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Number of limit cycles of a certain Liénard equation

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Abstract

In what follows we recall the basic notions of the theory of limit cycles of plane analytic vector field and illustrate them in the case of the Liénard equation. The main purpose of this treatise is to (re)prove the fundamental result of G.S.Rychkov from 1975 that the Liénard equation with an even degree 4 polynomial coefficient at the first derivative has at most 2 limit cycles. We complement this result with our numerical study of the dependence of the number of limit cycles of this special equation on two natural parameters.

to our parents

Acknowledgements

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Contents

Introduction

In this master thesis we study the so called Liénard equation $\ddot{x}+f(x)\dot{x}+x=0$, a non-linear differential equation with $f(x)$ a polynomial. There is a vast amount of literature about this equation which may be considered as fairly well studied. Any second order differential equation corresponds in a natural way to a planar vector field and so we may consider this thesis as a study in the geometry of differential equations. There are many interesting questions and unsolved problems within this area of research. One of the most difficult among these has been Hilbert's 16th problem, which may be posed as three different questions:

1. *Does a polynomial vector field in the plane have only a finite number of limit cycles?*

2. *Is the number of limit cycles of a polynomial vector field bounded above by a constant depending only on the degree of the polynomials?*

Let us denote this constant by $H(n)$, where n is the degree of the vector field. The third problem is then:

3. *Give an upper bound to the constant H(n).*

The first of these problems have been answered affirmatively by Il'yashenko and Écalle independently (see [2] p.115). The second, and consequently also the third, problem is still unsolved. It has been suggested that in order to make some progress concerning these questions one might study some special case of a planar vector field. Liénard's equation is an example of such a vector field. One can vary the degree and the coefficients of the polynomial $f(x)$ and try to deduce some general results on the behavior of vector fields associated with the Liénard equation. This is actually a good way to get acquainted with the subject and to get a feeling for the deep underlying difficulties. We note that even though the Liénard equation is well studied, problem 2 above is still unsolved even for this specific family of vector fields.

We have restricted our attention to the specific equation

$$\ddot{x} + \mu(x^2 - a)(x^2 - b)\dot{x} + x = 0 \ .$$

One may scale the parameter b to 1 and so the behavior of this equation essentially depends on the 2 parameters μ and a. The goal has been to exhibit the main traits of the bifurcation curve with respect to the number of limit cycles, i.e how many periodic solutions does the differential equation have for given values of μ and a. This is done in chapter 4. In Chapter 1 we present the basic theory needed on second order differential equations, such as the stability concepts for equilibria. The main tools are Liapunov's first and second method. In Chapter 2 we present the definition of a limit cycle and its stability. We briefly introduce the reader to the theory of generalized rotated vector fields which will be needed essentially in chapter 4. Chapter 3 is devoted to the general Liénard equation with applications to the special case of van der Pol. Finally, in chapter 4 we apply all the theory presented in earlier chapters in our own study of the specific equation mentioned above.

It has been our aim to produce a self-contained introduction to this topic which may inspire to further reading in this very interesting and never ending research area.

Södermalm, 23 October 2006

Chapter 1

Preliminaries and Basic concepts

1.1 Ordinary differential equations and systems of first order equations

An ordinary differential equation is an equation of the form

$$f\left(\frac{d^n x}{dt^n}, \frac{d^{n-1}x}{dt^{n-1}}, \frac{d^{n-2}x}{dt^{n-2}}, ..., \frac{dx}{dt}, x, t\right) = 0. \tag{1.1.1}$$

If we to the above equation add the conditions $\frac{d^i x}{dt^i}|_{t=t_0} = b_i$, $i = 0, \ldots, n-1$ then we call it a differential equation with initial conditions , or briefly, an initial value problem. The number n above is called the *order* of the differential equation. If the function f is linear in $\frac{d^i x}{dt^i}$, $i = 1, ..., n$, then we say that the differential equation is *linear*. The reason for this is as follows. If $x_1(t)$ and $x_2(t)$ are two solutions of a linear differential equation, then so is $x(t) = x_1(t) + x_2(t)$ as can readily be seen by plugging in $x(t)$ into (1.1.1) and using the linearity of the differential operator and the function f. A differential equation which is not linear is said to be *nonlinear*. If the function f is independent of t, then we say that the differential equation is *autonomous*. In this paper we shall mainly be concerned with homogeneous second order, nonlinear, autonomous differential equations. It is however instructive to give some background theory for the linear equations as well since this theory, as we shall presently see, forms a natural basis for the theory of nonlinear equations.

Differential equations occur frequently in all kinds of scientific research. Since the differential operator measures the rate of change of "well behaved" functions it is only natural that we should use differential equations as models for various phenomena occurring in nature and in society. The thinking which leads to a differential equation acting as a model suggests at the same time that this differential equation can be solved. Putting some suitable conditions on the solution also renders it highly plausible that the solution is in fact unique. It is of course desirable to exhibit a mathematical proof which gives sufficient conditions for the existence of a unique solution to an arbitrary initial value problem. Beside the question of existence and uniqueness of solutions there is also the question about the dependency of the solution on the initial data. If we for instance drop a stone from a slightly different altitude, we expect the kinetic energy and the velocity to change very little. This is, roughly, what is meant by the solution being continuously dependent on the initial data. The following classical theorem gives a general condition under which we can be certain from a mathematical viewpoint that the solutions to a first order differential equation are well behaved.

Theorem 1.1.1. *(see Theorem 1 in [1], pp.34-38) Suppose that the function $f(x,t)$ satisfies a Lipschitz condition in a region $\Omega \subseteq \mathbb{R}^2$ around the point (x_0, t_0). Then there exists an unique solution in Ω for the equation $\dot{x} = f(x,t)$ such that $x(t_0) = x_0$. Moreover, the solution will be continuously dependent on the initial data.*

A Lipschitz condition is as following.

Definition 1.1.2. Let $f(x,t)$ be a real function on $\mathbb{R} \times \mathbb{R}$. We say that $f(x,t)$ satisfies a *Lipschitz condition* in $\Omega \subseteq \mathbb{R} \times \mathbb{R}$ if there exist a constant K such that

$$(x,t),(y,t) \in \Omega \Rightarrow |f(x,t) - f(y,t)| \leq K|x-y|$$

Example 1.1.3. Consider the function $f(x,t) = x^2t^2$. Let Ω be a bounded subset of $\mathbb{R}^2$. Then any element $(x,t) \in \Omega$ will satisfy $|x| \leq K_1,\ |t| \leq K_2$ for some constants K_1, K_2. We get

$$|f(x,t) - f(y,t)| = |t^2(x^2 - y^2)| \leq {K_2}^2|(x+y)(x-y)| \leq$$

$$\leq {K_2}^2(|x| + |y|)|x-y| \leq 2K_2K_1^2|x-y|$$

and so $f(x,t)$ satisfies a Lipschitz condition in Ω. On the other hand, if Ω is open and unbounded $f(x,t)$ will not satisfy a Lipschitz condition in Ω

We shall not give a proof of theorem 1.1.1, conveniently referring the reader to [1]. It is however worth noticing that definition 1.1.2 and theorem 1.1.1 is easily adapted for vector valued functions

$$\mathbf{f}(\mathbf{x},t) = (f_1(\mathbf{x},t), \ldots, f_n(\mathbf{x},t)),\ \mathbf{x} = (x_1, \ldots, x_n),$$

and it is in fact a "simple" matter to generalize theorem 1.1.1 to such functions. The key is to notice that the proof basically utilizes the general language of metric spaces, i.e the concepts of distance, convergence, uniform continuity etc. Notice that theorem 1.1.1 thus also becomes generalized to higher order "scalar" equations. Indeed, theorem 1.1.1 restated for vector valued differential equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ is a statement about *systems* of first order differential equations

$$\begin{cases} \frac{dx}{dt} = f_1(x_1, \ldots, x_n, t) \\ \vdots \\ \frac{dx}{dt} = f_n(x_1, \ldots, x_n, t). \end{cases} \tag{1.1.2}$$

But any higher order differential equation can be transformed into a system of first order equations by making the following change of variables. We consider equation (1.1.1) and put

$$\begin{cases} x = y_1 \\ \frac{dx}{dt} = y_2 \\ \vdots \\ \frac{d^{n-1}x}{dt^{n-1}} = y_n. \end{cases} \tag{1.1.3}$$

For the sake of clarity, let us suppose that

$$f(\frac{d^n x}{dt^n}, \frac{d^{n-1}x}{dt^{n-1}}, ..., \frac{dx}{dt}, x, t) = \frac{d^n x}{dt^n} + a_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + ... + a_1(t)\frac{dx}{dt} + a_0(t)x \ .$$

The first order system corresponding to the scalar equation is then given by

$$\begin{cases} \frac{dy_1}{dt} = y_2 \\ \frac{dy_2}{dt} = y_3 \\ \frac{dy_3}{dt} = y_4 \\ \vdots \\ \frac{dy_n}{dt} = -a_{n-1}(t)y_n - ... - a_1(t)y_2 - a_0(t)y_1. \end{cases} \tag{1.1.4}$$

Since we have a theorem for the "well behavedness" of solutions to first order systems we immediately get the same result for higher order ordinary differential equations. The change of variables (1.1.3) will be referred to as the *canonical* transformation.

The reader may complain about the fact that we have tacitly assumed at least one of the derivatives $\frac{d^i x}{dt^i}$ to be obtainable from the equation

$f(\frac{d^n x}{dt^n}, ..., \frac{dx}{dt}, x, t) = 0$ in a closed form. What is in fact needed is some kind of condition on f which ensures that one of the derivatives is at least implicitly defined as a function of the other derivatives by the equation $f = 0$. The following fundamental theorem gives us such a condition:

Theorem 1.1.4. *(Implicit function Theorem, see Theorem 9.28 in [11], pp.224-227) Let $f(x_1, ..., x_n)$ be a real valued function and suppose that all the partial derivatives $\frac{\partial f}{\partial x_i}$ exists and are continuous in some open set $U \subset \mathbb{R}^n$. Suppose further that $\frac{\partial f}{\partial x_j}|_p \neq 0$, $p = (a_1, ..., a_n) \in U$, for some $1 \leq j \leq n$. Then the equation $f(x_1, ..., x_n) = f(p)$ defines x_j implicitly as a function of the other $n-1$ variables in some neighborhood $V \subset U$ of p.*

Notice that this is a local theorem. In order for Theorem 1.1.1 to be generalized to higher order equations we thus need to require that at no point do all the partial derivatives of f vanish. We will make use of this theorem in some of the reasoning in chapter 3.

Since we are going to deal with second order differential equations the canonical transformation will actually make our problems concerning these equations equivalent to dealing with vector fields in the plane. This is indeed very convenient since it allows one to utilize a lot of plane geometric reasoning. A general 2-dimensional system is of the form

$$\begin{cases} \frac{dx}{dt} = f(x, y, t) \\ \frac{dy}{dt} = g(x, y, t). \end{cases} \tag{1.1.5}$$

If the functions $f(x, y, t)$ and $g(x, y, t)$ are not autonomous, then the vector field being defined by $(\frac{dx}{dt}, \frac{dy}{dt})$ will change with time. This means that if we drop a particle somewhere in the plane and let the vector field act on it, i.e the particle starts moving in the direction of the field, then the motion of the particle will depend not only on its initial position but also on the initial time. If we assume that the system is autonomous, then the motion of the particle will only be dependent upon its initial position since the vector field will be constant with respect to time. Some basic concepts:

1. By a *trajectory* we mean the geometrical curve in the phase plane[1] associated with a solution to the first order system (1.1.5). These will be denoted by the letter γ and if P denotes a point in the phase space, then γ_P denotes the trajectory passing through P. If we wish to emphasize time as

[1] The phase plane is merely the plane in which we are viewing the vector field. That is, the phase plane is the (x, y)-plane where $y = \frac{dx}{dt}$.

parameterizing the trajectory we write $\gamma(t)$.

2. If we on the other hand want to emphasize the "analytical" properties of solutions, viewing them as functions $x(t)$[2], then we will refer to the *motion* of the system. This distinction is thus mainly used to give clear signals where we wish to use geometrical as opposed to analytical reasoning, but in the end a *motion* and a *trajectory* is of course the same thing.

3. A trajectory will sometimes be called an *orbit*. If an orbit passes through a point P, then the *forward orbit* from P, denoted by $O(P,+)$ or γ_P^+, is defined to be the part of γ_P lying after P with increasing time (that is, if $\gamma_P(t_0) = P$, then $\gamma_P^+ = \{\gamma_P(t) | \ t \geq t_0\}$). Analogously we define the *backward orbit* from P to be the part of γ_P lying before P with increasing time and we denote it by $O(P,-)$ or γ_P^-.

4. A point in the phase plane is called *critical*, or an *equilibrium*, if it satisfies $\dot{x}(t) = \dot{y}(t) = 0$ for some value of t. A trajectory in an autonomous field passing through a critical point at some point t_0 will have to stay there for all $t > t_0$. But this is independent of the direction of time and so no trajectory which contains regular (i.e noncritical) points will pass through a critical point. The critical points are thus trajectories corresponding to trivial solutions.

One important property which is characteristic for the geometry of the plane is described in the following theorem

Theorem 1.1.5. *(Jordan's lemma) Let δ be a simple closed path in $\mathbb{R}^2$. Then δ divides the plane into two open, disjoint, connected sets. One of them is bounded and the other one is unbounded.*

Suppose that a trajectory γ forms a simple closed curve. Theorem 1.1.1 tells us that in any polynomial vector field trajectories cannot cross or even lie tangential to each other. This means that no orbit can cross γ from the inside out or from the outside in . In the plane there is thus a certain rigidity which we loose when going into higher dimensions. The motions being continuously dependent on the initial data means geometrically: let P be a regular point such that $\gamma_P(0) = P$. Then, for any $\epsilon > 0$, $T > 0$, there exist $\delta > 0$ such that $|\gamma_Q(t) - \gamma_P(t)| < \epsilon$ for all $0 \leq t < T$ where Q is an arbitrary

[2] a solution to a system is in general of the form $(x(t), y(t))$, but all the actual systems which will be studied later on will come from ordinary differential equations and so we will always have $y(t) = \dot{x}(t)$ (perhaps after some topological transformation) which means that the solution is completely given by $x(t)$

regular point at a distance less than δ from P. Roughly speaking, the vector field is locally pointing in the same direction.

Consider the 2-dimensional autonomous system

$$\begin{cases} \frac{dx}{dt} = P(x,y) \\ \frac{dy}{dt} = Q(x,y) \end{cases} \tag{1.1.6}$$

where $P(x,y)$, $Q(x,y)$ are polynomials. The slope of the trajectories is given by

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)} \ .$$

It is of course an algebraic matter to find the zeroes of P and Q, which thus helps one to identify all the critical points and the location of the trajectories local extreme points (i.e where $\frac{dy}{dx}$ or $\frac{dx}{dy}$ vanish). We can also introduce a change of variables transforming (1.1.6) into polar coordinates (r,θ). One may then look for regions in the phase plane where $\frac{dr}{dt}$ is of constant sign (trajectories either moving closer to or farther away from the origin) or where $\frac{d\theta}{dt} = 0$ (trajectories consist of straight rays) respectively $\frac{d\theta}{dt}$ is of constant sign (trajectories "spiral" around the origin). Introducing the polar coordinates

$$\begin{cases} x(t) = r(t)\cos\theta(t) \\ y(t) = r(t)\sin\theta(t) \end{cases}$$

we obtain the relations

$$\begin{cases} \frac{dx}{dt} = \frac{\partial x}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial x}{\partial \theta}\frac{\partial \theta}{\partial t} = \dot{r}\cos\theta - \dot{\theta} r\sin\theta \\ \frac{dy}{dt} = \frac{\partial y}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial y}{\partial \theta}\frac{\partial \theta}{\partial t} = \dot{r}\sin\theta + \dot{\theta} r\cos\theta \end{cases}$$

Substituting into (1.1.6) yields

$$\begin{cases} \dot{r}\cos\theta - \dot{\theta} r\sin\theta = P(r\cos\theta, r\sin\theta) \\ \dot{r}\sin\theta + \dot{\theta} r\cos\theta = Q(r\cos\theta, r\sin\theta) \end{cases}$$

Multiplying the first row by $\cos\theta$ and the second by $\sin\theta$ and adding the resulting equations we find

$$\dot{r} = P(r\cos\theta, r\sin\theta)\cos\theta + Q(r\cos\theta, r\sin\theta)\sin\theta$$

and similarly we can obtain the expression

$$\dot{\theta} = \frac{1}{r}\left[Q(r\cos\theta, r\sin\theta)\cos\theta - P(r\cos\theta, r\sin\theta)\sin\theta\right] \ .$$

1.2 2-dimensional linear systems and the concepts of stability

A linear homogeneous 2-dimensional system has the form

$$\begin{cases} \frac{dx}{dt} = ax + by \\ \frac{dy}{dt} = cx + dy \end{cases} \tag{1.2.1}$$

and so we can utilize matrix notation and write

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.2.2}$$

where A is the coefficient matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Setting $\mathbf{x} = (x, y)$ we can write this even more compactly as $\dot{\mathbf{x}} = A\mathbf{x}$. What information can we get about the motions for this system from the matrix A? Before we answer this question it is a good idea to characterize some of the behavior which a solution to any system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ may exhibit.

The local behavior of an analytical vector field is only interesting near critical points. If we are looking at a (small) domain consisting of only regular points, then for any proper subdomain there exists a topological transformation which maps the trajectories into straight lines (see [10], pp.30-32). The local behavior is thus trivial except near critical points. Assume that $P = (x_0, y_0)$ is an *isolated* critical point (we shall in fact always assume that critical points are isolated). Since a critical point corresponds to the zero vector the vectors lying close to it may point in any direction. It is intuitively clear that any motion initiated close to P may either stay close to it or tend away from it. It might of course happen that some motion initiated close to P first tends away from it for some time and then gets closer to P again. This gives some motivation for the following definition.

Definition 1.2.1.

1. An equilibrium point P is *stable* if there exists an $\epsilon > 0$ with the following property: for every $R < \epsilon$ there exists an r, $0 < r < R$, such that if $\gamma(0)$ is inside $\mathbf{B}(P, r)$[3], then $\gamma(t)$ is inside $\mathbf{B}(P, R)$ for all $t > 0$.

2. An equilibrium point P is called *attractive* if there exists $r > 0$ such that any trajectory which satisfies $\gamma(0) \in \mathbf{B}(P, r)$ also satisfies $\lim_{t\to\infty} |P - \gamma(t)| = 0$.

[3]We denote by $\mathbf{B}(\mathbf{x}, r)$ the open neighborhood of all points with distance $< r$ from $\mathbf{x}$.

3. An equilibrium point P is *asymptotically stable* whenever it is *stable* and *attractive*.

4. An equilibrium point P is *marginally stable* if it is *stable* but *not attractive*.

5. An equilibrium point P is *unstable*, *repelling*, *asymptotically unstable* if the change of variables $t \to -t$ renders it *stable*, *attractive* or *asymptotically stable* respectively.

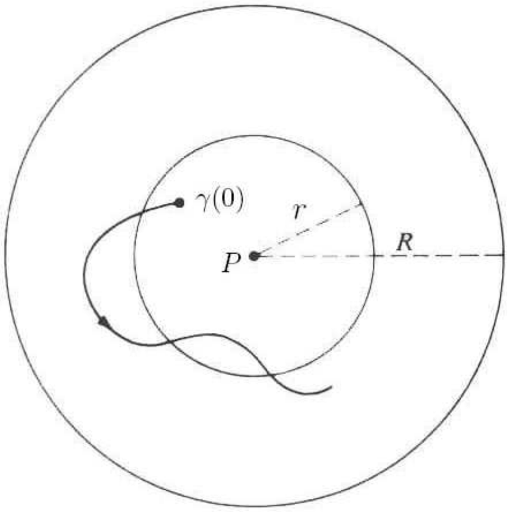
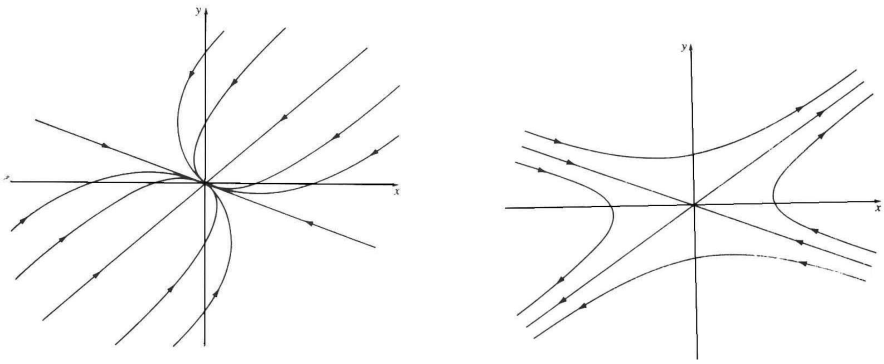


Figure 1.1: An illustration of the concept of stability. The geometrical meaning is that starting close to a critical point implies staying close to it.

We give some examples below of various systems with different kinds of equilibrium points. It is interesting to notice that the concepts of attractive equilibrium and stable equilibrium are independent of each other, as can be seen from example 1.2.4.

As long as we study a single isolated critical point from an abstract point of view we may always assume it is located at the origin. Before stating the main theorem connecting the behavior of trajectories near critical points in linear systems with the matrix defining that system we first make a more careful characterization of critical points than the ones given in definition 1.2.1.
If the trajectories lying close to the origin tend to it, or away from it, asymptotically along a set of straight lines through the origin, then the origin

is called a *node*. If the set of solutions falls into two categories, a set which tends to the origin and a set which tends away from the origin (asymptotically along a set of lines through the origin), then we say that the origin is a *saddle node*. If there are trajectories which spiral toward, or away from the origin then we call it a *focus*. Since different trajectories cannot have a common point of contact we see that the concepts of *node* and *focus* are mutually exclusive. Finally, if all motions are periodic, i.e if all trajectories are closed paths around the origin, then we call O a *center*.



(a) node (b) saddle node

Figure 1.2: Example of a node and saddle node respectively

In the following theorem we assume that the origin is the only critical point. In fact, this follows from the assumption that A is nonsingular.

Theorem 1.2.2. *(see [1], pp.262-266) Consider the first order linear system*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

where A is a nonsingular 2×2 matrix. Then the stability of the origin is completely determined by the signs of $Re(\lambda_i)$, where λ_i $i = 1, 2$, are the eigenvalues of A.

Case 1: *Both eigenvalues have negative real part. The origin will then be globally asymptotically stable. If the eigenvalues are real, then the origin will be a node. If they are complex the origin will become a focus (sink).*

Case 2: *Both eigenvalues have positive real part. The origin will then be asymptotically unstable in the whole. If the eigenvalues are real the origin will be a node. If the eigenvalues are complex it will be a focus (source).*

Case 3: *The eigenvalues are real and of opposite sign, in particular they are both nonzero. Then the origin will be a saddle point, i.e some trajectories tend to the origin asymptotically with increasing time and some tend to infinity.*

Case 4: *The eigenvalues are pure imaginary. Then the origin will be a center, i.e all trajectories are closed paths around the origin.*

For a complete proof of this theorem, see [1]. We can give a short argument for its validity making the simplifying assumption that the matrix A is diagonal. The system is then of the form

$$\begin{cases} \frac{dx}{dt} = ax \\ \frac{dy}{dt} = dy \end{cases}$$

$a, d \in \mathbb{R}$ and so the solutions are given by (e^{at}, e^{dt}). The eigenvalues are $\lambda = a$ and $\lambda = d$ and it is thus clear that the signs of these eigenvalues determine the stability of the origin. The same reasoning holds if A is diagonalizable, but not necessarily diagonal, since it can then be transformed to a diagonal matrix through a transformation XAX^{-1} where X is an invertible linear transformation. Such transformations are continuous and so X in fact defines a homeomorphism, i.e the stability properties of the origin are preserved. Another way of seeing this is by assuming that A has two linearly independent eigenvectors. If $v = (x, y)$ is an eigenvector of A then at the point (x, y) in the phase plane the directional derivative is given by λv. Since any vector of the form $c \cdot v$, where c is an arbitrary real constant, is an eigenvector we see that the lines through the origin spanned by the eigenvectors constitute trajectories (the origin splits these lines into two trajectories). Of course the sign of λ determines the direction of these trajectories. Any solution lying close to these orbit rays will tend to the same direction. If the eigenvalues are complex valued then in no point (x, y) of the plane will the vector field point to (cx, cy) for some real constant c since otherwise $v = (x, y)$ would be an eigenvector with real eigenvalue $\lambda = c$. By continuous dependency on the initial data we may conclude that all trajectories will necessarily spiral around the origin, either inwards or outwards.

Example 1.2.3. Consider the system

$$\begin{cases} \frac{dx}{dt} = ax - y \\ \frac{dy}{dt} = x + ay \end{cases}$$

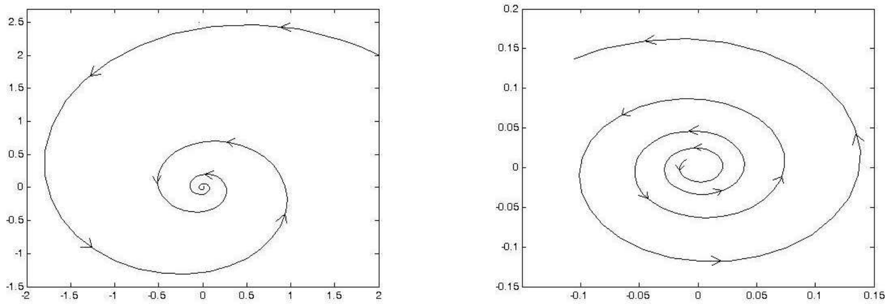
where a is a real parameter. The critical points satisfy $ax = y$. Inserted into the expression for $\frac{dy}{dt}$ we get $0 = x(1 + a^2)$ and so $x = 0$. This implies $y = 0$, which is to say that the origin is the only critical point. To determine stability properties we solve the equation

$$\begin{vmatrix} a-\lambda & -1 \\ 1 & a-\lambda \end{vmatrix} = (a-\lambda)^2 + 1 = 0 \ ,$$

whose roots are

$$\lambda = a \pm i \ .$$

The stability of the origin is then completely determined by the sign of a. If $a < 0$ ($a > 0$) we get a stable (unstable) focus. If $a = 0$ then the eigenvalues are pure imaginary and the origin becomes a center.



(a) a=-0.2

(b) a=0.1

Figure 1.3: The phase portrait for different values of a

Example 1.2.4. This example shows that the concepts of a stable critical point and an attractive critical point are independent of each other, even for autonomous systems. The system:

$$\frac{dx}{dt} = \frac{x^2(y-x) + y^5}{(x^2+y^2)(1+(x^2+y^2)^2)}, \ \frac{dy}{dt} = \frac{y^2(y-2x)}{(x^2+y^2)(1+(x^2+y^2)^2)}$$

was studied by Vinograd (see for instance [6][4]) and we refer the reader to Hahn for an analysis on the behavior of trajectories to this system (It utilizes polar coordinates in a very elegant way). In figure 1.4 we show a part of the phase-portrait. This picture indicates how trajectories starting arbitrarily close to the origin, which is the only critical point, may not stay arbitrarily

[4] pp.191-194

close to it at all times[5]. The origin is thus not stable, but as can also be seen from the phase-portrait it is attractive.

