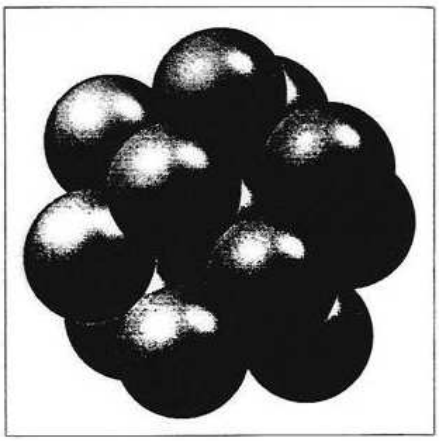
EXAMENSARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Kissing could be difficult: An analysis of the so called Kissing Problem



Thirteen spheres of equal size can simultaneously touch an inner sphere only if its radius is at least 4.6% longer than for the outer ones (Showed first time in 1951 by Kurt Schütte and Baartel Leendert van der Waerden.)

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Kissing could be difficult:
An analysis of the so called Kissing Problem

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The kissing problem

Introduction

Consider the following problem: How many spheres of same radius would simultaneously fit around a sphere of equal size provided that every outer sphere touched the inner one?

It is not very easy to solve this problem practically. Of course, one could try and hang a sphere on a thread and then try to attach spheres around it. This, however, would prove to be a rather difficult approach. As a matter of fact, as we will soon see, this problem is not that easily solved theoretically either.

A starting point would be to try and estimate an upper limit for how many spheres that could fit. A first theoretical, although very crude, attempt would be the following approach:

Every surrounding sphere has the largest possible plane cutting through its central point. The area of this plane is πr^2 which for a sphere of radius one equals π. Let us now see how many such planes that would fit on a sphere of a radius two which is the distance between the centre point of the inner sphere and the mid point of the surrounding ones. A sphere has a surface area of $4\pi r^2$ which with a radius of two equals 16π. Consequently, a maximum of 16 planes of outer balls would fit. Naturally, this method is very simplistic as it does not accurately handle the fact that the small planes are flat whereas the surface, to which they are compared, is bent. Furthermore, this attempt does not at all account for the packing question. It goes without saying that circular planes cannot be packed efficiently without area losses between them.

A less inaccurate method would be to create a ”supersphere” of radius 3, that is a sphere that would exactly contain our structure of an inner sphere surrounded by equal outer ones. We could then project a shadow of every surrounding sphere on the surface of the ”supersphere” by beaming light from the centre point of the inner sphere (see Figure 1).

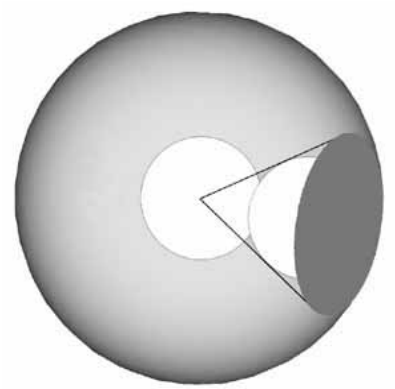

Figure 1

Since the surrounding spheres cannot overlap each other, neither can the projected shadows. Therefore we could calculate the area of such a shadow and compare it to the total area of the surface of the "supersphere". Let us derive the formula for the area of one such projected shadow:

One approach would be to use spherical coordinates (see Figure 2).

$x = \sin\varphi\cos\theta, y = \sin\varphi\sin\theta, z = \cos\varphi$

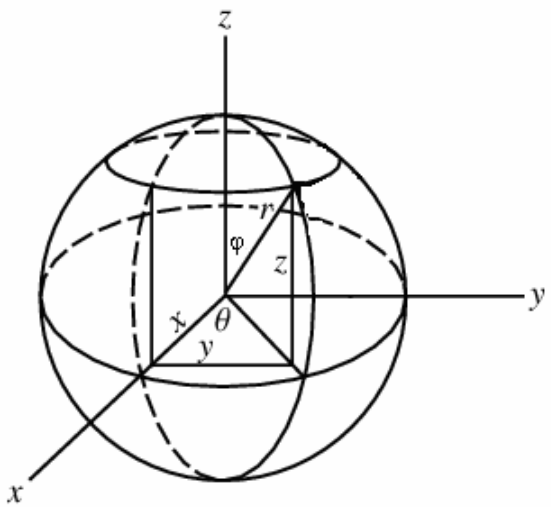

Figure 2

The area is given by the double integral: [1]

$$A = \iint_{0\le\varphi\le\pi/6,\ 0\le\theta\le 2\pi} r^2\sin\theta \, d\varphi d\theta$$

If we calculate the shadow that is made by a sphere placed on the" north pole", the limits of

[1] A. Persson, L.-C. Böiers, *Analys i flera variabler*, Studentlitteratur, Lund (2005), pp. 306-307.

the integral are 0 to 2π for θ and 0 to $\pi/6$ for φ. This is because the great circle of the inner sphere can be surrounded by exactly 6 spheres, each using up an angle of $\pi/3$ (see Figure 3). Consequently, φ will be half of that which is $\pi/6$.

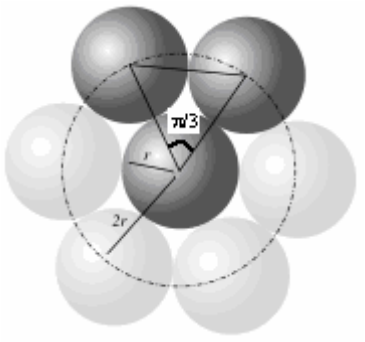

Figure 3

The area then becomes approximately 7.576.[2] Since the total area of the surface of the "supersphere" is given by the formula $4\pi r^2$, we get an area of roughly 113.097. As a result, a total of 113.097 / 7.576 ≈ 14.93 such shadows would fit on it. Naturally, we have the same problem with efficient packing as in the first method but at least we have narrowed down the upper limit of surrounding spheres to 14.

The problem of how many spheres that maximally can fit around a sphere of equal size, provided they all touch the inner one, dates back more than 300 years.[3] According to the legend, Sir Isaac Newton and his fellow mathematician, David Gregory, in 1694 argued about this maximum number. Whereas Newton claimed it was 12, Gregory would not discount the possibility of 13. Thorough studies into this topic, for example by Casselman, have not been able to substantiate these positions.[4] Nevertheless, it is still the prevailing view that it was Newton who argued for the correct solution.

One indication of the difficulty of the problem is the fact that it would take more than 250 years for anyone to come up with a solid proof. In 1953 Kurt Schütte and Baartel Leendert van der Waerden were the first ones to demonstrate a waterproof solution to the

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[2] Area = $\iint_{0\le\varphi\le\pi/6,\ 0\le\theta\le 2\pi} r^2\sin\varphi\, d\varphi d\theta = 3^2 * \int_{0\le\varphi\le\pi/6} \sin\varphi\, d\varphi \int_{0\le\theta\le 2\pi} d\theta = 9[(-\cos\pi/6 + \cos 0) * (2\pi - 0)] = 18\pi(1 - \sqrt{3}/2) = 9\pi(2 - \sqrt{3}) \approx 7.576$.

[3] G. Szpiro, *Kepler's Conjecture*, John Woley & sons, Hoboken, New Jersey (2003), p. 72.

[4] B. Casselman, "The difficulties of kissing in three dimensions", *Notices of the American Mathematical Society,* 51: 8 (2004), p. 884.

problem.[5] Three years later, in 1956, John Leech came up with a proof that was based on a different approach.[6] It is, however, Newton who has given the name to the number of maximum spheres in the problem. The number is often written with the symbol τ or κ, and is regularly referred to as ”Newton number” or ”kissing number”, ”kissing” as a metaphor for the contact points between the inner and outer spheres. Other used expressions are”contact number”,”coordination number” and ”ligancy” (the last two being used in chemistry). The actual problem is mostly known as the “Kissing number problem” or the “Problem of the thirteen spheres”.[7]

Moreover, it is important to mention that the discussion so far has been limited to a 3-dimensional space. The “Kissing number problem” is however much wider as it obviously could be considered for a space of any dimension. For a space of dimension 1, the “Kissing number” is two. It would simply be a unit segment on each side of a unit segment on a straight line. For a 2-dimensional space, the question boils down to how many circles one could place around a middle circle of same radius. It is not difficult to see that the solution is six. Since every outer circle, as has already been shown in figure 3, fits exactly within an inner angle of $\pi/3$. This also means that the outer circles touch each closest neighbour, making the structure solid with only one possible solution. As we will see later, this will not necessarily be the case for solutions in higher dimensions.

The aim of this paper is firstly to give an overview of the history of the “Kissing number problem”. The second and more important goal is to make a thorough and comprehensible analysis of Leech’s proof from 1956. The inspiration for doing this comes from two different sources. The first one is the fact that Leech’s proof was publicized in the first edition of Aigner and Ziegler’s famous book *Proofs from THE BOOK*.[8] This section, however, was then excluded in the following editions due to the large, and not so easy, degree of spherical geometry.[9] The second source for my inspiration is the following remark by Casselman:

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[5] K. Schütte, B.L.van der Waerden, ”Das Problem der dreizehn Kugeln”, *Mathematische Annalen* 125 (1953), pp. 325-334.
[6] J.Leech, ”The problem of the thirteen spheres”, *The mathematical gazette* 40 (1956), pp. 22-23.
[7] J. Conway, N. J. A. Sloane, *Sphere packings, lattices and groups*, Springer-Verlag, New York (1999), p. 21.
[8] M. Aigner, G. Ziegler, “The problem of the thirteen spheres”, *Proofs from THE BOOK*, Springer-Verlag, Berlin Heidelberg New York (1999), pp. 67-71.
[9] O. Musin, “The Kissing Problem in Three Dimensions”, *arXiv:math.MG/0410324 v3*, 1 Mar 2006.

> **It would be valuable if someone were to publish an account of Leech's proof that made it accessible to an elementary undergraduate course.**[10]

Judging from his comment, which came as late as 2004, there still exists a need for a detailed and simplified explanation of Leech's work. In the same paper Casselman also claims that there were errors in Aigner and Ziegler's analysis of Leech's proof.

My intention is that this paper will be able to shed some more light on Leech's compact solution. The aim is to do it in a way that does not require too much mathematical theory. As a direct consequence of dissecting and clarifying Leech's proof, the paper will also explain some of the basic concepts of spherical geometry. This is a mathematical discipline that according to several sources has almost been forgotten and where the last comprehensive educational book dates back to 1940.[11]

The third aim is to give some insight into the modern methods for finding Kissing numbers for any dimension. In this section, the close link between packing theory and Kissing numbers will be discussed. There will also be a short overview of the concept of error-correcting codes and its relevance for packing and Kissing theory.

The early proof attempts for n = 3 (three dimensions)

As has already been mentioned, Newton's view that it is impossible for 13 spheres to simultaneously touch an inner one has not been possible to verify. Let alone, do we have any mathematical proofs from the days of Newton. In fact, the earliest sincere attempts to solve the problem did not appear until the end of the 19th century. The fact that it took almost 200 years to come up with a proof proposal indicates that the solution is rather complex and that there were good reasons to believe that the number was greater than 12. According to George Szpiro, this belief was based not only on results like the one of the "supersphere" but also on the fact that there is an infinite amount of different solutions with 12 spheres (see App. 1).[12]

In 1869, a Swiss mathematician by the name of C. Bender sent a proof of the Kissing problem to the German journal *Archiv der Mathematik und Physik*. However, it would take

[10] Casselman, pp. 884-885.
[11] L. Kells, W. Kern, J. Bland, *Plane and spherical trigonometry*, McGraw-Hill Book Co (1940).
[12] Szpiro, p. 75.

as long as five years until his article was published.[13] The reason for this was apparently that the editor of the journal, Reinhold Hoppe, needed time to think about the problem himself.[14] Consequently, when Bender's proof was printed in 1874, it had an attached commentary by Hoppe.[15] Not only did Hoppe show that Bender's proof was flawed, he also presented his own solution. The problem with Bender's approach was that he started from a fixed structure of twelve spheres around an inner one. Using this assumption, he then showed that there would not be sufficient space to add an additional sphere. Because of his a-priori assumptions, his result was only valid for his specific starting structure where every outer sphere kisses exactly four spheres each. Therefore, his findings could no be generalised.

Hoppe, however, came up with a solution that did not require any specific assumption about how the spheres must be structured. His approach was to spin a net of polygons on the surface of the sphere, a method that bears many similarities to Leech's proof of 1956. After having done some calculations, he concluded that this net could only consist of 22 triangles. After this he showed that such a net does not fit on the surface of a sphere, why the Kissing number cannot be 13. Hoppe's proof was long regarded as being correct. It was not until Thomas Hales in 1994 pointed out his flawed triangulation algorithm to create the polygons of the net, that he lost the title of having solved the Kissing problem first.[16] The fact that it took so long for anyone to point out Hoppe's mistake may have to do with the fact that the larger part of his proof is correct. Later in this paper, in the analysis of Leech's proof, we shall take a closer look at Hoppe's approach.

