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On the Use of History in Calculus Education

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Abstract

Mathematics has history, but mathematical concepts, theorems and methods are often taught as if they were eternal truths independent of people and culture. The purpose of this paper is to show how calculus education can benefit with inspiration from the history of calculus.

There are two main parts in this paper. The first part deals with the history of calculus starting with functions and continuing with limits and continuity, differentiation, and integration. In the second part I suggest some reasons and methods for, as well as problems with, integrating the history of calculus with education.

Sammanfattning

Matematik har historia, men matematiska begrepp, satser och metoder undervisas ofta som om de vore eviga sanningar oberoende av människor och kultur. Syftet med den här uppsatsen är att undersöka hur undervisning av grundläggande analys kan förbättras med hjälp av inspiration från analysens historia.

Uppsatsen är tvådelad. I den första delen behandlas analysens historia utifrån utvecklingen av funktionsbegreppet, gränsvärden och kontinuitet, derivator och differentierbarhet, och slutligen integraler och integrerbarhet. I den andra delen föreslås anledningar till, metoder för och problem med att integrera analysens historia i undervisning.

Contents

1. INTRODUCTION 2

2. HISTORICAL CONTEXT 5

2.1 Functions and general development 5
- 2.1.1 Numerical tables, proportions and curves – early mathematics 5
- 2.1.2 The calculus of Newton and Leibniz 6
- 2.1.3 Euler and the concept of "function" 7
- 2.1.4 Pathological functions and generalisation in the 19th century 8

2.2 Limits and continuity 10
- 2.2.1 Early limits and continuity: motion and infinitesimals 10
- 2.2.2 The changing definitions of limit and continuity 11
- 2.2.3 Misconceptions about and problems with continuity 13

2.3 Derivation 13
- 2.3.1 Derivation in the 17th and 18th centuries 14
- 2.3.2 The rigorisation of derivation under Cauchy in the 19th century 16

2.4 Integration 17
- 2.4.1 Cavalieri and Wallis 18
- 2.4.2 Integration under Newton, Leibniz and the Bernoullis 19
- 2.4.3 Integration in the 19th century 20

3. USING THE HISTORY OF ANALYSIS TO IMPROVE EDUCATION 23

3.1 The "Why?" of using history in mathematics education 23
- 3.1.1 Some views from the literature 23
- 3.1.2 How does the use of history with education solve the three problems? 25
- 3.1.3 Epistemological frameworks 26

3.2 The "How?" of using history in mathematics education 29
- 3.2.1 The genetic approach to calculus 29
- 3.2.2 Outline for programme of calculus education 33

3.3 Potential problems 45

4. SUGGESTIONS FOR FURTHER RESEARCH 47

5. SUMMARY 48

WORKS CITED 49

1. Introduction

The mathematics education in Swedish post-compulsory (pre-university as well as university) education lacks almost any element of historical development. This is true as much of the goal of mathematical education as it is of the design of the courses. For instance, in the "goal" section of the course plans there is a complete lack of any mention of history or development of mathematics. It is only when discussing the criteria for the highest marks that Skolverket (the National Agency for Education) suggests that:

> The student gives examples of how mathematics developed, how it has been used throughout history and what its influence is in our time and in some different areas.[1]
>
> The student explains something of how mathematics is and has been affecting the development of our working and societal life and our culture.[2]

At the university level this absence is as conspicuous. Glancing through the course descriptions of mathematical departments at Stockholm and Gothenburg universities the only reference to history is an optional course in the development of mathematics. Obviously concerns about the history of mathematics do not figure in ordinary mathematics education.

This is also seen when one looks at the design of the courses in calculus. With few exceptions, the topics treated are in this sequence: function, limits and continuity, differentiation, and integration. Within these topics, new definitions and theorems are often introduced with barely any motivation. As I shall show below, this is in clear opposition to the historical development of calculus.

Regarding the explicit use of history, in Swedish textbooks history is sometimes mentioned by way of giving a face and a name to the concept taught. There are also brief biographical notices of prominent mathematicians. This approach does tend to humanize mathematics somewhat but it lacks the potential to show the students how mathematics came into being: how it is continuously created and why.

I believe that this leads to emphasis on memorization and routine solving of uninspiring exercises, with little attention to creativity, logic, and the balanced use of formal and informal reasoning. Without these core tools, or perhaps "aspects", of mathematics, students are unable to understand even the *necessity* of stringency in mathematical proofs. This gives rise to difficulties for the students when they attempt to understand how to use formal and informal reasoning when thinking about mathematics. In the worst case this might lead to the feeling that mathematics consists of formulas and is dry, shallow and inhuman.

Manya Raman discusses this when she examines how American textbooks in different levels of mathematical analysis present mathematical concepts and theorems. She studies textbooks in precalculus, calculus, and analysis, observing the differences between them. She

[1] "Eleven ger exempel på hur matematiken utvecklats och använts genom historien och vilken betydelse den har i vår tid inom några olika områden." Skolverkets criteria for "Väl Godkänt" in high school courses Matematics A – E. Nov 2006.

[2] "Eleven redogör för något av det inflytande matematiken har och har haft för utvecklingen av vårt arbets- och samhällsliv samt för vår kultur." Skolverkets criteria for "Mycket Väl Godkänt" in high school courses Matematics A – E. Nov 2006.

focuses on how the textbooks present the concept of continuity, but I consider her discussion to be equally valid concerning other topics in calculus.

From her criticism I have selected three main points to which I will return later:

1. There is little motivation for the introduction of new concepts, theorems and methods.
2. The change from informal to formal reasoning is sharp and unmotivated.
3. The role of the problems that the students are intended to solve is unclear.

There are many examples of these issues. One main, which will serve to illustrate the ideas ahead, is how teachers motivate the students to accept that a positive derivative implies an increasing function. The actual theorem is:

Main Theorem: Suppose f is differentiable in (a, b).

a. If $f'(x) \geq 0$ for all $x \in (a,b)$, then f is monotonically increasing.

b. If $f'(x) \leq 0$ for all $x \in (a,b)$, then f is monotonically decreasing.

c. If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant.

In Swedish high school textbooks today, this fact is presented excluding the cases where $f'(x) = 0$. The theorem is motivated intuitively, by pointing to the relationship between a positive derivative and an upward-tilted tangent, which implies that the graph is slanted upwards. One textbook states that:

> From the geometrical interpretation of the derivative as slope of a tangent follows that if the derivative is positive then the graph is rising.[3]

Then the textbooks attempts to make the argument more "precise"[4] by showing an increasing graph with tangents and stating that "Apparently one can use the derivative to decide whether a function is increasing or decreasing. Generally it is true that:" and then follows the statements of the theorem without any mention of the word "theorem".[5] Other textbooks skip even this basic motivation, and simply imply that because an increasing graph has a non-negative derivative,[6] the reverse is true as well. Of course, anyone familiar with the difference between "if" and "if and only if" would object to such reasoning. The fact that today's students do not object I consider due to their lack of understanding of basic logic and the necessity for formality in proofs. The formal proof of this theorem, however, is much more elaborate and, depending on level of desired strictness, requires knowledge of the Real Number system and the details of continuity. Different formal proofs will be outlined below.

Given the situation described above, my purpose is to suggest in what way the integration of the history of calculus into calculus education can improve the students' learning of calculus. I will first describe some essentials in the development of calculus. Subsequently I will discuss the potential benefits of using the history of calculus in education, as well as some ways in which one might approach such an integration. I will also attempt an outline of calculus education, ranging from the very elementary to the more theoretical and complex. Neither this outline nor any other sections on education are specifically intended for either

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[3] Björk, Brolin & Munther s 134: "Av derivatans geometriska tolkning som riktningskoefficient för en tangent följer att om derivatan är positiv så stiger grafen."

[4] Björk, Brolin & Munther s 134: "Vi ska nu försöka precisera detta resonemang."

[5] Björk, Brolin & Munther s 134: "Man kan tydligen med hjälp av derivatan avgöra om en funktion växer eller avtar. Allmänt gäller:"

[6] Since for an increasing graph f(x-h)-f(x) is positive, and the derivative is defined in terms of the secant as it approaches the tangent, the derivative is positive as well.

secondary or university level but rather are meant to be flexible to accomodate the particular goals and needs of individual educational institutions and situations.

I have limited my research to include only texts on the history of mathematics published in the last 40 years. It would have been interesting to compare different approaches to the history of mathematics throughout the last century, or even to use original texts from the mathematicians themselves, but the time constraints of this paper made that option too ambitious. My main source for the chapter on history is the anthology *History of Analysis* edited by Jahnke. Regarding education, I have relied primarily on the ICMI study *History in Mathematics Education*, which covers a broad spectra of issues related to this topic. Also Toeplitz' *The Calculus: a Genetic Approach* and Bressoud's *A Radical Approach to Real Analysis* have been very helpful not least in elucidating the practical aspects of integrating history in calculus education.

2. Historical context

In this section I will present some aspects of the history of calculus that I feel are pertinent to the education of calculus. I will use the traditional sequence of instruction; starting with functions and miscellaneous ideas, I will continue first to limits and continuity, then to derivatives and differentiation, and finally conclude with some history of the integral and integration.

2.1 Functions and general development

2.1.1 Numerical tables, proportions and curves – early mathematics

Though the concept of function is quite recent, the idea of one quantity being dependent on another is ancient. The Babylonians constructed tables of squares, cubes, and many more relationships between values.[7] One such table of reciprocals looked like this:[8]

2	30	16	3, 45	45	1, 20
3	20	18	3, 20	48	1, 15
4	15	20	3	50	1, 12
5	12	24	2, 30	54	1, 6, 40
6	10	25	2, 24		
8	7, 30	27	2, 13, 20		
9	6, 40	30	2		
10	6	32	1, 52, 30		
12	5	36	1, 40		
15	4	40	1, 30		

It tells us that, for instance, the reciprocal of 2 is 30/60, the reciprocal of 40 is $1/60 + 1/60^2$ and the reciprocal of 54 is $1/60 + 6/60^2 + 40/60^3$.[9]

The Greeks thought in terms of proportions, considering for example that given two strings with the shorter being half the length of the longer, the note produced by the shorter is the same sound but with a higher pitch than that of the longer.[10] The Greeks also studied the relationships of the sides of triangles to each other and to the angles in the triangle. In this way, they arrived at the elements of what now is called trigonometry.[11] Curves were also analysed by the Greeks, who interpreted curves in terms of kinematics. Thus, for instance, Archimedes thought of a spiral as being produced by a point moving on a half-line which in turn is rotating about its origin.[12] Following an understanding based on mechanics, the Greeks then formally proved the property being investigated.

[7] Kleiner, 1993
[8] Melville
[9] Kleiner 184
[10] Kleiner 184
[11] Kleiner 184
[12] Thiele 29

During the 13th and 14th centuries mathematics became an important tool in the study of natural phenomena. An important development is attributed to the philosopher Oresme, who was the first to represent physical processes in terms of graphs. For instance, when it came to graphing velocity, he represented time on the horizontal axis and for each moment of time drew vertical lines the height of which represented the velocity at that moment.[13] Then, in the 16th and 17th centuries men like Kepler and Galileo brought mathematics into many more questions in physics. The mathematical tools used were curves and proportions (similar to today's equations) to describe physical events and relations. At the same time analytic geometry was invented by Fermat and Descartes. They developed the art of representing curves by analytic expressions, which in turn led to the invention of an infinite number of curves where only a dozen had previously existed.[14] One could perhaps say that a shift was taking place; mathematics was once again becoming independent of the natural sciences and pursued for its own rewards.

2.1.2 The calculus of Newton and Leibniz

Newton and Leibniz are two names intimately associated with the mathematics of the 17th and 18th centuries. Commonly it is stated that these two men invented the calculus. Guicciardini points out that this simplification is unrealistic, and discusses their work instead in terms of three major contributions: reducing a myriad of problems to the two cases of quadrature and tangents, realizing the inverse relationship between these, and the creation of algorithms in general and especially for calculating differentials and integrals.[15] Some of these contributions will be discussed elsewhere in this paper; for now I want to focus on foundational questions having to do with notions of "functions" – though the term is premature at this stage.

Newton's contribution to what later would be called functions was primarily his recognition of the fact that infinite series are useful for describing curves, in particular difficult ones hard to handle directly in their closed form.[16] He interpreted curves and other geometric objects at first in terms of fluents and fluxions, moments and time – thus continuing the trend of interpreting mathematical objects through intuition related to physical processes. Newton imagined that geometrical objects were created by the movement of other such objects through the process of *flow*.[17] Thus, a curve was generated by the movement of a point in space, and a plane was generated by the movement of a line. The generated quantity was what Newton referred to as the *fluent*, and the instantaneous speeds were the *fluxions*.[18] The *moments* were infinitely small additions to the fluent generated in infinitely small intervals of time.[19] These moments were infinitesimals, and Newton operated with them haphazardly – sometimes dividing by them and at other times discarding them because of their supposed equality with zero.[20] Though at first Newton employed algebraic symbols and equations freely, later he decided, partly because of his doubts concerning infinitesimals, in favour of geometry. He abandoned infinitesimals, and insisted that all mathematical objects be easily interpreted in concrete terms.[21]

[13] Kline 210, 211
[14] Kline 210, 211
[15] Guicciardini 73
[16] Guicciardini 76
[17] Guicciardini 78
[18] Guicciardini 78
[19] Guicciardini 78
[20] Guicciardini 80
[21] Guicciardini 84, 98

Leibniz shared neither Newton's emphasis on geometry nor his use of flow and other metaphors from the physical world. Instead, he preferred algebraic expressions and was very careful with notation, effectively creating the notation of derivation and integration that we use today. He conceived not of functions but of variable quantities related to one another. He was also interested in their differentials, which are the variables' infinitely small increments. Simplifying matters somewhat, it is also possible to contrast Leibniz' approach to Newton's by noting that where Newton's variables varied in time and space, Leibniz' variables were thought to vary only over sequences of values infinitely close to each other. This would put Leibniz' approach a little ahead of its time, but it should be noted that both men changed their interpretations repeatedly.[22] In general, it can be said that their approaches were in most ways equivalent.[23]

2.1.3 Euler and the concept of "function"

During the early part of the 18th century calculus was still regarded as relating primarily to geometry. Then, during the middle decades of the century, there was a shift towards implicit algebra, and Euler in his *Introductio* referred to quantities in the sense of numbers rather than in terms of geometrical quantities.[24] Finally, in the late 18th century, Lagrange made his calculus explicitly algebraic. At the same time, less focus was given to physical problems and more to pure analysis, uninvolved with applications to geometry and the natural world.[25]

This development also brought changes in the ontological basis of analysis. Mathematics, according to Euler, was the science of quantity, but what was meant by this term changed during the 18th century. With Euler, "quantity" referred to "that which is capable of increase or diminution".[26] Jahnke gives examples such as money, area, and speed.[27] Increasingly, "quantity" was made more abstract and in calculations represented by letters of the alphabet. With this development came the possibility of including in the notion of "quantity" objects such as square roots of negative numbers, whose relation to the concrete world is far from obvious.