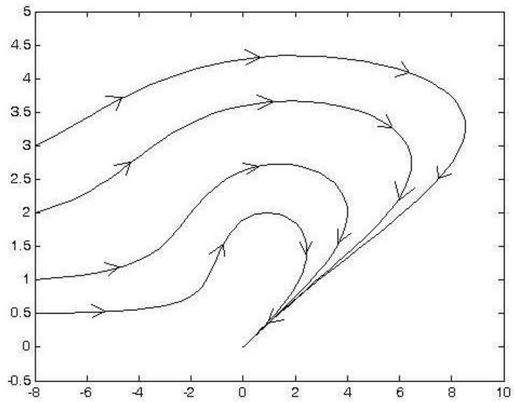


Figure 1.4: The phase portrait of the system of Vinograd

What is important to observe in theorem 1.2.2 is that an analytical problem about the qualitative behavior of solutions has been reduced to an algebraic problem of finding the eigenvalues of the matrix A. This may however not be so surprising considering the assumption of the linearity of the system. On the other hand, it is clear that this very assumption limits the applicability of the theorem a great deal. What is needed is some kind of generalization to nonlinear systems in order to get a powerful tool for qualitative studies of solutions. Moreover, we see that the qualitative behavior obtained by theorem 1.2.2 is *global*, being valid in the whole plane. In other words, there cannot be any isolated closed trajectories in linear systems. Isolated periodic solutions is an inherently *non*linear phenomenon.

1.3 Stability in nonlinear systems, Liapunov's methods

Before discussing the generalization of theorem 1.2.2 to nonlinear systems we introduce a useful concept which will be utilized further. Let $h(x,y)$ be a continuously differentiable real-valued function. Suppose we have a vector field $(\dot{x},\dot{y}) = (P,Q)$ where $P(x,y)$ and $Q(x,y)$ are real-valued polynomials and so we are considering the system (1.1.6) of section 1.1. The

[5]For a proof of this fact see the reference above

derivative of $h(x, y)$ *with respect to system* (1.1.6) is defined by

$$\left(\frac{dh(x,y)}{dt}\right)_{(1.1.6)} = \frac{\partial h(x,y)}{\partial x}\dot{x} + \frac{\partial h(x,y)}{\partial y}\dot{y} \quad . \tag{1.3.1}$$

Whenever we evaluate the function $h(x, y)$ along a trajectory $\gamma(t)$ of the system, its derivative with respect to t will be exactly (1.3.1). Notice that (1.3.1) can be written as

$$\left(\frac{dh(x,y)}{dt}\right)_{(1.1.6)} = \nabla h(x,y) \cdot (P, Q)$$

where $\nabla h(x, y) = (\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y})$ is the gradient of $h(x, y)$ and the dot, "$\cdot$", represents the usual scalar product in $\mathbb{R}^2$. Recall that $\nabla h(x, y)$ is a normal vector to the level curves $h(x, y) = C$ and so $\nabla h(x, y) \cdot (P, Q) = 0$ at some point (x_0, y_0) if and only if the vector $(P(x_0, y_0), Q(x_0, y_0))$ is tangent to the level curve $h(x, y) = h(x_0, y_0)$. In other words, a curve $h(x(t), y(t)) = C$ in the phase-plane is a trajectory if and only if the derivative of $h(x, y)$ with respect to the system is identically zero. In case $h(x, y) = C$ is not a trajectory the sign of $\nabla h(x, y) \cdot (P, Q)$ determines the direction of the field across the level curve $h(x, y) = C$ since $\nabla h(x, y) \cdot (P, Q) \geq 0$ if and only if the angle between gradient and the vector field is $\leq \frac{\pi}{2}$ and so useful information may still be contained in this expression. In fact, if $\nabla h(x, y) \cdot (P, Q) \leq 0$ and $h(x, y) \geq 0$ in a region around the origin one can deduce that the origin is stable.

Theorem 1.3.1. *(Liapunov's direct method, see [13] pp.467-469) Let* $O = (0, 0)$ *be an isolated equilibrium point for the system (1.1.6). Suppose there exists a function* $V(x(t), y(t))$ *in the region* $\Omega = \mathbf{B}(O, R)$*,* $R > 0$*, which satisfies the following conditions*

1. $V(x, y)$ *is continuous and has continuous first partial derivatives.*
2. $V(x, y) \geq 0$ *in* Ω *and equality holds only at the origin.*
3. $\left(\frac{dV(x,y)}{dt}\right)_{(1.1.6)} \leq 0$ *in* Ω *for all motions* $(x(t), y(t))$.

Then the equilibrium is stable. Further, if $\left(\frac{dV(x,y)}{dt}\right)_{(1.1.6)} < 0$ *for every trajectory in* $\Omega \setminus \{O\}$ *then the origin is asymptotically stable.*

Proof. We start with the first claim. Since $V(x, y)$ is continuous and non negative it must have a positive minimum m on the compact set $\partial\mathbf{B}(O, R)$[6].

[6] For any region Ω we denote its boundary by $\partial\Omega$

Because of continuity at the origin we can also find a small region $\mathbf{B}(O,r)$ with $r < R$ such that $V(x,y) < m$ in $\mathbf{B}(O,r)$. Let $\gamma = (x_1(t), y_1(t))$ be any trajectory which lies partially inside $\mathbf{B}(O,r)$, say $\gamma(t_0) \in \mathbf{B}(O,r)$ for some time t_0. Since $\left(\frac{dV(x,y)}{dt}\right)_{(1.1.6)} \leq 0$, $V(x_1(t), y_1(t))$ will be strictly smaller than m for all $t > t_0$. In other words, γ cannot cross $\partial\mathbf{B}(O,R)$ for any $t > t_0$, and therefore stability is proven.
Next, assume $\left(\frac{dV(x,y)}{dt}\right)_{(1.1.6)} < 0$ in $\Omega \setminus \{O\}$. We must prove that $V(x_1(t), y_1(t)) \to 0$ as $t \to \infty$. If the trajectory γ is equal to the origin there is nothing to prove. If γ is nontrivial then $V(x_1(t), y_1(t)) > 0$ and $\left(\frac{dV(x_1,y_1)}{dt}\right)_{(1.1.6)} < 0$. It follows that $V(x_1(t), y_1(t))$ is monotonically decreasing. It is also bounded from below by 0 and so the limit $\lim_{t\to\infty} V(x_1(t), y_1(t))$ exists and is equal to $m_1 \geq 0$ for some real number m_1. Assume $m_1 > 0$. We can then find a region $\mathbf{B}(O,r_1)$, $r_1 < r < R$, such that $V(x,y) < m_1$ whenever $(x,y) \in \mathbf{B}(O,r_1)$. Since $\left(\frac{dV(x_1,y_1)}{dt}\right)_{(1.1.6)} < 0$ and continuous it must have a maximum value, which is negative, in the compact region $(\partial\mathbf{B}(O,r) \cup \mathbf{B}(O,r)) \setminus \mathbf{B}(O,r_1)$. Denote this maximum as $-k$. Considering V as a function of t along γ we have

$$V(t) = V(t_0) + \int_{t_0}^{t} \frac{dV}{ds} ds$$

and since the forward orbit $O(\gamma(t_0), +)$ is contained in $(\partial\mathbf{B}(O,r) \cup \mathbf{B}(O,r)) \setminus \mathbf{B}(O,r_1)$ for all $t > t_0$ we get the inequality

$$V(t) \leq V(t_0) - k(t - t_0)$$

But this implies $V(t) \to -\infty$ as $t \to \infty$ which contradicts the assumption $V \geq 0$. So $m_1 = 0$ and we conclude that $V(x_1(t), y_1(t)) \to 0$ as $t \to \infty$. Since $V(x,y) = 0$ by assumption implies $(x,y) = O$ we have shown that γ tends to the origin. $\square$

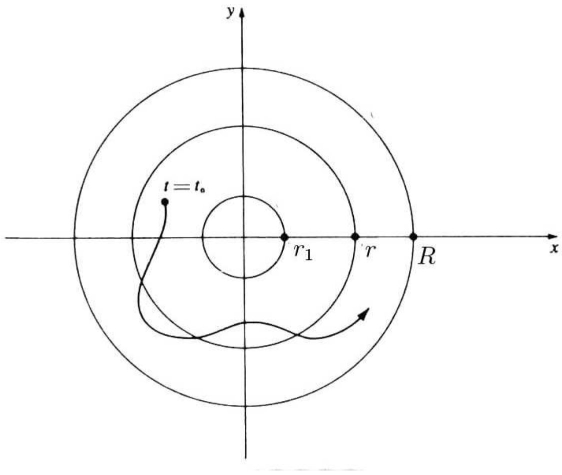


Figure 1.5:

Any function satisfying properties 1-3 in theorem 1.3.1 is called a *Liapunov* function in honor of the Russian mathematician P.L. Liapunov who introduced them in the beginning of the 20th century. The physical interpretation of theorem 1.3.1 is that $V(x, y)$ acts as an energy function on the system. The origin is viewed as an equilibrium and the question is whether this equilibrium is a desirable state to be in. This is the case if the energy of the equilibrium state is minimal. If $\dot{V} < 0$ then the "neighboring" states are dissipating energy and since the origin represents a local minimum for $V(x, y)$ the neighboring states are actually evolving towards the equilibrium.

Theorem 1.3.1 is very nice, but a closer examination shows it to be somewhat limited. In a typical physical situation the energy function $V(x, y)$ is likely to come from our interpretation of the problem at hand. But this is the same as saying that we already know in advance the behavior of the system close to an equilibrium state and theorem 1.3.1 merely becomes a mathematical confirmation of a known physical fact. If we on the other hand have no access to a physical reasoning which reveals the stability of an equilibrium then setting out to find the function $V(x, y)$ is probably a difficult problem. Theorem 1.3.1 can be said to be somewhat self fulfilling. If we wish to make a general mathematical investigation which gives us a convenient tool for deciding the stability of an equilibrium we might for instance pose the following question: *Up to what degree does the linear part of a general system determine its behavior near a singular point?* The answer is given by the following theorem which is the natural generalization of theorem 1.2.2:

Theorem 1.3.2. *(Liapunov's indirect method, see Theorems 7 and 8 in [1], pp.277-281) Consider a nonlinear system*

$$\begin{cases} \frac{dx}{dt} = f(x,y) \\ \frac{dy}{dt} = g(x,y) \end{cases}$$

and assume $f, g \in C^1(\Omega)$, where Ω is a region in $\mathbb{R}^2$. Further assume (x_0, y_0) be an isolated critical point of the system in Ω. Set

$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x_0,y_0)}$$

If the eigenvalues of A are distinct and have nonzero real parts, then the local stability of the critical point (x_0, y_0) is of the same type as that for the linear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

That is, the local behavior of the nonlinear system around (x_0, y_0) is as in cases 1-3 of Theorem 1.2.2.

If one of the eigenvalues of the Jacobian is equal to zero then no useful information is obtained and a deeper analysis has to be made. One may then for instance try to construct a suitable Liapunov function in order to determine the stability properties of the origin. Also notice that the results of this theorem are *local* in contrast to theorem 1.2.2 where we had a *global* behavior for the trajectories.

Chapter 2

Limit cycles

As we pointed out at the end of section 1.2, isolated periodic motions are an inherently nonlinear phenomena. This chapter is devoted to the definition of a limit cycle and some of the most rudimentary theory surrounding this concept. This theory is very rich and fascinating. It is striking how far one can get with simple analytic techniques together with clear geometric and topological reasoning.

2.1 The concept of a limit cycle and its stability

It is of course clear what a periodic motion $x(t)$ is. A function is called periodic if there exist a real number $T > 0$ such that $x(t + T) = x(t)$ for all t. The least such positive number T is called the *period* of $x(t)$. It is also rather clear that any periodic motion, i.e a solution to a first order differential system in the plane, is periodic if and only if its trajectory is a closed path in the phase plane. We want to distinguish between the the situation with infinitely many closed trajectories filling up an annulus region in the plane from the existence of an isolated closed trajectory. "Isolated" here means isolated with respect to other closed orbits so that there exist a small region around the isolated orbit where all other trajectories are not closed (i.e they correspond to non-periodic motions). It may be intuitively clear that for this to happen the surrounding trajectories will have to tend to the periodic motion, either as $t \to \infty$ or as $t \to -\infty$. The first step in proving this is made by introducing the concepts of forward and backward limit sets of an orbit.

Definition 2.1.1. Let γ_P be the trajectory through $P \in \mathbb{R}^2$. The *forward limit set of* γ_P is the set
$\Omega_P^+ = \{q \mid \lim_{k\to\infty} \gamma_P(t_k) = q$ *for some sequence* $t_1 < t_2 < \dots,$ *s.t* $t_k \to \infty\}$.
The *backward limit set* is the set
$\Omega_P^- = \{q \mid \lim_{k\to\infty} \gamma_P(t_k) = q$ *for some sequence* $t_1 > t_2 > \dots,$ *s.t* $t_k \to \infty\}$.
We also write Ω_γ^+ and Ω_γ^- for Ω_P^+ and Ω_P^- respectively. The *limit set of* γ is simply $\Omega_\gamma = \Omega_\gamma^+ \cup \Omega_\gamma^-$.

For a trajectory which tends to the origin we clearly have $\Omega_\gamma^+ = \{O\}$. We have the following interesting topological result about limit sets.

Theorem 2.1.2. *(see Theorem 3.252 in [10], p. 21) Forward (backward) limit sets are either empty or consist of whole trajectories.*

Proof. It is enough to prove that if $q \in \Omega^+(p)$ then $\gamma_q \subset \Omega^+(p)$. Let $\gamma_q(t_0) = q$. Since $q \in \Omega^+(p)$ there exists an unbounded monotone increasing sequence $\{t_n\}$ such that $\gamma_p(t_n) \to \gamma_q(t_0) = q$. This means that for increasing n the sequence $\{p_n = \gamma_p(t_n)\}$ comes arbitrarily close to q. Due to continuous dependence on the initial data the trajectories $\gamma_{p_n}(t)$ may stay arbitrarily close to the trajectory $\gamma_q(t)$ for any finite time interval $t_n < t < t_n + \bar{t}$. This is the same as saying that $\gamma_p(t_n + \bar{t}) \to \gamma_q(t_0 + \bar{t})$. Since any point of γ_q is of the form $\gamma_q(t_0 + \bar{t})$ we have proved that $\Omega^+(p)$ is made up of whole trajectories. The reasoning for Ω^- is completely analogous. $\square$

Theorem 2.1.3. *(see [10], p. 22) If γ is a closed trajectory, then $\Omega_\gamma = \gamma$.*

Proof. Let $\gamma(t)$ be a closed trajectory, i.e $\gamma(t)$ is periodic with period T, say. This means that $\gamma(t + nT) = \gamma(t)$ for all integers n and so clearly $\gamma \subset \Omega(\gamma)$. On the other hand, since γ is a closed trajectory it must contain all its limit points. We conclude that $\gamma = \Omega(\gamma)$. $\square$

These two theorems help us to formalize the concept of an isolated closed trajectory.

Definition 2.1.4. Let γ be a closed orbit. Suppose there exist $\epsilon > 0$ such that for any point $P \in \{Q \in \mathbb{R}^2 | d(Q, \gamma) < \epsilon\}$[1], we have either $\Omega_P^+ = \gamma$ or $\Omega_P^- = \gamma$. Then we say that γ is a *limit cycle*.

Before examining the behavior of limit cycles and trajectories near limit cycles we introduce the concept of an *invariant set*. Let $S \subset \mathbb{R}^2$. We say that S is *positively invariant* if for any point $p \in S$ we have $\gamma_p(t) \in S$ for all

[1] $d(A, B) = \inf_{x\in A, y\in B} |x - y|$

$t \geq t_0$ where $\gamma_p(t_0) = p$. S is called *negatively invariant* if the transformation $t \to -t$ renders it positively invariant. If S is both positively and negatively invariant it is simply called *invariant*. For example, any trajectory γ is invariant. If a critical point is asymptotically stable then there exists some positively invariant neighborhood surrounding this critical point.

We intend to show that a limit cycle must surround at least one critical point.

Definition 2.1.5. Let δ be a simple closed curve in $\mathbb{R}^2$ not passing through any critical point of the planar vector field $(P(x,y), Q(x,y))$. For any point (a,b) on δ, let $(r\cos(\theta), r\sin(\theta))$ be the polar representation for the vector $(P(a,b), Q(a,b))$. As we move along δ, the angle θ will vary continuously (if P and Q are continuous). The *rotation number* for δ is defined to be the integer n such that the variation of θ is $2\pi n$ as we make one complete turn around δ.

Since making one turn around δ means coming back to the starting point the total variation of θ must be an integer multiple of 2π. We can now prove the next result.

Theorem 2.1.6. *A limit cycle in a continuous vector field must surround a critical point.*

Proof. Let γ be a limit cycle in a planar vector field. It is intuitively clear that the rotation number for γ is either $+1$ or -1[2]. Suppose that it is $+1$, the reasoning for -1 being completely analogous. Suppose that γ does not surround a critical point. Then we can continuously deform γ (viewing γ as a simple closed curve) without crossing any critical point. Such a deformation will change the rotation number in a continuous fashion, and since it is an integer it must be preserved under continuous deformations of γ not crossing any critical point. Moreover, we can deform γ into an arbitrarily small circle C_r. Since this circle will not contain any critical points, and since the vector field is assumed to be continuous, all vectors in the region bounded by ∂C_r will "point in the same direction". That is, the rotation number of C_r is equal to zero. This however contradicts the fact that γ and C_r should have the same rotation number. Thus γ must surround a critical point. $\square$

Before we further examine the conditions for the existence of closed orbits we make some comments on the behavior of trajectories lying close to a limit

[2]As one thinks a little bit about this fact it actually seems obvious. Nonetheless it requires a rigorous proof which we leave out.

cycle. In connection to this it is natural to introduce the so-called successor function, also commonly referred to as the Poincaré return map in honor of its inventor. We first make the following definition.

Definition 2.1.7. Let AB be a line segment in $\mathbb{R}^2$ with a vector field (P, Q). If AB is nowhere tangent to the vector field, then we say that AB is a $cross-section$ for (P, Q).

Notice that any two vectors of the field (P, Q) which crosses AB must cross it in the same direction, since otherwise by continuity there would be a point on AB at which it is tangent to the field.

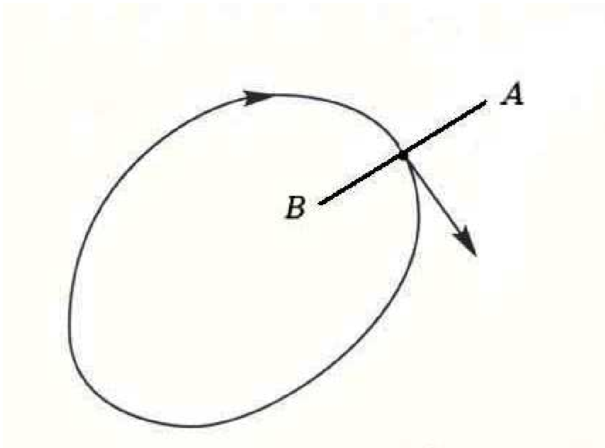


Figure 2.1: A cross-section

Suppose that we have a limit cycle (denoted by L) of a continuous vector field (P, Q). By uniqueness and continuity of solutions we have that any trajectory starting close enough to L will stay close to it in a finite time interval. Also the flow of the vector field will approximately point in the same direction as L close enough to L. We can thus construct a small cross-section AB through L at any point on L (take for instance the normal vector). If a trajectory $\gamma(t)$ crosses AB in p_0 at $t = t_0$, then there will exist a least time $t_1 > t_0$ such that $\gamma(t)$ crosses AB again, in p_1, at $t = t_1$. For a fixed cross-section AB we can thus define a map $\rho : AB \to AB$ which takes p_0 to p_1. The point p_1 is called the *succeeding* point of p_0 and ρ is called the $successor\ function$. The successor function inherits the smoothness of (P, Q), and so if the vector field is of class C^r then so is ρ.
The successor function can be used to define the concept of stability for a limit cycle. It is clear that any point p_0 on AB lies on a limit cycle if and only if $\rho(p_0) = p_0$.

Definition 2.1.8. Let $q \in L \cap AB$ (L a limit cycle and AB a cross-section through L). Let n_p denote the distance between a point $p \in AB$ and q. L is called *stable* (*unstable*) if $\exists \epsilon > 0$ such that $n_{p_0} > n_{p_1}$ ($n_{p_0} < n_{p_1}$) for any point $p_0 \neq q$ on AB with $d(p_0, q) < \epsilon$[3].

Analogously we can define *external stability* and *internal stability* (*external/internal instability*) by only considering the part of AB which lies in the exterior or the interior of L respectively. If a limit cycle is externally stable and internally unstable (or vice versa), then it is called $semi - stable$. It is clear from the definition that trajectories lying close to a stable limit cycle will tend closer to it with increasing time, and the opposite for unstable limit cycles. It may seem somewhat intuitively clear that this is the only type of closed orbits which can occur in a continuous, or at least once continuously differentiable vector field (assuming that the origin is not a center). However, there do exist other cases. For instance, we may have a situation where there are infinitely many closed orbits in a bounded region D of the plane not completely filling up D.

Definition 2.1.9. Let L be a closed orbit with a cross-section AB through L. Let n_p be as in definition 2.1.8. Then L is called a *compound limit cycle* if there for any $\delta > 0$ $\exists p_1, p_2 \in AB$ such that $0 < n_{p_i} < \delta$, i=1, 2, and $\rho(p_1) = p_1,\ \rho(p_2) \neq p_2$.

It is clear that a case of this type is much more difficult to analyze and so it is desirable to have conditions under which a compound limit cycle does not exist. We have here only introduced the successor function in a somewhat intuitive fashion. A deeper analysis must be made in order to get useful information from it. That is, we need a formula for ρ or for the numbers n_p. Such formulas can be obtained although we will not present them here. The interested reader may consult [3][4]. There the authors also prove the following important result.

Theorem 2.1.10. *(see Theorem 2.1 in [3], p.201) Let (P, Q) be a analytic vector field. Then (P, Q) cannot contain any compound limit cycle.*

The vector fields we are going to study, in chapters 4 and 5, will all be polynomial. We can thus exclude the occurrence of compound limit cycles. Moreover, the number of limit cycles (*isolated* closed trajectories) is always finite for such fields.

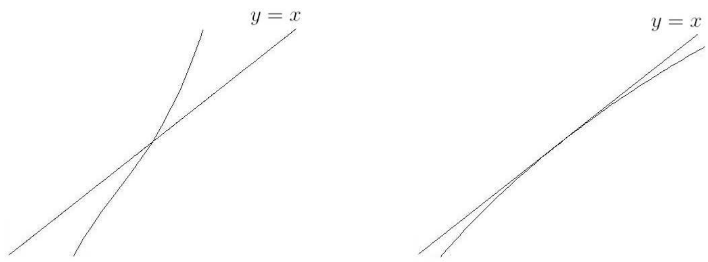
[3] As in a footnote above, $d(x, y)$ is the distance between x and y.

[4] pp.199-203

We can use the successor function in order to define the multiplicity of a limit cycle.

Definition 2.1.11. Let L be a limit cycle in a continuous vector field (P, Q) with a cross-section AB through L and let n_p be as in definition 2.1.8. Consider the function $\sigma(n_p) = n_p - n_{p_1}$, where $p_1 = \rho(p)$. It is clear that $\sigma(0) = 0$. The *multiplicity* of L is defined to be the multiplicity of the root of $\sigma(n) = 0$ at $n = 0$.

If the parameters in the expressions for $P(x, y)$ and $Q(x, y)$ are slightly perturbed then a simple limit cycle will only be slightly deformed without its stability being changed. Simple limit cycles thus form a topological structure which is stable under continuous variations of the parameters. Multiple limit cycles on the other hand may disappear or split into several new cycles, which is called a bifurcation. We make some further investigations on this topic in section 2.3.



(a) A limit cycle of multiplicity 1 (b) A limit cycle of higher multiplicity

Figure 2.2: The graph of $\sigma(n)$ is plotted against the line $y = x$

We have seen that a limit cycle must surround a critical point and so no cycles can be completely contained in a simply connected region consisting of regular points. On the other hand, given a critical point we may find a Liapunov function verifying its local stability properties. If $V(x, y)$ is such a function then we may be able to compute the maximal region Ω around the critical point where the derivative of V with respect to the given system is strictly negative. No limit cycle can be strictly contained in Ω either. We introduce one more negative criterion due to Ivar Bendixson.

Theorem 2.1.12. *(see Theorem 12 in [1], p.298) Let (P, Q) be planar vector field of class C^1. If there is a closed region Ω such that $div(P, Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \geq$*

0 *(or* ≤ 0*) in* Ω*, but not identically zero, then there are no closed orbits contained in* Ω*.*

Proof. Suppose there exists a closed trajectory γ in Ω and denote by D the region bounded by γ. Green's theorem yields

$$\int\int_D div(P, Q)dxdy = \int_\gamma Pdy - Qdx = 0 \ .$$

However, since $div(P, Q)$ is not identically zero and otherwise of constant sign we arrive at a contradiction. $\square$

The divergence of the vector field thus seems to play an important role in the nature of limit cycles. Its significance is indeed brought out by the following theorems.

Theorem 2.1.13. *(see [7], p.238) Let L be a limit cycle in the vector field* (P, Q)*. If we have*

$$\int_L div(P, Q)dt < 0 \ (> 0)$$

then L must be a stable (unstable) limit cycle.

Corollary 2.1.14. *If L is a semistable or a compound limit cycle, then*

$$\int_L div(P, Q)dt = 0$$

This theorem will be important in chapter 4.

2.2 The Bendixson theorem

We now turn our attention to the question of finding criteria for the existence of Limit cycles. One such method has already been implicitly mentioned. If one can find a cross-section AB in the vector field on which the return function ρ is defined then we can look for fixed points of ρ on AB since these correspond to closed trajectories. This of course demands that one has some kind of formula for ρ which may be analyzed. However, one may not actually need to consider the return function but simply any real valued function f defined on AB which is continuous and attains the value 0, say, only at points lying on a limit cycle. One then only has to estimate the values of this particular function at the end points of AB. If these are negative on one side and positive on the other one can deduce the existence of a point $p \in AB$ such that $f(p) = 0$. If f is differentiable on AB one may also be able to calculate the exact number of limit cycles.

The following theorem gives a geometric criterion for the existence of a limit cycle.

Theorem 2.2.1. *(Bendixson, see Theorem 11 in [1], pp.295-297) Let*

$$\begin{cases} \frac{dx}{dt} = P(x, y) \\ \frac{dy}{dt} = Q(x, y) \end{cases}$$

be an autonomous system of differential equations in the plane. Further let K be a compact subset of the plane which does not contain any equilibrium points. Assume that $\gamma(t)$ is a trajectory such that $\gamma(t) \in K$ for all $t \geq 0$. Then one of the following holds:
1) γ is closed,
2) $\gamma(t)$ tends spirally towards a closed trajectory as $t \to \infty$.

Proof. Since K is compact there exists an accumulation point p such that $\gamma(t_k) \to p$ for some subsequence $t_k \to \infty$. Let $\gamma_p(t)$ be the trajectory which satisfies $\gamma_p(0) = p$. It can be proven that $\gamma(s + t_k) \to \gamma_p(s)$ as $k \to \infty$ for all s, although we shall not do it here, conveniently referring the reader to [1][5]. The convergence is uniform on compact time intervals. Especially it follows that $\gamma_p(s) \in K$ for all s. This means that for any point q on γ_p there are points on γ which lie arbitrarily close to q (see fig. 2.3).