Besides Bender and Hoppe, two other mathematicians are mentioned for their attempts to solve the Kissing problem. The first one is Siegmund Günther who in 1875 came to the "exciting" conclusion that he neither could prove nor denounce the possibility of a solution entailing 13 outer spheres.[17] The second attempt was produced by Boerdjik in 1952. As Hales showed later, his approach included erroneous assumptions which entirely invalidated his solution.[18]

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[13] C.Bender, "Bestimmung der grössten Anzahl gleich grosser Kugeln, welche sich auf eine Kugel von demselben Radius, wie die übrigen, auflegen lassen", *Archiv der Mathematik und Physik*, 56 (1874), pp. 302-306.
[14] Szpiro, p. 82.
[15] R.Hoppe, "Bemerkung der Redaction", *Archiv der Mathematik und Physik*, 56 (1874), pp. 307-312.
[16] T. Hales, "The status of the Kepler conjecture", *The Mathematical Intelligencer*, 16 (1994), pp. 49-50.
[17] Szpiro, p. 85.
[18] Hales, p. 50.

Leech's proof

Before starting to explore Leech's proof (Appendix 3), it is interesting to briefly mention the two first solutions to the Kissing problem. They appeared simultaneously in a joint paper by Schütte and van der Waerden in 1953. The paper starts with van der Waerden's proof but the writers themselves underline that the proof by Schütte has priority since, although it is rather difficult, it gives the necessary tools for other analyses. In fact, the paper was really only a piece of a larger work by the authors. Already in 1951 van der Waerden wrote two joint papers about how to find the smallest possible sphere in order to fit a certain number of outer spheres around it.[19] It was then it was proposed (still to be proved) that the best possible configuration for 13 spheres was the one shown on the cover page, a 1-4-4-4 structure (the outer spheres structured in three rings with four spheres and one by itself) with an inner sphere of radius close to 1.046. The proofs of the Kissing problem that were then published in 1953 were therefore to some extent a special case of the previous findings.

Whilst the two first proofs showed that a sphere which is circumscribed by 13 unit spheres must have a radius greater than one, Leech's approach is to prove that a unit sphere cannot be touched by 13 outer spheres simultaneously. In his proof, which is not even two pages long and regarded as very elegant, Leech does not mention Hoppe's results. This fact is maybe a little surprising considering that the two methods bear similarities both in terms of approach and mathematical tools used. As many commentators have pointed out, Leech's article is rather difficult to understand. The reasons are twofold; firstly he uses a difficult language which is not always easy to follow, secondly the mathematical content is very compact as "certain details which are tedious rather than difficult are omitted".[20] Below follows a detailed explanation on all the steps involved in Leech's proof. Whenever relevant, geometrical figures are presented in order to give the reader a better understanding of the crucial spherical geometry used.

1. The aim of Leech's proof

The aim of the proof is to show that a centrally positioned unit sphere cannot simultaneously

[19] W. Habicht, B. L. van der Waerden, "Lagerung von punkten auf der Kugel.", *Mathematische Annalen* 123 (1951), pp. 223-234., K. Schütte, B. L. van der Waerden, "Auf welcher Kugel haben 5, 6, 7, 8, oder 9 Punkte mit Mindestabstand Eins Platz?", *Mathematische Annalen* 123 (1951), pp. 96-124.
[20] Leech, p. 22.

be touched by 13 similar spheres. The problem formulation implies that none of the outer spheres overlap which means that they must be apart by a great circle distance of at least $\pi/3$ as seen in Figure 3.

2. Criteria for placing points on the surface of the inner sphere

Every one of the 13 points on the surface of the inner sphere represents the tangent point, the ”kiss”, between it and an outer sphere. Consequently, the minimum distance between any two points must be $\pi/3$.

3. Decision rule for creating a network on the surface of the inner sphere

Leech uses an algorithm for connecting two such points on the surface of the central sphere. Two points shall be connected by a line if the distance between them is greater or equal to $\pi/3$ and shorter than arccos1/7. The later representing a value of roughly 0.45π. Why does he choose such an upper limit? One reason is that, with a limit lower than $\pi/2$, there is no possibility of two connecting lines crossing each other. The minimal quadrilateral will have all sides equalling $\pi/3$ and therefore never have both its diagonals shorter than the upper limit arccos1/7. This follows from the fact that the symmetrical quadrilateral has both diagonals exactly of length $\pi/2$ (See Appendix 2, page 43 for calculation).Would one compress it by shortening one of the diagonals to lie below the limit, the length of the other diagonal would increase. Therefore, we could not have a situation where both diagonals cross.

Symmetrical spherical quadrilateral **Compressed spherical quadrilateral**

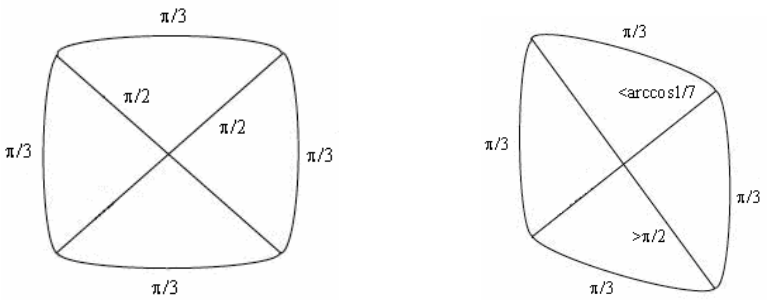


Figure 4

(It is always problematic to depict spherical polygons in two dimensions. Therefore, the reader must try to imagine that the bent or straight lines, making up the polygons in the figures presented in this paper, in fact are parts of great circles around a sphere.)

4. The network is connected and consists of spherical polygons

After having connected all points on the sphere according to the above mentioned algorithm, we will have a network of polygons whose vertices consist of these points. We can assume that the network is connected. Would this not be the case, for example would we have two "islands" of networks, we could move the nets closer to each other until the distance between one point of each island is less than the upper limit. Moreover, we can assume that no point exists with only one edge. If this were the case, the point could be moved, still attached to one edge, until it came at a distance to another point that is less than the upper limit. This means that each point has at least two edges, ensuring that all points are vertices of a polygon. In practise the movement of a point of course would mean rolling an outer sphere on the surface of the central one.

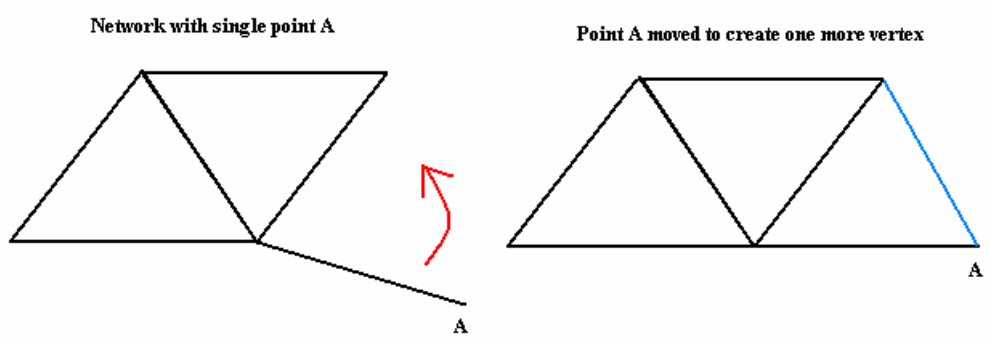


Figure 5

5. Every point of the network can be connected to a maximum of five edges

Every point, being a vertex of at least two polygons of the network, can at most be connected to five edges. The reason for this is the upper limit of the algorithm that only allows lengths of the edges of the network up to, but not including, arccos1/7. An elaboration on this constraint in Leech's network is found in the following chapter.

After having constructed a network according to his algorithm, Leech now continues with showing that no such network can exist for 13 points.

6. Calculation of minimum area for every type of polygon

Leech now calculates the minimum area for each polygon that might exist in the network (for area calculation see Appendix 2):

***Equilateral Triangle**:*

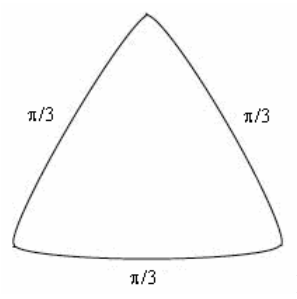


Figure 6

The smallest spherical triangle in the network is the one of all sides of length π/3. Since the surface is on a unit sphere, the area of this triangle is simply the sum of its angles minus π. Every angle, α is given by the formula:

$$\alpha = \arccos\left(\frac{\cos \pi/3 - \left(\cos \pi/3\right)^2}{\left(\sin \pi/3\right)^2}\right) = \arccos\left(1/3\right)$$

The area is therefore: $3\arccos\left(1/3\right) - \pi \approx 0.5513$

Equilateral quadrilateral:

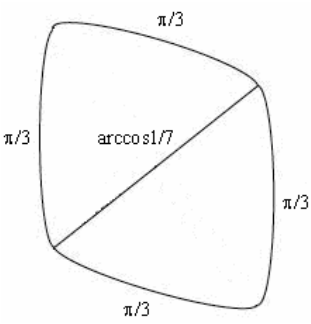


Figure 7

The equilateral quadrilateral with the smallest area is one of equal sides π/3 and where one of the diagonals is arccos1/7 (would we make it shorter, the quadrilateral would break down into two triangles). Its area is equal to that of two triangles with two sides of length π/3 and the third arccos1/7. Each triangle would have two angles, α, given by:

$$\alpha = \arccos\left(\frac{\cos\pi/3 - \cos\pi/3\cos\left(\arccos 1/7\right)}{\sin\pi/3\sin\left(\arccos 1/7\right)}\right) = \pi/3$$,and the third angle, β, given by:

$$\beta = \left(\frac{\cos\left(\arccos 1/7\right) - \left(\cos\pi/3\right)^2}{\left(\sin\pi/3\right)^2}\right) = \arccos\left(-1/7\right)$$

Therefore the total area of this quadrilateral is: $2\left(2\pi/3 + \arccos\left(-1/7\right) - \pi\right) \approx 1.3339$

E*quilateral pentagon*:

Symmetrical pentagon **Compressed minimal pentagon**

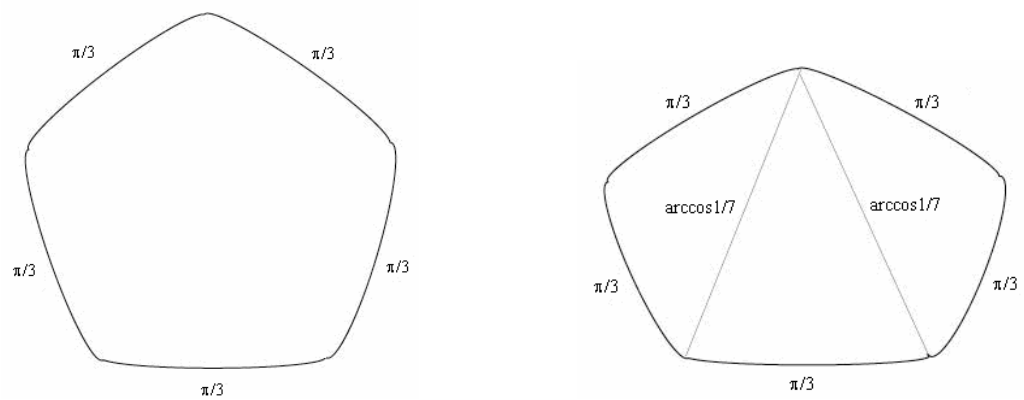


Figure 8

The smallest equilateral pentagon has equal sides of π/3 and two coterminous diagonals of length arccos1/7 (Would we compress it more, the diagonals would be shorter and, according to the algorithm, they would have to make edges why we would get 3 triangles instead of the pentagon.). This pentagon can be broken down into three triangles where two have two sides of length π/3 and one of length arccos1/7 and the third triangle with one side of length π/3 and two of length arccos1/7. Therefore the total area equals the area of the above calculated quadrilateral plus the area of the third triangle. The angles of this triangle are:

Two angles, α, where $\alpha = \arccos\left(\frac{\cos\left(\arccos \frac{1}{7}\right) - \cos\frac{\pi}{3}\cos\left(\arccos \frac{1}{7}\right)}{\sin\frac{\pi}{3}\sin\left(\arccos \frac{1}{7}\right)}\right) = \arccos\left(\frac{1}{12}\right)$

One angle, β, where $\beta = \arccos\left(\frac{\cos\frac{\pi}{3} - \left(\cos\left(\arccos \frac{1}{7}\right)\right)^2}{\left(\sin\left(\arccos \frac{1}{7}\right)\right)^2}\right) = \arccos\left(\frac{47}{96}\right)$

Total area of the pentagon = $1.334 + \left(2\arccos\left(\frac{1}{12}\right) + \arccos\left(\frac{47}{96}\right) - \pi\right) \approx 2.2262$

Leech stops at the pentagon as no larger polygons could fit in the network.