The infinitesimal and differential calculus of Newton and Leibniz was being reinterpreted as well. Euler argued that it is not the actual increments, differentials, of the variables that are interesting, but instead what should be examined are the ratios between different variables' differentials.[28] It proved difficult to operate with differentials in this way, with none being considered independent. Therefore, for the sake of easing calculations, mathematicians came to view some variables as independent and some as dependent. In the words of Jahnke:

> It became more and more accepted that one should calculate with functions and their "derivatives" rather than with variable quantities and their differentials.[29]

Kleiner suggests that the concept of function was introduced during the 18th century because there were by then enough examples of functions from which to give abstract generalisation.[30] What then did "function" actually mean? Johann Bernoulli was the first to

[22] Guicciardini 95, 96
[23] Guicciardini 96
[24] Jahnke 106, 107
[25] Jahnke 106
[26] Jahnke 107
[27] Jahnke 107
[28] Jahnke s 108
[29] Jahnke s 108
[30] Kleiner 187

use this term in mathematics, referring to arbitrary expressions containing variables relating to curves.[31] Leibniz liked this usage, and together they discussed how to designate functions by symbols.[32] Euler used the definition of Bernoulli, and gave it, in his 1748 *Introductio* as:

> A function of a variable is an analytic expression composed in any way whatsoever of the variable quantity and numbers of constant quantities.[33]

Here, *analytic expression* refers to all expressions formed by applying finitely or infinitely many times the algebraic operations. Euler also categorized functions into *algebraic* and *transcendental*, and algebraic functions into further subcategories.

Already in the 18th century some limitations were seen with this concept of function. One such limitation was that Euler wanted the solutions to the vibrating string problem to include those solutions which were not formed by a single analytic expression, in his words: "discontinuous" functions, such as curves drawn freely by hand.[34] The controversy about this problem continued for the last half of the 18th century, but the issue of which functions to admit is perhaps even more obvious in another of Euler's problems – the partial differential equation $\frac{\partial u(x,y)}{\partial x} = 0$ which admits any $f(y)$ regardless of shape and coefficients.[35] This example shows the full scope of Euler's 1755 revised definition of function:

> Those quantities that depend on others in this way, namely, those that undergo change when others change, are called functions of these quantities. This definition applies rather widely and includes all ways in which one quantity can be determined by others.[36]

General though this definition might seem, Euler continued to refer the term *function* only to those functions which he had included in his earlier definition.[37] Nevertheless, Jahnke finds it likely that this later definition of Euler's influenced later generalisations of the function concept.[38]

2.1.4 Pathological functions and generalisation in the 19th century

The 19th century was the century of rigorisation, in which analysis was given the foundation that we know today. Cauchy's *Cours d'Analyse* of 1821 was the first sign of this process, and in it Cauchy presented the concept of function explicitly and exclusively as the dependence of some variables upon others:

> If variable quantities are so joined between themselves that, the value of one of these being given, one can conclude the values of all the others, one ordinarily conceives these diverse quantities expressed by means of one among them, which then takes the name independent variable; and the other quantities expressed by

[31] Jahnke 114
[32] Jahnke 114
[33] Jahnke 114
[34] Jahnke s 124
[35] Jahnke s 127
[36] Jahnke s 126
[37] Jahnke 127
[38] Jahnke 127

> means of the independent variable are those which one calls functions of this variable.[39]

But once again, the formal definition does not tell us everything about how the concept of function was used. Cauchy immediately after the definition remarks that functions can be categorized as either explicit or implicit; explicit meaning that the equations giving the relations between the functions and the variables are algebraically solved and implicit meaning that these equations are not algebraically solved.[40] This implies that Cauchy still thought of functions as given by analytic expressions.

At the same time, Fourier was being careful not to assume anything about analytic expressions pertaining to functions. In 1822 he wrote:

> In general, the function *f(x)* represents a succession of values or ordinates each of which is arbitrary. An infinity of values being given to the abscissa x, there are an equal number of ordinates *f(x)*. All have actual numerical values, either positive or negative or nul. We do not suppose these ordinates to be subject to a common law; they succeed each other in any manner whatever, and each of them is given as it were a single quantity.[41]

Yet speaking of the convergence of Fourier series, Fourier assumed that arbitrary functions are continuous, which is not required by the definition of function. It was only with Dirichlet that continuity as well as analytic expression parted from the concept of function.[42] Dirichlet's definition was:

> y is a function of a variable x, defined in the interval $a<x<b$, if to every value of the variable x in this interval there corresponds a definite value of the variable y. Also, it is irrelevant in what way this correspondence is established.[43]

Another development was the change in desired generality. Whereas Euler had relied on the generality of algebra, assuming analytic expressions to be in some way meaningful everywhere, Cauchy insisted that such expressions be valid only where they are defined. Also, Gauss thought that algebraic formulas should only be used under the right conditions and with suitable limitations.[44] One example of the previous condition of calculus is that infinite series were employed carelessly with no concern as to their potential divergence.[45] Reasoning against such use of series, Cauchy moved calculus out of the generality of algebra. Much of this development was spurred by the development of Fourier series. These coincide with the function that they represent only on certain intervals, which necessitated a closer look at how functions, and the relationships of functions to each other, are limited to intervals.[46]

With Weierstrass, in the second half of the 19th century, came further rigorisation. Where Cauchy had used long-winding and vague language, Weierstrass insisted on more formal symbolic language now commonly referred to as his "epsilonic" style.[47] He also made important steps towards basing calculus on the real number system, which he properly

[39] Lützen 156
[40] Lützen 157
[41] Lützen 157
[42] Lützen 158
[43] Kleiner 204
[44] Kleiner 173,174
[45] Lützen 161.162
[46] Kleiner 198,199
[47] Kleiner 185

constructed. Weierstrass also worked with "pathological" functions, which are functions that seem very strange. For instance, a function that is everywhere continuous but nowhere differentiable was, based on these properties, deemed pathological. This trend of inventing new and strange functions was very different from former years, when functions had been invented from the modelling of physical processes. Some mathematicians were highly critical of this development. For instance, Poincaré commented that:

> In former times when one invented a new function, it was for a practical purpose; today one invents them purposely to show up the defects in the reasoning of our fathers and one will deduce from them only that.[48]

But the pathological functions did serve to show that Dirichlet's concept of functions was too general to be useful for the foundation of analysis.[49] Increasingly, mathematicians were obliged in their theorems and proofs to explicitly state the (sometimes numerous) assumptions. The simple and elegant statements of the past were replaced by complicated formulations reminiscent of legal jargon.

The 20th century brought rescue to the almost extinct infinitesimals and divergent series. Robinson constructed in the early 1960s a field extension[50] of $\Re$ in which infinitesimals were included. He was then able to rigorously prove many of the theorems used by Cauchy and others who had employed infinitesimals.[51] Other mathematicians generalised functions so that some non-differentiable functions could be differentiable.[52] This development might seem to mean that the rigorisation of the 19th century was unnecessary. Yet this so-called non-standard analysis rests on the rigorous foundations set up during the 19th century. What is spectacular about the developments of the 20th century is not that it cancels the work of Cauchy and Weierstrass but that it shows us that mathematics need not be predestined to develop in only one direction but is subject, like so many other things, to the creative impulses of the human mind.

2.2 Limits and continuity

2.2.1 Early limits and continuity: motion and infinitesimals

From antiquity limits have been intimately connected with physical processes. Zeno was one of the first to create an infinite series[53] and he did this partially to illustrate the problems with applying mathematical concepts such as "discrete" and "continuous" to intuitive physical processes.[54] Yet later, with graphs, came a new geometrical interpretation of limits. Newton, with his theory of fluxions and fluents, conceived of limits in terms of flow of variables through geometrical objects.[55] It seems that, not heeding the many voices of dissent, mathematicians until the 19th century at least privately thought in terms of motion and physical processes. This is not surprising especially considering that for large periods of time mathematics was intimately concerned with the applied sciences and the methods were, after all, surprisingly effective.

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[48] Poincaré (1899) quoted in Lützen 187, 188
[49] Lützen 188
[50] That is, a field encompassing a smaller field.
[51] Lützen 191
[52] Lützen 190, 191
[53] In the Achilles and tortoise paradox, for example.
[54] Kline 35
[55] Guicciardini 82

One aspect of 17th century mathematics blocking the development of rigour was the use of infinitesimals, already extensively used by the Greeks who already in antiquity had great doubts about their validity.[56] Infinitesimals were infinitely small quantities that were used alternatively as non-zero and zero quantities. Newton and Leibniz both disbelieved the existence of infinitesimals, and considered their work lacking mathematical rigor because of them, but commented repeatedly that infinitesimals were just a convenient way of denoting variables whose limits are zero. They agreed with each other that if one exchanged infinitesimals for limits the calculus would have solid foundations.[57] Newton, disenchanted with algebraic analysis, abandoned infinitesimals but Leibniz, whose analysis later developed into the one of today, continued to use infinitesimals freely. One example of how infinitesimals were used during the 17th and 18th century is the following calculation of the derivative of $y = x^2$:

$$\dot{y} = \frac{(x+dx)^2 - x^2}{dx} = \frac{2xdx + dx^2}{dx} = 2x + dx = 2x$$

where *dx* is an infinitesimal. The division above is possible because *dx* is not identical to zero but the removal of the last term is allowed because *dx* is considered to be zero. Not surprisingly, this approach gave rise to a number of contradictions and questions about rigor.

2.2.2 The changing definitions of limit and continuity

A prize problem was proposed in 1784 asking for an explanation of how it is possible that the contradictory theory of infinitesimals has given so many correct theorems, and for a mathematical principle to substitute instead of the infinitesimals. The answer came from Simon Lhuiller, who, like d'Alembert, defined limits as the value such that a variable can be made to differ from the value by an arbitrarily small amount. Though he also proved the product and division theorems for limits, introduced the notation "*lim*", defined *dy/dx* in the modern sense as the limit of the difference quotient, and remarked upon the important fact that variables need not monotonously approach the limit; Lhuiller's work did not become influential. One reason is that the definition of limits in terms of variables, physically and geometrically intuitive though it was, still carried some uncertainties. What could it mean, for instance, for a quantity to approach a given limit? It was not until limits were defined in terms of functions that the major contributions were achieved. Another reason, according to Grattan-Guinness, was simply that Lhuiller had written poorly and laboriously.[58]

Also in the 18th century, Euler gave a definition of continuity in which a function was continuous if it was given by a single analytic expression and discontinuous otherwise.[59] This understanding of continuity meant that for Euler,

$$f(x) = \{x^3 \text{ for all } x > 0 \text{ and } -x \text{ for all } x < 0\}$$

is a discontinuous function, while

$$f(x) = 1/x$$

[56] Toeplitz 61, 62
[57] Guicciardini 97
[58] Grattan-Guinness 101,102
[59] Kleiner 200

is continuous. When Fourier showed that many functions like the above could be represented by Fourier series, which are single analytic expressions, it became obvious that with Euler's definition of continuity some functions were continuous as well as discontinuous simultaneously. This motivated Cauchy and others to try to find a better definition of continuity.[60]

Whereas Euler thought of continuity as relating to algebra and the analytic expression of a function, Cauchy considered continuity by observing the graph of the function. Perhaps because Cauchy was very critical of the understanding of functions as global (see section 2.1.4 above), he was also open to interpreting continuity as a local, rather than global, quality.[61] Abrogast had previously investigated different ways in which the Euler-continuity could be broken, and he argued that one such way was by discontiguity of the function, by which he means jumps in the graph.[62] Cauchy had already observed that the proof of the fundamental theorem of calculus depended on the contiguity (or "no-jumps") property of some functions, so he knew that this property was worth investigating further.[63] It therefore became the focal point for Cauchy's understanding of continuity.

The breakthrough came with the understanding that limits need to be applied to functions of variables instead of to the variables themselves. Cauchy's definition of limit was:

> When the values successively attributed to a particular variable approach indefinitely a fixed value, so as to finish by differing from it by as little as one wishes, this latter is called the *limit* of all the others.[64]

While this definition seems to imply that the variable is in motion, Lützen points out that Cauchy seems always to have thought of variables in sequences s_n with n going to infinity.[65] Also in Cauchy's other definitions, the definition of continuity for example:

> The function *f(x)* will remain continuous with respect to x between given limits, if between these limits an infinitely small increase of the variable always produces an infinitely small increase of the function itself.[66]

It seems that Cauchy is thinking in terms of two variables where one changes in response to the other. Lützen therefore reaches the conclusion that Cauchy was already interpreting the limit in the way that we do today, giving meaning to statements like $f(x) \to a$ for $x \to b$ but not to statements like $x \to a$ by themselves.[67]

The phrasing in these definitions still includes terms like "infinitely small increase", and does not specify the order of increasing the variable or the function. As mentioned above, it was not until Weierstrass' re-interpretation of the variable as a letter symbolising any one of a set of values that the intuitive notions of time, motion and infinitely small quantities were eliminated from the calculus. During his time as a high school teacher Weierstrass formalised the definitions of continuity and limits into the ε, δ -notation that we are familiar with today.[68]

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[60] Lützen 165
[61] Lützen 164
[62] Lützen 165
[63] Lützen 165, 166
[64] Grattan-Guinness 109, 110
[65] Lützen 162
[66] Grattan-Guinness 110
[67] Lützen 162, 163
[68] Kline p 950-956

2.2.3 Misconceptions about and problems with continuity

At the same time continuity was a very different concept then from what it is today. Originally continuity was a taken for granted property of functions, though no definition of function ever implied anything of the sort. Then, as the use of the concept of function became more inclusive, continuity was considered a property of some functions but not others. What this property meant was still somewhat ambiguous. Kleiner lists some misconceptions which seemed as natural to mathematicians of the 19th century as they must do to students today:

- Continuity was confused with the idea of traceability, the ability to draw a curve without lifting the pen from the paper. This was remedied by the invention of pathological functions which met the formal requirements of the definition of continuity but failed to be traceable. One such function is $f(x) = x\sin(1/x)$ around $x = 0$.
- Another misconception was that continuity was dependent on the Intermediate Value Property which was the property of some functions defined on closed intervals to assume every value intermediate the values at the endpoints. Again a pathological function, $f(x) = \{\sin(1/x)$ for $x \neq 0$, and 0 for $x = 0\}$, showed that a function having the Intermediate Value Property on any closed interval may still fail to be continuous.[69]
- Continuity was believed to imply differentiability. This assumption was disproved to the mathematical community by Weierstrass, who introduced his everywhere continuous and yet nowhere differential function.
- Uniform continuity and uniform convergence were not fully developed concepts in the 19th century. Cauchy believed himself to have proven that infinite series of continuous functions were themselves continuous.[70] Once again a pathological function, $f(x) = \frac{1}{2}x$ over $[0, \pi]$,[71] this time provided by Abel, served as counterexample.[72]
- Also problems of clarity with upper and lower limits, particularly important for the concept of integral, took a long time to be resolved.