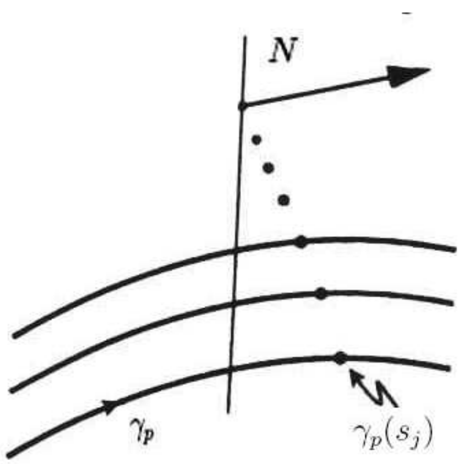
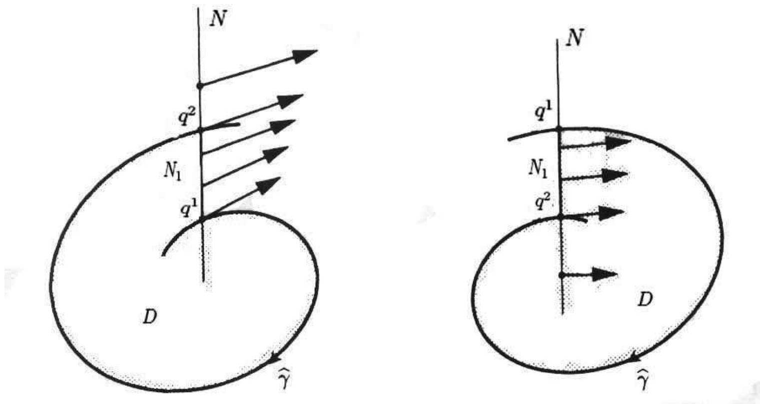


Figure 2.3:

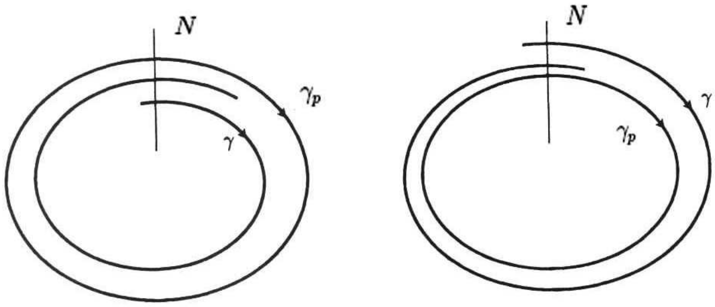
[5] see Theorem 9 p.63

We first prove that γ_p is a closed path.
The same argument as above for $\gamma(t)$ gives that there exists a point $q \in K$ such that $\gamma_p(s_j) \to q$ for some sequence $s_j \to \infty$. Since K does not contain any equilibrium points $(P(q), Q(q)) \neq (0,0)$. Now put through q a small cross-section L. Since all trajectories passing close to q will cross L we have that $\gamma_p(s) \in L$ for infinitely many s. Let $q^1 = \gamma_p(s^1)$ and $q^2 = \gamma_p(s^2)$ be two intersection points of γ_p and L such that q^2 is the succeeding point of q^1. We will show that $q^1 = q^2$. It will then follow that γ_p is closed.

Assume $q^1 \neq q^2$. Let D be the region in the plane which is bounded by the path γ_p between q^1 and q^2 and by the line segment L_1 of L which lies between q^1 and q^2. For every point on L_1 the vector field points in the same direction (either in or out from D), since our system is continuous and there are no equilibrium points in K.



Let us consider the case where the vector field points outwards. This means that all trajectories which intersect L_1 are oriented outwards from D. Since no trajectory of the system can intersect γ_p it follows that no trajectory can go into D. Hence if $\gamma(t)$ is for some time τ outside of D it stays there for all $t > \tau$. Consequently it is impossible for $\gamma(t)$ to come arbitrarily close to q^1. On the other hand, if $\gamma(t)$ is inside of D then either it stays there for all time, in which case it cannot come arbitrarily close to q^2, or it goes out from D after some finite time τ, in which case it again cannot come arbitrarily close to q^2. Since this is a contradiction we must have $q^1 = q^2$. Thus there exist a closed trajectory γ_p. Suppose now that $\gamma \neq \gamma_p$. Since L is a cross-section and since every point on γ_p is a limit point of points on γ, we must have that L and γ intersect infinitely many times. Since γ cannot intersect itself we must have that γ spirals towards γ_p, either from the inside or from the outside.



□

The Bendixson theorem is very important. Its limitation lies in the fact that although it gives us a positive indication for the existence of a limit cycle and although it gives us a hint about this limit cycles location, it does not tell us how many cycles there are in the bounded annular region. However, the following useful results are direct consequences of the proof. Call two limit cycles *consecutive* if one lies in the interior of the other and no closed orbits are located in the region bounded by the two limit cycles. $L_1 \subset L_2$ will henceforth mean that the closed orbit L_1 lies inside the closed orbit L_2. Suppose L_1 is externally unstable and is consecutive to L_2, $L_1 \subset L_2$. Then we must have that L_2 is internally stable. Otherwise the trajectories lying close to L_1 and exterior to it will tend to L_2 and the trajectories lying close to L_2 and interior to it will tend to L_1. We may then use the Bendixson theorem to deduce the existence of a closed orbit between L_1 and L_2 which contradicts to the fact that they are consecutive. In short, consecutive limit cycles must have different stability types on the facing sides. This means that even if we from the Bendixson theorem cannot conclude the number of limit cycles within an invariant annular region we can deduce that the number of cycles must be even, assuming that they are all simple.

When applying the Bendixson theorem the difficulties lies in finding suitable closed curves. In certain parts of the phase plane simple geometric reasoning might be enough, only using straight lines or perhaps circle arcs. When the situation is a bit more complicated one may try to produce functions $h(x, y)$ such that

$$\left(\frac{dh(x,y)}{dt}\right)_S = \nabla h(x,y) \cdot (P, Q)$$

is of constant sign, where S is the system under consideration. By the remarks made in 1.3 the vector field will then cross the level curves of $h(x, y)$ in one direction only.

2.3 Rotated vector fields

In this section we present some bifurcation theory for planar vector fields. The main results will be necessary for a complete proof of Rychkov's theorem in chapter 4. In this section we closely follow the presentation of [3], pp.203-212.

Consider the system

$$\frac{dx}{dt} = P(x, y, \alpha), \frac{dy}{dt} = Q(x, y, \alpha) \tag{2.3.1}$$

where α is a parameter. Throughout this section we assume that the vector field (P, Q) has only isolated critical points and that

$$P(x, y, \alpha),\ Q(x, y, \alpha),\ \frac{\partial P}{\partial \alpha},\ \frac{\partial Q}{\partial \alpha} \in C^0(\mathbb{R}^2 \times I)$$

where I is a closed interval for the values of the parameter α. We wish to present some results on the possible changes in the topological structure of such a system as α varies. These results will be used later in chapter 4. By $(2.3.1)_{\alpha_0}$ we denote the system (2.3.1) with the value α_0 for the parameter α. If the topological structure for system $(2.3.1)_{\alpha_0}$ is left unchanged under small perturbations of α_0, then α_0 is called a *regular value* for the parameter and the system $(2.3.1)_{\alpha_0}$ is called *structurally stable* with respect to perturbations of α. If α_0 is not a regular value, i.e the topological structure changes for small perturbations of α at α_0, then we say that α_0 is a *bifurcation value* and the change in topological structure is called a *bifurcation*. We shall only be concerned with bifurcations of limit cycles. As mentioned earlier, a limit cycle will be structurally stable if its multiplicity is 1. As we come to a bifurcation value for α the limit cycles will gain higher multiplicity and they may for instance disappear or split into several other limit cycles as α changes. These phenomena form a fascinating subject in the area of bifurcation theory. However, it is in general very difficult to make such an investigation without adding some conditions on the system (2.3.1). We will confine ourselves to so called generalized rotated vector fields. We first give a rough explanation of this concept followed by a suitable definition.

Consider a fixed vector in the field (2.3.1). As α varies, this vector may change its direction and norm. By a rotated vector field we mean a field where any vector is rotating in one definite direction as α varies. If we give a polar representation of a vector, $(r \cos\theta, r \sin\theta)$, this condition may be expressed as $\frac{d\theta}{d\alpha} \geq 0$ (or ≤ 0). If this rotation never stops completely, then there will exist a least real number T such for any $0 < \alpha_1 < \alpha_2 < T$ we have

$$0 \leq \int_{\alpha_1}^{\alpha_2} \frac{d\theta}{d\alpha} d\alpha \leq \pi \ .$$

We thus introduce the following definition.

Definition 2.3.1. Suppose that as α varies in $[0, T]$ the critical points of the vector field $(P(x, y, \alpha), Q(x, y, \alpha))$ remain unchanged and at all regular points
1) $\frac{d\theta}{d\alpha} \geq 0$, and $\frac{d\theta}{d\alpha}$ not identically equal to zero
2) for any two points $\alpha_1 < \alpha_2$ in $(0, T)$

$$0 \leq \int_{\alpha_1}^{\alpha_2} \frac{d\theta}{d\alpha} d\alpha \leq \pi$$

Then $(P(x, y, \alpha), Q(x, y, \alpha))$ are called *generalized rotated vector fields*.

Notice that rotated vector fields are defined with respect to an interval $[0, T]$. Also, strictly speaking, a generalized rotated vector field is a whole family of vector fields.
There is another way of defining generalized rotated vector fields. The main idea behind these fields is that as the parameter α varies the trajectories will change. So if we have two distinct values of α, $\alpha_1 < \alpha_2$, we wish for the vectors $(P(x, y, \alpha_1), Q(x, y, \alpha_1))$ and $(P(x, y, \alpha_2), Q(x, y, \alpha_2))$ not to be linearly dependent along a whole trajectory in any of the systems $(2.3.1)_{\alpha_i}$. This can be stated as

$$\begin{vmatrix} P(x, y, \alpha_1) & Q(x, y, \alpha_1) \\ P(x, y, \alpha_2) & Q(x, y, \alpha_2) \end{vmatrix} \geq 0, \ (or \ \leq 0)$$

Notice that since the determinant measures the area in the parallelogram spanned by the two vectors $(P(x, y, \alpha_1), Q(x, y, \alpha_1))$ and $(P(x, y, \alpha_2), Q(x, y, \alpha_2))$ *with orientation*, the above requirement will actually be sufficient for the concept of rotated vector field. We thus make the following definition, which is equivalent to the one introduced above.

Definition 2.3.2. Suppose that as α varies in (a, b), the critical point of the vector fields $(P(x, y, \alpha), Q(x, y, \alpha))$ remains unchanged, and for any fixed point $p = (x, y) \in \mathbb{R}^2$ and any parameters $\alpha_1 < \alpha_2$ in (a, b) we have

$$\begin{vmatrix} P(x, y, \alpha_1) & Q(x, y, \alpha_1) \\ P(x, y, \alpha_2) & Q(x, y, \alpha_2) \end{vmatrix} \geq 0, \ (or \ \leq 0) \tag{2.3.2}$$

where equality cannot hold on an entire closed orbit of the system. Then $(P(x, y, \alpha), Q(x, y, \alpha))$ are called *generalized rotated vector fields*.

Example 2.3.3. Let $F(x)$ be a polynomial with a root at the origin and consider the system

$$\begin{cases} \frac{dx}{dt} = y - F_\alpha(x) \\ \frac{dy}{dt} = -x \end{cases} \tag{2.3.3}$$

where

$$F_\alpha(x) = F(x) + \alpha r(x), \ \ with \ r(x) = \begin{cases} 0 \ if \ |x| < c \\ sgn(x)(|x| - c)^2 \ if \ |x| \geq c \end{cases}$$

Here c is some positive constant. By inspection we find the origin to be the only critical point for all α and so the critical points remain unchanged as α varies. Further we have to verify that the determinant in (2.3.2) is of the same sign for all fixed x and y and $\alpha_1 < \alpha_2$. Notice that

$$\begin{vmatrix} y - F(x) + \alpha_1 r(x) & -x \\ y - F(x) + \alpha_2 r(x) & -x \end{vmatrix} = |xr(x)|(\alpha_2 - \alpha_1) \geq 0 \ .$$

If all orbits cross the lines $|x| = c$ then the above inequality cannot be an equality on an entire orbit and so these systems will form a generalized rotated vector field.

The behavior of limit cycles in a rotated vector field as the parameter varies is fairly systematic. We shall see that any structurally stable limit cycle will either contract or expand as α in (2.3) changes monotonically. If L is a limit cycle in a rotated vector field which is semi stable then it will split into at least two cycles, one stable and on unstable respectively. In order to prove this we first present a few lemmas which clearly point in this direction.

Lemma 2.3.4. *Let L_0 be a smooth simple closed curve, parameterized by $x = \phi(t)$, $y = \psi(t)$ and suppose that L_0 is positively oriented. If on L_0 we have,*

$$H(t) = \begin{vmatrix} \phi'(t) & \psi'(t) \\ P(\phi(t), \phi(t)) & Q(\phi(t), \psi(t)) \end{vmatrix} \geq 0, \ (or \ \leq 0)$$

then as t increases, the orbits of the system

$$\frac{dx}{dt} = P(x, y), \ \frac{dy}{dt} = Q(x, y)$$

cannot move from the interior (exterior) of the region G bounded by L_0 to the exterior (or interior) of G.

Proof. We shall only consider the case outside the parenthesis, the other case being treated similarly. Let θ be the angle formed by the tangent vector at the point on L_0 and the vector field $(P(x,y), Q(x,y))$, see figure 2.4.

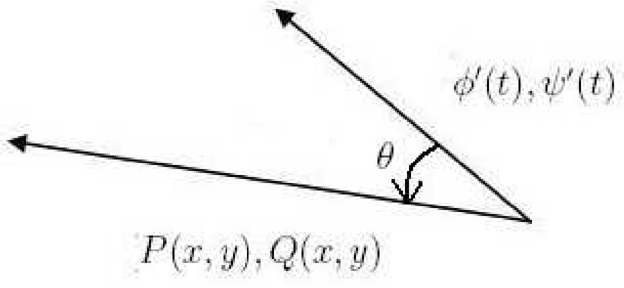


Figure 2.4:

By writing the scalar product of $(P(x,y), Q(x,y))$ and $(\phi'(t), \psi'(t))$ in two different ways we get

$$\cos\theta(t) = \frac{P(\phi(t),\psi(t))\phi'(t) + Q(\phi(t),\psi(t))\psi'(t)}{\sqrt{\phi'(t)^2+\psi'(t)^2}\sqrt{P^2(\phi(t),\psi(t))+Q^2(\phi(t),\psi(t))}}$$

We may interpret $-\theta$ as the angle formed between the vector field and L_0 in the negative direction. Since $\cos\theta = \cos(-\theta)$ and $\cos(-\theta + \frac{\pi}{2}) = \sin\theta$ we see that $\sin\theta$ is given by rotating the vector $(\phi'(t), \psi'(t))$ by $\frac{\pi}{2}$ radians in the positive direction. But this rotation of (x,y) amounts to the transformation $(x,y) \to (-y,x)$ and so we find

$$\sin\theta(t) = \frac{H(t)}{\sqrt{\phi'(t)^2+\psi'(t)^2}\sqrt{P^2(\phi(t),\psi(t))+Q^2(\phi(t),\psi(t))}}. \tag{2.3.4}$$

From (2.3.4) and $H(t) \geq 0$ we get $\sin\theta(t) \geq 0$ and so $0 \leq \theta(t) \leq \pi$. If $0 < \theta < \pi$ then the lemma is true (take into account that L_0 is positively oriented). Suppose there is some point $(\phi(t_0), \psi(t_0)) \in L_0$ with $\theta(t_0)$ equal to zero or π and such that the trajectory through $(\phi(t_0), \psi(t_0))$ passes from the interior of L_0 to the exterior. Since the solutions of our system depend continuously on the initial data there will be a neighborhood N_0 around

$(\phi(t_0), \psi(t_0))$ such that all solutions through points on $L_0 \cap N_0$ move from the interior to the exterior of L_0. This situation is however impossible. For if we assume that $\theta(t)$ is equal to zero or π for all trajectories in this neighborhood, then $L_0 \cap N_0$ will be an integral curve and so is part of a trajectory itself which clearly contradicts the assumptions that trajectories moves across $L_0 \cap N_0$. On the other hand, if $0 < \theta(t) < \pi$ for some points in $L_0 \cap N_0$ then at these points the trajectories will again point inwards. This concludes the proof. $\square$

Lemma 2.3.5. *Consider the systems*

$$\frac{dx}{dt} = P_1(x, y), \frac{dy}{dt} = Q_1(x, y) \tag{2.3.5}$$

$$\frac{dx}{dt} = P_2(x, y), \frac{dy}{dt} = Q_2(x, y) \tag{2.3.6}$$

where P_i, $Q_i \in C^0(G \subseteq R^2)$, *i=1,2. Suppose that for* $(x, y) \in G$

$$\begin{vmatrix} P_1(x, y) & Q_1(x, y) \\ P_2(x, y) & Q_2(x, y) \end{vmatrix}$$

does not change sign, then the closed orbits of (2.3.5), and (2.3.6) either coincide or do not intersect.

Proof. Let L_i:$x_i = \phi_i(t)$, $y_i = \psi_i(t)$, $i = 1, 2$, be closed orbits of (2.3.5) and (2.3.6) respectively. Without loss of generality we may assume that L_1 is positively oriented. From system (2.3.5) we get that

$$\phi_1'(t) = P_1(\phi_1(t), \psi_1(t)),\ \psi_1'(t) = Q_1(\phi_1(t), \psi_1(t))\ .$$

Since the determinant in the lemma never changes sign we immediately get that

$$\begin{vmatrix} \phi_1'(t) & \psi_1'(t) \\ P_2(\phi_1(t), \psi_1(t)) & Q_2(\phi_1(t), \psi_1(t)) \end{vmatrix} \tag{2.3.7}$$

never changes sign. From the previous lemma we deduce that L_1 and L_2 cannot intersect. If L_1 and L_2 are tangential at some point but do not coincide we will get a contradiction with the previous lemma. Suppose for instance that L_1 lies inside of L_2. The trajectories of system (2.3.6) depend continuously on the initial data and so we will get trajectories passing from the exterior to the interior of L_1 and then passing to the exterior again (see fig. 2.5). All other cases (L_2 lies inside L_1 and L_1 lies beside L_2) are treated similarly. This means that L_1 and L_2 either coincide or never intersect.

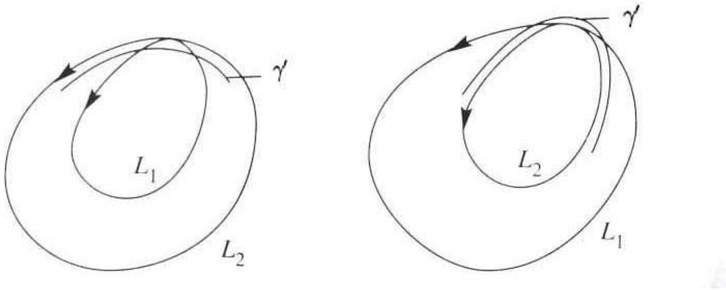


Figure 2.5:

□

Lemma 3.3.5 basically says that if we have two limit cycles in two different systems corresponding to two different values of the parameter α then one of them must surround the other. We now only need to show that the outer (or inner) cycle always corresponds to increasing values of α in order to conclude that the limit cycles contract and expand monotonically with monotonic changes of α.

Theorem 2.3.6. *Let $(P(x, y, \alpha), Q(x, y, \alpha))$ be generalized vector fields, satisfying inequality (2.3.2) for the case outside the parenthesis. Suppose that for $\alpha = \alpha_0$, L_{α_0} is an externally stable limit cycle for the system (2.3.1), in the positive (or negative) orientation. Then the following holds:*

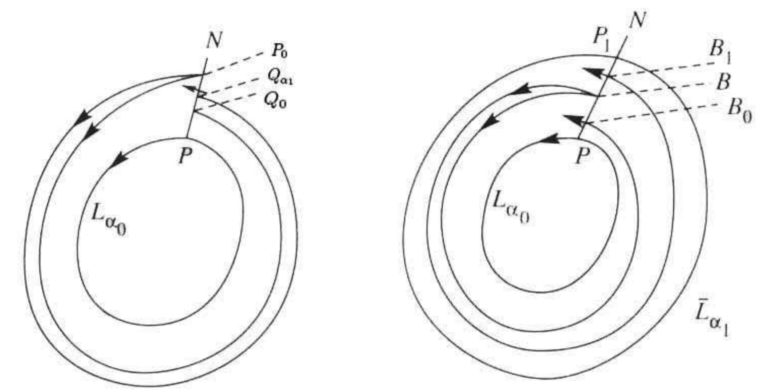
1) For any arbitrarily small $\epsilon > 0$, there exists $\alpha_1 < \alpha_0$ (or $\alpha_0 < \alpha_1$) such that for any $\alpha \in (\alpha_1, \alpha_0)$ (or $\alpha \in (\alpha_0, \alpha_1)$), there is at least one externally stable limit cycle L_α and one internally stable limit cycle L'_α for the system (2.3.1) in an exterior $\epsilon -$ neighborhood of L_{α_0}. (Here, L_α may coincide with L'_α).

2) There is an exterior $\delta -$ neighborhood of L_{α_0}, (with $\delta \leq \epsilon$), such that the neighborhood is filled with closed orbits $\{L_\alpha\}$ of the system, $\alpha \in (\alpha_1, \alpha_0)$ (or $\alpha \in (\alpha_0, \alpha_1)$).

3) When $\alpha > \alpha_0$ (or $\alpha < \alpha_0$), there are no closed orbits of the system in the exterior $\delta -$ neighborhood of L_{α_0}.

Proof. Suppose that L_{α_0} is positively oriented. Choose an arbitrary point P on L_{α_0} and let PN be the outer normal for L_{α_0} of P. The assumption that L_{α_0} in externally stable ensures that for sufficiently small $\epsilon > 0$ there will be no critical point or other closed orbit of the system in the $\epsilon -$ *neighborhood* of L_{α_0}. Also, we have that PN intersected with this $\epsilon - neighborhood$ forms a cross-section of the system. Further, choose some point $P_0 \in PN$ which lies close enough to P for the positive semi-orbit through P_0 to lie entirely in the $\frac{\epsilon}{2} - neighborhood$ of α_0. Let Q_0 denote the succeeding point of P_0. Clearly $Q_0 \in PP_0$. Since the solutions of system (2.3.1) depend continuously on α, we can choose a $\alpha_1 < \alpha_0$ such that $\alpha_0 - \alpha_1$ is so small that for any $\alpha \in [\alpha_1, \alpha_0]$, PP_0 will still be a cross-section for the system (2.3.1). Let Q_{α_1} be the succeeding point of P_0 with respect to the system $(2.3.1)_{\alpha_1}$. Because of the conditions in the definition of generalized vector fields we must have that $Q_{\alpha_1} \in Q_0P_0$ and that L_{α_0} will form the boundary of a Bendixson region together with the trajectory between P_0 and Q_{α_1} and the cross-section connecting P_0 and Q_{α_1} (see fig. 2.6). The Bendixson theorem now settles the first part.

We now show that there exists $\delta > 0$ such that any point in the exterior $\delta - neighborhood$ of L_{α_0} lies on a limit cycle L_α for some $\alpha \in (\alpha_1, \alpha_0)$. Let L'_{α_1} be an internally stable limit cycle which lies closest to $L_{\alpha 0}$. Since the point P was chosen arbitrarily it is enough to show that any point on the cross-section PP_1, where P_1 is the point of intersection between L'_{α_1} and PN, lies on some limit cycle L_α. Let $B \in PP_1$ be arbitrary and denote by B_0 the succeeding point of B with the respect to the system $(2.3.1)_{\alpha_0}$, and B_1 the one with the respect $(2.3.1)_{\alpha_1}$. B_0 must lie on PB and B_1 on BP_1 since L_{α_1}, L_{α_0} are both stable and lie on different sides of B. Because of continuity of the successor function on the parameter α, B will have to be a fixed point for some α as it varies from α_1 to α_0. This proves part 2) of the theorem (see fig. 2.6).



(a) (b)

Figure 2.6:

As for the third claim we just have to notice that for $\alpha > \alpha_0$ the succeeding point of P_0 will lie below Q_0. Since P_0 is chosen "arbitrarily" this holds for any point between P and P_0 (that is, if T lies on PP_0 and T_0 is the succeeding point of T with respect to α_0 then the succeeding point of T with respect to $\alpha > \alpha_0$ will lie below T_0). This finishes the proof. □

Theorem 2.3.7. *Let $(P(x, y, \alpha), Q(x, y, \alpha))$ be generalized vector fields, satisfying inequality (2.3.2) for the case outside the parenthesis. Suppose that for $\alpha = \alpha_0$, L_{α_0} is an internally stable limit cycle for the system (2.3.1), in the positive (or negative) orientation. Then the following holds:*

1) For any arbitrarily small $\epsilon > 0$, there exists $\alpha_2 > \alpha_0$ (or $\alpha_2 < \alpha_0$) such that for any $\alpha \in (\alpha_0, \alpha_2)$ (or $\alpha \in (\alpha_2, \alpha_0)$), there is at least one externally stable limit cycle L_α and one internally stable limit cycle L'_α for the system (2.3.1) in an interior $\epsilon-$neighborhood of L_{α_0}. (Here, L_α may coincide with L'_α).

2) There is an interior $\delta-$ neighborhood of L_{α_0}, (with $\delta \leq \epsilon$), such that the neighborhood is filled with closed orbits $\{L_\alpha\}$ of the system, $\alpha \in (\alpha_1, \alpha_0)$ (or $\alpha \in (\alpha_0, \alpha_1)$)

3) When $\alpha < \alpha_0$ (or $\alpha > \alpha_0$), there are no closed orbits of the system in the interior $\delta-$ neighborhood of L_{α_0}.

Proof. The proof is completely analogous to the previous theorem. □

These two theorems show how the simple limit cycles in a rotated vector field is evolving. As the parameter varies they will either contract or expand depending on the orientation of the cycle and on the direction of change for the parameter. There are analogous results for unstable cycles. For a fix orientation they will contract (expand) whenever the stable cycles expand (contract). This is how two consecutive simple limit cycles may collapse into a semistable one. The bifurcation pattern for a semistable limit cycle in a rotated vector field is described in the following theorem.

