident with at most one edge from N and one edge from M. Thus some connected component must be of type 3, wich is an augmenting path.This gives a contradication of our assumption that there is not an augmenting path.

□

4. Matching In Bipartite Graph

4.1. **Definition.** A graph $G(V, E)$ is called bipartite if $V = A \cup B$ whith$A \cap B = \phi$ and $A \neq \phi$, $B \neq \phi$, and every edge of G is of the form a, b with $a \in A$ and $b \in B$. If each vertex in A is joined with every vertex in B, we have a complete bipartite graph. In this case, if $|A| = m, |B| = n$, the graph is denoted by $K_{m,n}$.
Let's now turn our attention to matching in bipartite graphs. Consider the following example: if we have a set of teachers and a set of subjects, assume that every teacher is qualified to teach only some subjects. What condition would guarantee that each teacher take a suitable subject which she(he) qualified.
This is a typical problem in bipartite matching and we can model this situation using a bipartite graph, where each vertex in A represents a teacher, each vertex in B represents a subject, and edge $(t, s) \in E$ means that teacher t is qualified for subject s.

Theorem 2. *(König's Theorem) If G is a bipartite then the maximum size of matching $\mu(G)$ = the minimum size of a covering $\beta(G)$.*

Proof. □

5. The Marriage Theorem

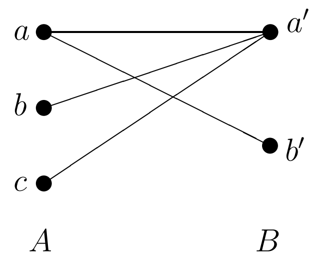
5.1. **Introduction.** If a group of men and women may date only if they have previously been introduced, then a complete set of dates is possible if and only if every subset of men has collectively been introduced to at least as many women, and vice versa.
"Do you call this a marriage?" asked somebody who was interested in understanding what I am writing about? Yes this theorem is about mathematical matching, which the mathematicians call it marriage. It is a mathematical marriage not a social one. Dont thing a bout the social definition of marriage, for the Marriage theorem has some assumptions that can only exist in mathematics. It assumes that any man would be happy to marry a woman who wants to marry him which is very polite and idealistic. It assumes that one-man matches/marries one and only one woman, which is not the case in all philosophies. Some philosophies or religions assume that one man can marry up to 4 women, and I hope that there is some philosophy which allows one woman to get marry at least two men. It assumes a marriage between a man and a woman, which is also not always the case nowadays.
"In this case call it Catholic Marriage theorem" somebody suggested.

Now the question is : When can one marry off the men to the women, and is there an algorithm, a simple set of rules, that can be followed, that will find such a pairing when one exists in a reasonable amount of time, even for large numbers of men and women?

The ideas that lie behind this marriage theorem were discovered independently in several different contexts. The Hungarian graph theorist Dénes König (1894-1944), the British group theorist Philip Hall and others, all found their way to involve the ideas in matching. Hall had come across his theorem in 1935. Hall's marriage theorem is an important result in combinatorics. It gives necessary and sufficient conditions for a maximal matching to exist between the two sets of vertices of a bipartite graph.

5.2. **Hall's Theorem.**

Definition 4. If $G(A \cup B; E)$ is a bipartite graph and a matching M of G saturates all the vertices in A, then M is called a complete matching (also called a perfect matching).
Consider the following bipartite graph:



Does it have a complete matching ?
It obviously does not. We have three vertices in A which are only connected to tow vertices in B. Since $3 > 2$ so no complete matching can exist.
Let us formulate the following definition of range:

Definition 5. let $G = (A \cup B, E)$ be a bipartite graph, and let $X \subset A$. The range of X, denoted $N(X)$, is defined to be $N(X) = \{v \in B$ such that (u, v) is an edge for some $u \in X\}$.

Observation The following is a necessary condition for a bipartite graph $G = (A \cup B, E)$ to have an A-complete matching: $|N(X)| \geq |X|$ for all $X \subseteq A$.

Proof. We prove that if we have complete matching then $|N(X)| \geq |X|$. Let M be a complete matching which saturates every vertex in A, then every vertex in A has a neighbor vertex in B. It means that for a subset $X \subseteq A$ every vertex in X is end of an edge in M and the other end is in B. Thus since edges in M are disjoint, $|X| \leq |N(X)|$. □

Hall's theorem says that whenever a bipartite graph satisfies the condition just mentioned then it has a complete matching,i.e. the condition above is not only necessary but also sufficient. To be precise:

Theorem 3. *(Hall's Theorem) A bipartite graph G with bipartition $A \cup B$ has a complete matching of A if and only if for every set $X \subseteq A$, $|N(X)| \geq |X|$.*

The "only if" part is proved above , so we will give several different proofs of the "if" part of this theorem.