The development of the concept of limits proved to be crucial to the development of the calculus. Already Newton and Leibniz were convinced that the rigour of calculus could be given by the theory of limits. What was needed as well was unambiguous language and notation. This trend of careful notation was introduced by Leibniz and developed into the formal ε, δ -notation by Weierstrass.

2.3 Derivation

The history of derivatives starts with the history of tangents. These were used by the Greeks primarily for the description of objects when the objects were easier to analyse in terms of

[69] Kleiner 201
[70] Grattan-Guinness 120, 121
[71] This function has fourier series representation $\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ldots$, which has discontinuities for exery $x = (2m+1)\pi$
[72] Jahnke 178

tangents than in terms of area.[73] For instance, the tangent to the circumference of a circle is simply and statically given as the line which is perpendicular to the radius.[74] Likewise, the Greeks calculated tangents for ellipses, parabolas, and other curves.

So while in antiquity the tangents were primarily conceived of as static lines, relating to fixed geometric objects[75], with the advent of graphs of physical processes mathematicians became interested in tangents as a way to measure instantaneous velocity, acceleration and much more pertaining to physics.[76] Already by 1637 Fermat had developed a method of finding the extreme values of an algebraic expression *I(x)* which was very similar to the methods of today. He considered that infinitely nearby such a point *I(x)* would be constant. With *e* being an infinitesimal, Fermat set $I(x+e) = I(x)$. He then cancelled the common terms, divided by *e* and then cancelled all terms including *e*. The remaining equation was solved for x, giving the x-coordinate of the critical point.[77] Because the method yielded correct results, Fermat was not worried about the inconsistent use of *e*.[78]

2.3.1 Derivation in the 17th and 18th centuries

As mentioned above, Newton thought of curves in terms of fluents and fluxions. The instants of time, denoted by *o*, were combined with the fluxions $\dot{x}$ into $\dot{x}o$ which would then represent the incremental increases or moments. Then, given an algebraic expression, for instance $yx = 1$, Newton would proceed as follows:

He included the moments in the expression: $(y + \dot{y}o)(x + \dot{x}o) = yx + y\dot{x}o + x\dot{y}o + \dot{x}\dot{y}oo = 1$

Because $yx = 1$, the expression reduces to: $\dot{x}o/x + x\dot{y}o + \dot{x}\dot{y}oo = 0$

Dividing through by *o,* Newton arrived at: $\dot{x}/x + x\dot{y} + \dot{x}\dot{y}o = 0$

Cancelling the remaining term containing *o* and shifting terms, he arrived at: $\frac{\dot{y}}{\dot{x}} = -1/x^2$

Concerning the vanishing of terms containing *o*, Newton later had this to say:

> Ultimate ratios in which quantities vanish are not, strictly speaking, ratios of ultimate quantities, but limits to which the ratios of these quantities, decreasing without limit, approach, and which, though they can come nearer than any given difference whatever, they can neither pass nor attain before the quantities have diminished indefinitely.[79]

This seems to imply that Newton had an understanding of the difference quotient in terms of the limit of a quotient, much like we think of it today.

[73] Thiele 32
[74] Thiele 32
[75] Thiele 32
[76] Kline 342
[77] Van Maanen 49
[78] Kline 348
[79] Kline 365

While Newton dealt with infinitesimals for the purpose of calculating fluxions, Leibniz was interested in the ratios of the infinitesimals themselves.[80] He handled the differentials *dy* and *dx* directly, and thought of them as infinitesimal differences between two close values of *y* and *x.* During this time, Leibniz also correctly calculated (but gave no proofs) the differentials of sums, differences, products, quotients, powers and roots of functions. He attempted some explanations, but his writing was so muddled, fragmented and difficult to comprehend that it was only with the work of the Bernoulli brothers, who were taken with Leibniz' ideas, that his calculus took intelligible form.[81]

Other developments took place at about the same time. Michel Rolle stated (without proof) in 1691 what is now called Rolle's Theorem[82] which I will have reason to mention again further on. Newton and Raphson developed the Newton-Raphson method for the approximation of roots of $f(x)=0$.[83] The Bernoulli brothers used the second differentials in a theorem concerning the radius of the curvature of a curve,[84] and by Johann Bernoulli and l'Hospital was developed a method of calculating the limit of a fraction whose numerator and denominator both approach zero.[85]

Yet the main contribution of the time was to formally establish the relationship, until then only intuitively suspected, between differentials and integrals. Leibniz, influenced by the work of Barrow, argued at first for the inverse relationship of differentials and integrals by reasoning that "But $\int$ is a sum and *d* a difference."[86] Newton's approach was more empirical; he was led to believe in the inverse relationship by calculating the rate of change for areas under curves and finding them to be equal to the expression of the curve itself.[87] In 1669 Newton also proved that the derivative of the integral of *y* equals *y*, as well as the reverse.[88]

The calculus of the 17th century did not escape criticism. Among those who could not accept the contradictions and lack of formal proofs was Bishop Berkeley, who already in the title of his criticism spoke his mind: *The Analyst; or a Discourse Addressed to an Infidel Mathematician. Wherein it is examined whether the object, principles, and inferences of the modern Analysis are more distinctly conceived, or more evidently deduced, than religious Mysteries and points of Faith.* Berkeley was aware that this calculus led to correct results, and commented that the mathematicians arrive

> Though not at Science, yet at Truth, for Science it cannot be called, when you proceed blindfold and arrive at the Truth not knowing how or by what means.[89]

Among those who heeded this criticism were Mclaurin, of the English school, whose response was to strip Newton's theories of fluxions of all references to infinitesimals and return it to the Archimedean method of exhaustion and the proofs by double contradiction. True to English mathematics following Newton, McLaurin stuck to reasoning based on geometry and intuitive concepts of motion in time.[90] On the continental side, where mathematicians were busy developing the calculus of Leibniz, d'Alembert had the idea to

[80] Kline 379
[81] Kline 378
[82] Kline 381
[83] Kline 381
[84] Kline 382, 383
[85] Kline 383
[86] Kline 374
[87] Guicciardini 76
[88] Guicciardini 76, 77
[89] Bos 89
[90] Jahnke 127, 128

base differential analysis on limits, but he continued to operate with differentials in the way of his predecessors.[91]

It was not until the late 18th century, already in the middle of the age of algebraic analysis, that the terms "derivative", "derived function" and "primitive function" were introduced by Lagrange.[92] He was adamant that the derivative was a function in its own right, instead of just a ratio of differentials. Because of the limited idea of the concept of function at the time, Lagrange felt sure that most functions could be expressed by power series. He stated that it was possible to give the derivatives of a function by looking at the coefficients of the power series representation.[93] Lagrange put much work into founding the differential analysis on power series, but eventually the expanding concept of function made power series as a foundation impossible.

2.3.2 The rigorisation of derivation under Cauchy in the 19th century

In the 19th century differentiation was more rigorously described. Cauchy stated in 1823 the following about derivatives:

> When the function $y = f(x)$ is continuous between two given limits of the variable x, and one assigns a value between these limits to the variable, an infinitely small increment of the variable produces an infinitely small increment in the function itself. Consequently, if we then set $\Delta x = i$, the two terms of the *difference quotient*
>
> $$\frac{\Delta y}{\Delta x} = \frac{f(x+i) - f(x)}{i}$$
>
> will be infinitesimals. But while these terms tend to zero simultaneously, the ratio itself may converge to another limit, either positive or negative. This limit, when it exists, has a definite value for each particular value of *x*; but it varies with *x* …. The form of the new function which serves as the limit of the ratio $\frac{f(x+i) - f(x)}{i}$ will depend upon the form of the given function $y = f(x)$. In order to indicate this dependence, we give to the new function the name derivative and we designate it using a prime, by the notation y' of $f'(x)$.[94]

Here Cauchy seems to create the difference quotient for a random continuous function, and only afterwards reflect that that the limit might or might not exist. According to Lützen, Cauchy rarely stated his assumptions about the functions he dealt with, and when he did he often assumed continuity and then proceeded to differentiate.[95] Later on, the invention of pathological functions of course necessitated a clearer distinction between differentiability and continuity. One reason for the difficulties in separating differentiation and continuity was because of the un-rigorous use of infinitesimals. As Grattan-Guinness points out, some

[91] Jahnke 128
[92] Jahnke 128
[93] Jahnke 128
[94] Lützen 159
[95] Lützen 169

mathematicians believed that sharp corners in continuous functions could be interpreted as infinitely tight smooth curves.[96]

Cauchy bases the derivative upon the difference quotient, rejecting Lagrange's proposal about finding the derivative by calculating the power series. Cauchy objected that not only might a power series not exist, even if it does exist it might not converge, and even if it does converge it need not converge to the correct function.[97]

Cauchy also tried to rigorously prove important theorems like the Intermediate Value Theorem and the mean value theorem. His proofs are not today viewed as rigorous particularly because he lacked the necessary distinction of terms such as continuity and uniform continuity, as well as an understanding of the real numbers.[98]

2.4 Integration

The oldest problems of analysis are problems concerning the calculation of lengths, areas and volumes.[99] From antiquity until the 17^{th} century the calculation of areas was done geometrically by either transforming an object into another whose area was more easily calculated, or by exhausting or filling the figure with objects such as many-sided polygons whose areas were easily given. The Greeks considered the problem of finding the area of a figure solved only when they, using only simple geometrical tools, could create a square having the same area as the figure; hence the term "quadrature".[100] But they ran into problems when trying to calculate the areas of circles, ellipses, and similar figures for which they instead used approximation. Some thought that the circle, because it can be approximated as closely as one likes by polygons on both sides, must have an area of the same type as the polygon.[101] The atomistic worldview at the time hardly allowed for the existence of infinite decimal expansions, and even less so for different types of quantities. In any case, the solving of areas demanded an ingenuous new method for each new figure; there was no general algorithm. The solving of areas and volumes, particularly by using the method of exhaustion, did have some striking similarities with modern integration. But the differences are greater: besides the lack of algorithms there were also no limits and exhaustion was not used to actually arrive at the quantities themselves, but rather to prove statements about proportions.

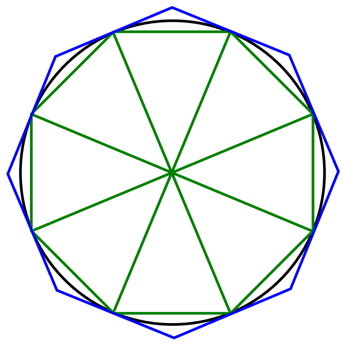
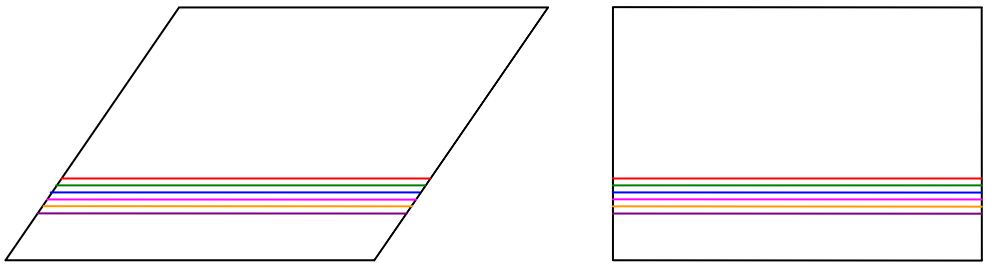


Illustration of a Greek approach to calculating area

[96] Grattan-Guinness 121
[97] Lützen 169
[98] Grattan-Guinness 113,114
[99] Thiele 14
[100] Thiele 15
[101] Thiele 17

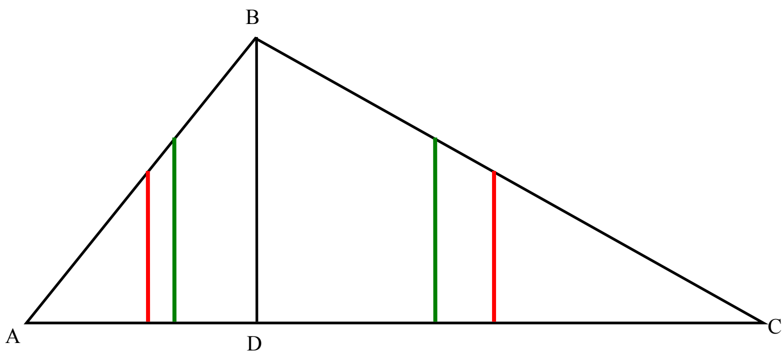
2.4.1 Cavalieri and Wallis

In the 17th century several novel methods of integration were developed or invented. One of these, the method of indivisibles, is credited to Cavalieri who published this method in 1647.[102] The method of indivisibles builds on the idea that a geometrical object of two or three dimensions, for instance a parallelogram, can be thought of as consisting of indivisible lines or planes. By comparing these lines to the lines of another object, the area of which is easier to calculate, Cavalieri was able to give the area of the first object.[103] The case of the parallelogram is illustrated below:



Here, corresponding lines in the two figures are equal; hence the figures have equal areas.

Useful though it was, there were some serious potential problems with Cavalieri's method. When applied to some figures it would yield an answer entirely wrong. This was because of difficulties with comparing one infinity to another. It seems that Cavalieri’s idea can be rephrased to say that if there is a one-to-one relationship between the equivalent indivisible lines of a geometric object, then these objects have equal area.



Although the corresponding lines are equal, the triangle on the left does not have the same area as that on the right.

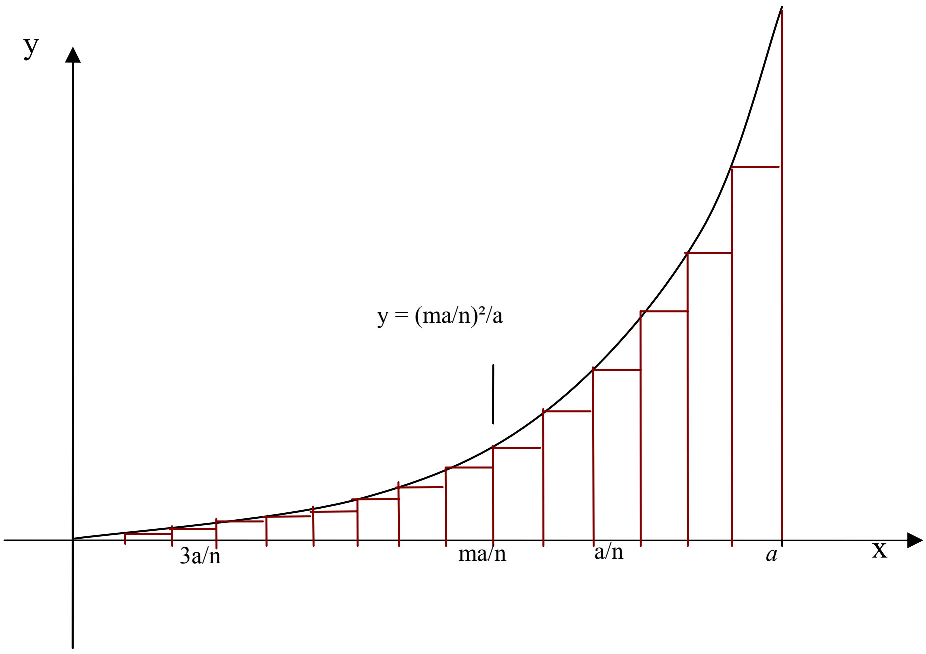
Observe, for instance, the figure above. Let us define the left triangle as smaller than the right one. Yet for each line in each triangle there is a corresponding line in the other triangle, so according to Cavalieri the triangles should be equal. This kind of problems arises from the difficulties with infinities, which were not sufficiently understood at the time.[104]
A different method, the unrigorous *Arithmetical integration*, was developed John Wallis who relied primarily of the convergence of infinite series. Van Maanen gives an example of how Wallis would calculate the area under the curve $y = x^2/a$ between $x = 0$ and $x = a$.