7. The larger polygon – the more inefficient the network becomes

The next step in Leech's proof calculates the difference between the area of each minimal n-gon (where n denotes number of vertices) compared to the (n-2) triangles that could substitute it. This difference, which is a measure of the n-gon's inefficiency, Leech calls "excess".

Excess for minimal equilateral quadrilateral:

The quadrilateral could be replaced by two minimal triangles why the excess is:

$1.3339 - 2 \times 0.5513 \approx 0.231$

Excess for minimal equilateral pentagon:

The pentagon could be replaces by three minimal triangles why the excess is:

$2.2262 - 3 \times 0.5513 \approx 0.572$

The larger n-polygon we have, the larger its excess will be:

A first observation about the excess is that it always is positive. Moreover, it is clear that the higher n-gon we look at, the larger the excess will be. Since any polygon admissible by Leech's algorithm cannot have any of its diagonals shorter than arccos1/7, any deformation of the polygon into triangles will always produce triangles with at least one side of length arccos1/7. Since the area of this triangle is larger than the one of the minimal triangle which

have all sides of length $\pi/3$, the excess for polygons grows the higher n is. In other words, if we start with an n-gon with a certain excess, on margin we have to add a triangle with a side of length no shorter than arccos1/7 to create an (n+1)-gon. The excess for this polygon must be equal to the excess for the n-gon plus the excess created by the added triangle.

8. Using Euler's formula to create possible networks

At this point of the proof, Leech uses Euler's formula from 1750. The formula specifies the relationship between vertices, edges and faces of a polygon. The formula looks like this:

If G is a connected plane graph with V vertices, E edges and F faces, then

(1) V - E + F = 2

The formula is also true for polyhedra on spheres in three dimensions as each polyhedron can be drawn as a connected plane graph in two dimensions. One way to see this is to imagine a balloon that is punctured and then stretched on the plane

(A simple example of the formula would be the cube. It has 6 faces, 8 vertices and 12 edges)

To start with, Leech writes the formula slightly differently:

(2) 2V - 4 = 2E - 2F

He then realises the relationship:

(3) 2E = (3*number of triangles) + (4*number of quadrilaterals) + (5*number of pentagons) + (……..) + (n*number of n-gons)

Why does the last equation hold? Well, first of all a triangle has three edges, a quadrilateral four and so forth. Second, in three dimensions every edge will be an edge of two polygons, one on each side. Therefore the sum on the right side of equation **(3)** will double count the total number of edges. This explains why the total number of edges, E, must be multiplied by two on the left side. Now let us define a notation that will be helpful:

F_n stands for number of polygons with n edges. Consequently, F_3 is the number of triangles, F_4 is the number of quadrilaterals and so on. Given this notation we have the

following equality:

(4) $\mathbf{F = F_3 + F_4 + F_5 + \ldots + F_n}$, Using this notation in **(3)** we get:

(5) $\mathbf{2E = 3F_3 + 4F_4 + 5F_5 + \ldots + nF_n}$

Putting the right hand side of **(5)** into **(2)** gives us:

$\mathbf{2V - 4 = (3F_3 + 4F_4 + 5F_5 + \ldots + nF_n) - 2(F_3 + F_4 + F_5 + \ldots + F_n)} \Leftrightarrow$

(6) $\mathbf{2V - 4 = F_3 + 2F_4 + 3F_5 + \ldots + (n-2)F_n}$

We also know that the area of the network, A, is at least the sum of the number of minimal polygons multiplied by their area:

(7) $\mathbf{A \geq 0.551F_3 + 1.334F_4 + 2.226F_5 + \ldots}$

Leech then skilfully reshuffles the right hand side in the following way:

$\mathbf{A \geq 0.551(F_3 + 2F_4 + 3F_5 + \ldots + (n-2)F_n) + 0.231F_4 + 0.572F_5 + \ldots}$

It is important to note that every F_n in the infinite sum on the right side of the brackets is multiplied by a coefficient which is exactly the excess number for that particular n. As was explained on page 12, the excess is always positive and grows with a higher n.

Now the sum which is multiplied by 0.551 equals, according to **(6)**, 2V-4 why we can write:

(8) $\mathbf{A \geq 0.551(2V - 4) + 0.231F_4 + 0.572F_5 + \ldots}$

Since the total area of the network cannot exceed the area of the sphere, 4π, we get the inequality:

(9) $\mathbf{0.551(2V - 4) + 0.231F_4 + 0.572F_5 + \ldots \leq 4\pi}$

First we look at the theoretical maximum for V. This is attained with $F_n = 0$, for all n>3:

(10) $\mathbf{0.551(2V - 4) \leq 4\pi \Leftrightarrow V \leq 13.25}$

From this we can deduce that no more than 13 vertices could be placed on the sphere. This is therefore to say that no more than 13 spheres could kiss the inner sphere. Let us then assume that V = 13 and apply it into **(9)**:

$\mathbf{0.551(2 * 13 - 4) + 0.231F_4 + 0.572F_5 + \ldots \leq 4\pi \Leftrightarrow}$

(11) $\mathbf{0.231F_4 + 0.572F_5 + \ldots \leq 0.438}$

This inequality only holds if $F_4 = 0$ or $F_4 = 1$ and with $F_n = 0$, for all n>4

The conclusion from this is that the only possible network that may include 13 vertices either:

1) consists of only triangles, or 2) consists of triangles and one quadrilateral

9. Analysis of the possible networks

Case 1: Only triangles.

Equation **(5)** showed us that $2E = 3F_3 + 4F_4 + 5F_5 + \ldots + nF_n$

Therefore, only having triangles in the network gives us:

(12) $\mathbf{2E = 3F_3 \Leftrightarrow F_3 = 2/3E}$

Using **(12)**, **(4)** and the fact that V = 13, Euler's formula, **(1)**, gives us:

(13) $\mathbf{13 - E + 2/3E = 2 \Leftrightarrow 1/3E = 11 \Leftrightarrow E = 33}$ which in **(12)** also gives us $\mathbf{F_3 = F = 22}$

Leech's analysis has produced a network with 13 vertices, 22 triangles and 33 edges. Having 33 edges implies that there are 33 * 2 = 66 ends that must be connected to the 13 vertices. This means that each vertex on average will be connected to 66 / 13 = 5.08 edges. Leech

earlier showed that any vertex only can be connected to a maximum of 5 edges (see point 5, p. 9). Therefore the existence of a network of only triangles is contrary to the underlying assumptions.

Case 2: One quadrilateral and triangles.

Using the same approach as for analysing the first network we get:

(14) $\mathbf{2E = 3F_3 + 4F_4 \Leftrightarrow 2E = 3F_3 + 4 \Leftrightarrow F_3 = 2/3E - 4/3}$

Putting this into Euler's formula, **(1)**, gives us:

$\mathbf{13 - E + (F_3 + F_4) = 2 \Leftrightarrow 13 - E + (2/3E - 4/3 + 1) = 2 \Leftrightarrow 1/3E = 32/3 \Leftrightarrow E = 32}$ which put into **(14)** gives us:

(15) $\mathbf{F_3 = 2/3 * 32 - 4/3 = 20}$

Leech then makes an inspection of the network consisting of 13 vertices, 20 triangles, 1 quadrilateral and 32 edges. The average edge connections per vertex is 64 / 13 = 4.92 which does not, as opposed to the previous structure, violate any of the underlying assumptions. The network will have 12 vertices connected to 5 edges each and one to 4. The one with 4 could be a vertex of a triangle or the quadrilateral.

Leech does not choose a theoretical approach to prove that the two networks are impossible. Instead he says: "I know of no better proof of this than sheer trial."[21] He then starts constructing a network with a quadrilateral as the starting polygon with one triangle abutted to each of its sides. After this he examines the two cases:

1) Every vertex of the quadrilateral is connected to a fifth edge and
2) Three of the vertices of the quadrilateral are connected to a fifth edge and one only to four.

[21] Leech, p. 23.

The vertex, A, of a triangle with 4 connections **The vertex, A, of a quadrilateral with 4 connections**

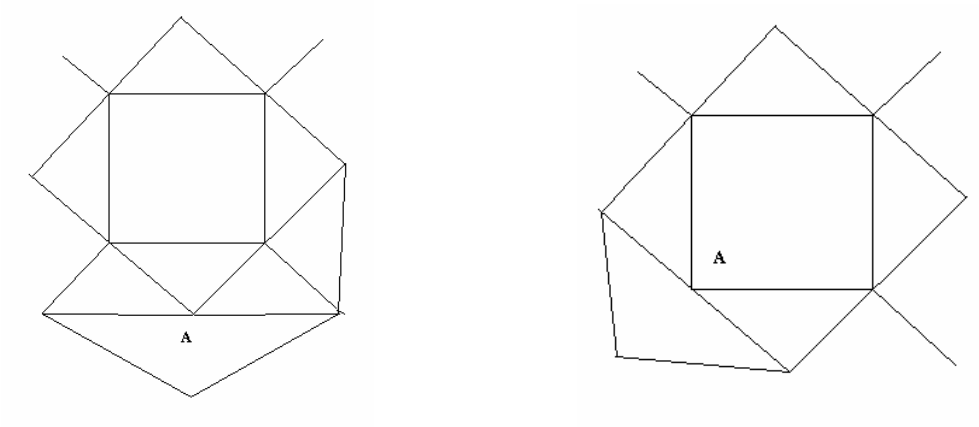


Figure 9

He then tries to practically construct the defined polyhedron but concludes that "it is immediately found to be impossible to complete the network". This means that he has been able to prove that neither of the only two possible configurations, 1) 13 vertices, 22 triangles and 33 edges and 2) 13 vertices, 20 triangles and 1 quadrilateral, can exist. The impossibility of these configurations to exist implies that 13 spheres of equal size cannot simultaneously touch an inner sphere of equal size. Therefore, the Kissing number in three dimensions is 12.

Analysis of Leech's proof

Let us now take a closer look at Leech's proof and try to understand some of its beauty. Hoppe failed with his proof because of a faulty algorithm to create the net between the contact points on the surface of the inner sphere. In the case of Leech, we shall see that his algorithm was created with great care in order to simplify the solution.

It is obvious why the lower limit for being able to connect two points is set to $\pi/3$, but why did he chose an upper limit of arccos1/7? One explanation, as has already been mentioned, is the desire to have the possibility of the existence of quadrilaterals with sides equal to $\pi/3$. However, it also means that the net cannot have any triangle with longer sides than arccos1/7. This is a very important aspect since it guarantees, as we will soon see, that the possible network of 22 triangles cannot exist. On the other hand, Leech does not want to get too many different combinations of possible network solutions. Had he chosen an upper limit closer to $\pi/3$, the excess area of the minimal equilateral quadrilateral and pentagon would have been much smaller. For example, an upper limit of arccos1/3 gives the possibility

of 5 different solutions; only triangles, triangles plus one quadrilateral, triangles plus two quadrilaterals, triangles plus three quadrilaterals and finally triangles plus one pentagon. He then would have needed to show that each one of these networks could not exist.

One may wonder why Leech put the upper limit at arccos1/7 instead of letting it be even closer to $\pi/2$, for example $< \pi/2$. At a first glance, this would seem to be sufficient upper limit as it would not allow the most obvious structure with one vertex being connected to 6 others; one sphere on the "north pole" connected to 6 spheres centred on the "equator". However, as the following analysis shows, there exists a configuration that puts all the 6 spheres closer than $\pi/2$ to the vertex.

Analysis of configuration with 1 vertex connected to 6 edges

Imagine the below depicted unit sphere. C1 represents a circle around the Z-axis which is at a distance of $\pi/3$ from the "north pole", NP. C2 represents a circle around the Z-axis which is at a distance of arccos1/7 from NP. C3 and C4 are great circles going through NP and separated by an angle of $\pi/3$.