Proof. 1. (using König's theorem) We prove that if we cannot match A into B then the Hall's condition fails.
If there is no compete matching in G, then there is a vertex in A which is not in the maximum matching M. Let X be the subset of A which has all the vertices in the matching M. Well... if there is no complete matching then Konig's theorem implies that there is an edge cover W such that $|W| < |A|$.
Let $W = A' \cup B'$ with $A' \subseteq A$ and $B' \subseteq B$. Then $|A'| + |B'| = |W| < |A|$.
Let $X = A \setminus W$.We prove that $N(X) \subseteq W \cap B$. Indeed, let $b \in N(X)$. Then there exist a vertex $a \in X$ such that ab is an edge. As $X = A \setminus W$, vertex a does not lie in W. So, as W is a vertex cover, $b \in W$. Moreover, as $a \in X \subseteq A$, we have also that $b \in B$. Hence, $b \in W \cap B$. So we proved that $N(X) \subset W \cap B$, as required. So
$| N(X) | \leq |W \cap B| = |W| - |W \cap A| = |W| - [|A| - |A \setminus W|] = |W| - |A| + |A \setminus W| = |W| - |A| + |X| < |X|$. So X is a subset of A with $|N(X)| < |X|$.

□

Proof. 2.(using Berge's theorem)
Suppose M is maximum and does not saturate A. We will find a set X that violates the Hall's condition. Let u be an M-unsaturated vertex in A. Consider the set U of vertices that are on M- alternating paths starting in u and let $X = U \cap A$, $Y = U \cap B$. Note that all the edges in M have the first endpoint in U and the other in U. We will find a set X such that:
(a) $|X| = |Y| + 1$
(b) $|N(X)| = Y$
To prove (a) note that u is M-unsaturated and M is maximum, thus by Berge's theorem there is no augmenting path which means that every vertex in $U - (u)$ is M-saturated. Thus all vertices in $X - u$ and Y are matched by edges from M which are on the M-alternating paths starting at u, so we have $|X| - 1 = |Y|$.
Now we prove (b) in this way:
Again all vertices from $U - u$ are M-saturated and than $T \subseteq N(X)$.To see that $N(X) \subseteq Y$, we have to prove that there is an odd M-alternating path from u to every vertex in $N(X)$. Let $w \in N(X)$ be a vertex. We prove that $w \in Y$. Now two things can happen:
1. uw is an edge in G, then we have an odd alternating path from u to w which means that $w \in T$ and $N(x) \subseteq T$.
2. There is no such edge between u and w. Since $w \in N(X)$, there is an edge say sw for some $s \in X - u$. Since u is the only vertex in X which is not M-saturated it means that there is an M-alternating path from u to s. Both end points (u,s) of this path lie in X which means that the alternating path has an even length and so the last edge on it is from M. Consequently we can extend this path with an edge and

get an odd M-alternating path from u to w. From (a) and (b) we have found a set X with $|N(X)| < |X|$ which contradicts Hall's condition. Thus we must to have a complete matching.

□

Proof. 3. This proof is based on induction.
Suppose that Hall's condition holds and we show that a matching exists. We use induction on the cardinality of A (the number of all vertices in A). If $|A| = 1$ then Hall's condition implies that the only vertex which is in A has at least one neighbor in B, and so a matching exists. Now suppose that the result is true for $|A| \leq n$. (Using the strong version of mathematical induction). We will prove that it is true for $|A| = n + 1$. Consider a bipartite graph$G$ whose input set A has cardinality $n + 1$. There is tow possibilities: either every proper subset $X \subset A$, the cardinality of $N(X)$ is greater than the cardinality of X, or there exists a proper subset $X \subset A$ such that $|N(X)| = |X|$.

Case 1: Choose any vertex $x \in A$ and any $y \in N\{x\}$ (we know that every $x \in A$ has at least one element in $N\{x\}$ by hypothesis). Then we define a new bipartite graph G^* with two new sets. The first set is $A - \{x\}$ and the another set is $B - \{y\}$, and we remain all the edges in the original graph G, except the edges which are incident to x and y (I mean we take away all the edges are related to x and y). Now the our new bipartite graph G^* satisfies the hypothesis $|N(X)| \geq |X|$, because in case 1 every proper subset $X \subset A$ has $|N(X)| > |X|$ or we can write $|N(X)| \geq |X| + 1$, and by deleting a vertex y from $N(x)$ we still have $|N(X)| \geq |X|$. By the induction hypothesis we have a perfect matching in G^*. We can extend this matching by adding the edge xy and in this case we have a perfect matching in the original graph G.

Case 2: We have $|N(X)| = |X|$ for a proper subset $X \sqsubseteq A$, we want to show that there is a complete matching in A. Construct two new bipartite graphs namely G^* and G^{**} with input sets: X and $A - X$, output sets $N(X)$ and $B - N(X)$ and the same edges which are in the original graph G. So we have $G^* = (X, N(X)), E^*$ and $G^{**} = (A - X, B - N(X), E^{**})$. Now we show that the both bipartite have complete matchings and then the joins of these tow bipartite G have a complete matching.
We know that both vertex sets X and $A - X$ have cardinality no greater than n because $X \subset A$ and $A - X \subset A$. Thus, the induction hypothesis will give us the existence of perfect matching in G^* and G^{**} provided it is shown that the hypothesis $|N(X)| \geq |X|$ for each of these graphs. For G^*: we take any subset $X^* \subseteq X$, then the range set$N^*(X^*)$ in the graph G^* coincides with $N(X^*)$ in G (we have the same edges which are in G). Consequently G^* have a complete matching.