[102] Van Maanen 58
[103] Van Maanen 58, 59
[104] Toeplitz 60

First, Wallis would split the interval into n smaller intervals, each having the length a/n. He would then sum the areas of the small rectangles formed by sides a/n and $(ma/n)^2/a$, obtaining:

$\frac{a}{n}\left(\frac{a}{n^2}+\frac{4a}{n^2}+\frac{9a}{n^2}+\frac{16a}{n^2}+\frac{25a}{n^2}+\frac{36a}{n^2}+...+\frac{n^2a}{n^2}\right)$ which is equal to $\frac{n+1}{n}a^2\left(\frac{1}{3}+\frac{1}{6n}\right)$.

Letting n go to infinity Wallis arrived at $\frac{a^2}{3}$ which we recognize as the integral of y.[105]



Graph illustrating Wallis' method of integration

Wallis expanded this technique to many more curves and published the results in his work, the *Arithmetica infinitorum*, in 1656. It was this method, rather than the previous geometrical methods, that influenced Newton and Leibniz and developed into the integration of today.

2.4.2 Integration under Newton, Leibniz and the Bernoullis

Newton used primarily two methods for integration; he changed variables so that the expression was reduced to one in his table of fluents, or, if this proved too difficult, he used series-expansion and integrated term by term.[106] He did have some intuitive grasp of the importance of the convergence of series, but did not formalise his ideas on the subject.[107] At the same time, Leibniz and other mathematicians were struggling to understand how to move from the sum of rectangles under a curve to the area under the curve. Popular at the time was to consider the area as a sum of y-values, but some also considered the area to be a sum of

[105] Van Maanen 66
[106] Guicciardini 81
[107] Kline 361

infinitesimal rectangles.[108] In 1680, Leibniz describes the calculation of area as the sum of rectangles and comments that the remaining triangles are negligible because they are "infinitely small compared to the rectangles".[109] The Bernoulli brothers continued to develop the Leibnizian style of integration in the 18th century, but where Leibniz had relied on infinitesimal rectangles, the Bernoulli's defined the integral as the inverse of differentiation.[110]

2.4.3 Integration in the 19th century

This definition proved unhelpful, however, when it came to obtaining the coefficients of Fourier series. The problem was that differential calculus could hardly apply to non-differentiable functions. Instead, Fourier chose to obtain the coefficients by using the definite integral which he interpreted as the area underneath a curve.[111] In 1822 Fourier introduced the notation that we have today, with the bounds of integration below and above the integral:[112] $\int_a^b f(x)dx$

Cauchy agreed with Fourier that the basis of integration needs to be the definite integral, but instead of interpreting it in terms of area (in fact, later Cauchy defined area, arc length and volume in terms of the integral[113]) Cauchy preferred to define it as the limit of a "left sum".

> Suppose that the function $y = f(x)$ is continuous with respect to the variable x between the two finite limits $x = x_0$ and $x = X$. We designate by $x_1, x_2, ..., x_{n-1}$ new values of x placed between these limits and suppose that they either always increase or always decrease between the first limit and the second. We can use these values to divide the difference $X - x_0$ into elements $x_1 - x_0, x_2 - x_1, ..., X - x_{n-1}$ which all have the same sign. Once this has been done, let us multiply each element by the value of $f(x)$ corresponding to the left-hand end point of that element [….] and let
>
> $$S = (x_1 - x_0)f(x_0) + (x_2 - x_1)f(x_1) + ... + (X - x_{n-1})f(x_{n-1})$$
>
> be the sum of the products so obtained [….] if we let the numerical values of these elements decrease while their number increases, the value of S ultimately […] reaches a certain limit that depends uniquely on the form of the function $f(x)$ and on the bounding values x_0, X of the variable x. This limit is what is called a *definite integral*.[114]

We need not here go into the details of his definition except to notice that the emphasis is on the existence (rather than on the use) of the integral, and that this is the first time that the integral is defined in terms of a limit. Cauchy also went on to prove that the sum converges to

[108] Kline 374, 375
[109] Kline 376, 377
[110] Lützen 170
[111] Lützen 170
[112] Lützen 170
[113] Kline 958
[114] Lützen 159, 160

the integral and to prove the fundamental theorem of calculus (assuming continuous derivatives).[115] He also defined integrals for functions discontinuous at isolated points, as well as for integrals over unbounded intervals. His method in both cases was to take the limit as the variable approached the discontinuity and infinity. [116]

While Cauchy showed (without proper rigour because of the absence of the notion of uniform continuity) that his definition of the integral is meaningful for continuous functions, Fourier was asking whether the integrals used for finding the coefficients of Fourier series made sense for *all* functions. This necessitated a further discussion about the meaning of the integral. In his work on trigonometric representation, Riemann included, more or less as an appendix, his thoughts on the integral.[117] His definition was similar to Cauchy's. He partitioned the interval (a, b) in the way of Cauchy, and then constructed the sum

$$S = \sum_{i=1}^{n} \delta_i f(x_{i-1} + \delta_i \varepsilon_i)$$

where δ_i is $x_i - x_{i-1}$ and ε_i is a rational number between 0 and 1. Riemann then stated that if, taking δ_i diminishing to 0, the above sum approaches the same limit A no matter how the partition is created or what values of ε_i are chosen, then the integral exists and

$$\int_a^b f(x)dx = A.$$ [118]

In this way, Riemann introduced a whole new class of functions: the integrable functions. He also went on to prove necessary and sufficient conditions for integrability.[119] As a finishing touch, Riemann amazed the mathematical community by presenting an integrable function discontinuous on infinitely many points in any finite interval.[120]

Darboux, whose definition of the integral in terms of the convergence of upper and lower sums was more precise than and directly equivalent to Riemann's, proved the fundamental theorem of calculus for his definition. The theorem was now stated as:

> If a function F on an interval $[a,b]$ is differentiable with a bounded and integrable derivative $f = F'(x)$, then $F(x) - F(a) = \int_a^x f(y)dy$ for all $x \in [a,b]$.[121]

This meant that Riemann's criterion of integrability was enough to arrive at the fundamental theorem of calculus. Nevertheless, some mathematicians argued that the existence of non-constant differentiable functions on $[a,b]$, whose derivatives have zeros dense in (a,b), shows that Riemann's notion of the integral is not general enough to allow differentiation and

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[115] Lützen 171
[116] Lützen 172
[117] Hochkirchen 264
[118] Hochkirchen 264
[119] Hochkirchen 265
[120] Hochkirchen 265, 266
[121] Hochkirchen 271

integration to be reversible.[122] The problem is that these functions, though they are differentiable, have derivatives which are not integrable. This problem inspired Lebesgue, who wrote:

> It thus seems natural to search for a definition of the integral which makes integration the inverse operation of differentiation in as large a range as possible.[123]

Lebesgue went on to create a more general notion of the integral, but his work is beyond the scope of this paper.

[122] Hochkirchen 271
[123] Hochkirchen 272

3 Using the history of analysis to improve education

I will now turn to the topic of why and how history should be integrated with education. First, I will describe some of the reasons proposed in the literature on the subject. I will then proceed to account for how the integration of history in education can ameliorate the three points of criticism mentioned in the background section. Following this, I continue with the question of how history can be used in teaching calculus. Starting by describing some general categories, I will gradually focus more on the indirect genetic approach. Finally, I will suggest an outline of calculus education and describe two books to which my approach is similar.

3.1 The "Why?" of using history in mathematics education

3.1.1 Some views from the literature

In what way may the history of analysis be useful in mathematical education? According to Barbin, the way in which integration of history with mathematics education works is that it first changes the way the teacher thinks of mathematics, then this teacher will change the way of teaching, and finally the students' view of mathematics will be influenced.[124] She therefore claims that due to the scope and complexity of this process, the evaluation of the approach to use history in mathematical teaching must so far be of a qualitative rather than a quantitative kind. While this may be true, there are some more or less fixed goals set by different educational institutions. I therefore hope that it should be possible as well as necessary to find ways to evaluate to what extent the integration of history into mathematical education helps students reach these goals. But until further developments give more quantitative evaluations, there is little to rely on other than the testimonies of teachers and students.

Specifically, several writers claim that using the history of calculus in its teaching makes both teachers and students aware of mathematics as a dynamic part of culture. Barbin quotes an article written by a group of French teachers:

> Mathematics becomes alive; it is no longer a rigid object. It is the object of enquiry, controversy, contains mistakes and uses methods of trial and error.[125]

and one of their students:

> …mathematics has for me passed from the status of a dead science to that of a living science, with an historical development and practical applications.[126]

Others point out that the role of the teacher changes dramatically. Instead of the traditional role of lecturer, a teacher who uses the history of calculus globally in his or her teaching may

[124] Barbin 63
[125] Barbin 67
[126] Barbin 67

act more as a guide, helping the students construct calculus by solving key problems.[127] Barbin puts it this way:

> This difference corresponds to a contrast in pedagogic style: that of the traditional teacher, where knowledge is handed out by the teacher, and a learning process based on mathematical activity by the student.

Tzanakis et al. give five main areas in which they feel that mathematics teaching may be improved by the use of history. Briefly, these areas are:

a. The learning of mathematics. This is in part essentially the main point of this paper, namely that the reorganizing of the curriculum to better reflect historical developments helps provide insights in and motivation for new concepts and methods. Other positive aspects are that using history may provide interesting curricular enhancements, create a natural bridge to other subjects, and help the students improve a variety of non-mathematical skills such as evaluating resources and documenting.[128]
b. The understanding of the nature of mathematics. By being exposed to original texts students may learn that mistakes, doubts, intuition, blind alleys and controversies are all integral parts of mathematics.[129] This might give the students more confidence in their own attempts to solve problems or understand tricky concepts.
c. The teacher's understanding of didactics. By understanding the historical development the teacher is better equipped to use a wide variety of relevant examples.[130] Another important point is, I think, that the teacher is more able to gage the difficulty of a given topic.[131] Often some concepts (like "function", for instance) may seem obvious to someone who has had time to digest and work with the concept for a long time, and the teacher might therefore need to be reminded of how difficult the concept was for brilliant mathematicians of the past in order to understand the difficulties experienced by the students.
d. The students' feelings towards mathematics. Seeing mathematics as described in point "b" above might give the students more tolerance with their own mistakes and confidence in their own attempts to solve problems or understand tricky concepts.[132] Students may also learn to be persistent with difficult problems and feel free to search for novel and creative solutions.[133] Interestingly, Tzanakis et al. do not mention it, but other authors are convinced that the students actually seem to like mathematics more when it is taught with its history.
e. Mathematics as within cultural context. Exposure to different types of texts and historical examples might give the students insight into how mathematics is tied with culture as a whole and how the mathematical communities have functioned historically. An interesting example given by Radford is the difficulties in the West, compared to the relative ease in China, with negative numbers. According to Radford, philosophical, by today's standards almost religious, differences were apparently very important in the understanding of negative numbers.[134] Regarding mathematical communities, examples like the priority dispute between Leibniz and Newton and the

[127] Barbin 64
[128] Tzanakis et al. 204.
[129] Tzanakis et al. 204, 205.
[130] Tzanakis et al. 206
[131] Tzanakis et al. 206
[132] Tzanakis et al. 206, 207
[133] Tzanakis et al. 207
[134] Radford 37, 38

resulting rift within the European mathematical community may not only illustrate problems with publication, but also serves to humanize mathematicians as well as mathematics itself.

3.1.2 How does the use of history with education solve the three problems?

In this paper I will not discuss further the alternative role of the teacher or the students' overall change in interest and appreciation of mathematics. Instead, I want to examine how integrating the history of calculus with calculus education solves the three problems presented in the background section above.

3.1.2.1 There is little motivation for the introduction of new concepts and methods

There is ample motivation historically for the introduction of new definitions, theorems and methods. For instance, one may progress from a narrow to a wide definition of function by pointing to different pathological functions and to the fact that functions can be expressed both by single analytic expressions and by multiple expressions over several intervals. This is, in fact, exactly what has happened historically from the Bernoullis and Euler to Cauchy, Fourier and Dirichlet.

Also the concepts of continuity and limits are easily justified by introducing the students to the problems encountered by using infinitesimals. Starting with the infinitesimals, one might ask the students to try to formalize their intuitive understanding of these concepts. Pointing to the discrepancies in alternatingly using infinitesimals as non-zeros and zeros to suit ones purpose, and perhaps quoting Bishop Berkeley, the students might be convinced that something stricter is necessary. The limit as motion is probably familiar to the students, and it is not difficult to argue that this implies a dependence on time. Instead of time, one might argue, we can look at an expression containing the variable and see what happens to the expression when the variable is allowed to change values. In this way we have followed Newton and Leibniz, through Lhuiller to Cauchy, Bolzano and Weierstrass.

Concerning continuity one might introduce Euler's definitions and then show that these definitions in terms of the functions' expressions makes no intuitive sense, as well as being contradicted by the fact that functions can be written in a variety of ways. The students might prefer Cauchy's definition of continuity and because they have already been convinced of the problems with infinitesimals they will readily wish to rephrase Cauchy's definition in terms of limits.

Regarding derivatives, I believe that students quite often wonder why they are considered important. Introducing students to a wide variety of physical problems involving tangents, such as problems of speed and distance, might provide motivation. The students may perhaps also be convinced that when faced with the problem of drawing a curve for which one does not know the algebraic expression, it can be very helpful to at every point know the slope of the curve.

The integral is easy to motivate by connecting it to area. Successive improvements can be motivated by connecting them to the developments in the theory of functions and limits. Finally the tables can be turned and area be defined in terms of the integral, much in the way of Cauchy.