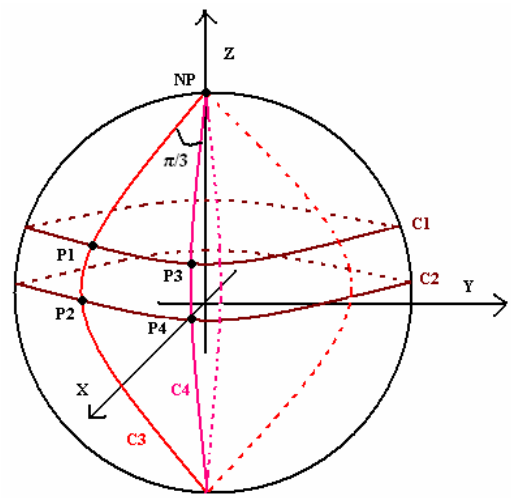


Figure 10

Now let us look at the spherical quadrilateral which is defined as the area between the intersections of C1, C2, C3 and C4 on the positive half of the X-axis. This figure will have the corners denoted P1, P2, P3 and P4. Let us calculate the largest distance possible between two points of this quadrilateral. This will of course be the diagonal of the figure, that is P1-P4 or P2-P3 (they are of equal length). Since the sphere is a unit one, the length of P1-P4 is

simply the angle P1-O-P4, where O is the middle point of the sphere. The easiest way to get this angle, let it be denoted α, is using vector algebra where we have the equation:

$$\cos\alpha = \frac{\vec{u}_1 \cdot \vec{u}_4}{|\vec{u}_1||\vec{u}_4|}$$

With $\vec{u}_1$ being the vector from origo, O, to P1 and $\vec{u}_4$ is the vector from origo to P4. Since the lengths of the vectors are 1, our formula becomes:

$$\cos\alpha = \vec{u}_1 \cdot \vec{u}_4$$

In order to calculate the scalar product we need the coordinates of P1 and P4. Using spherical coordinates (se page 2); $x = \sin\varphi\cos\theta, y = \sin\varphi\sin\theta, z = \cos\varphi$ and making the calculation as easy as possible by letting C3 lie on the XZ-plane, P1 gets the spherical coordinates:

$$\left(\sin\tfrac{\pi}{3}\cos 0, 0, \cos\tfrac{\pi}{3}\right) = \left(\tfrac{\sqrt{3}}{2}, 0, \tfrac{1}{2}\right)$$

P2 gets the coordinates:

$$\left(\sin\left(\arccos\tfrac{1}{7}\right)\cos\tfrac{\pi}{3}, \sin\left(\arccos\tfrac{1}{7}\right)\tfrac{\sqrt{3}}{2}, \cos\left(\arccos\tfrac{1}{7}\right)\right) =$$

$$\left(\frac{\sqrt{\frac{48}{49}}}{2}, \sqrt{\tfrac{48}{49}}\,\tfrac{\sqrt{3}}{2}, \tfrac{1}{7}\right) = \left(\tfrac{\sqrt{12}}{7}, \tfrac{6}{7}, \tfrac{1}{7}\right)$$

Since $\vec{u}_1$ is the vector from origo with the coordinates (0,0,0) to P1, $\vec{u}_1 = \left(\tfrac{\sqrt{3}}{2}, 0, \tfrac{1}{2}\right)$

$\vec{u}_4$, also starting at (0,0,0) is equal to $\left(\tfrac{\sqrt{12}}{7}, \tfrac{6}{7}, \tfrac{1}{7}\right)$

Going back to calculating the angle α between the two vectors we have that:

$$\cos\alpha = \left(\tfrac{\sqrt{3}}{2}\tfrac{\sqrt{12}}{7} + 0 + \tfrac{1}{2}\tfrac{1}{7}\right) = \tfrac{6}{14} + \tfrac{1}{14} = \tfrac{1}{2} \Rightarrow \alpha = \tfrac{\pi}{3}$$

Since the angle α is the length of the diagonal P1-P4, we can draw the conclusion that the longest possible distance between two points on the spherical quadrilateral P1-P2-P3-P4 is π/3.

Imagine that we in the above drawn figure add a third great circle, C5, that also goes through point NP and whose angle with C4 is π/3. Adding C5, we would cut the area between C1 and C2 into 6 identical pieces all of which would be of the same size as the quadrilateral P1-P2-P3-P4. Now consider the structure where NP represents one kissing point, and

therefore also a vertex, of Leech's network and that we would like it to be connected to 6 edges. The first restriction is that the other side of these edges can not lie closer to NP than the circle C1. Let us therefore create our first edge, E1, by placing a sphere at point P1 in our picture. After this we want to find a second point to create E2. We have the restriction that this point must be at least π/3 away from both NP and P1, as well as the overall restriction that all together 6 such points must be placed around NP. It is easy to see that the most efficient placing of these points will be at the diagonals of the 6 quadrilaterals that were defined by C1 through C5. Any other configuration would make at least one edge longer than arccos1/7. However, Leech's upper limit does not allow for C2 since the distance between two spheres can only be smaller than arccos1/7. This means that the diagonals for the 6 equal quadrilaterals created with a $C2 < \arccos 1/7$ will be just short of the necessary distance of π/3. This in turn means that the upper restriction in Leech's algorithm makes it impossible to have a vertex with 6 edges.

Using the knowledge from the above derived result, let us now construct the configuration having 1 vertex and 6 edges where the longest edge is as short as possible.

Configuration of 1 vertex and 6 edges where the longest edge is minimized

Let us start with a kissing point between the inner sphere and an outer sphere. In the picture below this vertex is denoted O. Put 3 spheres, A, B and C, symmetrically, around the sphere with kissing point O, so that they touch this sphere. According to Leech algorithm, we can draw edges between the kissing points in the following way:

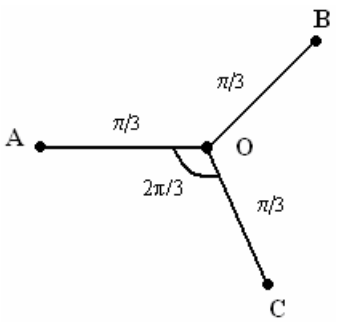


Figure 11

Now add another three spheres, D, E and F, to the above described configuration. The closest each of the new spheres can come to vertex O is when they lie on a distance of exactly π/3 from the closest points. This means that for example point F will be π/3 from points A and C.

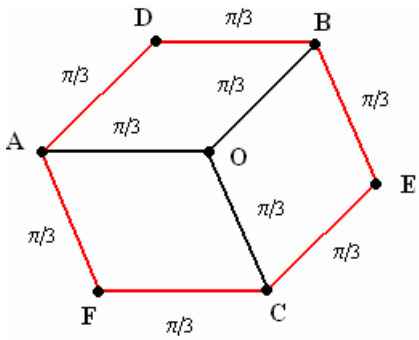


Figure 12

What we would like to know is what the distance of the edges between O and the new points D, E and F is. This distance is still unknown and is denoted by x.

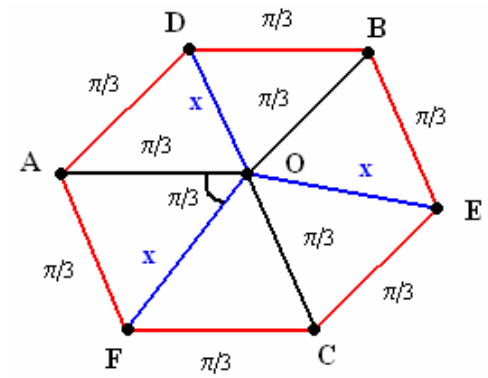


Figure 13

In order to calculate x, we concentrate on the spherical triangle OAF. Applying the dihedral formula we get the equation:

$$\pi/3 = \arccos\left(\frac{\cos \pi/3 - \cos \pi/3 \cos x}{\sin \pi/3 \sin x}\right) \Leftrightarrow 1/2 = \frac{1/2 - 1/2 \cos x}{\sqrt{3}/2 \sin x} \Leftrightarrow \sqrt{3}/4 \sin x = 1/2 - 1/2 \cos x$$

$$\Leftrightarrow 1 = \sqrt{3}/2 \sin x + \cos x \Leftrightarrow 1 - \cos x = \sqrt{3}/2 \sin x$$

Now let us quadrate each side of the equation. From this we get the equation:

$1+(\cos x)^2-2\cos x=\frac{3}{4}(\sin x)^2$, substitute $(\sin x)^2$ with $\left(1-(\cos x)^2\right)\Rightarrow$

$1+(\cos x)^2-2\cos x=\frac{3}{4}\left(1-(\cos x)^2\right)$, now substitute $\cos x$ with y $\Rightarrow$

$1+y^2-2y=\frac{3}{4}-\frac{3}{4}y^2 \Leftrightarrow y^2-\frac{8}{7}y=-\frac{1}{7}$

Solving this equation gives us the solutions: $y_1=\frac{1}{7}, y_2=1$

Since y stands for $\cos x$, where x is the distance we are seeking, we can reject the solution where y is 1. Consequently, the solution is that $\cos x$ = 1/7 which gives us that x = arccos1/7.

Now, it is most understandable why Leech chose such a seemingly odd upper limit as arccos1/7. He simply wanted as high a limit as possible under the restriction that using this limit no vertices with 6 edges could exist.

A look at some spherical polygons which do not exist in Leech's network

A final aspect of Leech proof which needs to be highlighted and discussed is his implied assumption that the optimal triangle, from the point of view to minimise the area of the net, for building an efficient network is the equilateral one with side length π/3. The following examples will illustrate that non-regular 3 and 4-sided spherical polygons that have smaller areas than Leech's equilateral ones, in fact at best are as efficient as the ones used by him.

Example 1:

Let us assume a spherical triangle ABC with two sides equal to π/3 (AB and BC) while AC is much longer than π/3. In reality this configuration would be possible as all Kissing points lie sufficiently apart from each other. Furthermore, the area of this triangle could be smaller than for the one of equal sides π/3 used by Leech, if the side AC is long enough.

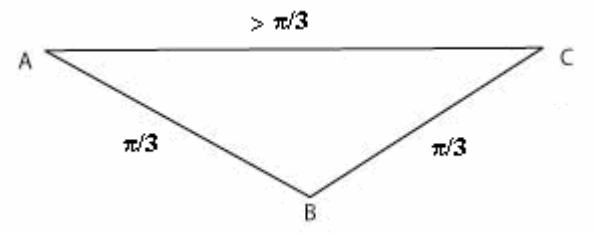


Figure 14

However, the long side AB must also be the side of another polygon on the other side. Let us assume that this is also a triangle. The vertex D of this triangle must lie at a distance of at least π/3 from all three points A, B and C.

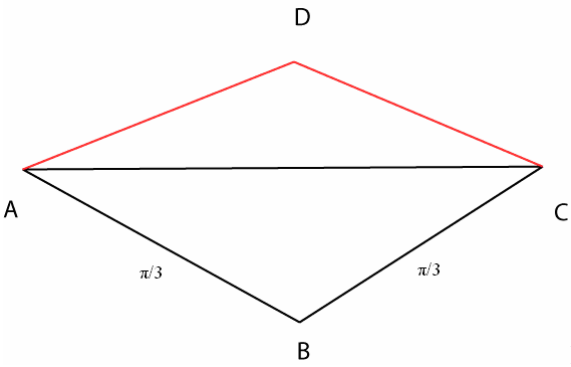


Figure 15

Let us now analyse the area of the combined triangles. A useful approach is to triangulate the 4-sided polygon by joining B and D instead of A and C.

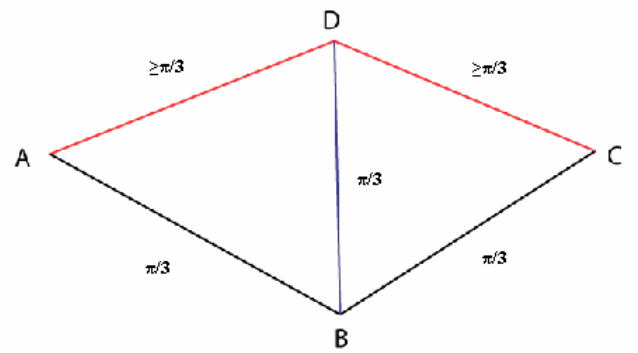


Figure 16

We see clearly that in the best case each triangle is equal to Leech's minimal one with all sides equal to π/3. If the initial distance AC is "too long", we will find that one or both of the distances AD and CD is longer than π/3. Hence, what seemed like an efficient triangle ended up belonging to, at best, the minimal quadrilateral with sides and diagonals of π/3.

Example 2:

Using the same technique, let us look at a possible 4-sided spherical polygon with an area smaller than two combined minimal triangles. We start with such a spherical polygon ABCD where AB, BC, and CD all are of length π/3 and AB being much longer. AD must be the side of a triangle why there must exist a vertex E which is at least at a π/3 distance from all points A through D.

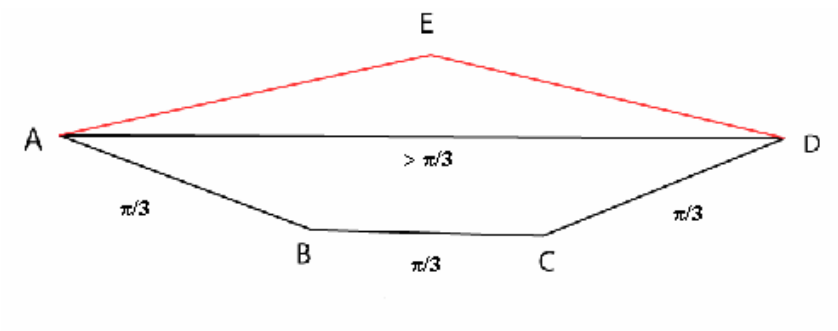


Figure 17

When analysing the combined polygon ABCDE we draw the lines BE and CE.