Now consider G^{**}. If there is a subset $X^{**} \subset A$ is disjoint from X, then $|N^{**}(X^*)| = |N(X \cup X^*)| - |N(X)|$, but $|N(X \cup X^*)| \geq |X \cup X^*|$ because $|X| \leq n \Rightarrow |N(X)| \geq |X|$, and the same for X^*. So $|(N^{**}(X^*)| \geq |(X \cup X^*)| - |A|$ or $|(N^{**}(X^*)| \geq |X^*|$. This means that G^{**} has perfect matching. The join of G^* and G^{**} which is G has complete matching. This proves the theorem.

□

Before going to the fourth proof, we will solve the following problem which will be usefull later.

Problem. Let G be a bipartite graph, with vertex sets V_1 and V_2. Let A and B be two subset of V_1

(i) Prove that

$$|N(A \cap B) \leq |N(A) \cap N(B)|$$

. (ii) Prove that

$$|N(A \cup B)| = |N(A) \cup N(B)|$$

Solution.

(i) It is enough to prove that:

$$N(A \cap B) \subset N(A) \cap N(B)$$

Suppose that $y \in N(A \cap B)$. Then there is $x \in A \cap B$ such that xy is an edge of G. But then $y \in N(A)$ and $y \in N(B)$ and so $y \in N(A) \cap N(B)$.

(ii) It is enough to show that

$$N(A \cup B) \subset N(A) \cup N(B)$$

Suppose that $y \in N(A \cup B)$. Then there is a vertex $x \in A \cup B$ such that xy is an edge of G. Thus $x \in A$ or $x \in B$. But then $y \in N(A)$ or $y \in N(B)$, so that $y \in N(A) \cup N(B)$.

Now suppose $y \in N(A) \cup N(B)$. But then $y \in N(A)$ or $y \in N(B)$, then there is a vertex $x \in A$ or$B$ such that xy is an edge of G. In this case $x \in A \cup B$ and $y \in N(A \cup B)$.

Back to Hall's theorem and the fourth proof

Proof. 4 Let G be a minimal graph with respect to the number of edges that satisfies Hall's condition $|N(X)| \geq |X|$.

We will prove that G is a matching with independent edges. Suppose at the moment that there is no such matching. Then there is two edges with a common vertex in B, a_1x, a_2x, where $a_1, a_2 \in A$ and $x \in B$. Now if we delete one edge of these tow the condition fails.

Take subsets $X_i \subset A$ such that $a_i \in X_i$ and a_i is the only vertex which has x as its neighbor, and

$$|N(X_i)| = |X_i|$$

Thus

$$|N(X_1) \cap N(X_2)| = |(N(X_1) - \{a_1\}) \cap (N(X_2) - \{a_2\})| + 1$$

$$\geq |N(X_1 \cap X_2)| + 1 \geq |X_1 \cap X_2| + 1$$

(we add 1 because when we delete a_i their neighbor namely x will be deleted then we have to add one).
But then

$$|N(X_1 \cup X_2)| = |N(X_1) \cup N(X_2)| = |N(X_1)| + |N(X_2)| - |N(X_1) \cap N(X_2)|$$

$$\leq |X_1| + |X_2| - |X_1 \cap X_2| - 1 = |X_1 \cup X_2| - 1$$

which gives a contradication.

□

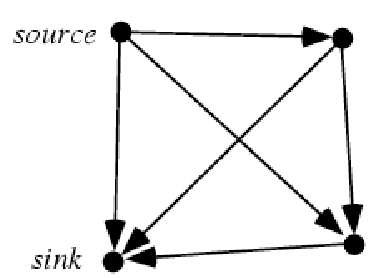
The last proof is based on Max-flow min-cut theorem, so to understand it we have to write some basic concepts and definitions about flow network.

6. Flow Theory

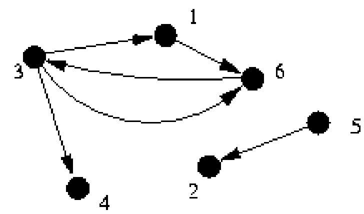
6.1. **Definition.** Directed graph: A directed graph (or digraph) $\overrightarrow{G} = (V, E)$ is a graph in which each edge receives a unique direction:
if there is an edge between u and v then either $(u, v) \in E$ or $(v, u) \in E$.

The World Wide Web is an example of a (directed) graph. The files are the vertices. A link from one file to another is a directed edge (or arc).

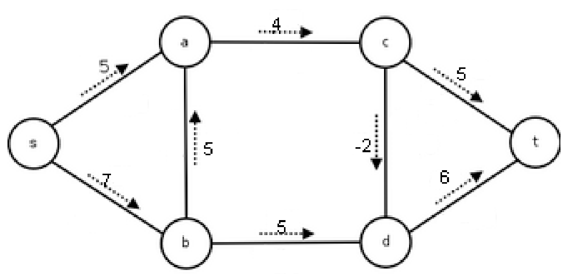
The following figure shows a directed graph with source and sink.