3.1.2.2 The change from informal to formal reasoning is sharp and unmotivated

The change from informal to formal reasoning can be achieved by attempting to be as formal as was possible at the time of the development of the concepts taught. Pointing out the vagueness of the language used, perhaps even showing the students texts from the past (maybe even citing Lhuiller as example) and asking them to interpret these texts, should convince the students of the advantages of formality. Cauchy's assumption that continuous functions also possessed the qualities of uniformly continuous functions, and other similar examples, can be given to show the need for clear definitions and the importance of adhering to formal reasoning even while using intuitive understanding. Thus, the striving to increased formality becomes a motivation for change, encouraging students to advance from intuitive definitions of things like limits to formal definitions, aware of why, when and how this happens in their own and the historical development of the understanding of mathematics.

3.1.2.3 The role of the problems that the students are intended to solve is unclear

Problems also take on a different role in this context. Instead of presenting students with the same problems as before the new concept was introduced, and ask them to solve it in a new way, problems can be used to illustrate the power of the concepts the students are asked to accept and understand.

One instance of this is the transition from early definitions of function to later definitions. The function defined as an algebraic expression relating two variables does not, for example, allow us to decide immediately whether a given graph corresponds to a function or not. Dirichlet's definition, on the other hand, does allow the standard way of identification by examining whether a vertical line will anywhere intersect the graph more than once. This improved usefulness of the new definition should be pointed out so that the students may also learn about what constitutes good theory and what does not.

Another example is of course, as previously stated, that of the definition of continuity. A definition in terms of traceability does not allow some graphs to be classified as continuous or discontinuous. With improved definition of continuity students should practice on such functions which elude the use of earlier definitions.

With derivatives the case is already quite acceptable in my opinion. Derivation is usually introduced with cumbersome tangents and limits, and the students are happy to exchange this for the standard rules of derivation. The use of derivatives for drawing graphs of unknown functions, for solving differential equations as well as for simplifying integration also let the students use their new knowledge on novel problems.

The same is true of integration, which must to many seem to be a wonderful tool allowing one to calculate areas under the strangest of curves, find the distance covered while accelerating, and even confirm the unintuitive formulas of volume and area of geometric figures.

3.1.3 Epistemological frameworks

I will now say something about the epistemological background and the different epistemological interpretations and frameworks behind using history in mathematics education. The first framework used was based on the concept of recapitulation, inspired by evolution and genetic heritage. In the last 20 years several more suggestions have been forwarded and I will mention two: the concept of epistemological obstacle, and the socio-cultural theory of Luis Radford.

The idea of recapitulation first stems from the late 19th century and is influenced by Darwin's theories of the evolution of species. The idea is that the individual's development (ontogenesis) recapitulates mankind's development (phylogenesis). According to Radford et al., a German biologist named Haeckel first applied this principle to psychology.[135] He claimed that "the psychic development of the child is but a brief repetition of the phylogenetic evolution."[136]

Piaget and Garcia proposed instead something a little less simplistic, namely that people's psychological development mirrors historical development. Their hypothesis is that historically people have progressed from the intra-operational stage of mental development (during which one can reflect on individual objects, but not on the connections between the objects), through the inter-operational (when one is able to perceive the interactions between objects) to the trans-operational stage (in which one can reflect abstractly on the interactions as parts of larger systems), and that the same thing happens in an individual acquiring knowledge.[137] Though Piaget and Garcia were sure that cultural context deeply influenced the meaning of knowledge, they firmly believed that the actual process of acquiring knowledge is hardwired into the biology of humans and therefore insensitive to cultural differences. "Society can modify the latter, but not the former" Piaget and Garcia said when speaking of the mechanisms of learning and the way in which the learned matter is conceived by the person.[138] Piaget's and Garcia's ideas have been interpreted by some researchers in mathematical epistemology as "ontogenesis mirrors phylogenesis", which in this context means that the development of an idea in an individual mirrors that idea's historical development.[139]

It is not at all obvious that this interpretation of Piaget's work is correct, especially since Piaget never used the word "recapitulation" himself. Also, a number of criticisms have been directed at the idea of "ontogenesis mirrors phylogenesis". One such criticism is simply that children today are neither like the children nor like the adults of past generations, so it is unclear why the development of their knowledge should mirror the past.[140] Also, the idea of recapitulation does not go well with the influential developmental psychologist Vygotsky's approach to epistemology. In his view culture has the power to influence the person's acquisition of knowledge by providing the person with different tools like language, computers, etc., which changes the mental functions of the person. Regarding recapitulation, Vygotsky had this to say: "We do not mean to say that ontogenesis in any form or degree repeats or produces phylogenesis or its parallel."

An alternative idea, that of the epistemological obstacle, has therefore by many been preferred to the idea of recapitulation. The concept of epistemological obstacle suggests that students today are faced with the same obstacles or difficulties as mathematicians of the past because of the inherent inadequacies of the mathematical knowledge they possess. The key here is the idea of inherent properties of knowledge; knowledge is understood as an objective entity with given strengths and limitations. Because knowledge is objective, the students will face the same difficulties as the mathematicians of the past, given that they are using the same knowledge.

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[135] Radford et al. 145
[136] Haeckel quoted in Radford et al. 145
[137] Radford et al. 146
[138] Radford et al. 146
[139] Radford 31
[140] Radford 32

The socio-cultural perspective disagrees with the concept of epistemological obstacle. According to Radford, knowledge can never be defined outside of the given cultural parameters. His claims that knowledge:

> "…can only be understood in reference to the rationality from which it arises and the way the activities of the individuals are imbricated in their social, historical material and symbolic dimensions."[141]

It is obvious even from Radford's definition of knowledge, "a culturally mediated cognitive praxis resulting from the activities in which people engage",[142] that is nonsensical to assume that today's students can ever have the same knowledge as mathematicians of the past, and it makes even less sense that this knowledge would be used in the same way. Given different socio-cultural contexts the seemingly same knowledge will be used in different ways and have different strengths and weaknesses. Radford gives the example, already mentioned above, of how easily Chinese mathematicians dealt with negative numbers while Greek and Western mathematicians had great difficulties - though they all started out with the knowledge of positive numbers.[143]

Given that the student appropriates knowledge by *interiorising* cultural concepts and methods, the classroom thus becomes "a micro-scape of the general space of culture" in which the students gains understanding of mathematics by "a process of cultural intellectual appropriation of meanings and concepts along the lines of student and teacher activities."[144] Instead of the epistemological obstacle, Radford and other advocates of the socio-cultural perspective propose viewing history of mathematics as:

> a rather wonderful locus in which to reconstruct and interpret the past, in order to open new possibilities for designing activities for our students.[145]

Though I find the theory of the socio-cultural perspective very compelling, it seems unclear to me how the theory is put into practice. Radford is unclear on this point, and my interpretation from the above quotes is that the teacher uses history explicitly, bringing into light the very real and very different historical contexts of mathematics while using the concepts of the culture surrounding the students today. Thus, I suppose that one would analyse the mathematics of the Greeks by considering at length the culture of ancient Greece, but not hesitate to express the Greek mathematics with modern notation.

In my opinion, there is also something to be said for not going to extremes. The criticism of the epistemological obstacle is valid, of course, but though knowledge is understood differently in different cultures, it is also possible that the similarities between cultures and people across these cultures, gives rise in similarities of knowledge. Therefore it might not be far fetched to seek inspiration from mathematicians of the past in order to understand the difficulties of the students of today. Also, it seems to me that history has more to show us than what has been mentioned in the three perspectives outlined above. The benefits described in section 5 above do not fit nicely within any one, or even all, of the epistemological frameworks. It certainly seems possible to teach mathematics without an underlying solid epistemological theory.

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[141] Radford et al. 164
[142] Radford et al. 163
[143] Radford 37
[144] Radford et al. 164
[145] Radford et al. 165

3.2 The "How?" of using history in mathematics education

Integrating the history of mathematics with education can be done in several different ways. Tzanakis et al. consider three such ways. One is giving direct historical information, for instance by using whole books or mere snippets on the history of mathematics focused on teaching history rather than mathematics.[146] Another way is a teaching approach inspired by history,[147],which I shall elucidate below. The third way is by increasing mathematical awareness by using historical sources or other methods to further students' understanding of the intrinsic and extrinsic nature of mathematics.[148] There are many options afforded by this third way of using history. Tzanakis et al. list and in detail describe an impressive number of methods, including: using primary sources, research projects based on historical texts, historical worksheets, the use of historical packages, using mechanical instruments, solving historical problems, and experiential mathematical activities.[149]

Among these, the main categories commonly used in the literature[150] are the "global" and the "local" use of history. To integrate history globally is to design the course so that it parallels historical developments. In contrast, to integrate history locally means to use individual problems that have historic context. Whether globally or locally used, history may be presented explicitly, by reading historical texts or in other ways use materials from the past, or implicitly, where the teacher might be the only one in the classroom who knows that the education is influenced by the history of mathematics. In this paper, I am interested in the global semi-explicit integration of history with mathematics education. This is very similar to the "indirect genetic approach to calculus" presented in the ICMI study.[151]

3.2.1 The genetic approach to calculus

Otto Toeplitz, a German mathematician active in the first half of the 20th century, advocated teaching calculus by the "genetic" (which implies heritage and evolution) approach. His book, *The Calculus: A Genetic Approach* was published posthumously in German in 1949 and the first translation to English came in 1963. Since Toeplitz others have worked to popularize the genetic approach in different ways. Bressoud's *A Radical Approach to Real Analysis* seems to be a good and perhaps the only actual textbook strongly based on this approach.

The idea behind the genetic approach is to learn from history how mathematics has developed, and then use this knowledge in teaching. For instance, if the concept of function was at first quite simple and narrow, it makes sense that students first encounter functions in a similar way. Also, if the development of mathematics has been fuelled by the necessity arising from unsolvable problems, then the genetic approach to calculus should have as a main principle that "a subject is studied only after one has been motivated enough to do so, and learned only at the right time in one's mental development."[152] Because of this, the

[146] Tzanakis 208
[147] Tzanakis 208-211
[148] Tzanakis 208-213
[149] Tzanakis 228
[150] Barbin 64, 65
[151] Tzanakis et al. 208-211
[152] Tzanakis et al. 208

genetic approach is more focused on understanding the "why?" of mathematical theory rather than the "how?" of computational techniques.[153]

Tzanakis et al. propose a general schema of implementing the genetic approach to teaching calculus. They recommend that the teacher first should acquire a sufficient knowledge of history so that (s)he may identify the "crucial steps…key ideas, questions and problems which opened new research perspectives."[154] Then these crucial steps etc. are reformulated so that they are didactically suitable for the mathematical classroom. They are then presented to the students in sequence, each problem building on previous problems and increasingly difficult. Tzanakis et al. distinguish here between a "direct" and "indirect" genetic approach; the direct being an explicit use of history, the problems appearing in historical order not only for didactical purposes but also to illustrate the actual historical development of mathematics. The indirect method, on the other hand, uses history more implicitly and determines the sequence of problems more on didactical grounds than historical.[155]

The advantages of the indirect genetic approach to integrating history are several. One main is that it is very flexible regarding how much explicit historical material is imported into the classroom. Another advantage is that the teacher requires only a little training in the history of mathematics. Yet another advantage is that the teacher might from history be inspired to present the class with motivating problems connected to natural or social sciences, philosophy and religion. Thus the students would understand mathematics to be connected with and useful to some of their other interests, which in turn would increase the students' interest in mathematics. Victor Katz designed and for several years taught a course fitting the description of the indirect genetic approach. He claims that teaching calculus in this way

> […] helps to provide not only a motivation for its study but also a reason for the students to further explore the connections between their studies and the world around them.[156]

The many epistemological difficulties of course create some problems for the direct genetic approach which tries to be historically accurate and didactically sound. At the very least the teacher whose goal it is to be faithful to both history and didactics needs to work hard to fully understand the complex social-cultural contexts of mathematics and it probably requires considerable didactical skill to craft from this knowledge ideas that lend themselves to teaching within the restrictions on time and resources imposed on most teachers. The indirect genetic approach is not as vulnerable to the criticisms mentioned above. Since the goal is more modest than in the direct genetic approach, the teacher needs not be so concerned with historical accuracy or epistemological formality and can feel free to take and rework from history whatever suits the students' needs.

3.2.1.1 Classics of the genetic approach

Otto Toeplitz' *The Calculus: A Genetic Approach* was the first book designed to teach calculus with history in the "genetic" fashion which has also been my inspiration. Toeplitz' described his goal as different from that of an historian:

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[153] Tzanakis et al. 209
[154] Tzanakis et al. 209
[155] Tzanakis et al. 210
[156] Katz 243

> The historian – the mathematical historian as well – must record all that has been, whether good or bad. I, on the contrary, want to select and utilize from mathematical history only the origins of those ideas which came to prove their value…. It is not history for its own sake in which I am interested, but the genesis, at its cardinal points, of problems, facts, and proofs.[157]

Toeplitz worked on his *Calculus* for much of the 1920's and 30's, and before he died in 1940 he had finished the first semester part of the course, covering mathematical content roughly from the Greeks to (and including) Newton and Leibniz.[158]

Toeplitz starts the course with an investigation of the infinite, beginning with the Zeno's paradoxes and the Greek method of exhaustion, and continuing through the modern definition of limit and infinite series. He follows this chapter with one on the integral, starting with Archimedes' quadrature of the parabola and ending with the definition of, and some theorems on, the definite integral. Next comes a chapter on "differential and integral calculus" wherein Toeplitz describes tangents and derivatives, many techniques (such as integration by parts) of the calculus, as well as the fundamental theorem of calculus. The end chapter covers applications to problems of motion.

Throughout the book Toeplitz uses modern notation, at times seeming very anachronistic as he does so. For instance, he does not hesitate to describe Greek geometry using algebra, and does not in any way apologize for doing so.[159] He also does not in all or even most cases illuminate the historical background of a concept or method. An illustrating point here is the product rule, of which he states:

> …it would be interesting to tell how it was discovered; but here lies the thin line between history for its own sake and history for the sake of illuminating the development of mathematical thought. The history of the discovery of the fundamental theorem served to illuminate aspects which usually do not stand out clearly at all, and this could hardly have been achieved by any other method. About the product rule, however, there is nothing to be illuminated. Hence the history of its discovery does not concern us.[160]

Apparently, Toeplitz did not appreciate the "humanizing" effect of the history of mathematics in education. Though he often gives explicit descriptions of historical contexts, he does so only to elucidate the development of particular mathematical ideas.

Bressoud's *A Radical Approach to Real Analysis* is much more recent (1994) and the focus seems to be on introductory real analysis rather than calculus.

> It is designed to be a first encounter with real analysis, laying out its context and motivation in terms of the transition from power series to those that are less predictable, especially Fourier series, and marking some of the traps into which even great mathematicians have fallen.[161]

The course starts with infinite series and these figure as a unifying topic throughout the rest of the book. Different problems in analysis, such as questions about continuity and differentiation, are first introduced as questions concerning operations on or convergence of

[157] Toeplitz quoted in Köthe v
[158] Köthe vi
[159] See for instance Toeplitz 3, 4, 10
[160] Toeplitz 99
[161] Bressoud vii

infinite series. The questions are then answered, and the answers are used to enhance the understanding of infinite series.