Theorem 2.3.8. *Let $(P(x, y, \alpha), Q(x, y, \alpha))$ be generalized rotated vector fields, and let L_{α_0} be a semistable limit cycle of the system. When the parameter varies in the suitable direction, L_{α_0} will bifurcate into at least one stable and one unstable limit cycle. They will lie distinctly on the inside and outside of L_{α_0}. When α varies in the opposite direction, L_{α_0} disappears.*

Proof. Assume that the case outside the parenthesis in (2.3.2) holds. We can further assume that L_{α_0} is positively oriented, externally stable and internally unstable. The two previous theorems allow us to deduce the existence of two numbers α_1, α_2, both strictly less than α_0 and such that:
1) $\alpha \in (\alpha_1, \alpha_0) \Rightarrow \exists$ two limit cycles for the system $(2.3.1)_\alpha$ outside of L_{α_0}, one internally stable and the other externally stable. The externally stable will lie outside of the internally stable.
2) $\alpha \in (\alpha_2, \alpha_0) \Rightarrow \exists$ two limit cycles for the system $(2.3.1)_\alpha$ inside of L_{α_0}, one internally unstable and the other externally unstable. The externally unstable will lie inside of the internally unstable.
By assumption our systems are analytic and so we have finitely many limit cycles between the two cycles in 1) and 2) above. In 1) this means that we must have at least one stable limit cycle outside of L_{α_0}. If the two cycles in 1) are both unstable on the inside and outside respectively, then we can deduce the existence of at least one more limit cycle between the two already mentioned. If this would not be a stable limit cycle then we would again get two limit cycles, one externally stable and the other internally stable as in 1). This process must end because of the finiteness of the number of limit cycles mentioned above and so we get the existence of a stable limit cycle outside of L_{α_0}.
Similarly from point 2) we deduce the existence of a unstable limit cycle inside of L_{α_0} for $\alpha \in (\alpha_2, \alpha_0)$. Let $\bar{\alpha} = max(\alpha_1, \alpha_2)$. We immediately see that for $\alpha \in (\bar{\alpha}, \alpha_0)$ L_{α_0} splits into at least two limit cycles, one stable and one unstable each lying on different sides of L_{α_0}. On the other hand, from the two previous theorems we get, for $\alpha > \alpha_0$ (α close to α_0) there are no limit cycles outside of L_{α_0} and no limit cycles on the inside L_{α_0}. This proves the theorem. □

This theorem is the main result of this section. We will need it in a crucial way in chapter 4 when proving Rychkov's theorem.

Chapter 3

The Liénard equation

Consider the equation

$$\ddot{x} + \mu f(x)\dot{x} + g(x) = 0 \tag{3.0.1}$$

where $f(x)$, $g(x)$ are real polynomials. This differential equation is named after the French engineer Alfred Liénard, who in the year 1928 gave general conditions for the existence of a unique periodic solution to (3.0.1). Special cases of the Liénard equation arise frequently in scientific studies. In 1927 the Dutch scientist van der Pol studied the case

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \tag{3.0.2}$$

in his investigations of electrical circuits with resistent properties that change with the current. Equation (3.0.2) is called the van der Pol equation and we will present some result concerning properties of periodic solutions in section 3.4. It is interesting to note that a study by Rayleigh, which goes back to 1877, made in connection with the theory of oscillations of a violin string can actually be transformed into the van der Pol equation. Rayleigh derived the equation

$$\ddot{y} + \mu(\frac{1}{3}(\dot{y})^3 - \dot{y}) + y = 0$$

If one differentiates this equation with the respect to t and puts $\dot{y} = x$ then (3.0.2) is obtained.

What physical interpretation can we make of the Liénard equation?
Recall that the equation

$$\ddot{x} + x = 0 \tag{3.0.3}$$

models an ideal vibrating string, i.e a frictionless harmonic motion. In the mechanical interpretation x represent position, $\dot{x}$ velocity and $\ddot{x}$ represent acceleration. In order to solve equation (3.0.3) we introduce the change of variable $\dot{x}(t) = y(x)$. Thus $y(x)$ represents velocity as a function of position. Equation (3.0.3) is now transformed into

$$y\frac{dy}{dx} + x = 0$$

from which we get

$$ydy = -xdx$$

integrating both sides yields

$$\frac{x^2}{2} + \frac{y^2}{2} = C \tag{3.0.4}$$

where C is an arbitrary (positive) constant. We thus see that the solutions of (3.0.3) are concentric circles around the origin in the phase plane. The physical meaning of (3.0.4) is that energy is conserved during the motion, $\frac{y^2}{2}$ represents kinetic energy and $\frac{x^2}{2}$ represents potential energy. More generally, we may have $g(x)$ instead of x in (3.0.3) where $xg(x) > 0$ for all nonzero x. As before we get the solution

$$\frac{y^2}{2} + G(x) = C \tag{3.0.5}$$

where $G(x) = \int_0^x g(s)ds$. Note that $G(x) \geq 0$ (since $xg(x) > 0$) and so even though the curves (3.0.5) generally are not circles they are still closed and "concentric" around the origin. It is natural to introduce the function

$$E(x, y) = \frac{x^2}{2} + G(x)$$

as a kind of energy function for the Liénard equation. For the sake of simplicity we will henceforth put $g(x) = x$ and so from now on we define

$$E(x, y) := \frac{y^2}{2} + \frac{x^2}{2} \ .$$

The equation (3.0.3) models an ideal motion where no energy is lost. If one thinks of this motion as a vibrating spring one may want to take in account the frictional force inherent in the spring itself. The mathematical model then becomes

$$\ddot{x} + \mu\dot{x} + x = 0$$

where μ is the so called spring constant. If $\mu > 0$ then the term $\mu\dot{x}$ will have a damping effect on the motion of the spring and so its oscillations will be of smaller and smaller amplitudes. This of course corresponds to the origin being stable in the phase-plane. Since the equation is linear we can apply the theory presented in chapter 1. The eigenvalues are found to be

$$\lambda = -\frac{\mu}{2} \pm \sqrt{\frac{\mu^2 - 4}{4}} \ .$$

The real part is always negative which confirms the stability of the origin. In the Liénard equation

$$\ddot{x} + \mu f(x)\dot{x} + x = 0$$

the frictional term will generally depend directly on the position of the "particle" in motion and not be linear. The effect of this nonlinear term is in some respect dependent upon the size of μ. With the *weakly nonlinear regime* we will refer to the values $\mu \approx 0$ and with the *strongly nonlinear regime* we will refer to the values $|\mu| \approx \infty$.
We make the same change of variables as before, obtaining

$$y\frac{dy}{dx} + x = -\mu f(x)\frac{dx}{dt}$$

multiplying both sides by dx yields

$$ydy + xdx = -\mu f(x)\frac{(dx)^2}{dt} = -\mu f(x)\frac{(dx)^2}{(dt)^2}dt$$

and so integration gives

$$E(x,y) = C - \int_0^t \mu f(x) y^2 ds \ .$$

Interpreting $E(x,y)$ as the energy of the motion we see that if $f(x) > 0$ for all x energy will be lost and the motion will slow down. If $f(x) < 0$ for all x the motion will gain energy and so it will accelerate positively. If, on other hand, $f(x)$ is not of constant sign then we may observe oscillatory phenomena that is the so called self-excited oscillations.

3.1 Existence of limit cycles

From now on we assume that $f(x)$ in (3.0.1) is an even polynomial with leading coefficient 1 and henceforth we will refer to

$$\ddot{x} + \mu f(x)\dot{x} + x = 0 \tag{3.1.1}$$

as the Liénard equation. Let us make a few observations about this equation. If we go backwards in time, making the change of time variable $t \to -t$, (3.1.1) will be transformed to

$$\ddot{x} - \mu f(x)\dot{x} + x = 0$$

since $\dot{x}(-t) = -\dot{x}(t)$ and $\ddot{x}(-t) = \ddot{x}(t)$. In other words, the effect of going backwards in time is the same as changing the sign on μ and so we can restrict our attention to $\mu > 0$ without any loss of generality. The canonical transformation of (3.1.1) yields the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - \mu f(x)y. \end{cases} \tag{3.1.2}$$

Recall that a critical point is given by the condition $\dot{x} = 0$ and $\dot{y} = 0$. By inspection we find $y = 0$ and $x = 0$ at such a point and so the origin is the only equilibrium for the Liénard system. The linearized system is given by

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - \mu c y \end{cases} \tag{3.1.3}$$

where c is the constant term in $f(x)$. To study the stability of the origin we seek the eigenvalues of (3.1.3).

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\mu c - \lambda \end{vmatrix} = \lambda^2 + \mu c \lambda + 1 = 0$$

$$\lambda = -\frac{\mu c}{2} \pm \sqrt{\frac{\mu^2 c^2}{4} - 1}\ .$$

We immediately get: $c > 0 \Rightarrow$ the origin is asymptotically stable. The converse is not true. However, if $f(x)$ is strictly positive in a small interval $0 < x < \epsilon$ then it is still true that the origin is stable, as will be seen later on.

Another interesting observation is that since $f(x)$ is an even polynomial the transformation $(x, y) \to (-x, -y)$ will leave (3.1.2) invariant, i.e $\frac{dy}{dx} = \frac{d(-y)}{d(-x)} = \frac{-x-\mu f(x)y}{y}$. This means that if $(x(t), y(t))$ is a solution to the system (3.1.2) then so is $(-x(t), -y(t))$. This is an important symmetry. Let us now make a change of variables to polar coordinates, putting $x(t) = r(t)\cos\theta(t)$ and $y(t) = r(t)\sin\theta(t)$. The equations for the polar coordinates from section 1.1 becomes

$$\begin{cases} \dot{\theta} = -1 + \mu f(r\cos\theta)\sin\theta\cos\theta \\ \dot{r} = -\mu f(r\cos\theta)\sin^2\theta \ . \end{cases} \tag{3.1.4}$$

The equation for $\dot{\theta}$ contains the following information:

$$\forall R > 0, \ \exists \mu > 0 \ \textit{such that} \ r < R \Rightarrow \dot{\theta} < 0 \ .$$

In other words, for a given small $\mu > 0$, $\dot{\theta}$ will be strictly decreasing within some neighborhood around the origin which forces the solutions to spiral in the negative direction around the origin, either inwards or outwards depending on whether the origin is stable or unstable. On the other hand, when μ is large the trajectories of (3.1.2) will be much more irregular.
From the expression for $\dot{r}$ we make the following observation. If we again let c be the constant term of $f(x)$ then $c \neq 0$ implies that for small values of r, $\dot{r}$ must be strictly decreasing or increasing depending on the sign of c. This is a different way of observing how the stability of the origin is connected with $f(x)$.

Let us introduce a change of variables[1]. Notice that the Liénard equation can be written as

$$\frac{d}{dt}\left(\dot{x} + F(x)\right) + x = 0$$

where $F(x) = \int_0^x f(t)dt$. This suggests the change of variables

$$y = \dot{x} + F(x). \tag{3.1.5}$$

The corresponding system becomes

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -x \ . \end{cases} \tag{3.1.6}$$

We call the change of variables (3.1.5) the *Liénard transformation* and the corresponding system (3.1.6) the *Liénard system*. The systems (3.1.2) and (3.1.6) are equivalent in the sense that the transformation $(x, y) \rightarrow (x, y + F(x))$ is a homeomorphism. Also we notice that system (3.1.6) is still invariant under the transformation $(x, y) \rightarrow (-x, -y)$. It is much more convenient to work with the Liénard system than with the canonical system (3.1.2). However, the price for this mathematical convenience is the loss of our earlier physical interpretation. Let us make an analysis of the

[1]In the following investigation the value of μ is irrelevant and so put $\mu = 1$ for convenience until we explicitly state otherwise.

phase-plane for the Liénard system. In a given point (x, y) the slope is given by

$$\frac{dy}{dx} = \frac{-x}{y - F(x)} \ .$$

The trajectories for the Liénard system will thus have a horizontal slope only when crossing the y-axis and a vertical slope only when crossing the curve $y = F(x)$. If we also look at the slope of the trajectories in the four different regions

$$\Omega_1 = \{(x, y) \in \mathbb{R}^2 | x > 0, \ y > F(x)\}$$
$$\Omega_2 = \{(x, y) \in \mathbb{R}^2 | x > 0, \ y < F(x)\}$$
$$\Omega_3 = \{(x, y) \in \mathbb{R}^2 | x < 0, \ y < F(x)\}$$
$$\Omega_4 = \{(x, y) \in \mathbb{R}^2 | x < 0, \ y > F(x)\}$$

we see that in Ω_1 and Ω_3 the slope is negative, while in Ω_2 and Ω_4 it is positive. Also notice that $\dot{x} > 0$ ($\dot{x} < 0$) when $y > F(x)$ ($y < F(x)$). From all this we conclude that in the phase-plane for the Liénard system the trajectories always move spirally around the origin and there are no irregularities of the type which appeared in the phase-plane for system (3.1.2) when μ is large. Thus the Liénard transformation "smoothens out" the trajectories. This simplifies the geometrical reasoning a great deal.

Let us suppose that there exists a limit cycle for some Liénard system. By the above remarks it can only cross the y-axis twice. Let $(0, y_+)$ and $(0, y_-)$ be these two points, $(0, y_+)$ lying above the origin. Due to the symmetry in the Liénard system we must have $|y_+| = |y_-|$. If not, then by symmetry we would have another limit cycle crossing the y-axis at the points $\bar{y}_+ = |y_-|$ and $\bar{y}_- = -y_+$. But this clearly implies that these two limit cycles intersect, which is impossible by uniqueness of solutions. On the other hand, if a trajectory crosses the y-axis in two different points $(0, y_+)$, $(0, y_-)$ such that $|y_+| = |y_-|$ then this trajectory will be a limit cycle. This follows also due to symmetry, since the arc trajectory in the half plane $\{x \geq 0\}$ connecting the two points on the y-axis need only be reflected in the origin in order to exhibit the full trajectory which is readily seen to be closed.
Let us consider the energy function $E(x, y)$ along any arc trajectory in the half plane $\{x \geq 0\}$. Notice that $E(x, y)$ is actually a potential and so the integral $\int_\gamma dE$ only depends on the starting point and end point of γ. If γ denotes the half-trajectory which goes from $(0, y_+)$ to $(0, y_-)$ we have that

$$\int_\gamma dE = \frac{1}{2}\left(y_-^2 - y_+^2\right) . \tag{3.1.7}$$

In other words, a necessary and sufficient condition for the half trajectory γ to be part of a limit cycle is that

$$\int_\gamma dE = 0. \tag{3.1.8}$$

We can interpret this as a conservation of energy result for limit cycles in the Liénard system. Notice that

$$\frac{dE}{dt}(x,y) = x\dot{x} + y\dot{y} = xy - xF(x) - xy = -xF(x)$$

and so

$$dE = -xF(x)dt\ .$$

Suppose that $f(x)$ is strictly negative for some small interval $0 < x < \epsilon$. Then we must have $F(x) < 0$ in this interval. If we let γ be an arc trajectory in the half plane $\{x \geq 0\}$ which crosses the x-axis within the interval $0 < x < \epsilon$ then by the analysis made above γ will be contained in the strip $0 \leq x < \epsilon$. The integral

$$\int_\gamma dE = \int_\gamma -xF(x)dt = \int_\gamma F(x)dy$$

will then necessarily be positive since both $F(x)$ and dy are negative along γ. But this implies that $|y_+| < |y_-|$ and we may conclude that the origin is unstable. The same reasoning shows that the origin is stable if there exists an interval $0 < x < \epsilon$ where $f(x)$ is strictly positive, as was claimed earlier. Another important observation is that any trajectory which crosses the y-axis far away from the origin must spiral inward towards the origin as time increases. To see this we can again change variables to polar coordinates and analyze the sign of $\dot{r}$. The expression for $\dot{r}$ in the Liénard system is

$$\dot{r} = -\mu F(r\cos\theta)\cos\theta\ .$$

Since the leading term of $F(r\cos\theta)\cos\theta$ has the form $\frac{r^{2n+1}}{2n+1}(\cos\theta)^{2n+2}$ it is always ≥ 0 and so $\dot{r}$ must be negative for large enough r. This means that we can always find a circular region D_R around the origin such that the vector field points inwards along the boundary ∂D_R (with the two points on the y-axis as the only exception where the vector field is tangential to the circular boundary). This gives us the possibility of using the Bendixson theorem to deduce the existence of a limit cycle when the origin is asymptotically unstable.

To sum up:

Theorem 3.1.1. *If there exists an interval $I : 0 < x < \epsilon$ such that $f(x)$ is negative on I then the Liénard equation will possess at least 1 periodic solution.*

3.2 Uniqueness of limit cycles

Under which general conditions can we deduce the existence of a unique limit cycle? This question has been extensively studied and there are many theorems in the literature which gives a positive answer.

Theorem 3.2.1. *(Liénard's Theorem, see Theorem 4.2 in [3], pp.221-223) Consider the Liénard system:*

$$\begin{cases} \frac{dx}{dt} = y - F(x) \\ \frac{dy}{dt} = -x. \end{cases}$$

If $F(x)$ satisfies the following conditions, then the system has a unique stable limit cycle.
1) *There exists a unique $a > 0$ such that $F(a) = F(-a) = 0$ and $F(x) < 0$ for $0 < x < a$,*
2) *$F(x)$ is odd and $F(\infty) = \infty$,*
3) *$F(x)$ is monotone increasing for $x \geq a$.*

Proof. A necessary and sufficient condition for the existence of a limit cycle is that

$$\int_\gamma dE = \frac{1}{2}\left(y_-^2 - y_+^2\right) = 0 \tag{3.2.1}$$

along some "half" trajectory γ. But $dE = F(x)dy$ and $F(x) < 0$ when $0 < x < a$ and so there exists some circular region around the origin which is negatively invariant. By the reasoning of section 4.1 we may deduce the existence of at least one limit cycle and also conclude that any limit cycle L will cross the lines $|x| = a$. We now show uniqueness. Recall that along any limit cycle L we have

$$\int_L dE = 0 \tag{3.2.2}$$

and

$$\int_L dE = \int_L -xF(x)dt = \int_L F(x)dy\ .$$

Assuming the existence of two limit cycles, $L_1 \subset L_2$, we will show that

$$\int_{L_1} F(x)dy > \int_{L_2} F(x)dy \tag{3.2.3}$$

which together with (3.2.2) this will imply uniqueness.
Divide L_1 and L_2 into segments according to figure 3.1.

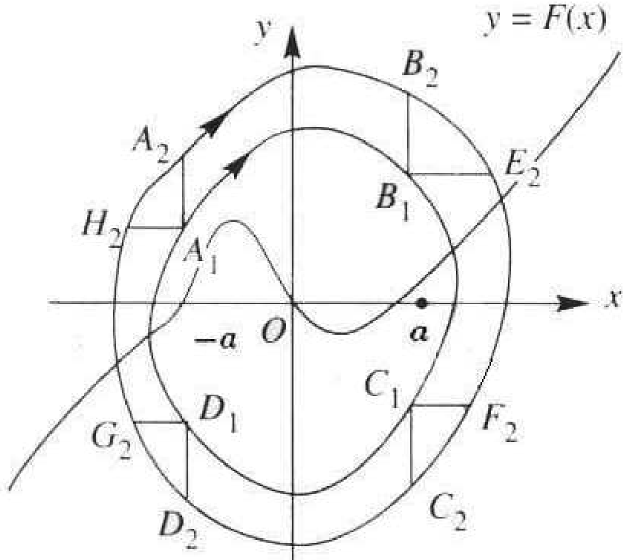


Figure 3.1:

Notice that:

$$\int_{A_2B_2} F(x)dy < \int_{A_1B_1} F(x)dy. \tag{3.2.4}$$

Indeed, let the segments A_1B_1, A_2B_2 be parameterized by $y = y_1(x)$ respectively $y = y_2(x)$. We have

$$\int_{A_iB_i} F(x)dy = \int_{A_iB_i} -xF(x)dt = \int_{-a}^{a} \frac{-xF(x)}{y_i(x) - F(x)}dx$$

so

$$\int_{A_1B_1} F(x)dy - \int_{A_2B_2} F(x)dy = \int_{-a}^{a} \left(\frac{-xF(x)}{y_1(x) - F(x)} - \frac{-xF(x)}{y_2(x) - F(x)} \right) dx =$$

$$= \int_{-a}^{a} \frac{-xF(x)(y_2(x) - y_1(x))}{(y_2(x) - F(x))(y_1(x) - F(x))}dx \ .$$

Since $y_2 - y_1 > 0$ and $y_i - F(x) > 0$, $i = 1, 2$ and since $-xF(x) > 0$ when $-a < x < a$ we get

$$\int_{-a}^{a} \frac{-xF(x)(y_2(x) - y_1(x))}{(y_2(x) - F(x))(y_1(x) - F(x))}dx > 0$$

so (3.2.4) holds.
On the other hand, we see that

$$\int_{C_1D_1} F(x)dy > \int_{C_2D_2} F(x)dy \tag{3.2.5}$$

since if we this time let $y_1(x)$ respectively $y_2(x)$ represent C_1D_1 respectively C_2D_2 we get

$$\int_{C_1D_1} F(x)dy - \int_{C_2D_2} F(x)dy = \int_a^{-a} \frac{-xF(x)(y_2(x) - y_2(x))}{(y_2(x) - F(x))(y_1(x) - F(x))}dx =$$
$$- \int_{-a}^{a} \frac{-xF(x)(y_2(x) - y_2(x))}{(y_2(x) - F(x))(y_1(x) - F(x))}dx \ .$$

From $y_i - F(x) < 0$, $-xF(x) > 0$ and $y_2 - y_1 < 0$ (3.2.5) is implied. We proceed by showing

$$\int_{y_1}^{y_2} F(x_1(y))dy > \int_{y_1}^{y_2} F(x_2(y))dy. \tag{3.2.6}$$

Let $x_1(y)$ and $x_2(y)$ represent the segments B_1C_1 and E_2F_2 and put $y_1 = y_{B_1} = y_{E_2}$, $y_2 = y_{C_1} = y_{F_2}$. In this representation we get

$$\int_{B_1C_1} F(x)dy - \int_{E_2F_2} F(x)dy = \int_{y_1}^{y_2} (F(x_1(y)) - F(x_2(y)))dy \ .$$

Noticing that $y_1 \leq y \leq y_2 \Rightarrow x_1(y) > x_2(y) \geq a$ and that $F(x)$ is strictly increasing for $x \geq a$ we see that

$$\int_{y_1}^{y_2} (F(x_1(y)) - F(x_2(y)))dy > 0 \tag{3.2.7}$$

which is equivalent to (3.2.6). For the remaining segments on L_2 we simply notice that $-xF(x) < 0$, which of course implies that

$$\int_{B_2E_2 \cup F_2C_2 \cup D_2G_2 \cup H_2A_2} -xF(x)dt < 0. \tag{3.2.8}$$

(3.2.4) – (3.2.8) implies (3.2.3) and we are done.

□

The method of proof is very interesting. It relies on the fact that the functional $I(y) = \int_{\gamma_{(0,y)}} dE$ is strictly decreasing in $y > 0$ and that $I(y) = 0$ is a necessary and sufficient condition for $\gamma_{(0,y)}$ to be a closed trajectory. A similar approach will be used later when proving Rychkov's theorem that when $f(x) = (x^2 - a)(x^2 - b)$ there exist at most two periodic solutions. It is noteworthy that the behavior of $F(x)$ in the interval $0 < x < a$ may be quite arbitrary. On the other hand the monotonicity of $F(x)$ for $x > a$ is a necessary requirement as will be shown by example 3.3.3.

3.3 Number and nature of limit cycles when $\mu \to 0$

We will now present an investigation on the number of limit cycles for the Liénard equation when $0 < \mu \ll 1$ (i.e. when μ is very close to zero). This beautiful result was obtained by Poincaré and leads straight to conjecture that for all polynomial of degree $2n$ or $2n+1$ there can be at most n limit cycles. The investigation itself is actually much more general, not only concerning the Liénard equation, but any equation of the form $\ddot{x} + \mu f(x, \dot{x}) + x = 0$ where $f(x, \dot{x})$ is assumed to be a polynomial. We pose the problem as follows. Let $f(x, \dot{x})$ be a polynomial such that $f(x, y) = ax + by + \textit{higher order terms}$, $b \neq 0$. If $\mu = 0$, the equation $\ddot{x} + x = \mu f(x, \dot{x})$ collapses into the harmonic oscillator $\ddot{x} + x = 0$ and every solution is then periodic. These periodic solutions will be concentric circles in the phase plane, assuming we set

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -x - \mu f(x, \dot{x}). \end{cases} \tag{3.3.1}$$

Problem: *To which circles do the periodic solutions of (3.3.1) tend as $\mu \to 0$?* We are thus seeking those values of α such that a periodic solution of (3.3.1) tend to the circle $x^2 + y^2 = \alpha^2$ as $\mu \to 0$.

First a small remark about the conditions we impose on $f(x, y)$. It does not have to be a polynomial, it suffices to require it to be analytic with a Taylor series expansion in the form $f(x, y) = ax + bx + \textit{higher order terms}$ with $b \neq 0$. Why $b \neq 0$? The answer is given by inspection of the linearized model of (3.3.1) which takes the form

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -x - \mu a x - \mu b y. \end{cases}$$

We calculate the eigenvalues in order to check the stability of the system

$$\begin{vmatrix} -\lambda & 1 \\ -\mu a - 1 & -\mu b - \lambda \end{vmatrix} = \lambda^2 + \mu b \lambda + 1 + \mu a = 0$$

$$\lambda = \frac{-\mu b}{2} \pm \sqrt{\frac{\mu^2 b^2}{4} - 1 - \mu a}\ .$$

The origin, which is the only equilibrium for the system (3.3.1), will then necessarily be asymptotically stable (or unstable) as $\mu b > 0$ (or $\mu b < 0$). If we require $b \neq 0$ we thus ensure that the system (3.3.1) will be nondegenerate for $\mu > 0$ in the sense that there wont exists an infinite number of cycles around the origin when μ small.

So we consider

$$\ddot{x} + \mu f(x, \dot{x}) + x = 0. \tag{3.3.2}$$

Let $x_{\mu,\eta}(t)$ be the solution to (3.3.2) satisfying $x_{\mu,\eta}(0) = 0$, $\dot{x}_{\mu,\eta}(0) = \eta$, $\eta > 0$. In the phase plane this motion will correspond to a trajectory which passes through the point $(0, \eta)$ at $t = 0$. If this motion lies close to a periodic solution it will return to the positive y-axis in a least time $T > 0$ such that $x_{\mu,\eta}(T) = 0$, $\dot{x}_{\mu,\eta}(T) = \eta'$. If $\eta' = \eta$ then $x_{\mu,\eta}(t)$ is periodic with period T. We can write $T = 2\pi + \tau(\mu, \eta)$ where $\tau(\mu, \eta)$ is analytic and satisfies $\tau(0, \eta) = 0$ for all $\eta > 0$. Let $x(t)$ be a function satisfying

$$x(t) = \eta \sin t + \mu \int_0^t f(x, \dot{x}) \sin(t - u) du \ .$$

A simple calculation shows that

$$\dot{x} = \eta \cos t + \mu f(x, \dot{x}) \sin(t - t) + \mu \int_0^t f(x, \dot{x}) \cos(t - u) du =$$

$$= \eta \cos t + \mu \int_0^t f(x, \dot{x}) \cos(t - u) du \ ,$$

and that

$$\ddot{x} = -\eta \sin t + \mu f(x, \dot{x}) \cos(t - t) - \mu \int_0^t f(x, \dot{x}) \sin(t - u) du = -x - \mu f(x, \dot{x}) \ .$$

Thus $x(t)$ must satisfy (3.3.2). Moreover, since $x(0) = 0$ and $\dot{x}(0) = \eta$ we conclude from uniqueness of solutions that $x(t) = x_{\mu,\eta}(t)$. We also observe that from continuous dependence on the initial conditions, $x_{\mu,\eta}(t)$ is continuous in μ and η and since $x_{0,\eta}(t) = \eta \sin t$, $\dot{x}_{0,\eta}(t) = \eta \cos t$ we have $x_{\mu,\eta}(t) = \eta \sin t + O(\mu)$ and $\dot{x}_{\mu,\eta}(t) = \eta \cos t + O(\mu)$ for small values of μ.