6.2. **Definition.** The incoming, or in degree of the vertex v is the number of edges in $\overrightarrow{G}$ that are incident into v, and this is denoted by $id(v)$.
The outgoing, or out degree of v is the number of edges in $\overleftarrow{G}$ that are incident from v, and this is denoted by $od(v)$.
Example: The vertex 6 in this figure has in-degree 2 and out-degree 1.

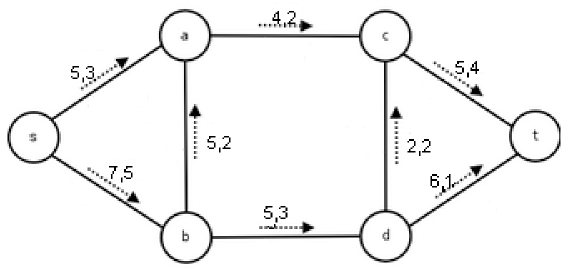


6.3. **Definition.** A network (directed network) $N = (V, E)$ is a connected directed graph where the following conditions are satisfied:
a) There exists a unique vertex $a \in V$ with $id(a)$, the in degree of a, equal to 0. This vertex a is called the source.
b) There is a unique vertex $z \in V$, called the sink, where $od(z)$, the out degree of z, equals 0.
c) The graph N is weighted, so there is a function from E to the set of nonnegative integers that assigns to each edge $e = (v, w) \in E$ a capacity, denoted by $c(e) = c(v, w)$.
Example: The graph in the following figure is a transpot network. Here vertex s is the source, the vertex t is the sink, and capacities are shown beside each edge. Since $c(s, a) + c(s, b) = 5 + 2 = 7$, the amount of the commodity being transported from s to t cannot exceed 7. $c(c, t) + c(d, t) = 2 + 4 = 6$, so the amount is restricted to be no greater than 6.



6.4. **definition.** A flow on the network is a function $f : E \to N$ that satisfies two rules:
a) For each $v \in V$ except the source a and the sink$z$ $\sum_{u,(u,v)\in E} f(u, v) = \sum_{w,(v,w)\in E} f(v, w)$
b) For any $e \in E$ $f(e) \leq c(e)$
Example: For the network in the following figure, the first number beside each edge e determines the capacity of the edge, and the second number assignes for possible flow f. The label on each edge e satisfies

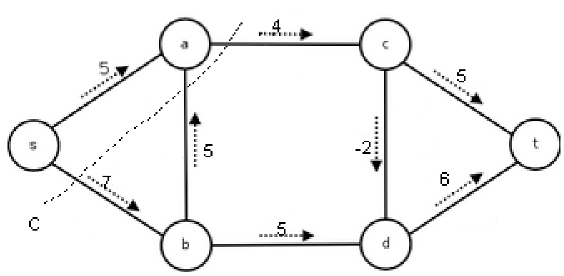
$f(e) \le c(e)$. The function f for in this figure satisfies the above properties, so it is a flow for the given network.



6.5. **Definition.** A partition of the vertices of the network into two sets, one containing the source s, called S, and the other the sink, called T, is called a cut. A cut can also be determined from, and is usually identified with, the set of arcs starting in S and ending in T. The capacity of a cut is the sum of the capacities on these arcs. A minimum cut is a cut with the minimum capacity.
Example: The dotted curve in the following figure indicates a cut for the given network. The cut C consists of the edges $\{s,b\},\{a,c\}$ and $\{a,b\}$. This cut partitions the vertices of the network into two sets $P = \{s,a\}$ and its complement $\overline{P} = \{b,c,d,t\}$, so C is denoted as $(P,\overline{P})$. The capacity of a cut, denoted $c(P,\overline{P})$, is defined by

$$c(P,\overline{P}) = \sum_{\substack{v\in P\\ w\in\overline{P}}} c(v,w)$$

the sum of the capacities of all edges (v,w), where $v \in P$ and $w \in \overline{P}$. In our example, $c(P,\overline{P}) = c(s,b) + c(a,c) + c(a,b) = 7 + 4 + 5 = 16$.



The following theorem is the basic theorem of Ford and Fulkerson which we will use to prove Hall's marriage

Theorem 4. *The Max-Flow Min-Cut theorem*
A value of a maximum flow in a network N is equal to the value of a minimum cut of N.

We will prove Hall's theorem using the above theorem.

Proof. 5. Let $V = A \cup B$, with $A = \{a_1, a_2, ..., a_m\}$ and $B = \{b_1, b_2, ..., b_n\}$. Construct a transport network N that extends graph G by introducing two new vertices a (the source) and z (the sink). For each vertex a_i, $1 \leq i \leq m$, draw edge a, a_i. For each vertex b_i, $1 \leq i \leq n$, draw edge b_i, z.Each new edge is given a capacity of 1. Let M be any possitive integer that exeeds $|A|$. Assign eagh edge in G the capacity M. The complete matching in G exists if and only if there is a maximum flow in N that uses all edges (a, a_i). Then the value of such maximum flow is $m = |A|$.