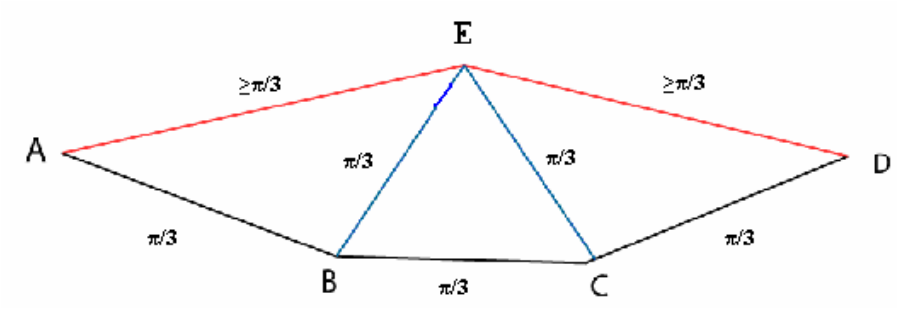


Figure 18

Again, it is seen that the area of the polygon is at best equal to three of Leech's minimal triangles. Had the AD side been "too long", AE and/or DE is longer than π/3 why the initial 4-sided polygon in fact contributed to a non-optimal packing.

A short comparison between Leech's and Hoppe's approaches

Undoubtedly, Leech's proof is very elegant, using a minimum of mathematics thanks to its powerful algorithm. Nevertheless, it is interesting to conclude this section of the paper by returning to the failed solution proposed by Hoppe. The argument about his deficient

triangulation algorithm is valid but is it really that important? Let us consider the following argument: If the Kissing number is 13, then there must exist a net of spherical polygons with 13 vertices, 33 edges and 22 faces. This, of course, follows from Euler's formula but also from the fact that if there is a solution with other spherical polygons than only triangles, these polygons can always be triangulated. Using this as a starting point, one could argue that it would be enough to prove that no such configuration with 22 triangles, with the restriction that each join must be at least $\pi/3$ long, can fit on the surface of a unit sphere. Hoppe failed with his algorithm (probably because he wanted to keep it as simplistic as possible) but no one has argued that his calculation, that proves the impossibility of spinning such a net on a unit sphere, is wrong. Therefore, it might be of interest to shortly describe his approach.

Hoppe's approach bears similarities to Leech's. He also spins a net and then analyses the result using Euler formula. From this he draws the conclusion that if such a net exists it must contain 22 spherical triangles. However, his approach starts with the fact that one of the triangles must have a vertex connected to 6 edges. Then he calculates the area of the minimal spherical figure that consists of 6 adjacent spherical triangles with one mutual vertex. He shows is that this figure is a 6-edged spherical polygon with all sides of length $\pi/3$ which contains a central point. When the central point is connected to the 6 vertices, we get 4 edges of length $\pi/3$ and 2 of length $\pi/2$.

Hoppe's spherical figure of smallest area containing a vertex, A, connected to 6 edges

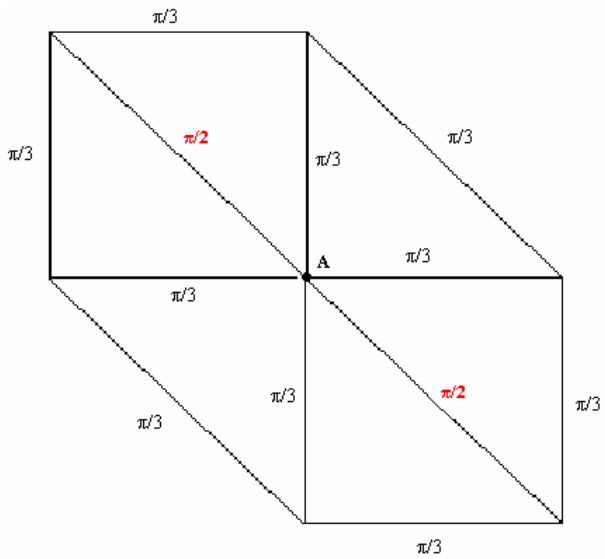


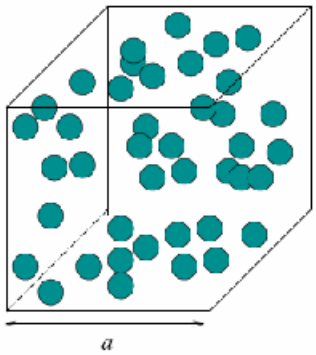
Figure 19

Hoppe then argues that the remaining 16 spherical triangles in the net cannot have a total area smaller than 16 times the triangle having all sides equal to $\pi/3$. He calculates the total minimum area possible and concludes that this is greater than the area of the surface on which it must fit. As we can see, in essence Hoppe's approach is very similar to Leech's. The largest difference is really the two algorithms; Hoppe's is ambiguous but also makes it necessary to calculate the smallest figure including 6 triangles with one common vertex, whereas Leech's elegant algorithm makes it possible to use simpler mathematical methods.

Newer methods to solve the Kissing problem for n ≥ 3

Before we start exploring more modern methods to solve the Kissing problem it is worthwhile discussing the relationship between this problem and the problem of optimal packing. In the latter problem, the goal is to pack many copies of a certain solid as efficiently as possible. The objective is to maximise the space that is occupied by the solids.

Mathematical definition of density, δ, in three dimensions



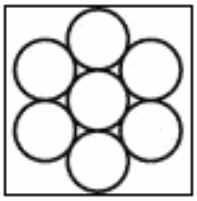
$$\delta = \lim_{a \to +\infty} \frac{\text{volume of balls}}{\text{volume of cube}}$$

Figure 20

When talking about packing problems one has to distinguish between lattice structures, that is structures that are perfectly symmetrical and general structures where the structures around two solids in the packing do not need to be identical. Another, rather obvious, aspect to consider is that the optimal structure may look different for a local compared to a general problem. An example of this in two dimensions would be to pack circles of radius 1 in a limited square with side length 6. The configuration that maximises density to this local problem is not the, in two dimensions generally most efficient, hexagon structure, which

would only fit 7 circles, but instead the quadratic configuration of equal columns side by side which fits 9.

The hexagonal structure only fits 7



The quadratic structure fits 9

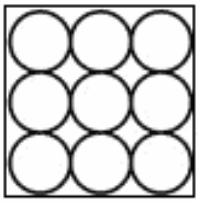


Figure 21

For three dimensions Gauss showed already in 1831 that the optimal lattice packing of spheres is the so called face-centred cubic (fcc) one.[22] This is actually nothing else but the usual packing used for example by fruit vendors when they pile fruits in a pyramidical shape. What is interesting to note is that the fcc packing can be built starting out in two different ways. Either one puts down straight columns next to each other and then fill the next layer by putting spheres in each hole created by four adjacent spheres, or one starts with a hexagonal pattern where the next layer puts spheres in the wholes created by three adjacent spheres.

The first layer of the face-centred cubic lattice as straight columns or as hexagons.

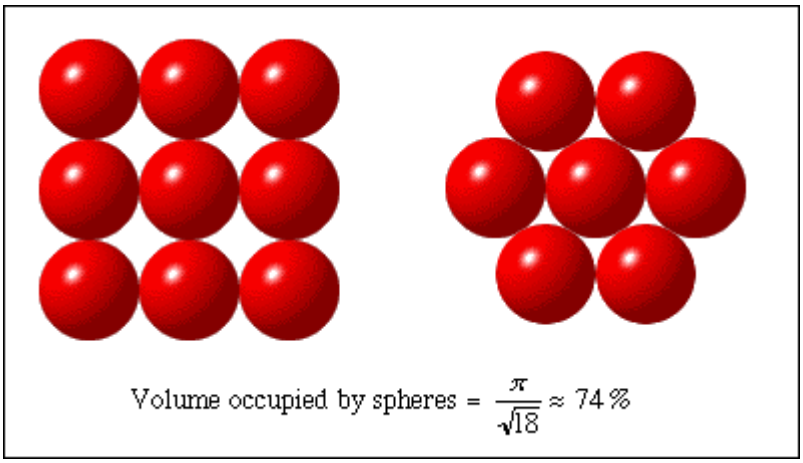


Figure 22

[22] F. Pfender, G. Ziegler, *Kissing numbers, sphere packings and some unexpected proofs*, http://www.ams.org/notices/200408/fea-pfender.pdf, 2007-01-17, p. 3.

Although these two packings were proven to have the same density, they have been regarded as different. However, if the fcc is looked at carefully, it is obvious that the two patterns just show different angles of the fcc packing. As a result, the 3-6-3 (the cuboctahedric structure in Appendix 1) and the 4-4-4 configurations, that are solutions to the Kissing problem in three dimensions, are in fact the same configuration looked at from different angles.

By rotating the 3-6-3 configuration found in fcc, one gets the 4-4-4 one:

The 3-6-3 configuration **The 4-4-4 configuration**

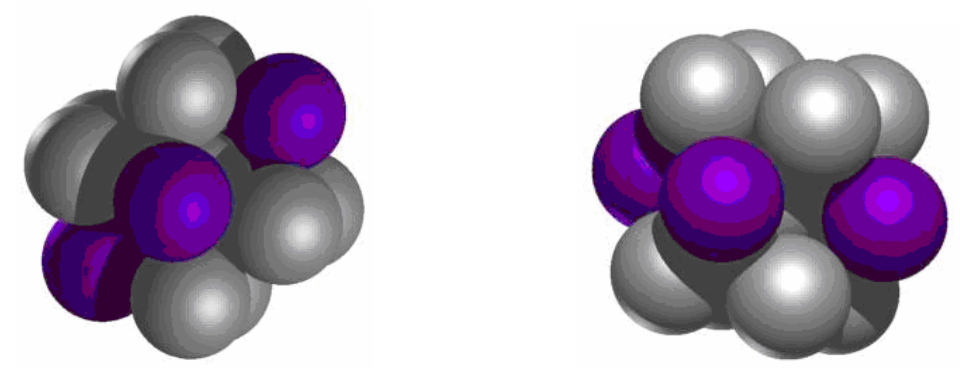


Figure 23

The icosahedric configuration (Appendix 1), however, could not be part of a lattice packing. It is obvious that using such a configuration as a part of a large packing would not create an identical configuration around every sphere. Whereas the central sphere of the icosahedric structure touches 12 spheres, none of these 12 spheres can touch as many as 12. The reason is that by definition all the neighbour spheres among the outer ones in the configuration do not touch each other.

Although Gauss proved that fcc was the most efficient lattice arrangement for spheres in three dimensions a long time ago, it would take mathematicians hard work and many years to prove that this was also the solution to the general packing problem, the so called Kepler conjecture.[23] The name comes from the fact that Johannes Kepler in 1611 proposed that the fcc lattice is the optimal sphere packing in three dimensions. When in 1998 Thomas Hales finally managed to prove that this was the case, it was a 250-page thick document which then took the mathematical community six years to digest. Only after this extensive scrutiny, his paper was considered to be correct and was published in a more digestible form in *Annals of*

[23] E.Klarreich, "Oddballs: It's easier to pack spheres in some dimensions than others", *Science News Online,* Week of Oct. 2, 2004; Vol. 166, No. 14, p. 219.

Mathematics.[24] However, there are still dissatisfied voices within the mathematical community claiming that Hales' proof is too much based on numerical data runs rather than on theoretical reasoning.[25]

Relationship between Kissing number and optimal packing

So what is the relationship between Kissing numbers, optimal lattice arrangements and optimal general packings? In two dimensions the most optimal packing is the hexagonal one which also entails the highest Kissing configuration with every circle being touched by six others. In three dimensions, as seen above, the optimal packing arrangement, the face-centred cubic one also gives the cuboctahedric solution to the Kissing problem. However, the packing approach does not say anything about the fact that an infinite amount of other solutions exist. In fact, as we shall see further on, the unique and "locked" configuration in dimension 2 is rather an exception. So far similar strong relationships have only been observed in dimensions 8 and 24 where the optimal packing structure also incorporates the highest Kissing configuration.