From the condition $x_{\mu,\eta}(T) = 0$ we get

$$0 = \eta \sin \tau(\mu, \eta) + \mu \int_0^{2\pi+\tau} f(x, \dot{x}) \sin(\tau - u) du = H(\mu, \eta, \tau). \tag{3.3.3}$$

Notice that this defines τ implicitly as a function of μ and η since $\frac{\partial H}{\partial \tau}|_{\mu=0} \neq 0$. We may thus express τ as a power series in μ and η and since $\tau(0, \eta) = 0$ we have:

$$\tau(\eta, \mu) = \mu B_1(\eta) + \mu^2 B_2(\eta) + ... \tag{3.3.4}$$

We will need this representation later.
What is the first approximation of τ in terms of small values of μ? We have already observed that

$$0 = \eta \sin \tau + \mu \int_0^{2\pi+\tau} f(x, \dot{x}) \sin(\tau - u) du \ .$$

we can split the integration interval and write

$$0 = \eta \sin \tau + \mu \int_0^{2\pi} f(x, \dot{x}) \sin(\tau - u) du + \mu \int_{2\pi}^{2\pi+\tau} f(x, \dot{x}) \sin(\tau - u) du \ .$$

The last term is of order $O(\tau^2)$, since the integration interval has length τ and $\sin(\tau - u)$ is of order $O(\tau)$ in that interval, and so we have

$$0 = \eta \sin \tau + \mu \int_0^{2\pi} f(x, \dot{x}) \sin(\tau - u) du + O(\tau^2) \ .$$

We can also expand $\sin(\tau - u) = -\sin u + O(\tau)$ in the integral above. This yields

$$0 = \eta \sin \tau - \mu \int_0^{2\pi} f(x, \dot{x}) \sin u du + \mu O(\tau) + \mu O(\tau^2) \ .$$

If we now expand $f(x(u), \dot{x}(u))$, using the fact that $x(u) = \eta \sin u + O(\mu)$ and $\dot{x} = \eta \cos u + O(\mu)$, we see that

$$0 = \eta \sin \tau - \mu \int_0^{2\pi} f(\eta \sin u, \eta \cos u) \sin u du + O(\mu^2) + \mu O(\tau) + \mu O(\tau^2) \ .$$

However, τ is of order $O(\mu)$ and so $\mu O(\tau) = O(\mu^2)$. Also, expanding $\sin \tau = \tau + O(\tau^3)$ we finally get

$$\eta \tau = \mu \int_0^{2\pi} f(\eta \sin u, \eta \cos u) \sin u du + O(\mu^2) \ .$$

In other words, the first approximation of τ is given by

$$\tau = \frac{\mu}{\eta} \int_0^{2\pi} f(\eta \sin u, \eta \cos u) \sin u du. \tag{3.3.5}$$

This is by itself quite a remarkable result, notice that this formula is valid for any $\eta > 0$. We will penetrate a bit deeper and see how we can find values for η such that $\dot{x}_{\mu,\eta}(2\pi+\tau(\mu,\eta)) = \eta$, i.e $x_{\mu,\eta}(t)$ becomes a periodic solution. The necessary and sufficient condition $\dot{x}_{\mu,\eta}(2\pi+\tau(\mu,\eta)) = \eta$ can be written as

$$\eta = \eta\cos\tau + \mu\int_0^{2\pi+\tau} f(x,\dot{x})\cos(\tau-u)du \ .$$

We proceed as before splitting the integration interval and write

$$\eta = \eta\cos\tau + \mu\int_0^{2\pi} f(x,\dot{x})\cos(\tau-u)du + O(\mu^2) \ .$$

Expanding $\cos\tau = 1 - \frac{\tau^2}{2} + O(\tau^4)$ and rearranging yields

$$\frac{\eta}{2}\tau^2 + O(\tau^4) = \mu\int_0^{2\pi} f(x,\dot{x})\cos(\tau-u)du + O(\mu^2) \ .$$

Further expanding $f(x,\dot{x})$ using $x(u) = \eta\sin u + O(\mu)$ and $\dot{x} = \eta\cos u + O(\mu)$ we see that

$$\frac{\eta}{2}\tau^2 + O(\tau^4) = \mu\int_0^{2\pi} f(\eta\sin u,\eta\cos u)\cos(\tau-u)du + O(\mu^2). \qquad (3.3.6)$$

Substituting the expression for τ given by (3.3.5) into (3.3.6) and dividing by μ we get the following

$$\frac{\eta}{2}\mu B_1(\eta)^2 + O(\tau^3) = \int_0^{2\pi} f(\eta\sin u,\eta\cos u)\cos(\tau-u)du + O(\mu) \ .$$

Letting $\mu \to 0$ and noticing that $\tau(\mu,\eta) \to 0$ we can deduce

$$0 = \int_0^{2\pi} f(\eta\sin u,\eta\cos u)\cos u du. \qquad (3.3.7)$$

The calculations above show that a necessary condition for a limit cycle to tend to the circle $x^2+y^2=\alpha^2$ is that $\eta=\alpha$ is a root of (3.3.7). We are thus led to study the equation

$$0 = \Phi(\alpha) = \int_0^{2\pi} f(\alpha\sin u,\alpha\cos u)\cos u du. \qquad (3.3.8)$$

Notice that since $f(x,\dot{x})$ is a polynomial in x and $\dot{x}$, $\Phi(\alpha)$ will be a polynomial in α. The integration is made over the full period 2π of the trigonometric functions, so all odd terms of the trigonometric functions in $f(\alpha\sin u,\alpha\cos u)\cos u$

will be canceled out. The degree of the trigonometric functions in the terms of $f(\alpha \sin u, \alpha \cos u) \cos u$ is of degree 1 higher than the degree of α, and so $\Phi(\alpha)$ will be an odd polynomial satisfying $\Phi(0) = 0$. This means that if α is a root of $\Phi(\alpha)$ then so is $-\alpha$. This is only natural since the solution $x_{0,\alpha,0}(t)$ is actually represented by the circle $x^2 + y^2 = \alpha^2$ in the phase plane.
Finally, if we make the assumption that α is a simple root of $\Phi(\alpha)$, i.e if $0 \neq \Phi'(\alpha)$, then the implicit function theorem will imply the existence of a unique function $\eta(\mu)$ such that $x_{\mu,\eta}(t)$ is a periodic function for small μ. This follows from the following calculation.

Let

$$K(\eta, \mu, \tau) = \eta(\cos \tau - 1) + \mu \int_0^{2\pi+\tau} f(x, y) \cos(\tau - u) du \ .$$

We may think of the function K as the difference between the starting point $(0, \eta)$ and its return point $(0, \eta_1)$ on the y-axis. The condition $K(\eta, \mu, \tau) = 0$ is equivalent to having a closed trajectory through $(0, \eta)$. Since all trajectories are closed when $\mu = 0$ we may write $K(\eta, \mu, \tau) = \mu K_1(\eta, \tau)$ where $K_1(\eta, \tau)$ is some analytic function[2]. Suppose η is a value close to a positive root α of $\Phi(\alpha)$. We can then express $K_1(\eta, \tau)$ as a double power series in $(\eta - \alpha)$ and μ

$$K_1(\eta, \mu) = (\eta - \alpha)A(\alpha) + \mu B(\alpha) + \dots \tag{3.3.9}$$

We will now show that $A(\alpha) = \Phi'(\alpha)$. We have that

$$A(\alpha) = \left(\frac{\partial K_1}{\partial \eta}\right)_{(\alpha,0)} = \left(\frac{1}{\mu}\frac{\partial K(\eta, \mu, \tau(\eta, \mu))}{\partial \eta}\right)_{(\alpha,0)} .$$

The chain rule yields

$$\frac{\partial K(\eta, \mu, \tau(\eta, \mu))}{\partial \eta} = \frac{\partial K(\eta, \mu, \tau)}{\partial \eta} + \frac{\partial K(\eta, \mu, \tau)}{\partial \tau}\frac{\partial \tau}{\partial \eta} .$$

We divide the calculation for a more comprehensive presentation.

$$\frac{\partial K(\eta, \mu, \tau)}{\partial \eta} = \cos \tau - 1 + \mu \int_0^{2\pi+\tau} \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta}\right) \cos(\tau - u) du$$

$$\frac{\partial K(\eta, \mu, \tau)}{\partial \tau}\frac{\partial \tau}{\partial \eta} = -\eta \sin \tau \frac{\partial \tau}{\partial \eta} + \mu \frac{\partial \tau}{\partial \eta}[f(x(t), y(t))]_{t=2\pi+\tau} +$$

[2] Of course τ is considered as a function of μ and η.

$$\mu \int_0^{2\pi+\tau} \left[\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial \tau} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \tau} \right) \frac{\partial \tau}{\partial \eta} \cos(\tau - u) - f(x,y) sin(\tau - u) \frac{\partial \tau}{\partial \eta} \right] du \ .$$

From (3.3.4) we have that $\frac{\partial \tau}{\partial \eta}$ evaluated at $\mu = 0$ equals zero and so we find

$$A(\alpha) = \frac{1}{\mu} \left(\frac{\partial K(\eta, \mu, \tau(\eta, \mu))}{\partial \eta} \right)_{(\alpha,0)} = \int_0^{2\pi} \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} \right) \cos u du \ .$$

But $\int_0^{2\pi} \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} \right) \cos u du = \Phi'(\alpha)$ and so we have shown that $A(\alpha) = \Phi'(\alpha)$. This means that if α is a simple root of $\Phi(\alpha)$, i.e $\Phi'(\alpha) \neq 0$, then we may use the implicit function theorem to deduce the existence of a unique analytic function $\eta(\mu)$ satisfying $K_1(\eta(\mu), \mu) = 0$ and $\eta(0) = \alpha$. We thus obtain a function satisfying $K(\eta(\mu), \mu, \tau(\eta, \mu)) = 0$. In other words, if α is a simple root of $\Phi(\alpha)$ then there are unique functions $\eta(\mu)$, $\tau(\eta(\mu), \mu)$ satisfying $x_{\eta,\mu}(2\pi + \tau) = x_{\eta,\mu}(0) = 0$, $\dot{x}_{\eta,\mu}(2\pi + \tau) = \dot{x}_{\eta,\mu}(0) = \eta$ in a small open interval around $\mu = 0$. This shows that if α is simple then there exists one and only one periodic solution $x_{\eta,\mu}(t)$ (η close to α) for every small enough $\mu > 0$.

We end this discussion by showing how the stability of the periodic solutions depends on $\Phi'(\alpha)$. Consider $K(\eta, \mu, \tau)$ and view it as the difference between a starting point $(0, \eta)$ and its return point $(0, \eta_1)$ on the y-axis. We assume that α is a simple root, η is close to α and that μ is *very* small. To be precise we assume that $0 < \mu < (\eta - \alpha)^2$. From (3.3.9) we see that we may express $K(\eta, \mu, \tau(\eta, \mu))$ as

$$K(\eta, \mu, \tau(\eta, \mu)) = \mu K_1(\eta, \mu) = \mu\big((\eta - \alpha)P(\eta) + \mu Q(\eta, \mu)\big) \ .$$

$P(\eta)$ is obtained by collecting all the terms in (3.3.9) which contains only powers of $(\eta - \alpha)$, and so we have $P(\alpha) = A(\alpha) = \Phi'(\alpha) \neq 0$. By the assumption that the order of μ is lower than the order of $(\eta - \alpha)$ and from the fact that for $(\eta - \alpha)$ small enough the sign of $P(\eta)$ is equal to that of $P(\alpha)$ we conclude that the sign of $(\eta - \alpha)P(\eta) + \mu Q(\eta, \mu)$ is the same as the sign of $(\eta - \alpha)P(\alpha)$. Assume now that $P(\alpha) = \Phi'(\alpha) > 0$. If $\eta > \alpha$ then $K(\eta, \mu, \tau)$ will be positive and so the trajectory $\gamma_{(0,\eta)}$ will tend outwards away from the limit cycle $\gamma_{(0,\alpha)}$. If $\eta < \alpha$ then $K(\eta, \mu, \tau)$ will be negative and $\gamma_{(0,\eta)}$ will spiral inwards away from $\gamma_{(0,\alpha)}$. Thus if $\Phi'(\alpha) > 0$ the limit cycle $\gamma_{(0,\alpha)}$ will be unstable. By the same reasoning we see that if $\Phi'(\alpha) < 0$ the limit cycle will be stable.
To sum up, we have proved the following theorem:

Theorem 3.3.1. *(see Theorem 5.5 in [7], pp.313-320) The "generating" circles $C_\alpha :\ x^2+y^2 = \alpha^2$ of the limit cycles of the equation $\ddot{x}+\mu f(x, \dot{x})+x = 0$*

correspond to the roots of $\Phi(\alpha)$. If $\Phi(\alpha_0) = 0 \neq \Phi'(\alpha_0)$ then there exists exactly one limit cycle L_μ which tends to C_{α_0} as $\mu \to 0$. Further, the stability of L_μ is determined by the sign of $\Phi'(\alpha_0)$. If $\Phi'(\alpha_0) < 0$ ($\Phi'(\alpha_0) > 0$) then the limit cycles tending to C_{α_0} is stable (unstable).

We have the following interesting corollary.

Corollary 3.3.2. *The Liénard equation $\ddot{x} + \mu f(x)\dot{x} + x = 0$ has at most n simple limit cycles where $2n$ or $2n+1$ is the degree of $f(x)$, for $0 < \mu \ll 1$.*

Proof. The equation

$$0 = \Phi(\alpha) = \int_0^{2\pi} f(\alpha \sin u)\alpha \cos^2 u du$$

is polynomial and of degree at most one more than the degree of $f(x)$. Since $\Phi(\alpha)$ is an odd polynomial with $\alpha = 0$ as a root the number of positive simple zeroes can be at most n. □

Notice that $f(x)$ is an arbitrary polynomial here, we do not require it to be even. By a suitable choice of the coefficients of $f(x)$ one can construct exactly j periodic solutions, $0 < j < n$, for a small enough value on μ.

Example 3.3.3. Let

$$f(x) = \frac{1}{5}x^6 - \frac{3}{8}x^4 + \frac{299}{1600}x^2 - \frac{99}{6400} .$$

It can be shown that the primitive function of $f(x)$ only has one positive root $x = c$, see figure 3.2. It will however not be monotone increasing for all $x \geq c$. This example will serve to show how the condition of monotonicity in Liénard's theorem is a necessary condition for the uniqueness of limit cycles. We have

$$\Phi(\alpha) = \int_0^{2\pi} \alpha f(\alpha \sin\theta) \cos^2\theta d\theta$$

which is found to be equal to

$$\frac{\pi}{64}(\alpha^7 - \frac{3}{5}\alpha^5 + \frac{299}{100}\alpha^3 - \frac{99}{100}\alpha) .$$

This polynomial can be factored as

$$\frac{\pi}{64}\alpha(\alpha^2 - 1)(\alpha^2 - \frac{9}{10})(\alpha^2 - \frac{11}{10}) .$$

According to the previous results the equation $\ddot{x} + \mu f(x)\dot{x} + x = 0$ will have 3 periodic solutions for small enough μ.

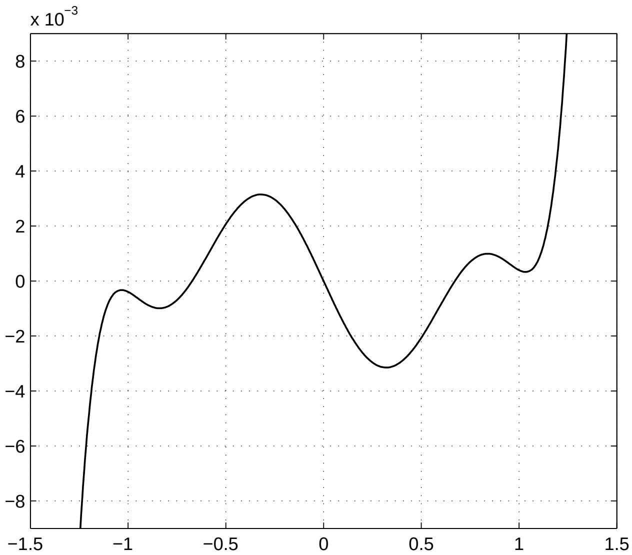


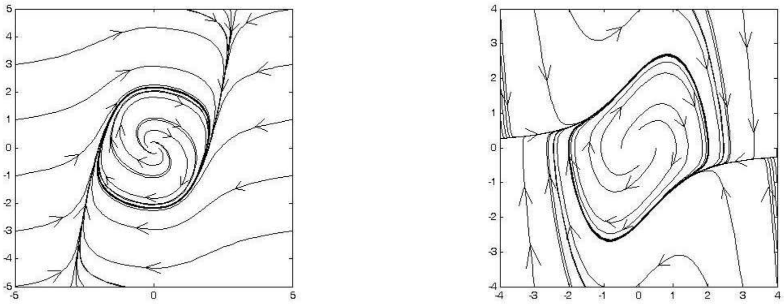
Figure 3.2: The graph of $y = \frac{1}{5}x^6 - \frac{3}{8}x^4 + \frac{299}{1600}x^2 - \frac{99}{6400}$

3.4 van der Pol's equation

We will now investigate a special case of Liénard's equation $\ddot{x}+\mu f(x)\dot{x}+x = 0$, namely the case when $f(x) = (x^2 - 1)$, $\mu > 0$. This equation is called the van der Pol equation and is considered to be important. Besides its historical importance the van der Pol equation serves as an excellent example of how one may attack the Liénard equation in general. Its equivalent Liénard system is

$$\begin{cases} \frac{dx}{dt} = y - \mu(\frac{x^3}{3} - x) \\ \frac{dy}{dt} = -x. \end{cases}$$

Notice that $F(x) = \mu(\frac{x^3}{3} - x)$ only has one positive zero at $x = \sqrt{3}$, is monotone increasing for $x > \sqrt{3}$ and negative for $0 < x < \sqrt{3}$. Obviously $F(\infty) = \infty$ and so we can use Liénard's theorem to deduce the existence of a unique stable limit cycle. This simplifies our reasoning a lot since we do not have to set out finding any cycles or try to bound the number of cycles. We simply know that any trajectory will either be the unique limit cycle, or it will spiral towards it in a nice and "smooth" fashion (the origin being excluded). We include the phase portrait of the two equivalent systems in figure 3.3 . Notice the difference in regularity with respect to $\dot{\theta}$ as observed in 4.1.



(a) The Liénard system (b) The canonical system

Figure 3.3:

We will first investigate the amplitude of this limit cycle in the strong and weak nonlinear regimes.

We begin with the case $\mu \to 0$. By Theorem 3.3.5 the amplitude is given by the roots of the polynomial

$$\Phi(\alpha) = \int_0^{2\pi} f(\alpha \sin u, \alpha \cos u) \cos u du \ .$$

Applying this to the van der Pol equation we find ourselves trying to solve

$$\Phi(\alpha) = \int_0^{2\pi} (1 - \alpha^2 \sin^2 u) \alpha \cos^2 u du = 0 \ .$$

Using the fact that $\cos^2 \alpha = \frac{1}{2} - \frac{\cos(2u)}{2}$, $\sin^2 u = 1 - \cos^2 u$ we can rewrite and split the integral as

$$\frac{1}{2}(\alpha - \alpha^3) \int_0^{2\pi} (1 + \cos 2u) du + \alpha^3 \int_0^{2\pi} \cos^4 u du \ .$$

The integral to the left is easy to solve

$$\frac{1}{2}(\alpha - \alpha^3) \int_0^{2\pi} (1 + \cos 2u) du = \frac{1}{2}(\alpha - \alpha^3) \left[u + \frac{1}{2} \sin 2u \right]_0^{2\pi} = \pi(\alpha - \alpha^3)$$

The right expression however is a bit complicated, but using Eulers formula and putting $\cos u = \frac{1}{2}(e^{ix} + e^{-ix})$ we get

$$\alpha^3 \int_0^{2\pi} \cos^4 u du = \frac{\alpha^3}{16} \int_0^{2\pi} (e^{iu} + e^{-iu})^4 du = \frac{\alpha^3}{16} \int_0^{2\pi} (e^{4iu} + 4e^{2iu} + 6 + 4e^{-2iu} + e^{-4iu}) du =$$

$$= \frac{\alpha^3}{16}\left[\frac{1}{4i}(e^{4iu} + \frac{2}{i}2iu + 6u - \frac{2}{i}e^{-2iu} - \frac{1}{4i}e^{-4iu})du\right]_0^{2\pi} =$$

$$= \alpha^3\left[\frac{1}{32}\sin 4u + \frac{1}{4}\sin 2u + \frac{3}{8}u\right]_0^{2\pi} = \frac{3\pi\alpha^3}{4} \ .$$

We conclude

$$\Phi(\alpha) = \pi\alpha(1 - \frac{\alpha^2}{4}) \ .$$

The equation $\Phi(\alpha) = 0$ has only one positive root at $\alpha = 2$. We also see that $\Phi'(\alpha) = \pi\left(1 - \frac{3\alpha^2}{4}\right)$. Since $\Phi'(2) = -2\pi < 0$ we have again confirmed that the limit cycle is stable. The amplitude of the limit thus tends to 2 as $\mu \to 0$.

Now let us turn to the case when $\mu \to \infty$.
We will use the theorem of Bendixson to show that the limit cycle tends to the closed curve shown in figure 3.4. For this purpose we first make a small change of variables for the parameter μ, setting $\epsilon = \frac{1}{\mu}$. The equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

takes the form

$$\epsilon\ddot{x} + (x^2 - 1)\dot{x} + \epsilon x = 0$$

after multiplying with ϵ. The Liénard system then becomes

$$\begin{cases} \epsilon\frac{dx}{dt} = y - (\frac{x^3}{3} - x) \\ \frac{dy}{dt} = -\epsilon x. \end{cases} \tag{3.4.1}$$

Theorem 3.4.1. *(see Theorem 17.2 in [7], pp.343-345) Let L_ϵ be the limit cycle of (3.4.1). As $\epsilon \to 0$, L_ϵ tends to the curve J in figure 3.4.*

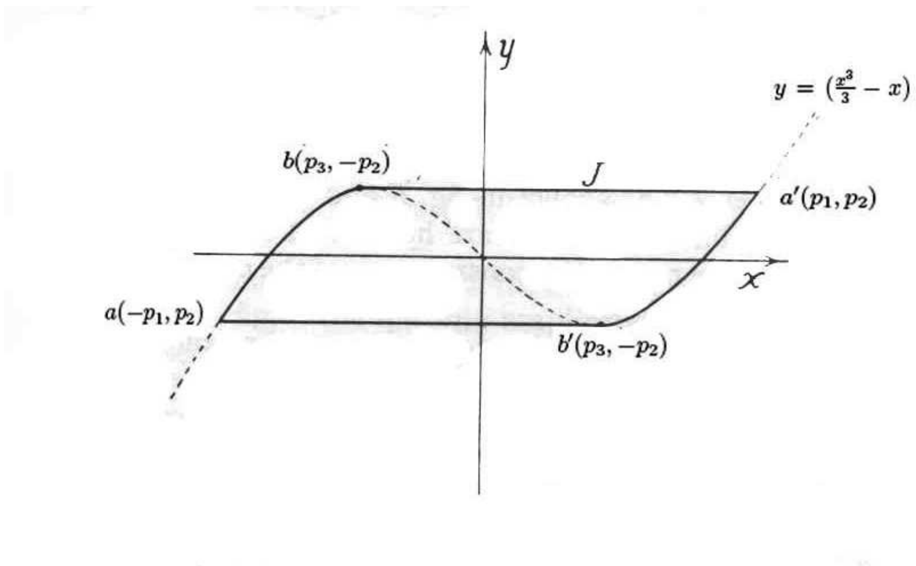


Figure 3.4:

Proof. As always when using Bendixson theorem the challenge is to construct an invariant annular region. Let $A - H$ respectively $A' - H'$ be the points given in figure 3.5. Here P and P' are always symmetric around the origin for any point P. The arc AB is a part of the curve $y = (\frac{x^3}{3} - x) - h$. Think of B as lying close to (but not at) the local minimum of this curve. The arc BC is a part of the tangent line to the curve $y = (\frac{x^3}{3} - x) - h$ at B. CD is just a horizontal line segment. DA' is a vertical line connecting D with the curve $y = (\frac{x^3}{3} - x) + h$. By symmetry we have constructed a closed curve $ABCDA'B'C'D'A$ which we denote by J_1.
Let H be the local minimum of the curve $y = (\frac{x^3}{3} - x)$. HE' is the just a horizontal line segment. Let the straight segment $E'F'$ be such that the slope is close to (but not equal with) zero. Here the point F' lies close to the curve $y = (\frac{x^3}{3} - x)$ but not on it. $F'G'$ is the vertical segment connecting F' with this curve. The arc $G'A'$ is simply a part of the same curve. By symmetry we have constructed a closed curve $HE'F'G'H'EFGH$ which we denote by J_2. We will now show that the annular region between J_1 and J_2 is positively invariant.

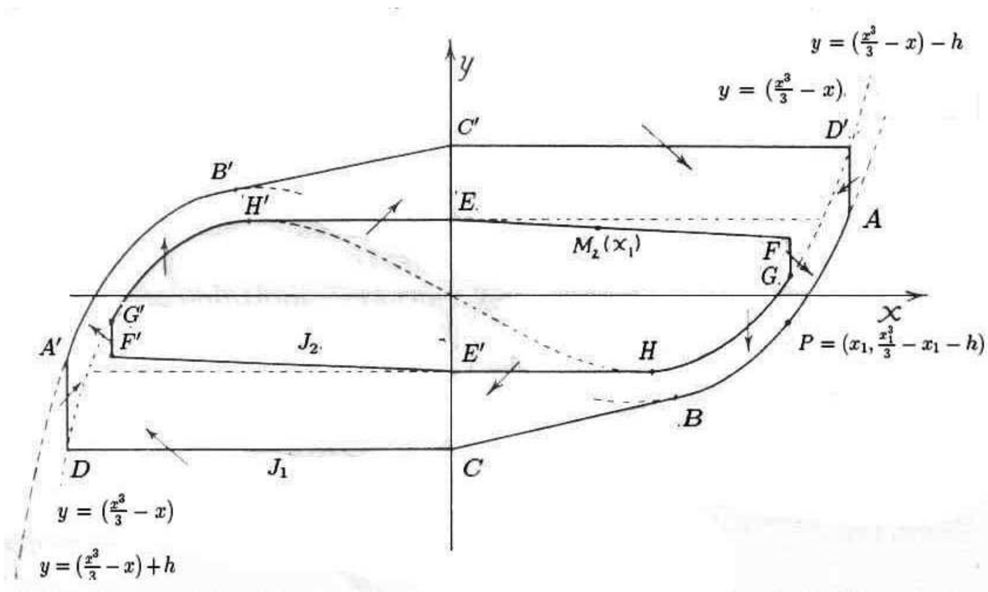


Figure 3.5:

Let us start by showing that our curve J_1 is such that at every point on J_1 the vector field points inward. This is clear from earlier discussion on the arcs CD, DA'. We turn to the arcs AB and BC. Let $P = (x_1, \frac{x_1^3}{3} - x_1 - h)$ be a point on AB. At this point the vector field has the slope

$$\frac{dy}{dx} = \frac{dy}{\epsilon dx} = \frac{-\epsilon x_1}{y - \frac{x_1^3}{3}} = \frac{\epsilon^2 x_1}{h} \ .$$

Note that on the arc AB the slope of the tangent is given by $x^2 - 1 \geq x_B^2 - 1$. So by choosing ϵ small enough the slope of the vector field will be smaller than that of AB at every point on AB ($\frac{\epsilon^2 x_1}{h} < x_B^2 - 1 \leq x_1^2 - 1$). Since we know that the field points leftwards under the curve $y = (\frac{x^3}{3} - x)$ it will thus point inward relative to J_1 on AB. On BC we note that the slope of any vector is $\frac{-\epsilon^2 x}{y - (\frac{x^3}{3} - x)} \leq \frac{-\epsilon^2 x}{h}$ since $|y - (\frac{x^3}{3} - x)| \geq h$. It thus becomes clear that we can make the slope arbitrarily small for any vector on BC, i.e the vector field must point inward on BC when ϵ is small enough. By symmetry the region enclosed by J_1 is positively invariant. On J_2 we have a similar situation as on J_1 and we only need to check the direction of the flow EF. Let k denote the small distance between F and G and again notice that $|y - (\frac{x^3}{3} - x)| \geq k$. Thus for small enough ϵ we can make the slope of the vector field strictly smaller than that of EF for all points on EF. We know that above $y = (\frac{x^3}{3} - x)$ then the field points rightwards and so by symmetry we conclude that the flow points outwards with respect to J_2. We now only have to realize that we can make h and k arbitrarily small and let B lie arbitrarily close to the local minimum $y = (\frac{x^3}{3} - x)$. This concludes the proof. $\square$

It is now easy to find the amplitude of the van der Pol limit cycle when $\epsilon \to \infty$. Consulting the picture we see that we need to find the root to the equation $\frac{x^3}{3} - x = y_{max}$ where y_{max} is the local maximum of $y = \frac{x^3}{3} - x$. Local maximum appears at $x = -1$ which gives $y_{max} = \frac{2}{3}$, thus we need to solve $\frac{x^3}{3} - x = \frac{2}{3}$. It is easily verified that $x = 2$ is the desired root. The amplitude therefore tends to 2. Since the amplitude is tending to the same value as $\mu \to 0$ and $\mu \to \infty$ we can conclude that there is a maximal value for the amplitude in the range $0 < \mu < \infty$. This maximum is still unknown.