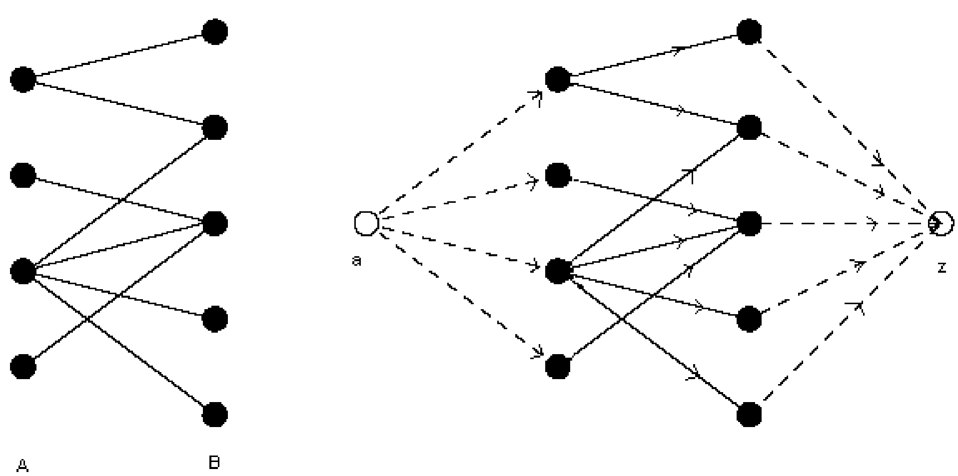


figure 1

We prove that there is a complete matching in G by showing that $c(P, P') \geq |A|$ for each cut (P, P') in N. So if (P, P') is an arbitrary cut in the transport network N, let us define $X = A \cap P$ and

$Y = B \cap P$. Then $X \subseteq A$ where we shall write $X = \{a_1, a_2, ..., a_i\}$ for some $0 \leq i \leq m$. now P consists of the source a together with the vertices in X and the set $Y \subseteq B$. In addition $\overline{P} = (A-X) \cup (B-Y) \cup \{z\}$. If there is an edge $\{a, b\}$ with $a \in X$ and $b \in (B-Y)$, then the capacity of that edge is a summand in $c(P, \overline{P})$ and $c(p, \overline{p}) \geq M > |A|$. If no such edge exist, then $c(P, \overline{P})$ is determined by the capacities of:
1. the edges from the source a to the vertices in $A - X$, and
2. $c(P, \overline{P}) = |A - X| + |Y| = |A| - |X| + |B|$.
With $Y \supseteq R(Y)$, we have $|Y| \geq |R(Y)|$, and since $|R(X)| \geq |X|$, it follows that $|Y| \geq |X|$. Consequently, $c(P, \overline{P}) = |X| + (|Y| - |X| \geq |A|)$. Therefore, since every cut in network N has capacity at least $|A|$ and the cut $(\{a\}, V - \{a\})$ achives a capacity of $|A'|$ (which is the complement to A), by theorem 4 any maximum flow for N has value $|A|$. such a flow will result in exactly $|A|$ edges from A to B having flow 1, and this flow provides a complete matching of A to B.

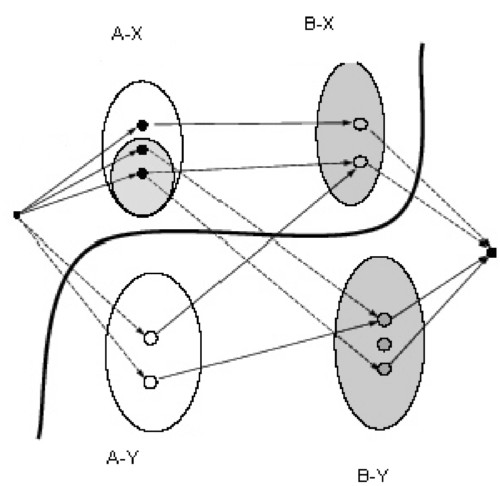


figure 2

□

7. Matching with defect

In general,there are problems where a bipartite graph has not a complete matching.This remarks leads us to the question of finding the maximal matching.

7.1. **Definition.** For a given bipartite graph $G(V, E)$ with V is partitioned as $A \cup B$, a maximal matching in G is one that matches as many vertices in A as possible with the vertices in B.

7.2. **Definition.** Let $G = (V, E)$ be a bipartite graph, where V is partitioned as $A \cup B$. If $X \subseteq A$, then $\delta(X) = |X| - |N(X)|$ is called the deficiency of X.The deficiency of a graph G, denoted $\delta(G)$, is given by $\delta(G) = max\{\delta(X)|X \subseteq A\}$.