The way to estimate Kissing numbers has been to establish a lower and an upper bound. The lower bound is typically the implied Kissing configuration in a known lattice structure. The upper bound uses a technique developed by Phillipe Delsarte in 1973. Based on his work, the mathematicians Andrew Odlyzko and Neil Sloane in 1979 developed a method to establish upper bounds for Kissing numbers. As regards Sloane, it is also worth mentioning that he together with John Conway wrote the book *Sphere Packings, Lattices and Groups* which according to Szpiro is regarded as "the bible of sphere packings".[26]

Consequently, an exact Kissing number can only be proved if the lower bound is the same as the upper one. For many years exact Kissing numbers were only known for dimension 1, 2, 3, 8 and 24. In 2003, however, Oleg Musin managed to prove that the number in dimension 4 is 24.[27] According to Pfender/Ziegler, Musin's trick to modify the so called Delsarte function in order to arrive at an upper bound lower than 25 is a "beautiful idea".[28] His enhanced method also manages to show that the number in 3 dimensions is

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[24] T. Hales, "A proof of the Kepler conjecture ", *Annals of Mathematics* 162:3 (2005), pp. 1065–1185.
[25] K. Chang, "In math, computers don't lie. Or do they?", *The New York Times*, 2004-04-06.
[26] Szpiro, p. 92.
[27] O. Musin, "The problem of the twenty-five spheres", *Russian Mathematical Surveys*, 58 (2003), pp. 794-795.
[28] Pfender, Ziegler, p. 8.

below 13. The following table shows the most recent intervals for Kissing numbers up to dimension 24. In order to see how the interval has narrowed over time, Coxeter's upper bounds up to dimension 8 from 1963 are shown as well (within brackets):

Lower and upper bounds for Kissing numbers[29]

Dimension	Optimal lattice	Optimal non-lattice	Upper bound
1	2		2
2	6		6
3	12		12
4	24		24 (26)
5	40		46 (48)
6	72		82 (85)
7	126		140 (146)
8	240		240 (244)
9	272	≥306	380
10	≥336	≥500	595
11	≥438	≥582	915
12	≥756	≥840	1416
13	≥918	≥1130	2233
14	≥1422	≥1582	3492
15	≥2340	≥2564	5431
16	≥4320		8313
17	≥5346		12215
18	≥7398		17877
19	≥10668		25901
20	≥17400		37974
21	≥27720		56852
22	≥49896		86537
23	≥93150		128096
24	196560		196560

Figure 24

Some interesting observations for:

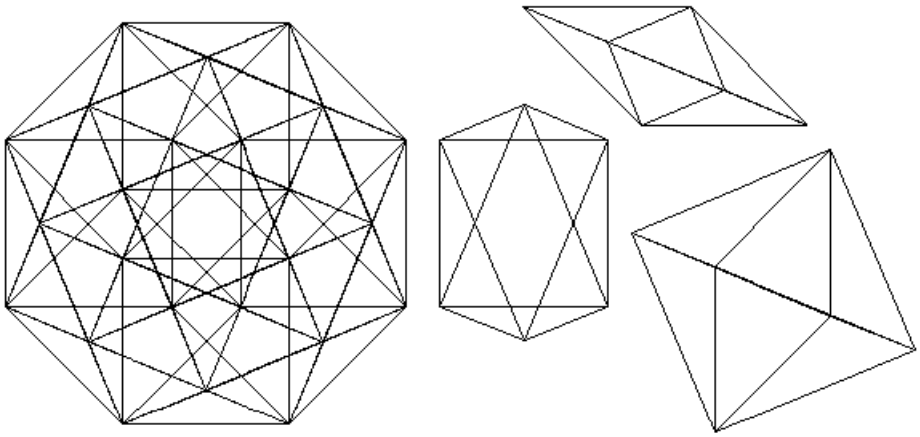
n = 4

As mentioned, it was not until 2003 that Musin managed to show that the Kissing number in 4 dimensions must be 24. In an interview he says it took him one and a half year of 16 hour days without weekends to arrive at the solution.[30] The configuration of the optimal Kissing structure is quite interesting. It is a 24-cell with a perfectly symmetric shape with 24 corners

[29] Based on information from *Mathword*, *Wolfram*, http://mathworld.wolfram.com/KissingNumber.html, 2007-01-17, and table maintained by Neil Sloane and Gabriele Nebe, http://www.research.att.com/~njas/lattices/kiss.html, 2007-01-17.

[30] Klarreich.

and 24 three-dimensional sides, each forming an octahedron. The interesting aspect of this polytope is that it can be regarded as a Platonic solid. In dimension three there are five Platonic solids but in dimension four there are the same five plus the 24-cell. One of the great experts in this field, H. Cohn, makes the following remark about this puzzling fact: "The 24-cell is this incredibly beautiful configuration that happens to fit perfectly in four dimensions, for reasons that are mysterious."[31]



24-Cell & its bounding octahedral cells

Figure 25

n = 8

This is the first dimension after dimension 2 where the Kissing configuration is symmetrical and locked tightly into place around the inner sphere. The lattice that is the densest packing and that gives this Kissing arrangement is called the E_8 lattice. The structure was found by Odlyzko/Sloane and, independently, by Vladimir Levenstein in the late 1970s.[32]

n = 9

This is the lowest dimension where a non-lattice packing is known to give a higher Kissing number than any lattice structure. The so called "P9a" packing contains spheres that are "kissed" by 306 other spheres whereas the densest lattice packing has a Kissing number of only 272.

[31] Ibid.
[32] Ibid.

n = 24

As in the case of dimensions 2 and 8, the Λ_{24}, called Leech lattice after its founder, gives a Kissing structure where the outer 196560 spheres lock tightly around the inner sphere. This structure that was found by Leech in 1964 has multiple symmetries. No matter how Λ_{24} is rotated around any axis and in any order, it still look the same.[33] Conway discovered that the symmetries of this lattice formed a so called "sporadic simple group" that had "eluded discovery" until then. The group, which is named after its founder, contains no less than 8,315,553,613,086,720,000 elements. The importance of this finding in the area of group theory enabled Conway to become a Fellow of the Royal Society in 1981.

The relevance of error-correcting code theory

The mathematics used in packing theory in n dimensions is closely connected to so called error-correcting codes.[34] The traditional use of such codes enables to correct a distortion of a message when being transmitted from one point to another. The following picture gives an example of a transmission of words where each word is encoded to a unique code word which is then transmitted.[35]

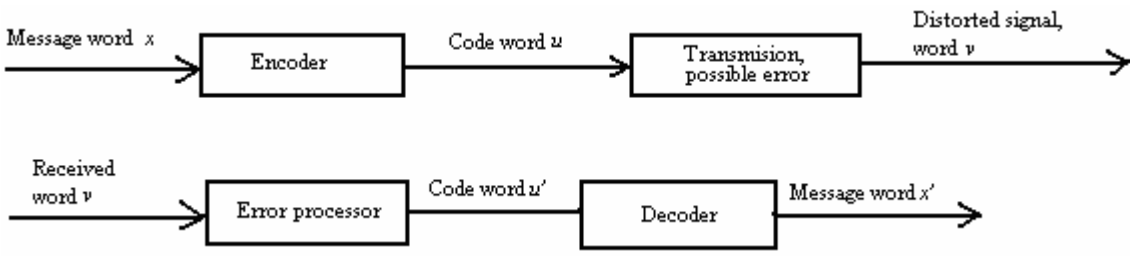


Figure 26

The code word may be distorted during transmission and, if so, the Error processor's role is to correct the distorted code word which is then sent to the decoder in order to be translated back to the initial word. When creating such a code, one makes sure that the allowed code words in the code are separated sufficiently from each other. This allows the decoder to

[33] Szpiro, p. 95.
[34] Klarreich.
[35] O. Pretzel, *Error-correcting codes and finite fields*, Clarendon press, Oxford (1992), p. 10.

correct any distortion up to a certain size.[36] Then the received message word *x'* would in fact be identical to the initially transmitted *x* in the above depicted transmission.

What would such a self-correcting code look like? Let us assume that the code words are defined in a two-dimensional space. First we have to introduce two definitions. The first one is weight which in short is the number of places the received code word differs from the sent one. In the specific case of a two-dimensional space, there are only two coordinates for each word why the weight maximally can be two. The second definition is "Hamming distance" which is the distance between the sent and received code words. In other words, the Hamming distance between *u* and *v* in our example is the length of the vector between the two in two dimensions. Let us now say that we would like our Error processor to always correct errors of maximum Hamming distance t, then we must make sure that our code is constructed in such a way that the distance between the code words is larger than 2t. This means that when an error of maximum t occurs, we would always know how to correct the error.

Two code words, *u* and *z*, in two-dimensional space

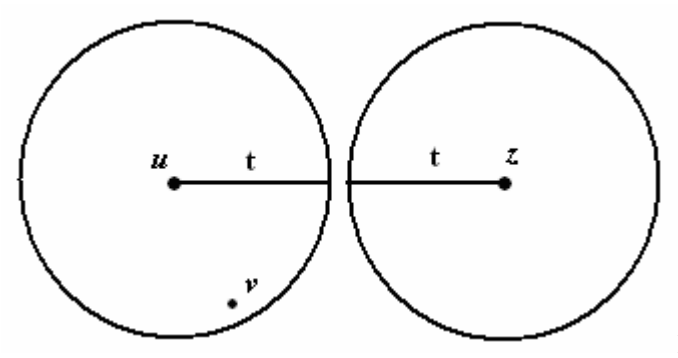


Figure 27

In the situation depicted in the figure above an erroneous code word v is received by the Error processor. Since the minimum Hamming distance between any two code words is >2t, the processor corrects the faulty code to *u* which is the only code word within a range of t.

The above mentioned example hopefully manages to show how error-correcting codes are connected to finding upper bands for sphere packings. The assumption that spheres in an n-dimensional space cannot overlap each other is equal to finding an efficient error-correcting

[36] Ibid., pp. 16-19.

code which can correct errors up to the length of the sphere's radius. What Delsarte found in 1973 was a method for relating size of the errors that a code can correct to the efficiency of the code, that is, the number of code words it can fit into a given space. Since the latter is directly linked to the density of the sphere packing, his method became very useful in creating efficient codes which would serve as upper bounds for sphere packing.

It is beyond the scope of this paper to go through the complex mathematical tools used in finding these efficient codes why the conclusion will be to suggest a few good papers that in detail go through the mathematics involved and all its crucial steps.

1. The award winning paper by Pfender and Ziegler goes through the Delsarte method in great detail. Furthermore, it explains how Musin managed to adapt the Delsarte method in order to find an even lower upper bound in four dimensions. The paper then continues with explaining the enhanced methods by Cohn, Elkies and Kumar to find overall lower upper bounds. Since the "most spectacular applications" of this method concerns the Leech lattice, the paper concludes by a detailed analysis of it.[37]
2. One of Musins last papers offers a solution of the Kissing problem in three dimensions using modern mathematical tools.[38] The author claims that: "This paper needs just basic calculus and simple spherical geometry."[39] However, he then uses mathematical tools like "Legendre polynomials", "Schoenenberg's theorm" and "Gegenbauer (ultraspherical) polynomials" which may seem difficult enough for the interested reader.
3. The last paper to be mentioned is written by Kurt Anstreicher. The aim of the paper is to find a solution to the Kissing problem in three dimensions using a method that is also applicable to higher dimension.[40] His approach is based on linear programming and "properties of the Delaunay decomposition associated with ***C***".[41] He explains terms like "spherical z-code", "Gegenbauer and Jacobi polynomials", "the spherical Voronoi cell" and "Delaunay triangulations".

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[37] Pfender, Ziegler, p. 10.
[38] Musin, 2006.
[39] Ibid., p. 2.
[40] K. Anstreicher, "The thirteen spheres: a new proof", *Discrete & computational geometry*, vol. 31 issue 4 (March 2004), Springer-Verlag NY, pp. 613-625.
[41] Ibid., p. 614.

Final remarks

Initially it does not look that difficult to solve the famous Kissing problem in three dimensions. However, as the actual time that went by before it was solved and the relative complexity of the solutions show, the problem cannot be called trivial. The aim of this paper has been to give the reader a good understanding of all the different aspects of the problem. In particular, my goal has been to explain the elegant proof proposed by Leech. By doing the latter, I hope to have managed to do what Casselman called for in his paper.

One may of course wonder if there are any practical applications for Kissing theory. The answer to this is undoubtedly, yes! To some extent, one may consider the Kissing problem as a local one within general packing theory which has many applications also outside the mathematical world. According to John Conway this theory, for example, is crucial in the fields of "optimal resource allocation and efficient phone switching"[42] Other applications that could be mentioned are, cryptosystems, crystal structures in chemistry and even planning of radio surgery. In mathematics, as we have seen with the finding of a "sporadic simple group", packing theory has lead to findings of importance to other areas of mathematics.

As could be seen in the table of upper and lower bands, the complexity of optimal packing grows exponentially with higher dimensions. Although a lot of achievements have been made up to this date, we still have reached exact Kissing numbers for only six dimensions. Therefore, we have interesting times ahead with hopefully new findings that will be able to narrow the bands of Kissing numbers in higher dimensions.