The level of difficulty of the text varies incredibly. The first chapter presupposes familiarity with Fourier series and partial differential equations. Yet only later is the student gently introduced to the ε, δ -notation and formal definition of continuity and differentiation. Bressoud recognizes this variation in difficulty, and explains that the aim is that

> …every student in the classroom and each individual reader striking out alone should be able to read through this book and come away with an understanding of analysis. At the same time, they should be able to return to explore certain topics at greater depth.[162]

The order of topics is not sequential; instead Bressoud alternates integration, differentiation and continuity with increasing exactification and difficulty. The impression is of a tightly held together and very interesting course. Yet I miss some key topics that I associate with a course in real analysis. For instance, there is nothing on topology in Bressoud's book, and nothing on the real numbers. *A Radical Approach* seems more as a guide to mathematical theory for engineers than as a comprehensive course in real analysis for students of pure mathematics.

What he does teach, Bressoud teaches magnificently. For every idea that he presents, Bressoud gives ample motivation. He uses Mathematica as a tool for exploration, letting the reader develop an intuitive idea of the problem or solution at hand before continuing to the formal mathematics. For instance, in my opinion Bressoud's handling of differentiation is masterful: he introduces the chapter with the Newton-Raphson method and shows numerically that it is unreliable for many functions.[163] This leads him into error estimation, which motivates the more complicated definition of derivative: $f'(a) = \frac{f(x) - f(a)}{x - a} + E(x, a)$ with $E(x,a)$ going to 0 as x goes to approaches a.[164] Bressoud states this definition in terms of ε, δ -notation, and, understanding that the epsilonics might be difficult for some students, he continues to explain the ε, δ -process of determining continuity, differentiability and limits by presenting it as a game between two opponents.[165] Bressoud also gives two of Cauchy's proofs of the mean value theorems and points out the dubious assumptions (particularly the assumption of uniform convergence) and faulty reasoning in these proofs. Bressoud states that:

> There is an intentional emphasis on the mistakes that have been made. These highlight difficult conceptual points. That Cauchy had so much trouble proving the mean value theorem or coming to terms with the notion of uniform convergence should alert us to the fact that these ideas are not easily assimilated. The student needs time with them.[166]

The exploration of Cauchy's proofs brings Bressoud to the main theorem used as an example in my paper, the statement that a positive derivative implies an upward graph. Apparently, Cauchy first used this idea to prove the mean value theorem

[162] Bressoud viii
[163] Bressoud 60-66
[164] Bressoud 68
[165] Bressoud 69-71
[166] Bressoud vii

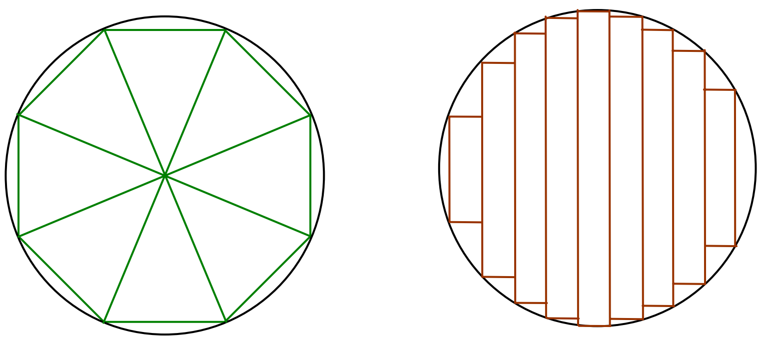
instead of the other way around. Bressoud bases his proof of the theorem of positive derivative on the existence of supremum, and presents the proof in an exercise.[167]

3.2.2 Outline for programme of calculus education

To illustrate how history can be used in the teaching of calculus I propose the following outline, inspired in part Katz and Bressoud, but to the greatest part consisting of my own ideas.

<u>3.2.2.1 Early integration – finding the areas of geometric objects</u>

Before being introduced to calculus, students frequently learn some geometry, especially the finding of areas. Building on this knowledge, one might introduce the students to a method of integration by approximating the area with polygons. The areas of polygons can be cumbersome to calculate, and the students will most likely find it difficult to creatively discover new constellations of polygons for each new problem – but the method seems intuitive enough. Because of the difficulties with using strange new polygons for each new problem, the motivation for a single method, perhaps that of using rectangles, arises. Then, one divides some area, the circle for instance,[168] into rectangles in such a way that it is not completely exhausted, but it is obvious to the students that if one divided the circle into more and smaller rectangles, the difference between the sum of the rectangles and the circle becomes as small as one wishes. Yet it is difficult to calculate the area of some objects,



Area of circle calculated first by polygons, then by rectangles

circles for instance, using rectangles because one does not know the height of the rectangles at a given point. "If only we had a way of expressing the circle in such a way that any point on its circumference is readily given by some formula" the students might wish. This of course opens the door to introducing circles and other figures as graphs of equations.

[167] Bressoud 117

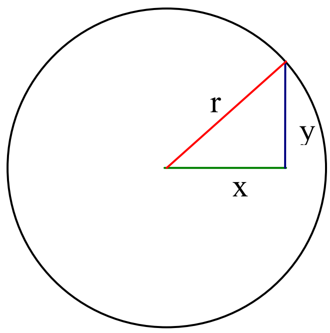
[168] The circle is a handy illustration only. In practice one should stay true to the principle that students should practice new methods on problems that were difficult or impossible to treat with old methods.

3.2.2.2 Interlude – functions and algebraic expressions

It is probably a good idea at this stage to stick to curves that can be motivated physically; otherwise the students' interest might wane because they do not see the relevance of the task. Also, when increasingly relying on algebraic symbols instead of geometric objects or arithmetic, one might discuss at length what the algebraic symbols are and what they stand for. This understanding took a very long time to develop and students can hardly be expected to intuit it after only a quick explanation. One might also take the opportunity to review the different quantities that one accepts and uses for calculations, starting from natural numbers and ending with either irrationals or complex numbers depending on the knowledge of the students.

When (re)-introducing the students to graphical interpretations of algebraic expressions there is the possibility of letting the students experiment. On one hand there are perhaps many ways of representing algebraic expressions graphically (see the methods of Oresme, for examples). Showing the students one or two such methods, and asking them to come up with own ideas, might be rather enjoyable – but probably the students have already been exposed to the Cartesian system and will therefore be unwilling to experiment further.

On the other hand, it is not obvious how to represent geometrical objects algebraically, and not all geometrical objects can easily be interpreted as functions. One might ask for an algebraic representation of the circle, for example, perhaps calling the students' attention to the Pythagorean theorem in order to elicit the formula $x^2 + y^2 = r^2$. Asking the students' to draw the circle from the formula, giving them a random r, will show them that using one of the variables as a dependent variable is much easier than not doing so. Yet it seems that easier geometrical objects like rectangles are more complicated to express algebraically. This is an interesting topic to explore.



Motivation for $x^2 + y^2 = r^2$

One might tentatively introduce Euler's first definition of function and ask the students to reflect on it. Given how much easier calculations become when one variable is dependent, the students might prefer Euler's second definition or Cauchy's definition. Also, the distinction (in the manner of Cauchy) between explicit and implicit functions may be useful to talk about so that the students are not discouraged when they later find that some functions are not readily given in terms of one dependent and one independent variable.

3.2.2.3 Integration and derivation of simple curves

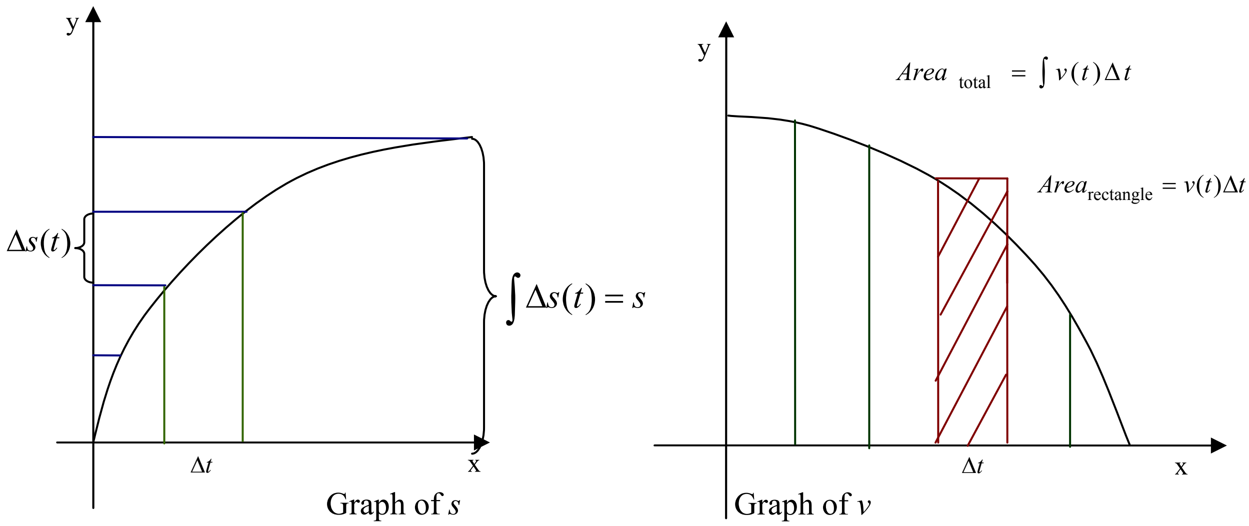
The students have also learned to solve polynomial equations, as well as graph these equations. One can now apply the method in 3.2.1.1 to find the areas under polynomial curves. This requires the ability to calculate sums so now is the time to introduce the new notation ($\sum_{x=1}^{n} x^m$) and spend some time exploring sums.

Once this is done, it is effectful to show what the area of the rectangles becomes for very high n's. The students will see that for such n's, the sum seems not to change very much as n increases. It is possible to mention an intuitive concept of infinite series as well as convergence at this point while remaining honest about the limitations of the approach at hand.

Leaving integration, it is time to discuss the problem of tangents. A good starting point is with the maximum of a curve. This point is interesting because it allows one to solve many real world problems of optimization. One may easily convince the students that this point coincides with a "flat" tangent, meaning that the slope is 0. This should be done as formally as possible and then one might show one method of finding such extreme points, perhaps by using Fermat's method described in the section on history above. Fermat's method is of course quite similar to setting the slope of the tangent to zero. $I(x+e) = I(x)$ when divided by *e* easily reduces to $\frac{I(x+e)-I(x)}{e} = 0$. Observing this connection with tangents, students might be inspired to wonder about further uses of tangents, which motivates an interpretation of derivatives, or slopes of tangents, as instantaneous rates of increase or decrease. Illustrations from problems of physics, biology and economics may be motivating. Newton's approach with infinitesimals can be used for the purpose of finding some derivatives, but Fermat's and Newton's methods must be accompanied by the observation that adding an "infinitely small quantity" is a dubious enterprise.

3.2.2.4 The Fundamental Theorem of Calculus

Having calculated the derivatives of common functions, the students can be asked to reflect on the relationship between derivatives and integrals and the possible reasons for their inverse relationships. One might mention Leibniz's observation that "$\int$ is a sum and *d* a difference." Examples from physics, particularly pertaining to distance and velocity, may be illuminating. The students should be happy to find that instead of the tedious and difficult work of calculating series, they can integrate by finding the primitive function.
This, the fundamental theorem of calculus, needs to be proven in some basic and intuitive way, leaving the rigor for later. One might also motivate the fundamental theorem of calculus by arguing geometrically from the graphs of *s* and *v* below.



Geometric motivation for the intuitive proof of the fundamental theorem of calculus

Start from the premise that $s'(t) = v(t)$ to show that this implies that s is the primitive function of v. We (and hopefully the students, by now) know that $\Delta s(t) \approx s'(t)\Delta t$. Together with the premise this means that $\Delta s(t) = v(t)\Delta t$. Now we also have from the graph of s that $s(t) = \int \Delta s = \int v(t)\Delta t$ and we are done.

This very intuitive approach lacks rigour and parallels the historical development of understanding the integral. Newton, Leibniz, and their predecessors first understood the fundamental theorem of calculus in these intuitive geometric ways. When the students have learned more about limits, continuity and convergence the proof should be improved. Now it is also appropriate to let the students practice on problems involving finding the integral of a curve whose analytic expression makes calculation by sum difficult.

At this point, Katz prefers to introduce power series. I am sceptical of this, especially since the students have just been relieved of one of the uses of series. Instead, I propose returning to differentiation and proving its many rules, including the chain rule and the product rule. After some examples of the power of these rules, it is time to impress the students with the tricks now available to solve some previously unsolvable integrals. As far as I understand, differentiation and integration developed in a similar interconnected manner under Newton and Leibniz.

3.2.2.5 Applications to graphing

Next could come the applications to graphing which is still being done on a global level, assuming that the function is valid everywhere. One might ask what information the derivative gives us about the graph of the function. Remembering the intimate connection between tangents and derivatives, the students will readily agree that a positive derivative means a graph everywhere pointing upwards. The theorem is the one I mentioned earlier:

Main Theorem: Suppose f is differentiable in (a, b).

d. If $f'(x) \geq 0$ for all $x \in (a,b)$, then f is monotonically increasing.
e. If $f'(x) \leq 0$ for all $x \in (a,b)$, then f is monotonically decreasing.
f. If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant.

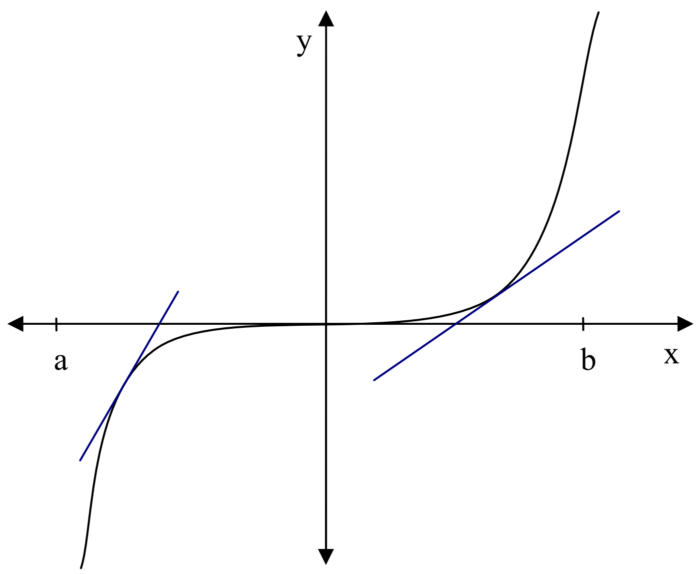


Illustration of main theorem

Taking the opportunity to increase the students' awareness of rigor, the teacher may contrast the intuitive understanding of the connection between positive derivative and upward pointing graph with the formal proof that the derivative at an extreme point is zero. Discussions may arise on whether the derivative can be zero even though the graph is strictly increasing. This would invite further reflection on what is meant by derivative, infinitesimals and limits. One might ask the students to formally motivate their intuitive understanding of the connection between the positive derivative and upward graph. This will lead to discussion on how to move from local to global statements.