Chapter 4

The case $f(x) = (x^2 - a)(x^2 - b)$

In chapter 3 we made some general observations on the Liénard equation and applied the theory to the special case of van der Pol. We shall now turn our focus to the equation $\ddot{x} + \mu f(x)\dot{x} + x = 0$ with $f(x) = (x^2 - a)(x^2 - b)$ where $a > 0$, $b > 0$. All the results which were obtained on the van der Pol equation will have their analogues, but proofs will be lengthier. Things are indeed complicated by the fact that we do not have the existence of a unique limit cycle. We shall see that depending on the values of μ and a in $f(x)$ we may actually exhibit either 0, 1 or 2 limit cycles. For each value of $\mu > 0$ there exists exactly two values for a such that there will only be 1 limit cycle, with multiplicity 2. The final aim of this chapter is to sketch the bifurcation diagram in the (μ, a)-plane which indicates this relation between μ and a.

We are thus studying the equation $\ddot{x} + \mu f(x)\dot{x} + x = 0$ and its equivalent system

$$\begin{cases} \frac{dx}{dt} = y - \mu F(x) \\ \frac{dy}{dt} = -x \end{cases} \tag{4.0.1}$$

with $f(x) = (x^2 - a)(x^2 - b)$. By scaling, $x \mapsto x\sqrt{b}$, this equation is transformed to

$$\ddot{x} + \mu'(x^2 - a')(x^2 - 1)\dot{x} + x = 0 \ ,$$

where $\mu' = \mu b^2$ and $a' = \frac{a}{b}$. Since this scaling of x preserves all topological structure there is no loss of generality in studying the transformed equation. Recall that changing sign on μ corresponds to changing the time direction and so we may assume $\mu > 0$. Also notice that if $a < 0$ then $f(x) = F'(x)$ will have only one positive zero at $x = 1$, implying that $F(x)$ will have one positive root and be strictly monotone for all x larger than that root. It will thus satisfy the conditions in the theorem of Liénard, the case already studied, and so it is no loss of generality to assume $a > 0$. These observations show that the bifurcation diagram depends essentially on μ and a where all the interesting information is contained in the first quadrant of the (μ, a)-plane.

From section 3.1 we see that the origin is asymptotically stable (due to the stability of the origin in the linearized system as in (3.1.3)) and so we can not use the Bendixson theorem to deduce the existence of at least 1 limit cycle. It is indeed easy to show that for certain values of a there are no limit cycles regardless of the value of $\mu > 0$. Consider the energy function $E(x, y)$. Its time derivative with respect to system (4.0.1) becomes

$$\frac{dE(x,y)}{dt} = -xF(x) = -x\left(\frac{x^5}{5} - \frac{a+1}{3}x^3 + ax\right) \quad .$$

For which values of a will $\frac{dE(x,y)}{dt} < 0$ for all x? If there exists such an a, then $E(x, y)$ will be a global Liapunov function and so the origin will be asymptotically stable in the whole. In order to find these values of a we need to solve the equation $F(x) = 0$.

$$F(x) = x\left(\frac{x^4}{5} - \frac{a+1}{3}x^2 + a\right)$$

putting $x^2 = t$ we find

$$\frac{t^2}{5} - \frac{a+1}{3}t + a = 0$$

with the roots

$$t_1 = \frac{5(a+1)}{6} + \sqrt{\frac{25(a+1)^2}{36} - 5a}, \; t_2 = \frac{5(a+1)}{6} - \sqrt{\frac{25(a+1)^2}{36} - 5a}$$

These roots will become a double root when the discriminant is zero. This happens exactly when

$$a^2 - \frac{26}{5}a + 1 = 0$$

i.e when $a = 5$ or $a = \frac{1}{5}$. Since the function $s(a) = a^2 - \frac{26}{5}a + 1$ is negative for $\frac{1}{5} < a < 5$, the roots of $\frac{t^2}{5} - \frac{a+1}{3}t + a = 0$ will all be complex for these values of a, and so are the roots of $-\left(\frac{x^5}{5} - \frac{a+1}{3}x^3 + ax\right) = 0$. If we for instance let $a = 2$ and evaluate at $x = 1$ we find

$$\frac{dE}{dt} = -\left(\frac{1}{5} - 1 + 2\right) < 0 \; .$$

We conclude that $\frac{dE}{dt} < 0$ along any trajectory of the system (4.0.1) when $\frac{1}{5} < a < 5$. Therefore there can be no limit cycles, regardless of the value of μ, in this interval of a.

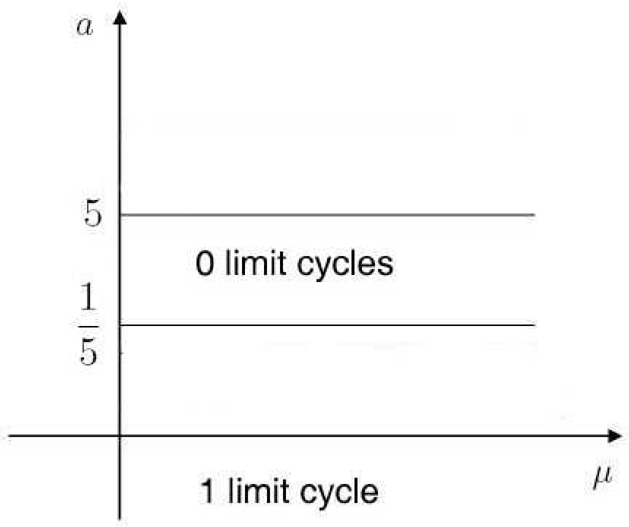


Figure 4.1: A first insight into the bifurcation diagram

We now turn our attention to the proof that system (4.0.1) has at most two limit cycles. Even if this result is not as "convenient" as having a unique limit cycle, it is really helpful to know the exact upper bound for the number of closed trajectories.

4.1 The number of limit cycles is at most two

The theorem stating that $\ddot{x} + \mu(x^2 - 1)(x^2 - a)\dot{x} + x = 0$ has at most two periodic solutions as $\mu > 0$ was obtained by Rychkov in 1975[1]. The proof uses a technique similar to that in the proof of Liénard's theorem and is also valid for a general class of functions $f(x)$ which behave like a polynomial of degree 5. From Liénard's theorem one knows that there is at most 1 limit cycle in the region $|x| \leq \max(1, \sqrt{a})$ (since in this strip the vector field satisfy all the conditions in Liénard's theorem) which is unstable if it exists. One

[1]see [12]

proceeds by showing that the integral of the divergence along limit cycles crossing the lines $|x| = \max(1, \sqrt{a})$ is monotone decreasing with respect to the amplitudes of the cycles, i.e if L_1 is inside L_2 then

$$\int_{L_1} div(y - F(x), -x)dt > \int_{L_2} div(y - F(x), -x)dt \ .$$

Since this integral is related to the stability of the limit cycles one can conclude that if there exists more than two limit cycles then two of them will have to be consecutive and both stable which is impossible. Before giving the full proof we need some extra lemmas. Notice that $div(y - F(x), -x) = -f(x)$.

Lemma 4.1.1. *(see Lemma 5.3 in [3], p.257) Suppose that the Liénard system has an arc of trajectory $\gamma : y(x)$ defined on $[\alpha, \beta]$. Then the integral of the divergence on γ is given by*

$$\int_{\gamma} -f(x)dt = sgn(y(\alpha) - F(\alpha)) \left(\ln \left| \frac{F(\beta) - y(\alpha)}{F(\alpha) - y(\alpha)} \right| + \int_{\alpha}^{\beta} \frac{(F(\beta) - F(x))x dx}{(F(\beta) - y(x))(F(x) - y(x))^2} \right)$$

Proof. Let us first assume that $y(x) - F(x) > 0$ (notice that if $y(x) - F(x) = 0$ at $x = \xi$ then the solution passing through $(\xi, F(\xi))$ cannot be written as a function $y(x)$ around $x = \xi$). We rewrite $-f(x)dt$ as $\frac{F'(x)dx}{F(x) - y(x)}$ and obtain the following

$$\int_{\gamma} -f(x)dt = \int_{\alpha}^{\beta} \frac{F'(x)}{F(x) - y(x)} dx = \int_{\alpha}^{\beta} \frac{F'(x) - y'(x) + y'(x)}{F(x) - y(x)} dx =$$

$$\Bigg[\ln |F(x) - y(x)| \Bigg]_{x=\alpha}^{x=\beta} + \int_{\alpha}^{\beta} \frac{y'(x)}{F(x) - y(x)} dx = \ln \left| \frac{F(\beta) - y(\beta)}{F(\alpha) - y(\alpha)} \right| +$$

$$+ \int_{\alpha}^{\beta} \frac{y'(x)}{F(x) - y(x)} dx \quad .$$

Since

$$\frac{dy}{dx} = \frac{x}{F(x) - y(x)}$$

and

$$\ln \left| \frac{F(\beta) - y(\beta)}{F(\alpha) - y(\alpha)} \right| = \ln \left| \frac{(F(\beta) - y(\beta))(F(\beta) - y(\alpha))}{(F(\alpha) - y(\alpha))(F(\beta) - y(\alpha))} \right| = \ln \left| \frac{F(\beta) - y(\alpha)}{F(\alpha) - y(\alpha)} \right| +$$

$$+ \ln \left| \frac{F(\beta) - y(\beta)}{F(\beta) - y(\alpha)} \right|$$

we get

$$\int_\gamma -f(x)dt = \ln\left|\frac{F(\beta)-y(\alpha)}{F(\alpha)-y(\alpha)}\right| + \ln\left|\frac{F(\beta)-y(\beta)}{F(\beta)-y(\alpha)}\right| + \int_\alpha^\beta \frac{xdx}{(F(x)-y(x))^2}\ .$$

Notice that the function $F(\beta)-y(x)$ satisfies

$$-\int_\alpha^\beta \frac{y'(x)dx}{F(\beta)-y(x)} = \ln\left|\frac{F(\beta)-y(\beta)}{F(\beta)-y(\alpha)}\right|$$

and so we get

$$\int_\gamma -f(x)dt = \ln\left|\frac{F(\beta)-y(\alpha)}{F(\alpha)-y(\alpha)}\right| - \int_\alpha^\beta \frac{y'(x)dx}{F(\beta)-y(x)} + \int_\alpha^\beta \frac{xdx}{(F(x)-y(x))^2}\ .$$

Finally, rewriting $y'(x)$ as $\frac{x}{F(x)-y(x)}$ again and putting the two integrands on a common denominator we get

$$\int_\gamma -f(x)dt = \left(\ln\left|\frac{F(\beta)-y(\alpha)}{F(\alpha)-y(\alpha)}\right| + \int_\alpha^\beta \frac{(F(\beta)-F(x))xdx}{(F(\beta)-y(x))(F(x)-y(x))^2}\right)\ .$$

If $y(x)-F(x)<0$, then consider $-\int_\gamma -f(x)dt$. $\square$

Lemma 4.1.2. *(see Lemma 5.4 in [3], pp.257-258) Suppose that the Liénard system has arc trajectories $\gamma_i : y_i(x)$ i=1, 2, $x\in[\alpha,\beta]$, $\alpha\geq 0$ which satisfy*
1. $y_2(x)>y_1(x)>F(x)$ or $y_2(x)<y_1(x)<F(x)$ for $x\in[\alpha,\beta]$
2. $F(\beta)-F(x)\leq 0$ (≥ 0), for $x\in[\alpha,\beta]$.
Then the following holds

$$\int_{\gamma_2} f(x)dt - \int_{\gamma_1} f(x)dt > 0\ (<0)\ .$$

Proof. We only show the case outside the parenthesis, the other case being treated analogously. By the previous lemma we get that

$$sgn(y(\alpha)-F(\alpha))\left(\int_{\gamma_2} f(x)dt - \int_{\gamma_1} f(x)dt\right) = \ln\left|\frac{F(\beta)-y_1(\alpha)}{F(\alpha)-y_1(\alpha)}\right| - \ln\left|\frac{F(\beta)-y_2(\alpha)}{F(\alpha)-y_2(\alpha)}\right| +$$

$$+\int_\alpha^\beta x(F(\beta)-F(x))\left(\frac{1}{(F(\beta)-y_1(x))(F(x)-y_1(x))^2} - \frac{1}{(F(\beta)-y_2(x))(F(x)-y_2(x))^2}\right)dx\ .$$

First assume that $y_2(x) > y_1(x) > F(x)$ for $x \in [\alpha, \beta]$. since $F(\alpha) \geq F(\beta)$ we see that $y_i(\alpha) - F(\alpha) \leq y_i(\alpha) - F(\beta)$, $i = 1,\ 2$ and so we easily get

$$1 \leq \frac{F(\beta) - y_2(\alpha)}{F(\alpha) - y_2(\alpha)} \leq \frac{F(\beta) - y_1(\alpha)}{F(\alpha) - y_1(\alpha)} \quad . \tag{4.1.1}$$

Also, since $y_i(x) > F(x) \geq F(\beta)$ on $x \in [\alpha, \beta]$, we find

$$\frac{1}{(F(\beta) - y_1(x))(F(x) - y_1(x))^2} - \frac{1}{(F(\beta) - y_2(x))(F(x) - y_2(x))^2} < 0. \tag{4.1.2}$$

From (4.1.1) and (4.1.2) and the assumption that $F(x) \not\equiv 0$ we immediately get

$$\int_{\gamma_2} f(x)dt - \int_{\gamma_1} f(x)dt > 0 \ .$$

Now assume that $y_2(x) < y_1(x) < F(x)$. The inequalities (4.1.1) and (4.1.2) now become

$$1 \geq \left|\frac{F(\beta) - y_2(\alpha)}{F(\alpha) - y_2(\alpha)}\right| \geq \left|\frac{F(\beta) - y_1(\alpha)}{F(\alpha) - y_1(\alpha)}\right| > 0 \tag{4.1.3}$$

and

$$\frac{1}{(F(\beta) - y_1(x))(F(x) - y_1(x))^2} - \frac{1}{(F(\beta) - y_2(x))(F(x) - y_2(x))^2} > 0. \tag{4.1.4}$$

From these two conditions we again obtain that

$$\int_{\gamma_2} f(x)dt - \int_{\gamma_1} f(x)dt > 0 \ .$$

□

Lemma 4.1.3. *Consider the Liénard system*

$$\begin{cases} \frac{dx}{dt} = y - F(x) \\ \frac{dy}{dt} = -x \end{cases}$$

where $F(x) = \int_0^x f(s)ds$ and $f(x)$ is an even, two times differentiable function. Let $x = \beta$ be the largest positive root of $f(x)$ (which is assumed to be simple) and assume that f is monotone increasing and that $f''(x) > 0$ for $x \geq \beta$. Further assume that $\gamma_i : x_i(y)$, $i = 1, 2$ are two arc trajectories in the half-plane $x \geq \beta$ with $x_1(y) < x_2(y)$. Then the following holds

$$\int_{\gamma_1} -f(x)dt > \int_{\gamma_2} -f(x)dt \ .$$

Proof. We first show that $\frac{f(x)}{x}$ is monotone increasing when $x \geq \beta$. We have

$$\frac{d}{dx}\left(\frac{f(x)}{x}\right) = \frac{xf'(x) - f(x)}{x^2} = 0$$

if and only if

$$xf'(x) - f(x) = 0$$

Let x_0 be such that $\frac{f(x_0)}{x_0} = f'(x_0)$ and let $P = (x_0, f(x_0))$ be a point on the graph of $y = f(x)$. Further let α_1 be the angle between the x-axis and the line segment OP (where O denotes the origin) and let α_2 be the angle between the tangent line of $y = f(x)$ through P and the x-axis. Then we have that

$$\tan\alpha_1 = \frac{f(x_0)}{x_0},\ \tan\alpha_2 = f'(x_0)$$

which yields $\alpha_1 = \alpha_2$ since $\frac{f(x_0)}{x_0} = f'(x_0)$ is assumed. But $f(x)$ is concave on the interval $x > \beta$ since $f''(x) > 0$ and so the graph of $y = f(x)$ must lie above its tangent line given by $y = f'(x_0)x$ at P. Again by concavity we may conclude that $y = f(x)$ does not intersect with its tangent line through P on the interval $\beta < x < x_0$. But this contradicts the fact that β is a root of $f(x)$ and so we must have

$$\frac{d}{dx}\left(\frac{f(x)}{x}\right) > 0$$

for all $x \geq \beta$, i.e $\frac{f(x)}{x}$ is monotone for all $x \geq \beta$.

From the above we conclude that the transformation

$$\begin{cases} X = \frac{f(x)}{x} \\ Y = y \end{cases}$$

on the range $\{x \geq \beta,\ y \in \mathbb{R}\}$ is a continuous bijection. Moreover, it is clear that this bijection has a continuous inverse on the range $\{X \geq 0,\ Y \in \mathbb{R}\}$, i.e it is a homeomorphism from the half-plane $\{x \geq \beta\}$ to the half-plane $\{X \geq 0\}$. We denote this homeomorphism by H.
We want to show

$$\int_{\gamma_1} -f(x)dt > \int_{\gamma_2} -f(x)dt.$$

It is clear that this is the same as proving

$$\int_{\gamma_1} \frac{f(x)}{x}dy > \int_{\gamma_2} \frac{f(x)}{x}dy.$$

The transformation H takes the arcs γ_i to the arcs γ_i^* with start and end-points lying on the Y-axis, say at $Y_{i,1}$ and $Y_{i,2}$. Moreover, if we parameterize γ_i^* by $X_i(Y)$ we see that the integrals

$$\int_{Y_{i,1}}^{Y_{i,2}} X_i(Y)dY$$

are equal to the integrals

$$\int_{\gamma_i} \frac{f(x)}{x} dy.$$

Since the integrals in the (X, Y)-plane are equal to the area between γ_i^* and the Y-axis with negative sign (we are integrating in the negative Y-direction), we immediately see that

$$\int_{Y_{1,1}}^{Y_{1,2}} X_1(Y)dY > \int_{Y_{2,1}}^{Y_{2,2}} X_2(Y)dY \ .$$

We have thus managed to show that

$$\int_{\gamma_1} -f(x)dt > \int_{\gamma_2} -f(x)dt.$$

□

We are now in a position to prove Rychkov's result that the number of limit cycles cannot exceed two.

Theorem 4.1.4. *(Rychkov's theorem,see Theorem 5.1 in [3], pp.259-261. See also the original paper [12]) Consider the Liénard equation and its equivalent system*

$$\begin{cases} \frac{dx}{dt} = y - F(x) \\ \frac{dy}{dt} = -x \end{cases} \tag{4.1.5}$$

and let $F(x)$ be an odd degree five polynomial like function. More precisely, let $F(x)$ satisfy the following conditions.
1)$F(x)$ is differentiable with continuous derivative.
2)$F(x)$ has two positive roots, $0 < \beta_1 < \beta_2$, and $F(x) \leq 0$ for $x \in [\beta_1, \beta_2]$.
3)$\exists \alpha$, $\beta_1 < \alpha < \beta_2$, such that $F'(\alpha) = 0$ and $F'(x) \leq 0$ for $\beta_1 < x < \alpha$.
4)$F(x)$ and $F'(x)$ are monotone increasing for $x > \alpha$ and $F(\infty) = F'(\infty) = \infty$.
Then the system (4.1.5) has at most two limit cycles.

Proof. By Liénard's theorem we can conclude the existence of at most one limit cycle in the region $|x| \leq \alpha$ which is unstable if it exists (the situation here is the same as the situation in 3.2.1 with $-\mu$ instead of μ). Assume the existence of two limit cycles, L and L', where L' is outside of L and both L and L' intersect the lines $|x| = \alpha$. We first show that the integral of the divergence along L and L' satisfies

$$\int_{L'} -f(x)dt < \int_{L} -f(x)dt \Leftrightarrow \int_{L'} f(x)dt > \int_{L} f(x)dt. \tag{4.1.6}$$

Since $f(x)$ is an even function, by symmetry we only need to estimate the divergence along the half cycles (see fig. 4.2):

$$\int_{A_2F_2} f(x)dt > \int_{A_1F_1} f(x)dt. \tag{4.1.7}$$

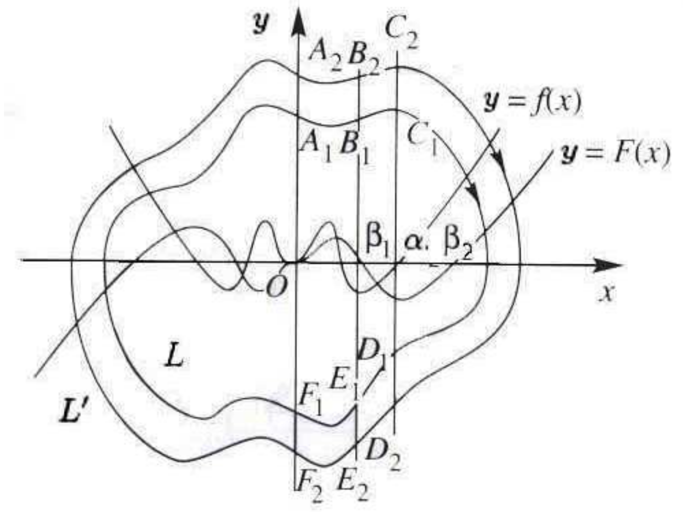


Figure 4.2:

We start with A_1B_1 and A_2B_2. Let $y_1(x)$ and $y_2(x)$ be the arcs A_1B_1 and A_2B_2 (E_1F_1 and E_2F_2 respectively). On the interval $0 \leq x \leq \beta_1$ the functions $F(x)$ and y_i, i=1, 2, satisfy the assumptions in lemma 4.1.2. We can thus deduce

$$\int_{A_2B_2} f(x)dt > \int_{A_1B_1} f(x)dt \tag{4.1.8}$$

$$\int_{E_2F_2} f(x)dt > \int_{E_1F_1} f(x)dt. \tag{4.1.9}$$

Now consider B_1C_1 and B_2C_2. Since $dt = \frac{dx}{y-F(x)}$ we get

$$\int_{B_2C_2} f(x)dt - \int_{B_1C_1} f(x)dt =$$
$$= \int_{\beta_1}^{\alpha} \left(\frac{f(x)}{y_2(x) - F(x)} - \frac{f(x)}{y_1(x) - F(x)} \right) dx \geq 0, \tag{4.1.10}$$

where $y_1(x)$ and $y_2(x)$ are parameterizations for the arcs B_1C_1 and B_2C_2 respectively. (4.1.10) holds because $f(x) \leq 0$ on the interval $[\beta_1, \alpha]$ and $y_2(x) - F(x) > y_1(x) - F(x)$. The reasoning on D_1E_1 and D_2E_2 is similar. Here we have $y_2(x) - F(x) < y_1(x) - F(x)$ but integration is now from α to β_1 and so we get

$$\int_{D_2E_2} f(x)dt - \int_{D_1E_1} f(x)dt =$$
$$= \int_{\alpha}^{\beta_1} \left(\frac{f(x)}{y_2(x) - F(x)} - \frac{f(x)}{y_1(x) - F(x)} \right) dx =$$
$$= \int_{\beta_1}^{\alpha} \left(\frac{f(x)}{y_1(x) - F(x)} - \frac{f(x)}{y_2(x) - F(x)} \right) \geq 0 \tag{4.1.11}$$

Finally consider C_1D_1 and C_2D_2. Since $f(x)$ and $F(x)$ satisfy the conditions in lemma 4.1.3 we conclude that

$$\int_{C_2D_2} f(x)dt > \int_{C_1D_1} f(x)dt \tag{4.1.12}$$

(4.1.6) now follows from (4.1.8)-(4.1.12).

Having proven (4.1.6) we make the following observations.
i) If we have a stable limit cycle L crossing the lines $|x| = \alpha$, then there cannot be any limit cycle outside of L. Indeed, the integral of the divergence along L is ≤ 0 and so if we had a limit cycle L outside of L then (4.1.6) would force it to be stable (see Theorem 2.1.13). But it is impossible to have two stable limit cycles lying next to each other.
ii) If we have a semistable limit cycle crossing the lines $|x| = \alpha$, then this limit cycle cannot be internally stable and externally unstable. To see that consider the system

$$\begin{cases} \frac{dx}{dt} = y - F_b(x) \\ \frac{dy}{dt} = -x \end{cases},$$

where

$$F_b(x) = F(x) + br(x),\ with\, r(x) = \begin{cases} 0\ if\ |x| < \alpha \\ (|x| - \alpha)^2\ if\ |x| \geq \alpha \end{cases} \quad for\, some\, b \in \mathbb{R}\,.$$

From example (2.3.3) we know that this is a generalized rotated vector field. This vector field is equal to (4.1.5) when $b = 0$ and it satisfies the conditions 1)-4) above. From Theorem 2.3.8 we know that if $0 < b \ll 1$ the supposed semistable limit cycle will split into at least two limit cycles, one stable and one unstable. Moreover, the stable limit cycle will lie inside the unstable one. This will however contradict (4.1.6) since the generalized rotated vector field satisfies conditions 1)-4).