[42] J. Conway, http://www.princeton.edu/~umpc/conway.html, 2007-02-13.

APPENDIX 1

Possible structures of 12 unit spheres kissing a central one

The easiest way to show that there is an infinite amount of different structures that solve the Kissing problem with 12 outer spheres is to show the following two configurations.

The icosahedron configuration:

12 outer spheres are placed around a central one on the vertices of an icosahedron

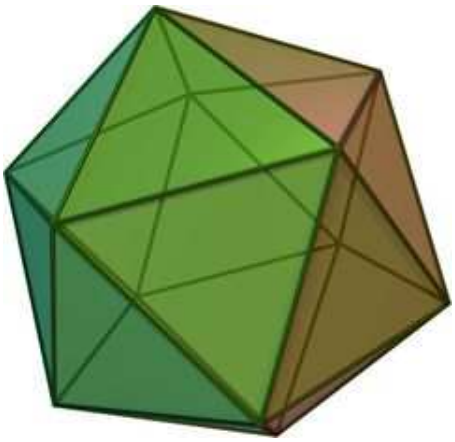

Figure 28

One possible arrangement is to place the outer spheres loosely at the 12 vertices of an icosahedron. The icosahedron is one of the five Platonic solids in three dimensions which means that it is perfectly symmetrical, consisting of 20 equal equilateral triangles and every of the 12 vertices being connected to 5 edges. It can be circumscribed by a sphere such that each vertex of the solid will be a point on the surface of that sphere. In this structure one sphere, the central one, will touch 12 others, whereas none of the surrounding ones will touch each other. The length, l, of every edge between two neighbour vertices is directly proportional to the radius, r, of the circumscribed sphere by the formula:[43]

$$l \equiv \frac{4r}{\sqrt{10+2\sqrt{5}}} \approx \frac{r}{0.95106}$$

For the unit sphere, this means an edge length of approximately 1.051 which is longer than the minimum distance π/3 needed between two kissing spheres that are neighbours on the

[43] Wikipedia, http://en.wikipedia.org/wiki/Icosahedron, 2007-02-02.

surface.(One might add that the great circle distance between two edges is obviously even longer than 1.051).

The cuboctahedron configuration:

12 outer spheres are placed around a central one on the vertices of a cuboctahedron

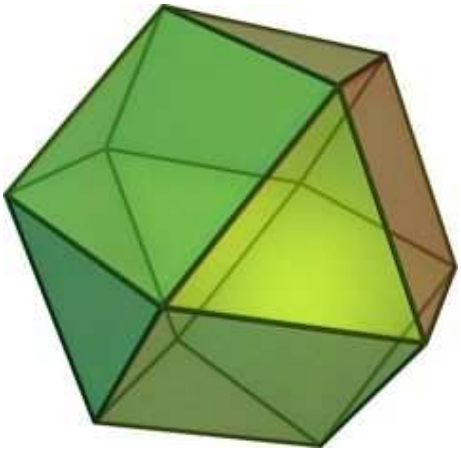

Figure 29

This configuration is achieved if the outer spheres are placed on the vertices of a cuboctahedron which is circumscribed by the unit sphere. The cuboctahedron is a polyhedron with 8 equilateral triangles and 6 squares with all the 24 edges of same length.[44] The 12 vertices are identical with 2 triangles and 2 squares meeting at each one. A cuboctahedron can be circumscribed by a sphere, so that all the 12 vertices lie on its surface. For a cuboctahedron circumscribed by the unit sphere, the great circle distance between two neighbour vertices is exactly $\pi/3$. This means that the kissing structure is rigid in so far that every outer sphere touches its closest neighbours.

As both Coxeter and Conway/Sloane have shown, it is possible to roll, without lifting, the 12 spheres on the surface of the inner sphere and thereby go from one to the other one of the two described configurations.[45] This means that there is an infinite amount of possible structures that are created during this process of transformation.

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[44] Wikipedia, http://en.wikipedia.org/wiki/Cuboctahedron, 2007-02-02.
[45] H.S.M.Coxeter, ”An upper bound for the number of equal nonoverlapping spheres that can touch another of the same size”, *Proceedings of Symposia in pure mathematics* 7, Providence: American Mathematical Society (1963), p. 54., Conway, Sloane, p. 29.

Conway and Sloane's suggestion on how to continuously move the contact points representing the cuboctahedral arrangement in order to arrive at the icosahedral one

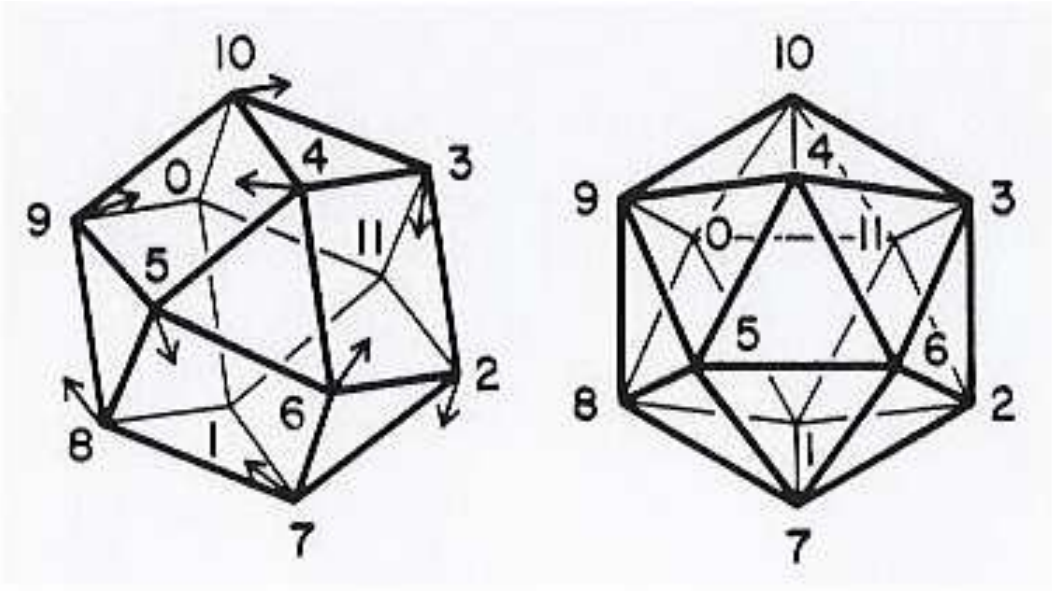

Figure 30

APPENDIX 2

Calculation of areas of spherical triangles

Calculating the area of a spherical triangle is quite simple as it merely requires knowledge of the angles of the triangle and the radius of the sphere on which the triangle lies.[46] The formula looks as follows:

Area of spherical triangle = (sum of angles - π)R^2

An easy example would be the triangle that has its vertices at the "north pole", the equator and the equator a quarter of a turn away from the second point. Obviously, every angle of this triangle is $\pi/2$ why the sum of the angles is $3\pi/2$. Consequently, its area is $(3\pi/2 - \pi)R^2 = (\pi/2)R^2$. If we compare this to the total area of a surface of a sphere, $4\pi R^2$, we see that the triangle's area according to the formula is 1/8 of the total surface. This is right as the triangle we defined takes up exactly 1/4 of the upper half of the sphere.

As we can see, the problem for calculating spherical areas does not lie in the basic formula but rather in the need to calculate the angles of every triangle. The way to do this is to understand the relationship between the triangle's arc angles, being directly proportional to the length of its sides, and its angles. Every polygon can be broken down into triangles whose angles can be calculated by the below specified method.

The dihedral angle

A *dihedral angle* is the angle of intersection of two planes. It is the measure of an angle having its vertex on the intersecting line and one side in each of the planes. The sides of the angle are perpendicular to the intersecting line.[47]

[46] http://www.applet-magic.com/sphere.htm, 2007-01-17.
[47] http://whistleralley.com/polyhedra/derivations.htm, 2007-01-17.

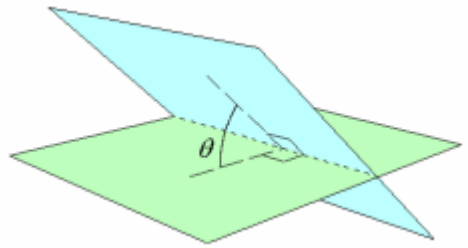

Figure 31

In the context of polyhedra, a dihedral angle is the angle of intersection of two adjacent faces. For each of the Platonic solids, there is only one dihedral angle, because all pairs of adjacent faces intersect at the same angle.

A vertex of a polyhedron is at the intersection of three or more faces. In the case of three, there is a convenient formula for the dihedral angles. If the three faces fit together at all (they might not), there is only one way. Take this example. Three polygons share the common vertex *A*. The polygon angles at *A* are α, β, and γ, which are not necessarily equal. The polygons shown here are triangles, but they could have any number of sides.

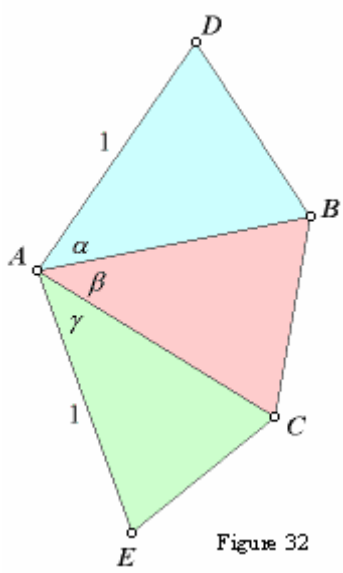

Figure 32

Start with the faces laid out in a net in a horizontal plane. The net is hinged on sides *AB* and *AC*. Points *D* and *E* swing up and meet at point *P*. Point *P* is projected downward to point *Q*, on the plane containing triangle ABC. In order for the points to meet, it is necessary for *AD* and *AE* to be equal, so they are arbitrarily given unit length here. It is clear that this forms a rigid structure, yielding only one solution for the three dihedral angles at the three edges. Actually, it is also possible to fold the faces downward and have them meet below, but that would simply be a reflection of what is seen here, so the solutions would be the same.

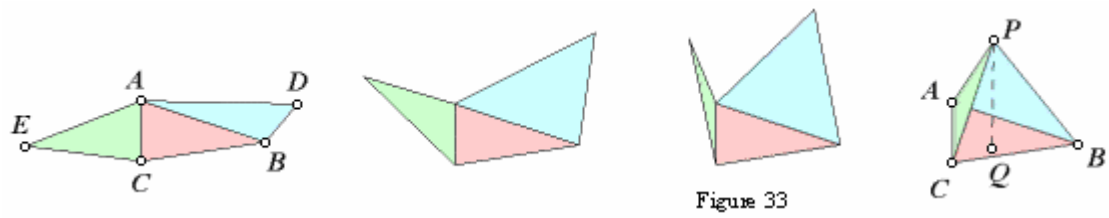

Figure 33

For this example, the dihedral angle at edge AB will be derived. Lay it back out into a flat net again. As the faces are folded, the vertex at point D revolves on axis AB and meets point P. This means that points D, P, and Q lie in a plane perpendicular to line AB. It follows that point Q lies on a line through D, perpendicular to AB. By the same reasoning, Q lies on a line through E, perpendicular to AC.

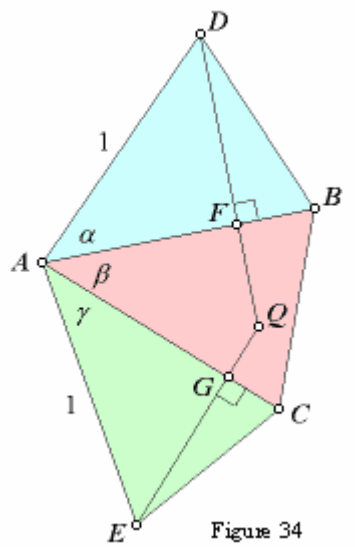

Figure 34

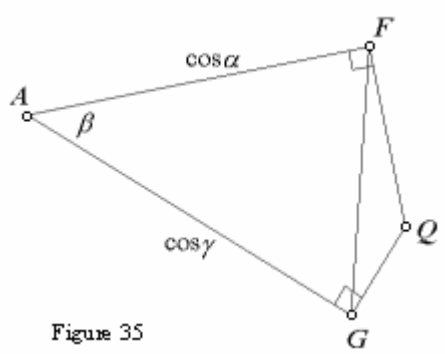

Figure 35

From right triangles ADF and AEG, it can be seen that $AF = \cos\alpha$, $AG = \cos\gamma$, and $FD = \sin\alpha$. The immediate objective it to derive the measure FQ. For now, concentrate on quadrilateral $AFQG$. It has opposite right angles, making it a cyclic quadrilateral, so angle $Q = \pi - \beta$.