For instance, if the students possess an understanding of derivatives in terms of limits, they will readily agree that given any starting point the function is increasing in a sufficiently small interval surrounding this point. Taking a second point on the fringes of this interval, the function is also increasing in a small interval surrounding this second point – but the problem is that this interval could be much smaller than the first one. To quote Cauchy, who thought in a very like manner:

> If one increases the variable *x* by insensible degrees from the first limit to the second, the function *y* will grow with it as long as it has a finite derivative with positive value.[169]

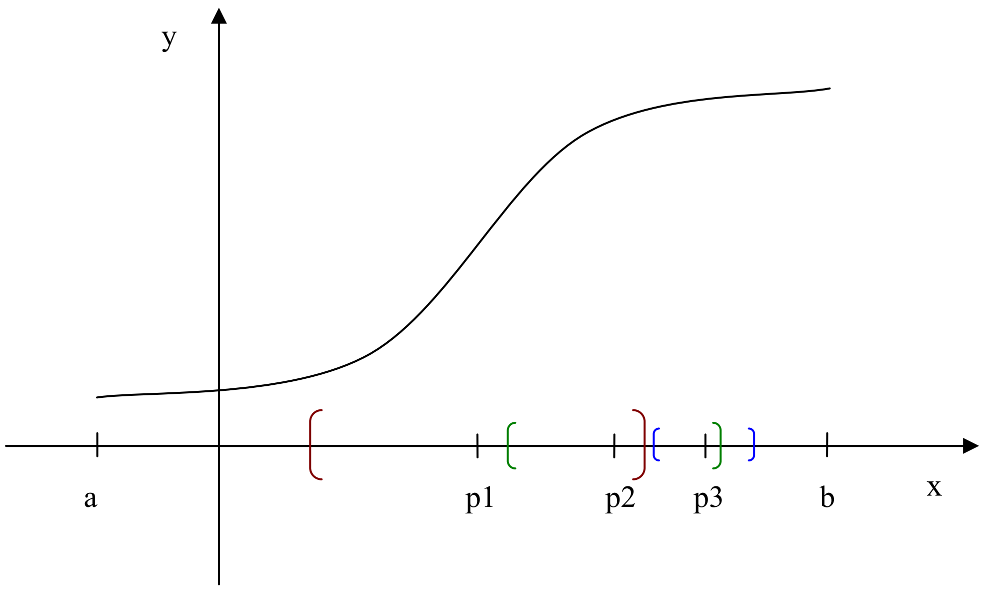


Illustration of progression of points within interval where derivative is positive

Progressing in like manner, the question becomes whether one ever arrives at an endpoint. This might give the students some insights into some difficulties of mathematical theory that they probably have had limited experience with before.

The teacher might then leave the motivation as intuitive as it is, stating that it is not proved with sufficient rigor and that the students will come back to it later. Another possibility is to describe the mean value theorem informally and without proof because the students will most likely find it intuitive. Later one may return to the mean value theorem in order to find out what that useful intuition actually meant and how it can be better motivated. The proof of the main theorem can be illustrated graphically and will then run something like this:

[169] Bressoud 83

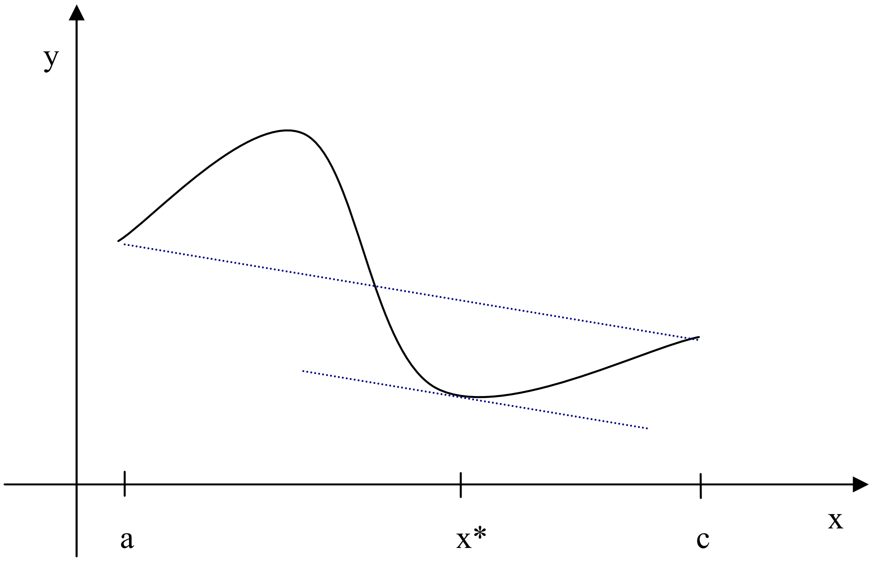


Illustration of proof using mean value theorem

Proof of a): Suppose that there are points x_1 and x_2 in (a, b) such that $x_1 < x_2$ and $f(x_1) > f(x_2)$. We know that there is a point $x^* \in (x_1, x_2)$ such that $f'(x^*) = \dfrac{f(x_2) - f(x_1)}{(x_2 - x_1)}$. But the left side is always positive, and the right negative, so that cannot be. We have reached a contradiction.
b) and c) are proven in a similar fashion.

Having talked about and worked with some problems concerning the properties of the derivative and their relationships to the properties of the function, it is now possible to introduce some differential equations. Following some easy problems, it could be interesting to show the students the problem of the vibrating string (or perhaps Euler's other problem, depending on the abilities of the students) and ask, as did the mathematicians at the time, what kind of solutions we allow for this problem. This leads into discussions about the ontology of the previous mathematics that the students have been taught. Until this point students have only been exposed to relatively well-behaved graphs, but now is the time to introduce them to some stranger ones, including obviously continuous and not continuous ones. This motivates a broad notion of function, as well as considerations of continuity and differentiation.

3.2.2.6 Improved definitions of functions

Until now the student's have been working with nice functions satisfying Euler's first definition. It is time to bring it to mind explicitly and discuss its properties in terms of globality or locality and the restrictions concerning the use of analytic expressions. The teacher might present the class with functions like $f(x) = \dfrac{1}{1-x} + \dfrac{1}{4-x^2}$ and $f(x) = \ln x$ to discuss where these functions are defined and where they are not, and what that means. In this way the students may follow Cauchy in deciding that the algebraic representations of functions should not be interpreted as general as algebra, but rather should be examined within the domains where these expressions make sense.

Pathological functions are obviously valuable in widening the understanding of function, and by examining these extremes the students should be encouraged to propose improved definitions by looking for the red thread of what it means for something to be a function. Following such attempts, the different definitions of Euler, Cauchy, Fourier and Dirichlet should be compared not just in terms of formal mathematical meaning but also of clarity of language, usefulness, etc. Given such a broad definition of function the students should readily agree, as did the mathematicians of the late 19th century, that the statement "*f* is a function" actually says very little about *f* and that other properties should be examined and explicitly stated. The pathological functions are good examples of how mathematics ceased to be restricted by the practicalities of the natural sciences where one usually only finds the kind functions of Euler's first definition. This motivates a discussion of the nature of mathematics and the need for concepts that are not based on practical experience.

3.2.2.7 The concepts of limit and continuity

At this point it is good to let the students themselves propose definitions of continuity. Some might, like Euler, suggest that continuity should rest on the form of the analytic expression. This can be questioned by examining the function $f(x) = \sqrt{x^2}$, which is continuous because it is written as an analytic expression, but which can also be written as $f(x) = |x|$ and is therefore discontinuous. In any case, students will probably want to say that it is continuous. Another good example is $f(x) = \frac{1}{x}$ which has a single analytic expression but, the students will probably agree, behaves oddly around 0. It seems that Euler's definition makes sense algebraically, but not visually. Others will prefer the geometric intuitive approach by which a function is continuous if its graph can be traced with a finger. The teacher can then show the graph of the function $f(x) = x\sin(\frac{1}{x})$, which is intuitively continuous though it cannot be traced without lifting the finger from the paper. Perhaps someone will bring up the property that a continuous function should pass through all intermediate values, but that the reverse isn't necessarily true can be shown by the counter-example, $f(x) = \{\sin(1/x)$ for $x \neq 0$, and 0 for $x = 0\}$.

Piecewise continuity can also be discussed in this context. One might propose that the difficulties encountered are all concerning *points* at which the function may be discontinuous. Thus continuity is a local phenomenon. Eventually, the teacher can define continuity by saying that a function is continuous in a point if the value of the function in this point coincides with the values to which the function approaches from both sides. This of course brings up the question of what "approaches" means and the time has come to formalize the limit.

Reviewing and clarifying the early historical, as well as the students', understanding of limit might help the students get a firmer grasp on this concept. I think that it would be good to discuss the limit as motion, giving many examples from the natural sciences, as well as Zeno's paradoxes. Given time one might even discuss the different interpretations of space and time as discrete or continuous, and relate these to Zeno's paradoxes and Aristotle's' rebuttals to Zeno.

In any case, students will need to be convinced of two kinds of limits. One is the limit of a variable, what we mean when we say "as x approaches 0…" This necessitates a firm understanding of what variables are and that they can be made to take on any value within the domain. Thus this first limit can be said to be decided by us, the teacher or the student. The second limit is that of a function as the independent variable is made to approach some given limit, and here I think that one can clarify by saying something like "just like previously we thought of a variable approaching a limit through time, now this function uses not time but instead it uses the independent variable. What happens to the function as the variable is made to approach some given value?"

Now we are ready for more formal definitions of continuity. Cauchy's definition may be presented and the students asked to use their understanding of limits to improve upon his definition. The "as little as one wishes" of Cauchy's definition may present a problem which the teacher may perhaps be able to solve by gently introducing the ε, δ - notation of Cauchy and Weierstrass. Depending on the level of understanding of the students one might also discuss global properties like uniform continuity and contrast them to continuity by using Abel's counter-example.

The students should also be encouraged to review previous proofs, such as the intuitive proof of the Fundamental Theorem, and see what tacit assumptions about functions were made then, and what the difficulties are without these assumptions. For instance, if the function is not continuous, can we still prove the fundamental theorem as easily as we did? If not, what is the problem? We can choose to either state the theorem for continuous functions, or to find some way to isolate the points of discontinuity, given these are not too dense.

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3.2.2.8 Reformulating the derivative and a new proof of the main theorem

Reinterpreting the derivative by using these new concepts of limit and continuity, it is easy to show that not all continuous functions have a derivative everywhere. If the derivative is defined as $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ it simply does not exist for $f(x) = |x|$ and many other functions. The step from this to the distinct concept of differentiability is slight.

In particular it can be instructive to show the students the power of these formal concepts in dealing with the theorem mentioned earlier about the relationship between a positive derivative and an increasing graph. One proof proper at this time is by interval encapsulation. This proof presupposes very little knowledge from the students, but relies on their understanding of the formal definition of the derivative and their appreciation of the concept of limits and convergence.

Alternative proof of the main theorem:

Assume that there exist points x_1 and x_2 such that $x_1 < x_2$ and $f(x_1) > f(x_2)$ on some interval in (a,b). Then suppose that this interval is within the half-interval closest to b (otherwise, take the midpoint in (a,b) as one endpoint of the suspicious interval), and denote this interval $[a_1, b]$. Continue to half the interval, encapsulating ever smaller intervals in like manner, $[a_1, b] \supset ... \supset [a_k, b_n] \supset ...$ which converges to a single point x*. But $f'(x^*) > 0$, which means that within a sufficiently small interval $(x^* - \delta, x^* + \delta)$ around x* the function is increasing. Yet encapsulation can be continued until the interval in which there exist points x_1 and x_2 such that

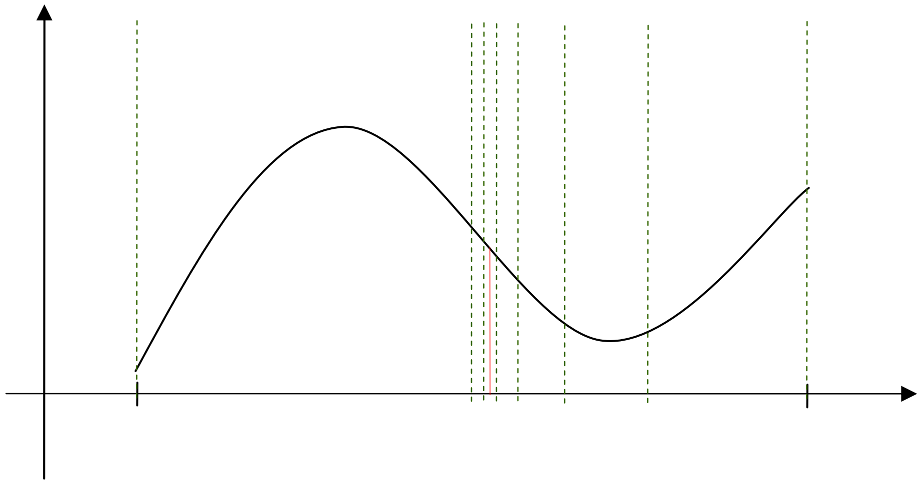


Illustration of proof by encapsulation

$x_1 < x_2$ and $f(x_1) > f(x_2)$ is within $(x^* - \delta, x^* + \delta)$, which again is a contradiction.

3.2.2.9 A proof based on a chain of theorems

Actually, the students now have everything they need to appreciate the way the main theorem is usually proved in textbooks in calculus. This introduces them to some powerful theorems in mathematics and gives them opportunity to exercise more formal and complicated mathematical reasoning. The theorems are also interesting because they are based on very different approaches, which are important for the students to appreciate if they are going to design proofs by themselves in the future. Having learned something about mathematical rigor in theorems and definitions one by one, students now get to see how the theorems that previously have been accepted on an intuitive basis now are proved; how one rests upon another; how together they make a rigorous chain of thought that leads to elaborate and elegant results.

Boundedness theorem

If *f*(x) is continuous on [a,b], then *f* is bounded on [a,b].

Proof:

Suppose f(x) is not bounded. Then for any number M there exists some x in $[a,b]$ such that f(x) > M. Suppose that there is such an x in the half-interval closest to b, and denote this interval $[a_1, b]$. Continue to half the interval, encapsulating ever smaller intervals $[a_1, b] \supset ... \supset [a_k, b_n] \supset ...$
This sequence of intervals converges to a single point, x*.[170] Let f(x*) = N. Then f being continuous gives that for all $x \in [x^* - \delta, x^* + \delta], |f(x) - N| < 1$. Thus *f*(x) is bounded within that given δ of x.