Theorem 5. *Let $G = (V, E)$ be bipartite with V partitioned as $A \cup B$. The maximum number of vertices in A that can be matched with those in B is $|A| - \delta(G)$.*

Proof. We use transport networks as in the proof of theorem (4). As in figure (1), let N be the network assosciated with the bipartite graph G. The result will follow when we show that:
(a) the capacity of every cut $(P, \overline{P})$ in N is greater than or equal to $|A| - \delta(G)$.
(b) there exsists a cut with capacity $|A| - \delta(G)$.
Let $(P, \overline{P})$ be a cut in N, where P is made up of the source a, the vertices in $X = P \cap A \subseteq A$, and the vertices in $Y = P \cup B \subseteq B$.[see figure 2].As in the proof of theorem 4, the subset X, Y may be ϕ.
1) If edge (x, y) is in N with $x \in X$ and $y \in B - Y$, then $c(x, y)$ is a summand in $c(P, \overline{P})$. Since $c(x, y) = M > |A|$, it follows that $c(P, \overline{P}) > |A| \geq |A| - \delta(G)$.
2) If no such edge in (1) exists, then $c(P, \overline{P})$ is determind by the $|A-X|$ edges from a to $A - X$ and the $|Y|$ edges from Y to z. Since each of these edges has capacity 1, we find that $c(P, \overline{P}) = |A - X| + |Y| = |A| - |X| + |Y|$. No edge connects a vertex in X with a vertex in $B - Y$, so $R(X) \subseteq Y$ and $|R(X)| \leq |Y|$. Consequently, $c(P, P\prime) = (|A| - |X|) + |Y| \geq (|A| - |X|) + |R(X)| = |A| - (|X| - |R(X)|) = |A| - \delta(X) \geq |A| - \delta(G)$.
Therefore in either case, $c(P, \overline{P}) \geq |A| - \delta(G)$ for every cut $(P, \overline{P})$ in N.
To complete the proof, we must establish the existence of a cut with capacity $|A| - \delta(G)$. Since $\delta(G) = max\{\delta(X)|X \subseteq A\}$, we can select a subset X of A with $\delta(G) = \delta(X)$. Let $P = \{a\} \cup X \cup R(X)$. Then $\overline{P} = (A - X) \cup (B - R(X)) \cup \{z\}$. There is no edge between the vertices in X and those in $B - R(X)$, so $c(P, \overline{P}) = |A - X| + |R(X)| = |A| - (|X| - |R(X)|) = |A| - \delta(X) = |A| - \delta(G)$.

□

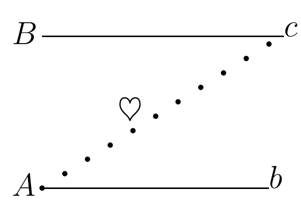
Example: Let $G = (V, E)$ be bipartite with V partitioned as $X \cup Y$. For each $x \in X$, $\deg(x) \geq 4$ and for each $y \in Y$, $\deg(y) \geq 5$. If $|X| \leq 15$, find an upper bound for $\delta(G)$.
(A subset A of X is said to have an upper bound c if $c \geq a$ for all $a \in A$). **Solution:** Let$\phi \neq A \subseteq X$ and let $E_1 \subseteq E$, where $E_1 = \{\{a, b\} | a \in A, b \in R(A)\}$. Since $deg(a) \geq 4$ for all $a \in A, |E_1| \geq 4|A|$. With $deg(b) \leq 5$ for all $b \in R(A)$, $|E_1| \leq 5|R(A)|$. Hence $4|A| \leq 5|R(A)|$ and $\delta(A) = |A| - |R(A)| \leq |A| - (4/5)|A| = (1/5)|A|$. Since $A \subseteq X$, we have $|A| \leq 15$, so $\delta(A) \leq (1/5)(15) = 3$. Consequently, $\delta(G) = max\{\delta(A) | A \subseteq X\} \leq 3$, so there exists a maximal matching M of X into Y such that $|M| \geq |X| - 3$.

8. The Stable marriage Problem

8.1. **Description of the problem.** Suppose that there are n boys and n girls. Each boy ranks all of the girls according to his preference, and each girl ranks all of the boys. There are no ties in anyone's rankings. For example the man M cannot like the woman w_1 and w_2 equally. Rankings are known at the start and stay fixed for all time.
Imagine you are a matchmaker and your job is to arrange n "happy" (stable) marriages.
By stable we mean that once the matchmaker has arranged the marriages, there should be no man who says to another woman "I love you more than the woman I was matched with", and the woman agree because she loves this man more than she loves her husbad. Solving the problem of stable marriage is an example of such new thinking.

Definition 6. A set of marriages is unstable if there is a boy and a girl who prefer each other more than their spouses.

For example, suppose that the man B is married to the woman c, and man A is married to b. But c likes A more than B and A likes c more than b. The situation is shown in Figure. So c and A would both be happier if they ran off together. We say that c and A are a rogue couple, because this is a situation which encourages roguish behavior.



Definition 7. A set of marriages is stable if there are no rogue couples (blocking pair) or, equivalently, if the set of marriages is not unstable.

In other words: A matching M is stable if there is no pair (m, w) of man m and a woman w satisfying the following conditions:
1. m and w are not married in M,
2. m prefers w to his current partner in M,
3. w prefers m to her current partner in M.
If this pair (m, w) exists, M is unstable and the pair (m, w) is called a blocking pair.

8.2. **Example.** (Marrying 3 men (A, B, C) to 3 women (a, b, c)).