In triangle AFG, use the law of cosines to solve for FG.

$$(FG)^2 = (AF)^2 + (AG)^2 - 2(AF)(AG)\cos\angle FAG$$
$$FG = \sqrt{(\cos\alpha)^2 + (\cos\gamma)^2 - 2\cos\alpha\cos\gamma\cos\beta}$$

Now, find $\cos\angle AGF$.

$$\cos\angle AGF = \frac{(AG)^2 + (FG)^2 - (AF)^2}{2(AG)(FG)} = \frac{(\cos\gamma)^2 + (\cos\alpha)^2 + (\cos\gamma)^2 - 2\cos\alpha\cos\gamma\cos\beta - (\cos\alpha)^2}{2\cos\gamma\sqrt{(\cos\alpha)^2 + (\cos\gamma)^2 - 2\cos\alpha\cos\gamma\cos\beta}}$$
$$= \frac{\cos\gamma - \cos\alpha\cos\beta}{\sqrt{(\cos\alpha)^2 + (\cos\gamma)^2 - 2\cos\alpha\cos\gamma\cos\beta}}$$

Use the law of sines to find FQ.

$$FQ = \frac{(FG)\sin\angle FGQ}{\sin\angle Q} = \frac{(FG)\cos\angle AGF}{\sin(\pi-\beta)} =$$

$$\frac{\sqrt{(\cos\alpha)^2+(\cos\gamma)^2-2\cos\alpha\cos\gamma\cos\beta}}{\sin\beta}\times\frac{\cos\gamma-\cos\alpha\cos\beta}{\sqrt{(\cos\alpha)^2+(\cos\gamma)^2-2\cos\alpha\cos\gamma\cos\beta}}$$

$$=\frac{\cos\gamma-\cos\alpha\cos\beta}{\sin\beta}$$

Going back to the folded faces, we see that FP and FQ are both perpendicular to AB, so the angle labeled θ is the dihedral angle. From the right triangle PQF, we get this:

$$\cos\theta = \frac{FQ}{FP} = \frac{FQ}{FD} = \frac{\cos\gamma-\cos\alpha\cos\beta}{\sin\alpha\sin\beta} \Leftrightarrow \theta = \arccos\left(\frac{\cos\gamma-\cos\alpha\cos\beta}{\sin\alpha\sin\beta}\right)$$

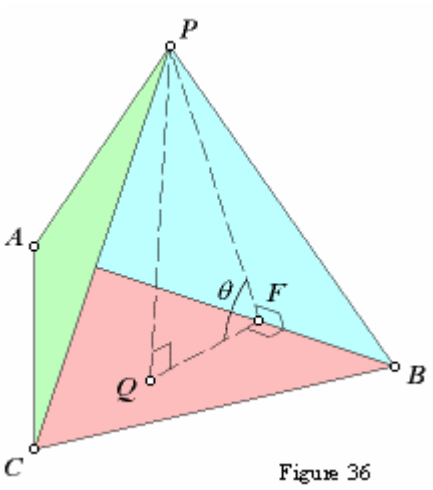

Figure 36

Application on a spherical triangle

The below depicted spherical triangle is drawn with the same notations as the above picture. A is the centre of a sphere. The planes ABP and ABC are perpendicular to the intersecting line AB on the surface of the sphere at point B. Therefore calculating the dihedral angle θ according to the formula above would give the angle of vertex B in the spherical triangle below.

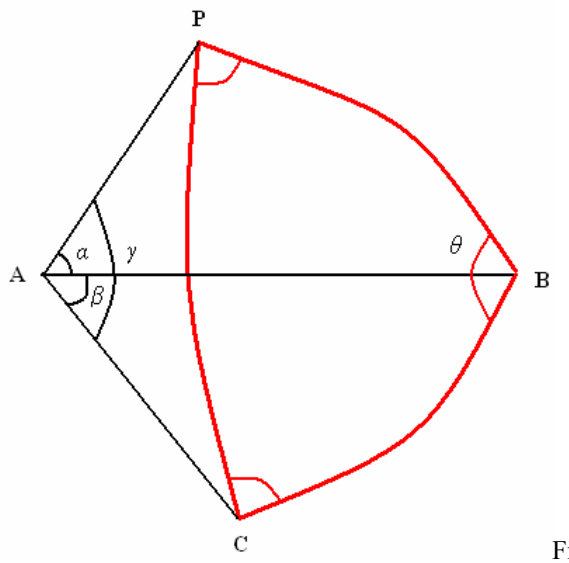

Figure 37

Calculation of diagonal of the symmetrical spherical quadrilateral on page 8

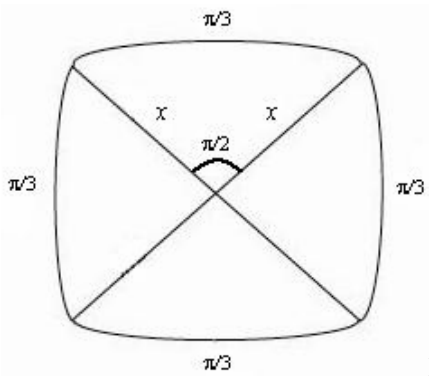

Figure 38

Let us use our dihedral formula on one of the four equal triangles with one side being π/3 and the other two x. If we can calculate x, we know the diagonal of the quadrilateral which is $2x$. We apply the above derived formula:

$$\pi/2 = \arccos\left(\frac{\cos \pi/3 - (\cos x)^2}{(\sin x)^2}\right) \Leftrightarrow \cos \pi/3 = (\cos x)^2 \;\Rightarrow\; x = \arccos\left(1/\sqrt{2}\right) \Leftrightarrow x = \pi/4$$

This proves that the diagonal of the symmetrical spherical quadrilateral is $\pi/2$.

APPENDIX 3, Leech's proof

THE PROBLEM OF THE THIRTEEN SPHERES

By John Leech

It was conjectured by David Gregory* that a sphere can touch thirteen non-overlapping spheres equal to it. In a recent paper† proofs have been given that this in fact is impossible. In the present paper I outline an independent proof of this impossibility, certain details which are tedious rather than difficult being omitted.

We consider the configuration of the points of contact of the sphere with those around it, or, equivalently, the configuration (on a sphere) of the centres of these spheres. In terms of this configuration, we prove that there cannot be thirteen points on a sphere such that the (great circle) distance between any two of them is at least $\frac{1}{3}\pi$. (We assume throughout that the sphere is of unit radius and that distances are measured on the surface of the sphere.)

Let us consider the network on the sphere formed by joining every pair of the points of the configuration whose distance apart is less than $\cos^{-1}\frac{1}{7}$. No two joins of this network cross, since any four points of the network form a quadrilateral of sides at least $\frac{1}{3}\pi$ whose diagonals cannot both be less than $\frac{1}{2}\pi$, the extreme case being that of the regular quadrilateral of side $\frac{1}{3}\pi$ whose diagonals are both exactly $\frac{1}{2}\pi$. The network therefore divides the sphere into polygons whose vertices are the points of the configuration. We may assume that the network is connected, since if it is in detached parts, these may be moved so as to bring vertices of each part within $\cos^{-1}\frac{1}{7}$ of another, and that no point of the configuration is within $\cos^{-1}\frac{1}{7}$ of only one other point, since any such point can be moved until it is within $\cos^{-1}\frac{1}{7}$ of another. Every angle of these polygons exceeds $\frac{1}{3}\pi$, the lower bound, unattained in the network, being for the spherical triangle of sides $\frac{1}{3}\pi$, $\frac{1}{3}\pi$, $\cos^{-1}\frac{1}{7}$, whose angles are $\frac{1}{3}\pi$, $\frac{1}{3}\pi$, $\cos^{-1}(-\frac{1}{7})$. Thus at most five joins of the network meet in any one vertex. The proof consists of showing by consideration of areas that for such a network to exist having thirteen vertices, it must divide the sphere into triangles except possibly for one quadrilateral, and that for topological reasons such networks do not exist.

The least area of a triangle of the network is that of an equilateral triangle of side $\frac{1}{3}\pi$. This has angles of $\cos^{-1}\frac{1}{3}$ and its area is $3\cos^{-1}\frac{1}{3}-\pi=0{\cdot}5513$.

The least area of a quadrilateral of the network is that of an equilateral quadrilateral of side $\frac{1}{3}\pi$, one of whose diagonals is of length $\cos^{-1}\frac{1}{7}$, this is of area $2(\cos^{-1}(-\frac{1}{7})+\frac{2}{3}\pi-\pi)=1{\cdot}334$.

The least area of a pentagon of the network is that of an equilateral pentagon of side $\frac{1}{3}\pi$ having two coterminous diagonals of length $\cos^{-1}\frac{1}{7}$, and area $2{\cdot}226$.

Thus the excess of the area of a quadrilateral over the minimum for two triangles is at least $0{\cdot}231$, and of a pentagon over three triangles at least $0{\cdot}572$. It is clear further that for $n\geqslant 5$ the excess of the area of an n-gon over the minimum area of $(n-2)$ triangles increases with n, since if on a side of any n-gon is abutted a triangle of minimum area, the resulting $(n+1)$-gon has to be deformed such that the original common side becomes a diagonal of length not less than $\cos^{-1}\frac{1}{7}$ with consequent increase of area.

By Euler's theorem on polyhedra,

$$V+F=E+2,$$

i.e.

$$\begin{aligned}2V-4&=2E-2F\\&=3F_3+4F_4+5F_5+\ldots-2(F_3+F_4+F_5+\ldots)\\&=F_3+2F_4+3F_5+\ldots,\end{aligned}$$

* In an unpublished notebook at Christ Church, Oxford.

† K. Schütte and B. L. van der Waerden, *Math. Annalen* 125 (1953) 325–334.

where F_n is the number of n-gons of the network. The area of the network exceeds

$$\begin{aligned}&0{\cdot}5513F_3+1{\cdot}334F_4+2{\cdot}226F_5+\ldots\\&=0{\cdot}5513(F_3+2F_4+3F_5+\ldots)+0{\cdot}231F_4+0{\cdot}572(F_5+\ldots)\\&=0{\cdot}5513(2V-4)+0{\cdot}231F_4+0{\cdot}572(F_5+\ldots).\end{aligned}$$

Now the area of the sphere is 4π, whence

$$\begin{aligned}2V-4&\leqslant 4\pi/0{\cdot}5513\\&=22{\cdot}8,\end{aligned}$$

and

$$V\leqslant 13.$$

Suppose now, if possible, that $V=13$. Then $2V-4=22$, and we have

$$4\pi\geqslant 0{\cdot}5513\times 22+0{\cdot}231F_4+0{\cdot}572(F_5+\ldots),$$

i.e.

$$0{\cdot}438\geqslant 0{\cdot}231F_4+0{\cdot}572(F_5+\ldots).$$

Thus $F_4=0$ or 1 and $F_5=\ldots=0$, and the network has to divide the sphere into triangles except possibly for the one quadrilateral.

In the case $F_4=0$, we have $F_3=\frac{2}{3}E$, and $13+\frac{2}{3}E=E+2$, giving $E=33$ and an average number of $66/13>5$ joins meeting in each vertex, which is contrary to requirements.

In the case $F_4=1$, Euler's theorem gives $F_3=20$, $E=32$, and 4 joins meet at some one vertex and 5 at each other. Despite the consistency of these numbers, there is in fact no polyhedron which has them. I know of no better proof of this than sheer trial. One attempts to construct the plane net of such a polyhedron beginning with the quadrilateral and assuming in turn that the vertex at which only four joins meet is or is not a vertex of the quadrilateral. In either case a triangle is abutted on to each side of the quadrilateral, and a fifth join drawn to 3 or 4 of the vertices of the quadrilateral in the respective cases. It is immediately found to be impossible to complete the network as required if the further ends of these joins and the further vertices of the triangles are not all distinct points, and very shortly afterwards to be impossible even if they are. This completes the proof.

The question of when one can construct nets having prescribed numbers of vertices and numbers of joins meeting in them is of some independent interest. Among other cases of impossibility are: 7 vertices, at most 4 joins meeting at each vertex, all triangles except for one quadrilateral or pentagon; 13 vertices, at most 5 joins meeting at each vertex, all triangles except for one pentagon. Among cases of possibility are: 13 vertices, at most five joins meeting at any one vertex, all triangles except for one hexagon or two quadrilaterals (the latter in several ways, in one of them the quadrilaterals abut and removal of their common side leaves the former), but the former of these is not the net of a true polyhedron as the hexagon has two sides in common with one of the triangles. There is no obvious criterion for whether it is possible to construct a net having prescribed regions and prescribed numbers of joins meeting in each vertex when these prescriptions are consistent with Euler's theorem. (In all these nets the region outside the net is counted as a region of the network; for practical reasons this region is usually chosen not to be one of the triangles.)

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