[170] Strictly speaking, this should be proven by using the supremum axiom but for present purposes the students can be asked to accept it as intuitive, understanding meanwhile that there is a potential gap in logic in the argument.

But we had narrowed down the interval of *f*'s unboundedness to any interval around x*, in particular to an interval within $[x^*-\delta, x^*+\delta]$. This gives a contradiction, so *f*(x) must be bounded.

This theorem uses an intricate proof by contradiction, which the students are familiar with but which may be further discussed at this point. One question that should be interesting at this point is whether proofs by contradiction are "good" proofs compared to proofs by construction, and what kinds of problems are suitable for which method of proof.

Another feature of this proof is the method of interval encapsulation. This reduces the problem from a question about a global, and quite intractable, property to something one can handle with the familiar epsilonics-based reasoning.

Extreme-value theorem

If *f*(x) is a continuous function on [a,b], then *f* has a maximum and a minimum on [a,b] .

Proof:

We know from the above that *f* is bounded. Thus there exists an M which is the least upper bound to *f*. Suppose that there is no x such that *f*(x) = M. Then observe the function $g(x) = 1/(f-M)$ which is continuous and therefore bounded. But because *f* can be arbitrarily close to M, *g* cannot be bounded. Therefore *f* must have a maximum, which is M. The argument to show that *f* also has a minimum is similar.

Here we see an elegant use of the previous theorem. This proof also works by contradiction, but the method is the creation of a wholly new function, *g*(*x*). Thus, the new question of extreme values was transformed into a question of boundedness to which we already had the answer.

Rolle's Theorem

Let *f* be continuous on [a,b] and differentiable on (a,b).
If *f*(a) = *f*(b), there exists a number $c \in (a,b)$ such that *f'*(c)=0.

Proof:

If *f* is constant then it is clear that *f'*(c)=0.
If *f* is not constant, suppose that there exists an x such that *f*(x)>*f*(a). Then there is a maximum of *f* on (a,b), and since the derivative of a function in a local maximum is either non-existent or zero (and the current function is differentiable on (a,b)), we have that there is a point c, maximum of *f* on (a,b), in which *f'*(c) = 0. Of course the reasoning is the same if instead there is only an x such that *f*(x)<*f*(a).

This proof has two interesting features. First, two cases are recognised and treated separately, in effect creating two narrow problems instead of one wide problem. The second feature is that it is a proof by construction – it shows that there is a point where the derivative is zero by giving us such a point. This should be contrasted to the previous proofs and discussed.

Mean-value Theorem

Let *f* be continuous on [a,b] and differentiable on (a,b).
Then there exists a $c \in (a,b)$ such that $f'(c) = (f(b)-f(a))/(b-a)$.

Proof:

Let $g(x) = f(x) - \dfrac{f(b) - f(a)}{b - a}(x - a)$

Then $g(x)$ meets the requirements stated in Rolle's Theorem, so there must exist a number *c* such that $g'(c) = f'(c) - \dfrac{f(b) - f(a)}{b - a} = 0$. This gives that for this c: $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ and we are done.

The proof of the Mean-value Theorem differs from the previous proofs in that one not only constructs a new function, but at the end returns to the first function again. One can argue that the auxiliary function is supportive in the sense of scaffolding surrounding a building under construction. It serves its purpose and is then removed.

Main Theorem

Suppose *f* is differentiable in (a, b).
a) If $f'(x) \geq 0$ for all $x \in (a, b)$, then *f* is monotonically increasing.
b) If $f'(x) \leq 0$ for all $x \in (a, b)$, then *f* is monotonically decreasing.
c) If $f'(x) = 0$ for all $x \in (a, b)$, then *f* is constant.

Proof:

The mean value theorem gives that for any a* and b* (where a*<b*) inside (a, b) there exists a c between a* and b* such that

$$f'(c) = \frac{f(b^*) - f(a^*)}{b^* - a^*}$$
$$\Rightarrow f(b^*) - f(a^*) = f'(c)(b^* - a^*)$$

From this all the statements of the theorem are readily derived.

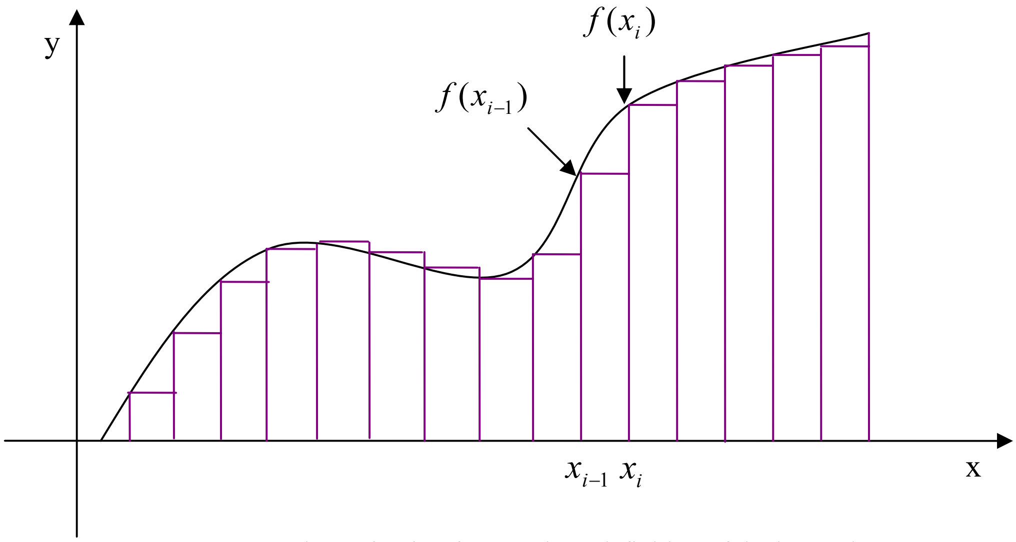
In this proof all that is done is that one uses a previous theorem and then manipulates the expressions so that they readily yield the desired properties. This "tinkering" technique is similar to that of solving algebraic equations, and similarities between proofs and the solving of other mathematical problems may be emphasized. This might be particularly helpful seeing as many students freeze in fear when they encounter problems starting with "Show that…" or "Provide a proof that…" If the students can learn to look at proofs as just another form of problem-solving perhaps this fear might dissipate.

3.2.2.10 A stricter definition of the integral and the fundamental theorem of calculus

Armed with a better understanding of limits and continuity, as well as perhaps a firmer conviction in the logic of the mean value theorem, students are now ready for a closer look at the integral and the fundamental theorem of calculus.

It makes sense to build on the students' understanding of area as many tiny rectangles and present them with Cauchy's definition of the integral:

$$S = \lim_{(x_i - x_{i-1}) \to 0} \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$$



Geometric motivation for Cauchy's definition of the integral

This is really very intuitive and probably the students will agree that one requisite for the definition to be reasonable is that the limit exists and does not depend on in what way one creates the rectangles. One property which ensures that this will be the case is the property of uniform continuity[171] because given uniform continuity it is true that for any ε, every partition $a = x_0 < x_1 < ... < x_n = X$ when it becomes increasingly finer will inevitably reach the point where the largest $x_i - x_{i-1}$ is smaller than the δ needed to satisfy the $f(x_i) - f(x_{i-1}) < \varepsilon$. This would be a good time to discuss integrability, but the students have not yet enountered functions which are integrable but not uniformly continuous. One can show that functions which are unbounded in their discontinuities are difficult to approach in this way, but any further discussion of integrability is too difficult at this level.

If the students can be convinced to replace the $f(x_{i-1})$ with $f(x_{i-1}^{\#})$ (where $x_{i-1}^{\#}$ is any number between x_{i-1} and x_i, the limit now has to exist for any choice of $x_{i-1}^{\#}$) in the definition of the integral they will be ready for the standard proof of the fundamental theorem of calculus. This proof is good for all integrable functions (in the above sense) and not just for the uniformly continuous functions for which the definition of the integral is easily seen to be well-defined.

Fundamental theorem of calculus

If *f* is an integrable function on $[a,b]$, and if $f(x) = F'(x)$ on this interval, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

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[171] Actually, Cauchy showed that the integral is well defined also for continuous functions which are not uniformly continuous, but his proof is, in my opinion, too difficult to present to the students at this stage.

Proof of fundamental theorem of calculus:

$$F(b) - F(a) = F(x_n) - F(x_{n-1}) + F(x_{n-1}) - ... - F(x_1) + F(x_1) - F(x_0) = \sum_{i=1}^{n} \left[F(x_i) - F(x_{i-1}) \right]$$

The mean value theorem gives us that there exists an $x_{i-1}^{\#}$ between every x_{i-1} and x_i so that $F'(x_{i-1}^{\#})(x_i - x_{i-1}) = F(x_i) - F(x_{i-1})$ which leads us to

$$F(a) - F(b) = \sum_{i=1}^{n} F'(x_{i-1}^{\#})(x_i - x_{i-1}) = \sum_{i=1}^{n} f(x_{i-1}^{\#})(x_i - x_{i-1})$$

Letting $x_i - x_{i-1}$ go to 0, we arrive at $F(b) - F(a) = \int_a^b f(x)dx$ and we are done.

Only the unusual length and complexity of the theorems involved in this and the previous section may be problematic, suggesting the use of the alternate proofs mentioned above. The important thing here is the progression from intuition to strict formality, mirroring the historical developments of analysis.

In this outline I have tried to present some concepts and theorems in relative detail, while others I have mentioned only in passing. The outline has followed early integration to differentiation, limits and continuity, finally returning to take a closer look at derivatives and integrals. Like the main theorem of this paper, other elements of the calculus can and should be presented several times to the students, with increasing formality and complexity. This means of course that one should not separate integration from differentiation from continuity, but rather that one should pay close attention to how the developments and needs of one field fuels and motivates progress in another.

3.3 Potential problems

Using the history of mathematics in the teaching of calculus and analysis does pose a few problems. One such problem is that of time, seeing as teachers who want to integrate history in their teaching must make room in an already busy educational plan. One solution used by those who use history in a local way is to simply replace some standard problems with problems from history and ask the students to do small projects on the famous historical problems. For those who use history globally the issue is more one of an immediate expenditure of time when redesigning the whole course, but should not require more than regular time and effort during the course itself. There are also other practical problems; in order to use history as inspiration for education the teacher needs to be well versed in the history of calculus, which means that teacher-training should put more focus on this subject. Another problem is that there is a lack of assessment of student's knowledge of the history of calculus. If there is no assessment, students might not be interested in learning something that they may consider superfluous or extra. This is primarily a problem when using history explicitly in the form of primary sources and the like, and does not influence the strategy I propose in this paper.

A different type of issue is that of the approach to history that is used when trying to implement history in this instrumental way to improve another discipline. The looming risk is that of Whiggism, the tendency to interpret history from a narrowly modern perspective. The teacher becomes an editor of history, classifying historical ideas as "relevant", "leading nowhere", or "an early version of today's integral". Even when one has zeroed in on the development or obstacle that one feels elucidates the situation of the students, one can never avoid the fact that mathematical, as well as all other, knowledge is shaped and given meaning within cultural contexts that are unique and can never be reproduced or even imitated in a classroom. Even within the explicit use of history, it is unavoidable to, at least in part, change the notation, language and formulation of old texts so as to be able to present the main ideas more effectively, and then the text will never be understood the same way by the students as it was by the mathematicians at the time the text was produced. Fried puts it this way:

> So, if one is a mathematics educator, one must choose: either (1) remain true to one's commitment to modern mathematics and modern techniques and risk being Whiggish, i.e., unhistorical in one's approach, or, at best, trivializing history, or (2) take a genuinely historical approach to the history of mathematics and risk spending time on things irrelevant to the mathematics one *has to* teach.[172]

This is one reason that I propose an implicit approach to incorporating the history of mathematics. By not stressing the explicit historical aspects, one avoids conveying to the students a history of mathematics that is simply not historical. Instead, one might present history explicitly once in a while, by showing the students old instruments of mathematics or occasionally studying old texts. Indeed, the interest of the teacher is in finding the gems of insight into the epistemology of the problems, not in being faithful to the study of history for its own sake.

[172] Fried, p 398

4. Suggestions for further research

In the future I hope that more systematic research will be done evaluating and meta-evaluating the effects of letting the history of mathematics influence the way of teaching mathematics. Qualitative studies, interviewing individual teachers and students, should be systematically collected and evaluated. Quantitative studies, with experimental classes or courses, should be evaluated in relation to the existing goals of the educational system.

In my opinion there is also a need for a thorough evaluation of the goals and standards set by Skolverket. It should be examined what the purpose of mathematics education is – whether it is to acquaint the students with mathematics only as a tool used by other sciences, or whether the wish is to show mathematics as a thing in itself, with a history, philosophical and even literary concerns: a human enterprise embedded in human culture.

Given the limits of this paper, I have only briefly discussed *some* means of integrating history with education. More work needs to be done on the "details" of implementing different aspects of and episodes in history within the many differing needs and interests calculus education.

5. Summary

The current state of education of calculus in Swedish upper-secondary and university levels is such that understanding the history of mathematics is not explicitly stated to be a goal, yet is expected of the students wishing to achieve the highest marks. The education seems to follow the traditional deductive sequence of function, continuity, derivation, and integration, repeated again and again with increasing rigor for every course taken. The main criticisms addressed in this paper are based on Manya Raman's analysis of American textbooks and include that there is insufficient motivation for the introduction of new concepts, that the shift to formal reasoning is not properly motivated or explained, and that the role of the practice problems is unclear.

My investigation into the history of calculus shows that the development has been very different from the way mathematics is taught today. Integration and derivation have been developed hand-in-hand and long before the concepts of continuity and function. Mathematics has for a long time been closely connected to the physical sciences, either by being motivated by problems in these sciences, or by providing a rigorous proof of theorems already guessed at by observing physical processes. There has always been a striving for rigor but the degree to which mathematicians have strayed beyond rigorous foundations has varied. Originally what later became the calculus was purely arithmetical, then geometrical, then algebraic, and finally all of these and something more than the parts. The problems worked on at all times were problems that stretched the abilities of mathematicians – they were problems previously unsolvable, and problems forcing further development.

Returning to the question of education, the reasons for including the history of mathematics in education are that the inclusion will humanize mathematics for the students and, more importantly for my purpose, improve the education by supplying the much-needed motivations, contexts, and a different sequence of topics. Following a discussion of various epistemological theories, I describe some methods of integrating history with education and focus on the genetic approach to calculus. I then propose some ideas on how such an education of calculus would look by giving an outline of calculus education and examining two courses based on the genetic approach. Finally, I briefly discuss potential difficulties, primarily issues with time and historical accuracy, and suggest that some further avenues of research should include meta-studies of qualitative research, as well as attempts at quantitative evaluation of the use of history in education relative local educational goals.

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