Men	*Order of preference*
A	c b a
B	b a c
C	b a c

Women	*Order of preference*
a	A B C
b	C A B
c	C B A

One unstable matching is (Aa, Bb, Cc) because A and b prefer each other more than the pair they matched with.
Let us take another matching, namely Ab, Ba, Cc. This marriage is unstable too because of b and C. But the matching Ac, Ba, Cb is stable because the list of the wives that each man prfers to his own:
A prefers c then b
B prefers b
C has his first choice
and the list of husbands that each woman prefers to her own:
a prefers A
b has her first choice
c prefers C, then B
We cannot improve upon $A's$ choice because neither c nor b prefers him to her husband. Neither we cannot improve $B's$ choice because b has already obtained her favorite partner. The matching considered is therefore stable.

8.3. **Example.** Consider that there are n boys and n girls. Each boy lists the girls in order of preference, and each girl lists the boys in order of preference. The goal is to find a stable pairing between all boys and girls. Stable in the sense that there is no boy - girl couple who are not paired, but who both prefer each other above their partner in the pairing. The number of ways to pair n boys and n girls is $n!$. (The first boy can be paired with n girls; once that girl has been chosen, the next boy can be paired with $n-1$ girls; etc. An so the total number

of choices is $n(n-1)(n-2)1 = n!$.) We know that there is at least one stable pairing. But that might be the only one, or maybe there are many stable pairing. Its very likely that the actual number of stable pairings is depending on the actual lists of preferences. Of course the largest possible number of stable pairings is $n!$.
(a) For each $n \geq 1$, give an example of lists of preferences for the boys and the girls so that only <u>one</u> of all the $n!$ possible ways of pairing them gives a stable pairing.
(b) Give an example of a lists of preferences for 2 boys and 2 girls so that all $2! = 2$ possible pairings are in fact stable.

Solution For convenience, we always denote the boys by $b_1, ..., b_n$ and the girls by $g_1, ..., g_n$. Also, given a person p, we use $L(p) = [...]$ to denote the preference list of that person. For instance, if $n = 3$, then $L(b2) = [g2, g1, g3]$ would indicate that boy b_2 prefers girl g_2 the most, then g_1 and girl g_3 the least.
Also, we use (b_i, g_j) to denote that boy b_i and girl g_j form a pair in some chosen pairing.
(a) For all $i = 1, ..., n$ set preference lists $L(b_i) = [g_i, ...]$ and $L(g_i) = [b_i, ...]$, where the dots indicate that it doesnt really matter what the lists after the first choice look like.
First notice that with these preferences the pairing $(b_1, g_1), (b_2, g_2), ..., (b_n, g_n)$ forms a stable pairing : each person is paired with the person he/she prefers most, hence nobody has another possible partner they prefer over their current partner.
But we also must show that the pairing above is the only stable pairing. So consider a pairing that is different from the one above. That means that at least one boy, say b_i, is not paired with his preferred girl g_i. So there is a pair (b_i, g_j) with $g_j \neq g_i$. Note that this also means that girl g_i is in a pair (b_k, g_i) with $b_k \neq b_i$. So both b_i and g_i have a partner in this pairing they prefer less than each other. So b_i, g_i will form an unstable pair.
(b) Choose $L(b_1) = [g_1, g_2], L(b_2) = [g_2, g_1], L(g_1) = [b_2, b_1]$ and $L(g_2) = [b_1, b_2]$. The both pairing $(b_1, g_1), (b_2, g_2)$ and $(b_1, g_2), (b_2, g_1)$ are stable pairings.

8.4. **The Fundamental Algorithm of Stable Marriage.** A method for computing a stable marriage is called a stable marriage algorithm. The algorithm we describe, due to D. Gale and H. S. Shapely, originally appeared in the American Mathematical Monthly in 1962 under the title "College Admissions and the Stability of Marriage.
Gale and Shapley devised a simple intuitive algorithm, now quite well known, for solving the classical one-to-one stable marriage problem.
They proved that, for any equal number of men and women, it is always possible to solve the stable marriage problem (SMP) and make all

marriages stable. They presented an algorithm to do so. A matching in this algorithm is a one-to one mapping between the two sexes.
The Gale-Shapley algorithm involves a number of ”rounds” (or ”iterations”) where each unengaged man ”proposes” to the most-preferred woman to whom he has not yet proposed, and she accepts or rejects him based on whether she is already engaged to someone she prefers. If she is unengaged, or engaged to a man lower down her preference list than her new suitor, she accepts the proposal (and in the latter case, the other man becomes unengaged again). Note that only women can switch partners to increase their happiness. The SMP is based on the idea of minimizing happiness. The unhappiness, however, is only relative, in example one pairing is either better or worse than another. No measurment of how much better or worse is done.