We can now consider two different situations. First assume that there exists a limit cycle C in the region $|x| \leq \alpha$ and a limit cycle L crossing the lines $|x| = \alpha$. Since the limit cycle C is unstable L has to be internally stable. From observation ii) above it must be externally stable as well. From observation i) we conclude that there exists no more cycles outside of L. Now assume that there is no limit cycle in the region $|x| \leq \alpha$ and that we still have a limit cycle L crossing $|x| = \alpha$. Then L has to be internally unstable since the origin is stable. If L is also externally unstable then we will have one more limit cycle outside of L which will be internally stable. From observations i) and ii) it will have to be stable and there will be no other limit cycles and so we again have exactly two cycles. If L is externally stable, then there can be no limit cycle outside of L. Otherwise consider the rotated vector field above. For $0 < b \ll 1$ the limit cycle L will split into two limit cycles, one unstable and one stable, and the stable one will lie outside the unstable one. But then there can be no more limit cycles outside of the stable one (again from observation i)). This concludes the proof. □

In the proof above we could have divided the integrals of the divergence in a simpler manner. Indeed, lemma 4.1.2 is applicable on the interval $x \in [0, \alpha]$ since $F(x) \geq F(\alpha)$ on this interval. This gives us the possibility of generalizing Rychkov's theorem somewhat:

Theorem 4.1.5. *Consider the Liénard system*

$$\begin{cases} \frac{dx}{dt} = y - F(x) \\ \frac{dy}{dt} = -x, \end{cases} \tag{4.1.13}$$

where $F(x)$ is a odd and continuously differentiable function satisfying the following conditions.
1)The derivative, $f(x) = F'(x)$ has exactly n positive simple roots $0 < \xi_1 < \xi_2 < ... < \xi_n$ and $F(x)$ has exactly n positive simple roots $\xi_1 < \alpha_1 < \xi_2 < \alpha_2 < ... < \xi_n < \alpha_n$.
2)Put $m_i = F(\xi_i)$, then $|m_i| < |m_j|$ for $i < j$. Also $F(\infty) = \infty$.

3)$f(x)$ is monotone increasing for $x \geq \xi_n$.
Then the system has at most n limit cycles.

Proof. The proof is by induction over n. Notice that $n = 1$ is settled by Liénard's theorem and $n = 2$ is settled by Rychkov's theorem. In the range $|x| \leq \xi_n$ there are at most $n-1$ limit cycles by induction hypothesis (in this range F may be considered as a polynomial with $n-1$ roots and negative leading term). Suppose $L_1 \subset L_2$ are two limit cycles crossing the lines $|x| = \xi_n$. By lemmas 4.1.2 and 4.1.3 we have that

$$\int_{L_1} -f(x)dt > \int_{L_2} -f(x)dt. \tag{4.1.14}$$

This means that if L_1 is a stable limit cycle then L_2 cannot exist. Also, by considering the generalized rotated vector field

$$\begin{cases} \frac{dx}{dt} = y - F_b(x) \\ \frac{dy}{dt} = -x \end{cases},$$

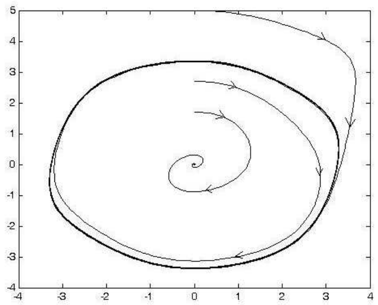
where

$$F_b(x) = F(x) + br(x),\ with\ r(x) = \begin{cases} 0\ if\ |x| < \xi_n \\ (|x| - \xi_n)^2\ if\ |x| \geq \xi_n \end{cases} \quad for\ some\ b \in \mathbb{R}$$

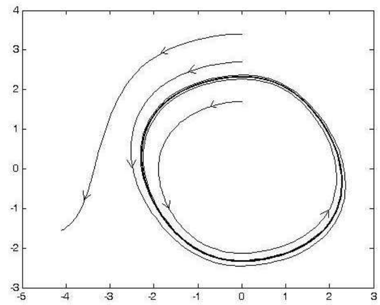
we can reason in exactly the same manner as in the proof of Rychkov's theorem to deduce that a semi stable limit cycle crossing the lines $|x| = \xi_n$ will have to be internally unstable and externally stable. Suppose we have exactly $n-1$ limit cycles in the region $|x| \leq \xi_n$. They will necessarily have to be simple limit cycles and so they will be alternately stable and unstable, the outermost being unstable. This means that L_1 is internally stable. It therefore has to be stable and so L_2 cannot exist. On the other hand, suppose we have less than $n-1$ limit cycles in the region $|x| \leq \xi_n$. Then L_1 can be either internally stable or internally unstable. If the former holds then by the above L_1 will be stable and L_2 wont exist. If L_2 is internally unstable then it may be either externally unstable or externally stable. If the former holds then we must have a cycle L_2 outside of L_1 since trajectories far away from the origin tends inwards in the phase plane. But L_2 will then be completely stable and no other cycles will exist. If L_1 is externally stable then it will bifurcate into two cycles in the generalized rotated vector field above, one unstable and one stable, the outermost being stable. But then again L_2 cannot exist outside of L_1. □

Even though this theorem does not completely solve the problem of the number of limit cycles for the Liénard equation it shows us what kind of polynomials may cause a "pathological" situation.

Rychkov's theorem gives us a lot of information concerning the bifurcation diagram. For instance we see that the points (μ, a) for which there exists a single limit cycle form a closed set, i.e a bifurcation curve, in the first quadrant in the (μ, a)-plane. This curve will divide the plane into a number of regions, each representing the existence of either 0 or 2 limit cycles in the phase plane. Crossing this curve corresponds to a saddle-node bifurcation (two cycles coincide and then disappear). In the next section we will apply the theory presented earlier to investigate the behavior of this bifurcation curve for small values of μ.



(a) Positive time direction. We see the stable limit cycle.



(b) Negative time direction. We see the unstable limit cycle.

Figure 4.3: The phase portrait for the equation $\ddot{x}+0.1(x^2-1)(x^2-7)\dot{x}+x=0$. In (a) the time direction is positive, while in (b) it is negative. Notice how different trajectories tend to different locations in the phase plane.

4.2 The weakly nonlinear regime

From the theory presented in section 3.3 we know that the number of simple limit cycles, when μ is small, is determined by the equation

$$\Phi(\alpha) = \int_0^{2\pi} f(\alpha \sin u, \alpha \cos u) \cos u du = 0 \ .$$

Since $f(x, y) = (x^2 - 1)(x^2 - a)y$ we are led to investigate

$$\Phi(\alpha) = \int_0^{2\pi} (\alpha^2 \sin^2 u - 1)(\alpha^2 \sin^2 u - a)\alpha \cos^2 u du \ .$$

When this equation has 2 simple roots the system will have 2 limit cycles for small μ. When the equation has no roots there will be no cycles. The bifurcation thus occurs when $\Phi(\alpha)$ has a double root. We thus seek the values of a which make α a double root. We first need to solve the equation $\Phi(\alpha) = 0$ in terms of a:

$$\Phi(\alpha) = \int_0^{2\pi} (\alpha^2 \sin^2 u - 1)(\alpha^2 \sin^2 u - a)\alpha \cos^2 u du =$$

$$= \int_0^{2\pi} \alpha^5 \sin^4 u \cos^2 u du - \int_0^{2\pi} \alpha^3 a \sin^2 u \cos^2 u du - \int_0^{2\pi} \alpha^3 \sin^2 u \cos^2 u du +$$
$$+ \int_0^{2\pi} \alpha a \cos^2 u du$$

$$\Phi(\alpha) = \frac{\alpha^5}{32}\left[\frac{1}{6}\sin(6u) - \frac{1}{2}\sin(4u) - \frac{1}{2}\sin(2u) + 2u\right]_0^{2\pi} -$$
$$(\alpha^3 a + \alpha^3)\left[\frac{1}{8}u - \frac{1}{32}\sin(4u)\right]_0^{2\pi} + \alpha a' \left[\frac{u}{2} + \frac{1}{4}\sin(2u)\right]_0^{2\pi} =$$
$$= \frac{\pi\alpha^5}{8} - (\alpha^3 a + \alpha^3)\frac{\pi}{4} + \alpha a \pi = \frac{\pi\alpha^5}{8} - \frac{\pi\alpha^3 a}{4} - \frac{\pi\alpha^3}{4} + \pi a \alpha =$$
$$= \frac{\pi\alpha}{8}(\alpha^4 - 2\alpha^2(a+1) + 8a) = 0. \tag{4.2.1}$$

The important case is of course when

$$\alpha^4 - 2\alpha^2(a+1) + 8a = 0. \tag{4.2.2}$$

Putting $\alpha^2 = t$ makes (4.2.2)

$$t^2 - 2t(a+1) + 8a = 0 \ .$$

We now have a simple second degree equation in t to solve. The solution is

$$t = (a+1) \pm \sqrt{a^2 - 6a + 1}$$

In order to find real solutions to (4.2.1) we assume that $\sqrt{a^2 - 6a + 1} \geq 0$. Solving this equation with respect to a yields

$$a = 3 \pm \sqrt{8} \ .$$

So $a^2 - 6a + 1$ is therefore negative whenever

$$3 - \sqrt{8} < a < 3 + \sqrt{8} \ .$$

There is no periodic motion satisfying $\ddot{x} + \mu(x^2 - 1)(x^2 - a)\dot{x} + x = 0$ when a lies in that interval, at least not for small values of μ. The situation is sketched in the following figure 4.4.

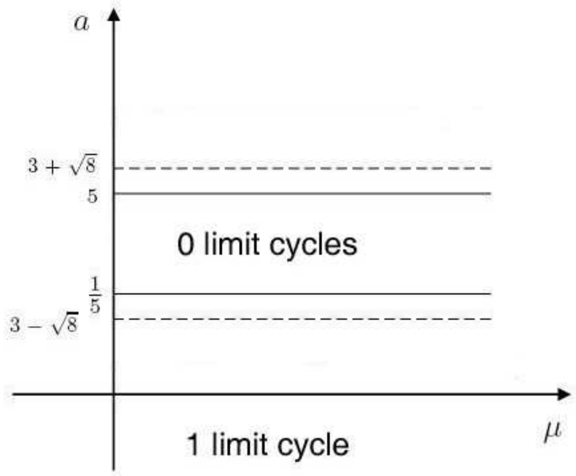


Figure 4.4:

Calculating $\Phi'(\alpha)$ and letting $a > 3 + \sqrt{8}$ ($a < 3 - \sqrt{8}$) we find

$$\Phi'(\alpha) = \frac{\pi}{8}(5\alpha^4 - 6\alpha^2(a+1) + 8a) \ .$$

If we for instance choose $a = 10$ we see that

$$\alpha_1 = \sqrt{5.3} - \sqrt{1.3} \quad \alpha_2 = \sqrt{5.3} + \sqrt{1.3} \ .$$

Evaluation of $\Phi'(\alpha)$ at these values yields $\Phi'(\alpha_1) > 0$ and $\Phi'(\alpha_2) < 0$. This confirms that the inner cycle is unstable while the outer is stable.

So far we only know that the bifurcation curve starts at $(0, 3 \pm \sqrt{8})$, but we know nothing about the behavior of the curve(s) after that. We made some numerical investigations for the upper part of the bifurcation curve. They are presented in the table below and are to be interpreted as follows: for a given value of μ we have listed the "lowest" value of a for which there exist two limit cycles.

μ	a_{num}	μ	a_{num}	μ	a_{num}
0.05	5.831	0.12	5.843	0.20	5.870
0.06	5.832	0.14	5.849	0.22	5.879
0.08	5.834	0.16	5.856	0.24	5.888
0.10	5.839	0.18	5.862	0.25	5.90

The numerical values are plotted here in the (μ, a)-plane.

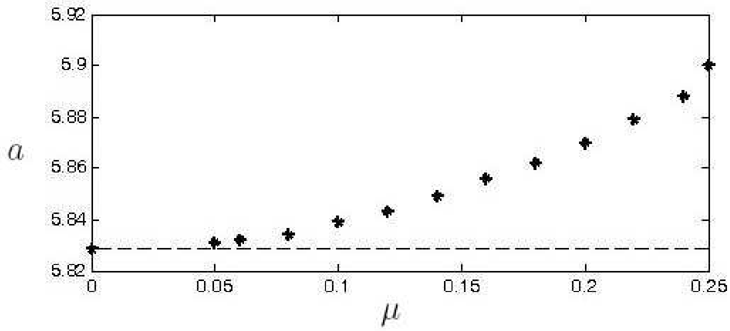


Figure 4.5:

It appears that the upper bifurcation curve at first is monotone increasing. The following questions arise: is it always monotone increasing? Is it bounded? Does the lower curve behave in the same manner?
The lower curve is of course bounded from below by $a = 0$ and from above by $a = \frac{1}{5}$, since if $a \leq 0$ we are in the case of Liénard's theorem and for $a = \frac{1}{5}$ we have no limit cycles by the remarks made in the beginning of section 4.1.

4.3 The strongly nonlinear regime

We now make a similar study of the number of limit cycles when μ is very large. The approach is analogous to the one made for the van der Pol equation. The difference is of course that we are no longer assured the existence of a single limit cycle and so we need to find some suitable conditions on a which guarantees the existence of two limit cycles as $\mu \to \infty$. We start by changing the parameter μ, putting $\mu = \frac{1}{\epsilon}$, and get the corresponding Liénard system

$$\begin{cases} \epsilon \frac{dx}{dt} = y - F(x) \\ \frac{dy}{dt} = -\epsilon x \end{cases}$$

where $F(x) = \int_0^x (s^2 - a)(s^2 - 1)ds$, $\epsilon \to 0$. We study the slope $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{-\epsilon^2 x}{y - F(x)} \ .$$

As $\epsilon \to 0$ the slope will tend to 0 at any point where $y \neq F(x)$. This means that the trajectories will consist of horizontal lines pieced up together. One important observation needs to be done here. If we are located below the curve $y = F(x)$ the trajectories will point to the left; if we are above they will point to the right. This means that there exist arc trajectories lying arbitrarily close to the curve $y = F(x)$. This is why the unique limit cycle in the van der Pol equation gets its peculiar shape for large μ (of course we must make the change of parameters $\mu = \frac{1}{\epsilon}$ in order to get a bounded trajectory). This reasoning indicates the following simple but important fact. If $m_1 < 0 < M_1$ represents the local maximum and minimum of $F(x)$ on the positive x-axis, then in order for us to have two limit cycles as $\mu \to \infty$ we must have $|m_1| > |M_1|$. The geometric reasoning behind this fact is fairly easy. First of all, any limit cycle will have to cross a local maximum/minimum as $\mu \to \infty$. As soon as a (horizontal) trajectory lying above all critical points hits the curve $y = F(x)$ it will cross it vertically and then lie arbitrarily close to it, i.e it will follow it downwards until it reaches the local minimum m_1. As it reaches m_1 it will move horizontally to the left until it reaches the curve $y = F(x)$ again. If $|m_1| < |M_1|$ then this will happen before it passes the local minimum on the negative x-axis. The result is that the trajectory crosses $y = F(x)$ vertically and then follows it into the origin, i.e the origin is asymptotically stable in the whole. On the other hand, it is easily seen that if $|m_1| > |M_1|$ then we can copy the arguments in the proof of Theorem 3.4.1 (the theorem stating the shape of the van der Pol cycle as $\mu \to \infty$) showing the existence of two limit cycles with the same peculiar shapes as in the van der Pol equation (see figure 4.6).

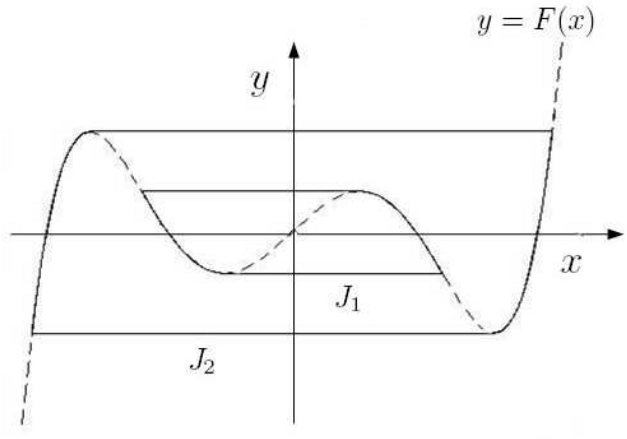


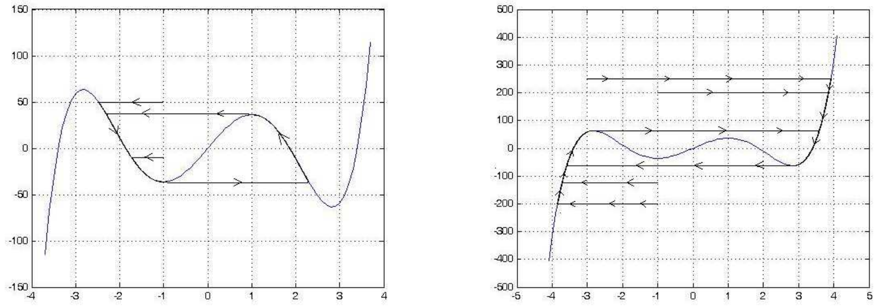
Figure 4.6:

We have thus sketched the proof of the following:

Theorem 4.3.1. *Consider the system*

$$\begin{cases} \epsilon \frac{dx}{dt} = y - F(x) \\ \frac{dy}{dt} = -\epsilon x \end{cases}$$

with $F(x) = \int_0^x (s^2 - a)(s^2 - 1)ds$. *If* $a > 1$ $(a < 1)$ *and* $|F(\sqrt{a})| > |F(1)|$ $(F(\sqrt{a})| < |F(1)|)$ *then there exist two limit cycles as* $\epsilon \to 0$. *Moreover, these limit cycles tend to the Jordan curves* J_1 *and* J_2 *shown in figure 4.6.*



(a) Negative time direction (b) Positive time direction

Figure 4.7: We have here plotted the phase portrait together with the curve $y = F(x)$, where $\mu = 7$. The straightening out of the trajectories does not require μ to be exceptionally large.

The previous paragraph shows that the values of a which make $|m_1| = |M_1|$ are of main interest. We of course assume that $f(x)$ is normalized to $f(x) = (x^2-1)(x^2-a)$ and so the critical points on the curve $y = F(x)$ occurs at $x = \pm 1$ and $x = \pm\sqrt{a}$. $|m_1| = |M_1|$ then means that $F(\pm 1) = F(\mp\sqrt{a})$. We solve $F(-1) = F(\sqrt{a})$ and find

$$a = \frac{7 \pm \sqrt{45}}{2} .$$

The two roots corresponds to $a < 1$ and $a > 1$ respectively. For any a between these two roots we have $|m_1| < |M_1|$ and so no limit cycles will survive as $\mu \to \infty$. This gives further interesting information about the bifurcation curve. We now know that it "starts" at the points $(0, 3 \pm \sqrt{8})$ and "ends" at the points $(\infty, \frac{7\pm\sqrt{45}}{2})$. Because of continuity the curve(s) must be bounded and so there exist values for a, say a_* and a^*, such that if $0 < a < a_*$ or $a^* < a$ we will have two limit cycles *for any value of* μ. Beside from searching the values a_* and a^* we are interested in the qualitative behavior of the bifurcation curve(s).

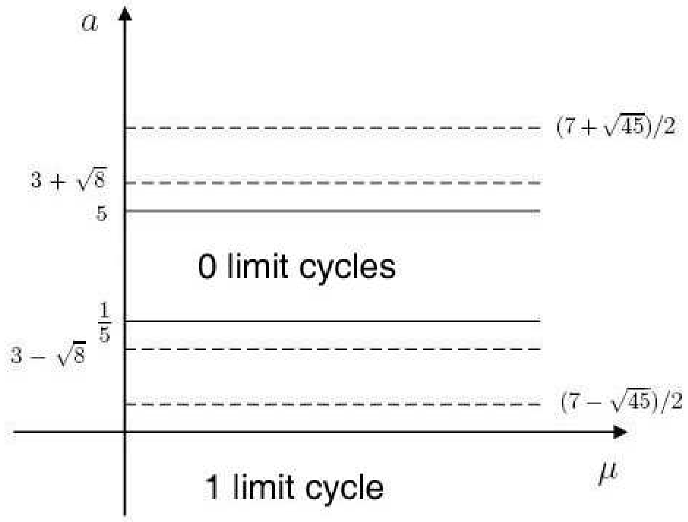


Figure 4.8:

4.4 An upper bound for the bifurcation curve

We want to find some conditions that ensure the existence of at least two limit cycles for the Liénard system

$$\begin{cases} \frac{dx}{dt} = y - F(x) \\ \frac{dy}{dt} = -g(x) \end{cases}$$

when $F(x)$ is an odd polynomial of degree five, and $g(x)$ is some odd function with $x = 0$ as the only real root. The method described here is due to N.G Lloyd. Remember that the only positive criterion for the existence of limit cycles is given by the theorem of Poincaré-Bendixson. We shall thus show how to construct a closed curve Γ with the property that the vector field points outward at every point of Γ. Γ will of course surround the origin, and since we know that the origin will actually be asymptotically stable for our choice of $F(x)$ this will yield the existence of a cycle inside Γ. Moreover, we have already pointed out that any motion in our system which begins sufficiently far away on the y-axis will have to "loose energy" and spiral inwards towards some neighborhood of the origin. We can thus use Bendixson's theorem again to conclude the existence of at least two limit cycles. By Rychkov's result that there are at most two limit cycles for our system, the conclusion is that there are exactly two limit cycles. In the following construction of Γ we will of course exhibit conditions providing the possibility of such a construction. These conditions will be discussed later.

Set $D_+ = \{(x,y)|y > F(x)\}$ and $D_- = \{(x,y)|y < F(x)\}$. If we can construct a curve Γ_+ in D_+ with the desired property, then we can by symmetry of the vector field construct a similar curve Γ_- in D_- with the same property. Γ will then be $\Gamma = \Gamma_+ \cup \Gamma_-$. Recall that if a curve is given by $\phi(x,y) = C$ where C is some real constant, then the sign of the time derivative of ϕ with respect to the system determines in which direction the vector field points on each point of $\phi(x,y) = C$. The idea is thus to exhibit some appropriate functions which we can use to define the curve Γ. In order to do this we need to introduce the following notation.

1. Let $x = a_1, a_3$ be the local maximum of $F(x)$ and $x = a_2, a_4$ be its local minimum $(a_1 < a_2 < 0 < a_3 < a_4)$
2. Put $c_i = F(a_i)$ and $\gamma_i = G(a_i)$ where $G(x)$ is the primitive function of $g(x)$
3. Let ξ_1 be the zero of $F(x)$ between a_1 and a_2 and ξ_2 that between a_3 and a_4

To construct Γ_+ we begin at the point $A = (a_1, F(a_1))$, the first local maximum of F. But first we introduce the following curves[2]:

$$\phi_1 \equiv E(x,y) = \frac{1}{2}c_1^2 + \gamma_1$$

$$\phi_2 \equiv E(x, y - c_3) = \frac{1}{2}(\eta - c_3)^2$$

$$\phi_3 \equiv \frac{1}{2}(y - F(x))^2 + G(x) = \frac{1}{2}(\nu - c_3)^2 + \gamma_3 \ .$$

Here η can be considered as a parameter and the value for the constant ν will be given below. Let B be the intersection point between ϕ_1 and $y = \eta$. By AB we denote the arc of the curve ϕ_1 from A to B. Set $C = (0, \eta)$ and let BC be the straight line connecting B and C. Set $D = (a_3, \nu)$, the intersection point of the curve ϕ_2 and the line $x = a_3$ (and so from this we can determine the value of ν). Set CD to be the arc of ϕ_2 connecting C and D. Let E be the intersection point of the curve ϕ_3 and the line $x = a_4$. DE

[2]Recall that we have defined the function $E(x,y)$ as $\frac{x^2}{2} + \frac{y^2}{2}$.

is the arc of ϕ_3 between D and E. Let $A' = (a_4, F(a_4))$ and put EA' to be the straight line connecting E and A' (see fig. 4.9). Finally, let Γ_+ be the union of all the five arcs introduced above.

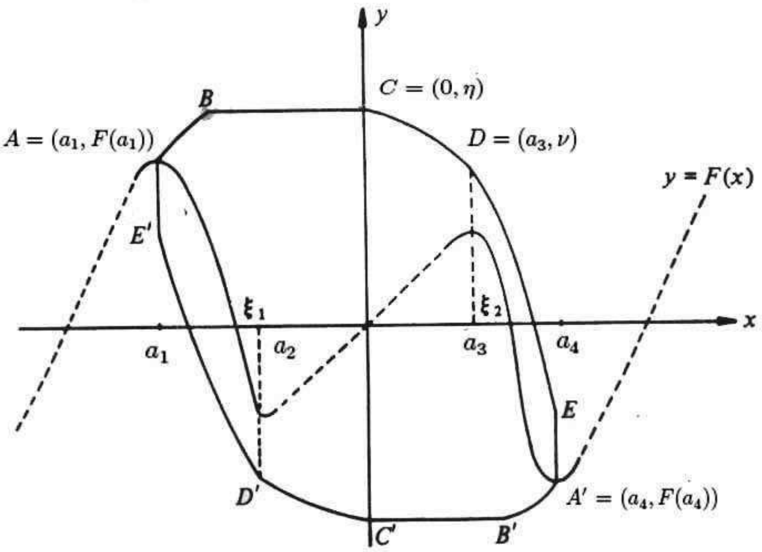


Figure 4.9:

Obviously we need some conditions on η for the points $A - E$ to lie above the curve $y = F(x)$. For instance we will see in the proof of the following theorem that we should have $x_B \leq \xi_1$. Other inequalities that need to be satisfied are:

$$(\eta - c_3)^2 > 2\gamma_3 \tag{4.4.1}$$

$$\frac{1}{2}(\nu - c_3)^2 + \gamma_3 - \gamma_4 \geq 0. \tag{4.4.2}$$

To see that (4.4.1) implies that D is above the curve $y = F(x)$ notice that ϕ_2 can be written as $(y - c_3)^2 = (\eta - c_3)^2 - 2G(x)$, so (4.4.1) implies $y - c_3 > 0$. Condition (4.4.2) implies in an analogous way that $y_E \geq y_{A'}$. We observe that $\frac{1}{2}(\nu - c_3)^2 + \gamma_3 = (\eta - c_3)^2$ and we can write

$$(\eta - c_3)^2 \geq 2\gamma_4. \tag{4.4.3}$$

We leave it to the reader to verify that $x_B \leq \xi_1$ can be expressed as

$$\frac{1}{2}c_1^2 + \gamma_1 - \frac{1}{2}\eta^2 \geq G(\xi_1). \tag{4.4.4}$$

We can eliminate η from (4.4.3) and (4.4.4) and obtain the inequality

$$\frac{1}{2}c_1^2 + \gamma_1 - G(\xi_1) \geq \frac{1}{2}(c_3 + \sqrt{2\gamma_4})^2. \tag{4.4.5}$$

Setting

$$\eta = c_3 + \sqrt{2\gamma_4}$$

we will have $E = A'$. If we construct Γ_- by symmetry exactly like Γ_+ we will get conditions analogous to those above. In particular, condition (4.4.5) becomes

$$\frac{1}{2}c_4^2 + \gamma_4 - G(\xi_2) \geq \frac{1}{2}(c_2 + \sqrt{2\gamma_1})^2. \tag{4.4.6}$$

Theorem 4.4.1. *(see [9]) Let $\Gamma = \Gamma_+ \cup \Gamma_-$ be as above with the conditions (4.4.5) and (4.4.6) satisfied. Then the vector field will point outwards at every point on Γ.*

Proof. It is clear that the arcs BC and EA' satisfies the theorem. On AB we have

$$\dot{\phi}_1 = \dot{E}(x, y) = -g(x)F(x) \geq 0$$

provided that $x_B \leq \xi_1$. On CD we have

$$\dot{\phi}_2 = \dot{E}(x, y - c_3) = (c_3 - F(x))g(x) \geq 0$$

because we require that D be above the curve $y = F(x)$ and $xg(x) > 0$. On DE we have

$$\dot{\phi}_3 = -f(x)(y - F(x))^2 \geq 0$$

since $F(x)$ is decreasing on DE. These three inequalities implies the theorem. □

We know that the origin is asymptotically stable and so it has a "radius of convergence", i.e there is a small circular boundary around the origin which bounds a positively invariant domain Ω. The region between Γ and $\partial\Omega$ is thus negatively invariant and due to the Bendixson theorem we get the existence of at least one limit cycle. On the other hand, trajectories crossing the y-axis far away from the origin spiral inwards and so we must have at least one limit cycle outside of Γ as well (using Bendixson's theorem again). We have thus deduced the existence of at least two limit cycles and using Rychkov's result yields the existence of exactly two limit cycles[3].