Algorithm

1. assign each person to be free;
2. while some man X is free
3. x := first woman on X's list; X proposes to x
4. if X is not on x's preference list then
5. delete X from x's preference list;
6. go to line 3
7. end if
8. if some man Y is engaged to x then
9. assign Y to be free;
10. end if
11. assign X and x to be engaged to each other;
12. for each each successor Y of X on x's list loop
13. delete Y from x's list;
14. delete x from Y's list;
15. end loop;
16. end loop;

This algorithm starts by setting all persons free and iterates until all the men are engaged.
A man always wanted to marriage his most-preferred woman (line 3). When a woman x receives a proposal from a man X, she accepts it if X is on her preference list. Otherwise, X deletes x from his preference list (line 5) and then a new proposal is started (line 6). Whether X is on x's preference list and x is already engaged to Y, she discards the previous proposal with Y and Y is set free (line 8-9). Afterwards, X and x are engaged each other (line 11). Woman x deletes from her preference list each man Y that is less preferred than X (line 13). Conversely, man Y deletes x from his preference list (line 14). Finally, if there is a free man, a new proposal is started. The algorithm terminates when all the men are engaged.
To understand this algorithm, we take an example.

Example

Men	Order of preference
A	b c a d
B	d c a b
C	b c a d
D	d b a c

Women	Order of preference
a	C B A D
b	D A B C
c	C A B D
d	A D C B

1. At the begining every person is free.
2. we take man A which has woman b as his first choice, A and b become engeged.
3. Take man B which has woman prefer most woman d. B and d become engaged.
4. Man C has woman b as his first choice, but b prefer A to C, so she refused C. we delete b from his list and go to his second choice which is woman c. c is not engeged so she accept him and we have now a new couple Cc and go to the last man D
5. Man D has d as his first choice. Woman d prefers D to her current partner B. She changes partners and B becomes the new suitor. Then d is removed from B's list.
6. The second choice for B now is c, but woman c is already has her love, so she dumps B.
7. The best choice remaining on poor B is a. She accept him because she prefers to has a partner instead of be alone.
8. We have now a stable list of couples which is shown in the table below.

Theorem 6. *At the end of the algorithm, everyone is engaged.*

Proof. The number of engaged men at the end will be equal to the number of engaged women at the end, because it could not be the case that there remains an un-paired couple (A, a), since at some point (A) would have proposed to (a), and she would have accepted. □

Theorem 7. *The solution obtained by the Algorithm is a stable matching.*

TABLE 1. the table our example

Pairing	Man	Woman	Result
0	A	b	accept
{Ab}	B	d	accept
{Ab, Bd}	C	b	not accept
{Ab, Bd}	C	c	accept
{Ab, Bd, Cc}	D	d	accept
{Ab, Cc, Dd}	B	c	not accept
{Ab, Cc, Dd}	B	a	accept
{Ab, Ba, Cc, Dd}			

Proof. We prove it by contradication.We Assume that the final matching is not stable, which mean that there exists parings (A, a) and (B, b) but (A, b) is a blocking pair. Since A prefers b to a, A would have proposed to b first, who must have rejected him, so b's current mate must be preferred to A This is a contradiction, since A is higher than B in b's preference list. Hence, the matching must be stable. □

In fact, the algorithm gives the optimal result for each man because each man has the best possible matching. The result constitutes the worset solution for the women

Proof. Suppose that Aa is a matching in the algorithm, but Ba and Ab are matchings in another stable matching where a prefers A to B. Thus A must prefer b to a, which contradicts the fact that the Aa is the best solution for A.

Definition 8. A persons optimal partner is the possible partner that person most prefers. A persons pessimal partner is the possible partner that person least prefers.

Notation

$aAb \iff A$ prefers a to b or b is not in A's list of references.

$AaB \iff a$ prefers A to B or B is not in a's list of references.

8.5. **Conflict of interest.** We generelize the result that the best for the men is the worst for the women in the following theorem which says that every other stable matching is better for one of the spouses and worse for the other.

Theorem 8. *If one stable matching contains the couple Aa, and another contains couples Ab and Ba, then either*

$$bAa \text{ and } AaB,$$

or

aAb and BaA

Proof. by the definition of stability, the situations of A and a cannot both worsen in the second solution. Therefore it remains to show that they cannot improve for the two at the same time.
Let $A = X_0$, $a = x_0$, $b = x_1$, and suppose that A prefers b to a, that X_0, it must be that x_1 has obtained a better choice.
Let X_1 be the husband of x_1 in the first stable matching. Thus must be that X_1 has obtained a better choice. Let x_2 be X_1's wife in the second stable matching. Thus $x_2 X_1 x_1$, etc.
We obtain the sequence
$X_0 x_0, X_1 x_1, X_2 x_2, ...$ in the first stable matching,
$X_0 x_1, X_1 x_2, X_2 x_3, ...$ in the second,
where
$x_{k+1} X_k x_k$ and $X_{k+1} x_{k+1} X_k$ for all $k \geq 0$. Since the number of persons is infinte, there exist integers j and k, $j < k$, such that $X_j = X_k$. Let j be the smallest integer such that $X_j = X_k$ and $k > j$. We have $x_j = x_k$. Furthremore, $j = 0$ since otherwise $X_{k-1} x_k = X_{k-1} x_j$ would appear in the second matching as well as $X_{j-1} x_j$. (From which $X_{j-1} = X_{k-1}$, contradicting the fact that j is the smallest integer with $X_j = X_k$). Thus $X_{k-1} = B$. Given that $X_k x_k X_{k-1}$, we have proved AaB. □

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