[3]One can show that Rychkov's theorem is valid without necessarily having $g(x) = x$, see for instance **??** pp.

Can we utilize the equation (4.4.5) to get some kind of estimate to the bifurcation curve of Liénard's system in the (a, μ)-plane? First notice that since $F(x)$ is odd we can rewrite (4.4.5) and (4.4.6) as

$$c_1^2 \geq 2G(\xi_1) + c_3^2 + 2c_3\sqrt{2\gamma_1}. \tag{4.4.7}$$

If we set $g(x) = x$ and $f(x) = \mu(x^2 - 1)(x^2 - a)$ we get $a_1 = -\sqrt{a}$, $a_2 = -1$ $a_3 = 1$ and $a_4 = \sqrt{a}$. We can find the root ξ_1 using Maple and then calculate c_i, γ_1 and $G(\xi_1)$, $i = 1,\ 3$ and obtain an inequality in a and μ by inserting these values in (4.4.7).

$$c_1 = -\mu\left(\frac{a^{\frac{5}{2}}}{5} - \frac{(a+1)a^{\frac{3}{2}}}{3} + a^{\frac{3}{2}}\right)$$

$$c_3 = \mu\left(-\frac{2}{15} + \frac{2a}{3}\right)$$

$$G(\xi_1) = \frac{5a}{12} + \frac{5}{12} - \frac{\sqrt{25a^2 - 130a + 25}}{12}$$

$$\gamma_1 = \frac{a}{2}$$

and so (4.4.7) becomes

$$\frac{5a}{6} + \frac{5}{6} - \frac{\sqrt{25a^2 - 130a + 25}}{6} + \mu^2\left(-\frac{2}{15} + \frac{2a}{3}\right)^2 +$$

$$+2\mu\left(-\frac{2}{15} + \frac{2a}{3}\right)\sqrt{a} - \mu^2\left(\frac{a^{\frac{5}{2}}}{5} - \frac{(a+1)a^{\frac{3}{2}}}{3} + a^{\frac{3}{2}}\right)^2 \leq 0\ . \tag{4.4.8}$$

The function $a(\mu)$ implicitly defined by (4.4.8) with the inequality replaced by equality represents the upper bound for the bifurcation curve. This function is shown in figure 4.10

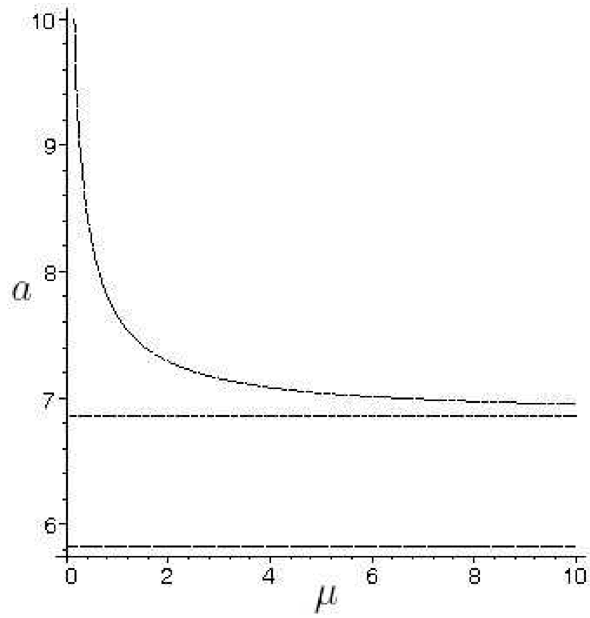


Figure 4.10: The upper bound found by Lloyd

As can be seen from the figure, the function $a(\mu)$ may only work as a good approximation for large μ. When $\mu < 1$, it is better to utilize the techniques presented in section 3.3. Dividing the relation (4.4.8) by μ^2 and letting $\mu \to \infty$ we get

$$\left(-\frac{2}{15} + \frac{2a}{3}\right)^2 - \left(\frac{a^{\frac{5}{2}}}{5} - \frac{(a+1)a^{\frac{3}{2}}}{3} + a^{\frac{3}{2}}\right)^2 = 0$$

The roots are given by

$$a = \frac{7 \pm \sqrt{45}}{2}\ .$$

The upper bound given by Lloyd is thus converging to the exact value of a as μ tends to infinity.

For the lower bound we have that $\sqrt{a} < 1$. We get $a_1 = -1$, $a_2 = -\sqrt{a}$, $a_3 = \sqrt{a}$ and $a_4 = 1$. From these we once again find the values of the relevant constants in (4.4.7) and get the following inequality in μ and a

$$\frac{5a}{6} + \frac{5}{6} - \frac{\sqrt{25a^2 - 130a + 25}}{6} - \mu^2\left(-\frac{2}{15} + \frac{2a}{3}\right)^2 +$$
$$+2\mu\left(\frac{a^{\frac{5}{2}}}{5} - \frac{(a+1)a^{\frac{3}{2}}}{3} + a^{\frac{3}{2}}\right) + \mu^2\left(\frac{a^{\frac{5}{2}}}{5} - \frac{(a+1)a^{\frac{3}{2}}}{3} + a^{\frac{3}{2}}\right)^2 \leq 0$$

Dividing by μ^2 and letting $\mu \to \infty$ we find the same equation as before.

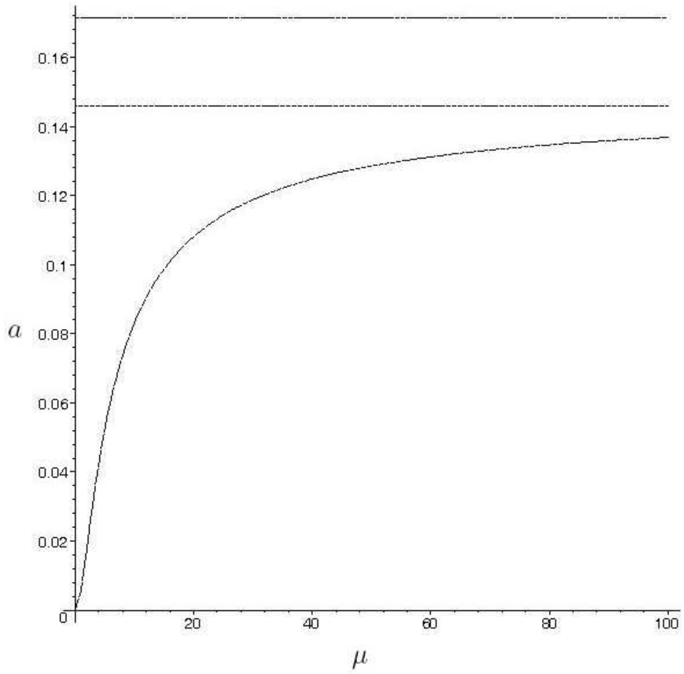


Figure 4.11: The lower bound found by Lloyd

In connection with the theoretical upper bound exhibited by Lloyd, one of his students, J.M. Abdulrahman (see [**?**]), also made some numerical investigations comparing how good this upper bound actually is. The results are given in the table below, where $\hat{a}(\mu)$ represents the theoretical upper bound and a_{num} represents the numerical value.

μ	$\hat{a}(\mu)$	a_{num}	μ	$\hat{a}(\mu)$	a_{num}
0.25	9.14	5.95	2	7.29	6.75
0.5	8.24	6.20	2.5	7.21	6.78
0.75	7.86	6.41	3	7.15	6.79
1	7.65	6.56	4	7.08	
1.5	7.41	6.89	5	7.04	

According to the numerical values shown here it seems that the bifurcations curve has a local maximum and is not strictly monotone. Indeed, for the values $\mu = 1.5,\ 2$ we see that a_{num} is first increasing and then decreasing. If this were correct one would like to explain this type of behavior. It would mean that for some values of a we would have 2 limit cycles for small μ which would disappear as μ increases and then appear again. This seems rather strange. We have made some numerical investigations of our own which we present in the following table.

μ	$\hat{a}(\mu)$	a_{num}	μ	$\hat{a}(\mu)$	a_{num}
0.25	9.14	5.9	2	7.29	6.75
0.5	8.24	6.13	2.5	7.21	6.78
0.75	7.86	6.41	3	7.15	6.79
1	7.65	6.56	4	7.08	6.82
1.5	7.41	6.69	5	7.04	6.83

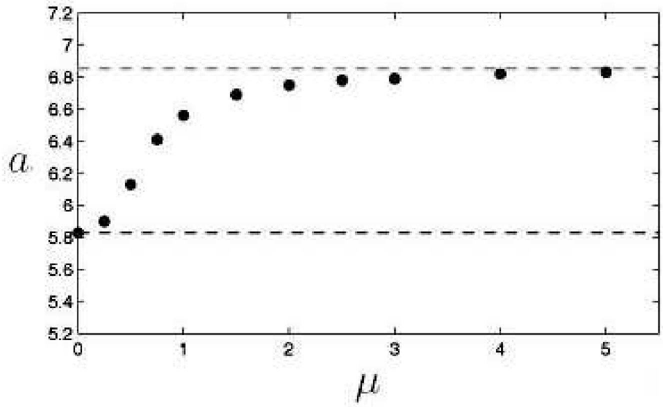


Figure 4.12:

We get almost the same result as Lloyd with one important exception. When $\mu = 1.5$ we obtained the value $a_{num} = 6.69$. It thus seems that the bifurcation curve is indeed monotone increasing. We also present our numerical investigations for the lower bifurcation curve:

μ	a_{num}	μ	a_{num}	μ	a_{num}	μ	a_{num}
1	0.171	18	0.163	35	0.153	80	0.148
5	0.170	22	0.160	40	0.152	100	0.147
8	0.169	25	0.158	50	0.151	200	0.146
12	0.167	28	0.157	60	0.150		
15	0.165	32	0.155	70	0.149		

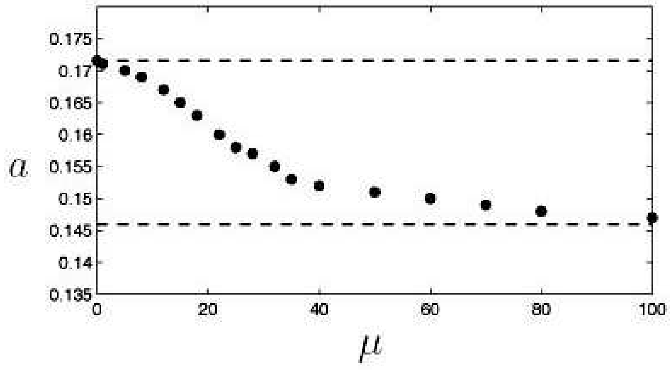


Figure 4.13:

We state the following conjecture:

Conjecture 4.4.2. *The bifurcation diagram in the (μ, a)-plane consists of two bifurcation curves, $a_{up}(\mu)$ and $a_{low}(\mu)$ where $a_{up}(\mu) > a_{low}(\mu)$. We know that*
1. $\lim_{\mu\to 0} a_{up}(\mu) = 3 + \sqrt{8}$ *and* $\lim_{\mu\to\infty} a_{up}(\mu) = \frac{7+\sqrt{45}}{2}$
2. $\lim_{\mu\to 0} a_{low}(\mu) = 3 - \sqrt{8}$ *and* $\lim_{\mu\to\infty} a_{low}(\mu) = \frac{7-\sqrt{45}}{2}$
We conjecture that the functions $a_{up}(\mu)$ and $a_{low}(\mu)$ are monotone increasing and decreasing respectively. We thus conjecture that the values a_* and a^* are given by $a_* = \frac{7-\sqrt{45}}{2}$ and $a^* = \frac{7+\sqrt{45}}{2}$.

Notice how this conjecture, if correct, renders Lloyd's theoretical upper bound completely superfluous. The bound $a = a^*$ will indeed be more accurate then the one exhibited by Lloyd.

4.5 A lower bound for the bifurcation curve

In section 4.4 we presented an upper bound for the bifurcation curve derived by Lloyd. We are now going to discuss the possibility of deriving strict lower bounds for the bifurcation curve(s). The method presented is due to Giacomini and Neukirch, see [4] and [5].

It is not difficult to derive some criterion ensuring the nonexistence of limit

cycles. We started this chapter by showing that for $\frac{1}{5} < a < 5$ there cannot exist any limit cycles regardless of the value of μ. This is of course, as we have seen, not a very precise result. It was based on the fact that the energy function $E(x, y)$ becomes a global Liapunov function for the system when a lies in the prescribed interval. Can we find a better function to do the same job but also yielding some relation between μ and a? We approach this question through the following problem: to find algebraic curves approximating the limit cycles of the Liénard system from within. Let us consider an easy example on the van der Pol equation.
Let $h(x, y) = y^2 + g_1(x)y + g_0(x)$ be a polynomial in two variables. We want the level curves $h(x, y) = C$ to represent some kind of approximation to the unique limit cycle in the van der Pol system. For the sake of simplicity we assume the value of μ is $\mu = 1$. Since the Liénard system is symmetric with respect to the origin we want the polynomial $h(x, y)$ to have the same symmetry property, i.e we require that $h(x, y) = h(-x, -y)$. This means that $g_1(x)$ will have to be an odd polynomial and that $g_0(x)$ will have to be even. Consider now the derivative of $h(x, y)$ with respect to the van der Pol system:

$$\dot{h}(x, y) = \frac{\partial h}{\partial x}(y - F(x)) - \frac{\partial h}{\partial y}x. \tag{4.5.1}$$

Suppose that we impose the condition that this derivative should be a function of x only,

$$\dot{h}(x, y) = R(x) .$$

Let L be the unique limit cycle of the van der Pol system. If we integrate $R(x)$ along this cycle we have

$$\int_0^T R(x(t))dt = \int_0^T \dot{h}(x(t), y(t))dt = h(x(T), y(T)) - h(x(0), y(0)) = 0$$

where T is the period of L. We thus see what the point is in requiring that $\dot{h}(x, y) = R(x)$. If $R(x) < 0$ for some interval $|x| < r$ then the above integral shows that L cannot be contained in this interval. The level curves $h(x, y) = C$ lying in the region $|x| < r$ will thus be completely contained in L. There will be a maximal value C^* such that $h(x, y) = C^*$ is the largest such level curve and this level curve will thus form an algebraic approximation to L. Notice that this should work for any Liénard system. Let us make the above computations explicitly for the van der Pol system.

$$\dot{h}(x, y) = \frac{\partial h}{\partial x}(y - F(x)) - \frac{\partial h}{\partial y}x =$$

$$y(g_1'(x)y + g_0'(x)) - F(x)(g_1'(x)y + g_0'(x)) - x(2y + g_1(x)) =$$

$$= y^2 g_1'(x) + y(g_0'(x) - F(x)g_1'(x) - 2x) - F(x)g_0'(x) - xg_1(x) \ .$$

In order to eliminate y we require that

$$g_1'(x) = 0, \ g_0'(x) - F(x)g_1'(x) - 2x = 0 \ .$$

Since $h(x,y)$ is supposed to be symmetric with respect to the origin we must have $g_1(x) = 0$. We thus obtain $g_0(x) = x^2 + c$. For convenience we put the integration constant equal to zero and find $h(x,y) = y^2 + x^2 = 2E(x,y)$. We notice that $R(x) = -2x^2(\frac{x^2}{3} - 1)$ has only 1 positive root at $x = \sqrt{3}$ and so the level curve $h(x,y) = 3$ is the largest one which lies within the region $|x| \leq \sqrt{3}$. This level curve is not a good approximation, but what happens if we let $h(x,y)$ be higher degree polynomial in y? Let us put

$$h_n(x,y) = y^n + g_{n-1,n}(x)y^{n-1} + ... + g_{1,n}(x)y + g_{0,n}(x)$$

and make the same requirements as before, i.e that $\dot{h}_n(x,y) = R_n(x)$. For example we can compute h and R for $n = 4$.

$$\dot{h}_4(x,y) = (y - F(x))\left(\sum_{j=0}^{3} g_{j,4}'(x)y^j\right) - x\left(\sum_{j=1}^{4} jy^{j-1}g_{j,4}(x)\right) =$$

$$= g_{3,4}'(x)y^4 + \sum_{j=1}^{3}\left(g_{j-1,4}'(x) - F(x)g_{j,4}'(x) - (j+1)xg_{j+1,4}(x)\right)y^j -$$
$$-F(x)g_{0,4}'(x) - xg_{1,4}(x) = R_4(x) \ .$$

We see that $g_{3,4}'(x) = 0$ and since $g_{3,4}(x)$ should be odd we must have $g_{3,4}(x) = 0$. We put $g_{4,4}(x) = 1$ and so we find $g_{2,4}(x)$ from $g_{2,4}'(x) = 4x$. We henceforth put all the integration constants equal to zero, so $g_{2,4}(x) = 2x^2$. We are now left with the following equations:

$$g_{1,4}'(x) = F(x)g_{2,4}'(x) + 3xg_{3,4}(x) \ ,$$
$$g_{0,4}'(x) = F(x)g_{1,4}'(x) + 2xg_{2,4}(x) \ .$$

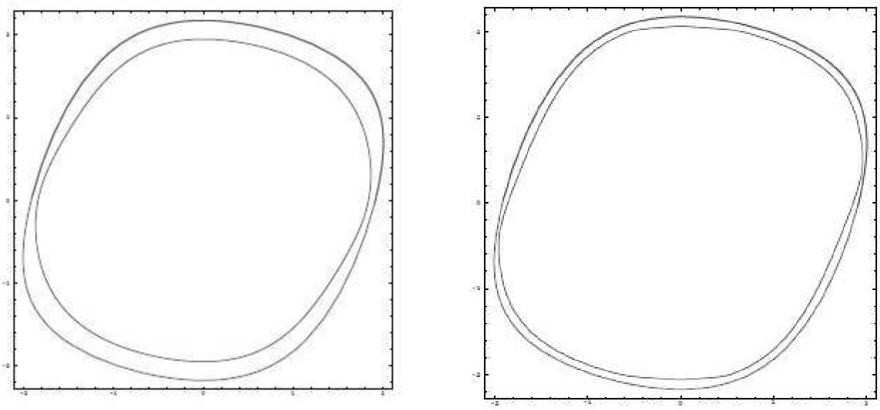
These are easily solved and we see that

$$g_{1,4}(x) = \frac{4}{15}x^5 - \frac{4}{3}x^3 \ , \ g_{0,4}(x) = \frac{1}{18}x^8 - \frac{4}{9}x^6 + 2x^4 \ .$$

This yields

$$R_4(x) = \frac{-4}{27}x^{10} + \frac{4}{3}x^8 - \frac{208}{39}x^6 + \frac{28}{3}x^4 \ .$$

The positive root of $R_4(x)$ is approximately given by $x = 1.824$, which is larger than $\sqrt{3} \approx 1.732$. The largest level curve is given by $h_4(x, y) = h_4(1.824, F(1.824))$.
Notice that the calculations above will go through for any value of n, although it will only be even n which yields closed level curves of $h_n(x, y)$. In all the cases we get a recursive system of first order differential equations which is always "trivial" to solve. What is interesting here is that the roots of $R_n(x)$ seem to converge in a monotone fashion to the amplitude of the limit cycle, which means that the level curves $h_n(x, y) = h_n(r_n, F(r_n))$, where r_n are the roots of $R_n(x)$, become better and better approximations of this cycle.



(a) $n = 6$ (b) $n = 16$

Figure 4.14: The inner closed curves are the algebraic approximations for the van der Pol cycle for different values of n, while the outer curves represent the cycle itself. Notice how the algebraic curves tend closer to the limit cycle as n increases.

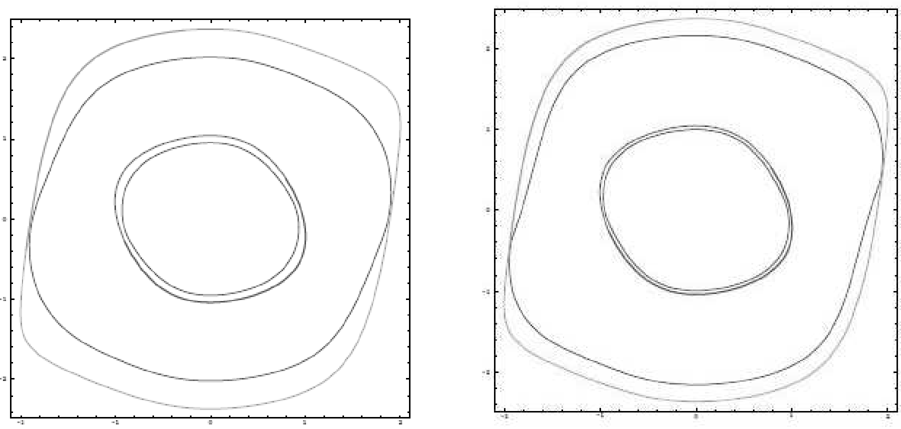
At all times the number of positive roots of $R_n(x)$ seems to be equal to one according to the research done by Neukirch, at least for the van der Pol system. Neukirch has also made several investigations on the connections between the roots of the polynomials $g_{j,n}(x)$, $R_n(x)$ and the amplitude of the limit cycles of the Liénard system. So far nothing has been proven, but some striking and highly interesting observations have been made. It seems that even for the Liénard systems where $F(x)$ is of higher degree than 3 the roots of $R_n(x)$ will correspond to different limit cycles and the level curves of $h_n(x, y)$ will form better and better algebraic approximations to these cycles. It may happen that the number of positive roots of the polynomials R_n exceeds the number of limit cycles but with increasing n the superfluous roots seem to disappear. In the article [4] the following conjecture is formulated

Conjecture 4.5.1. *(see [4]) Let l be the number of limit cycles of the Liénard system. Let r_n be the number of positive roots of $R_n(x)$ (with n even) of odd multiplicity. Then we have*
i) $l \leq r_n$ $\forall n$ even
ii) if $n' > n$, then $r_n - r_{n'} = 2p$, $p \in \mathbb{N}$

Guided by this conjecture and the results leading to it Giacomini and Neukirch (G&N) have provided the following interesting method to exhibit a lower bound for the bifurcation curve of some first order systems. Consider the system

$$\begin{cases} \frac{dx}{dt} = y - \mu F(x) \\ \frac{dy}{dt} = -x \end{cases} \tag{4.5.2}$$

with $F(x) = \int_0^x (s^2 - 1)(s^2 - a)ds$. $F(x)$ has 2 positive roots for any value of $a > 0$. We expect the number of positive roots to the polynomials $R_n(x)$ to be equal to 2. The experimental results of G&N supports this idea.



(a) $n = 6$ (b) $n = 16$

Figure 4.15: The algebraic approximations of the limit cycles of the Liénard system with $F(x) = 0.8x - \frac{4}{3} + 0.32x^5$ for different values of n. The inner closed curves are the algebraic approximations and the outer ones are the limit cycles.

These roots will of course depend on μ and a. If we pose the condition

$$R_n(x) = 0, \ \frac{d}{dx} R_n(x) = 0$$

we will get a relation between μ and a ensuring a double root for the polynomial R_n. This double root will then correspond to the existence of single limit cycle of multiplicity two. With increasing n this relation will hopefully make better and better approximations to the bifurcation curve in the μa-plane. These approximations will be lower bounds to the bifurcation curve since the level curves of $h_n(x, y)$ forms algebraic approximations to the limit cycles from within. In [5] G&N investigated the Liénard system where $F(x) = x^5 + \delta x^3 + \epsilon x$, which is slightly more general than the system studied here. The results were a sequence of lower bounds to the bifurcations curve.

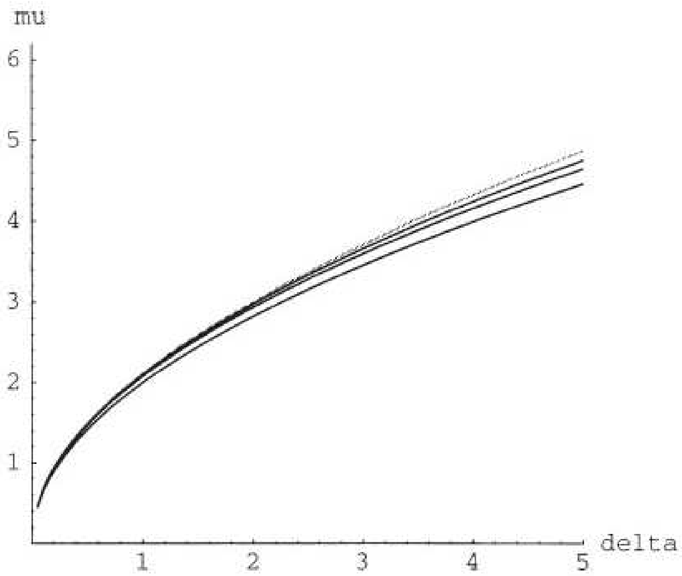


Figure 4.16: The dotted curve represent the numerical values for the bifurcation curve. We see how it is approached by the successive algebraic curves with increasing n.

Notice how general and flexible this method is. Potentially it can be applied to any polynomial vector field yielding good information about the existence, location and number of limit cycles and the bifurcation diagram of the relevant parameters. However, as we have already pointed out, nothing has so far been proven. This method is more experimental than it is deductive, at least at present date.

Conclusions

We have been studying the number and nature of periodic solutions of the Liénard equation $\ddot{x} + \mu f(x)\dot{x} + x = 0$. We have merely managed to settle a few specific problems concerning these. There are still a great number of questions to be answered and we shall point out some of them here.

To begin with, Poincarés method described in section 4.3 is indeed quite satisfying. If the roots of the polynomial $\Phi(\alpha)$ are all simple then this method gives a complete description of the position and number of limit cycles in the phase plane, but what happens in the occurrence of multiple roots? The problem becomes much more difficult and although there are some treatments on this subject (see [8]) there is still a lot to be done. Additional investigations concerning the monotonicity of the bifurcation curve is also in place and perturbation theory seems to be a good place to start at.

The aim of this thesis has been mainly to exhibit some qualitative features of the bifurcation curve in the (μ, a)-plane. Lloyd has provided a theoretical upper bound to this curve while Giacomini and Neukirch have provided a sequence of lower bounds. The obvious advantage of Lloyd's result over G&N's is that Lloyd has actually proved his result. On the other hand G&N gives us an interesting algebraic method which is much more flexible than that of Lloyd's. Numerical investigations seem to comply well with their theoretical suggestions obtained through this method. But it is far from obvious how to make this experimental method into a valid mathematical theory. One would also like to extend this method in order to get upper algebraic bounds converging towards the bifurcation curve. If it is possible to provide some rigorous theory showing how to yield successive algebraic approximations in the manner of G&N one may perhaps extend this theory to a wider range of polynomial vector fields.

As a final remark we wish to point out how both the results of Lloyd and G&N hinge crucially on Rychkov's theorem, i.e on the knowledge of the exact upper bound for the number of limit cycles. All the investigations concerning the location and shape of the cycles use the theorem of Bendixson in some way. As already pointed out this theorem may be efficient for finding cycles but it is useless in trying to find out how many limit cycles that actually occur in some region of the plane. It is thus a tool most powerful when we already have some information about the possible multiplicities of the limit cycles.

Index

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