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Finding numerical solutions to the Schwarz-Christoffel Equation

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Introduction

A Schwarz-Christoffel mapping is a conformal mapping from the upper half plane to a polygon. According to wikipedia; “Schwarz-Christoffel mappings are used in potential theory and some of its applications, including minimal surfaces and fluid dynamics.”

Searching the internet will produce results that would take a lifetime to exhaust. So let us only mention a few. There are many texts written about Schwarz-Christoffel transformation searching Amazon online bookshop will at the writing time produce 48 hits that contains “Schwarz-Christoffel” in its search word. Further there are computer programs that find these equations for different polygons. Let us only here mention the free Matlab Schwarz-Christoffel toolbox [5] written by Toby Driscoll. This is a big library that will find the equations for numerous polygons. We shall use this toolbox at the end to see an example in which we compare our results with the ones the toolbox produces.

In this text we will start by taking the general form of the equation and analytically develop this by numerous steps to something that can be implemented into a computer program. The second section will cover the idea of the implementation without going into specifics in the programming. This will be followed by a few examples of the use of this program, some tests and a little further work to solve related problems.

Further according to wikipedia; Karl Hermann Amandus Schwarz was born 1843 in Hermsdorf, Silesia then a part of Germany. (Nowadays Jerzmanowa, Poland.) He worked in Halle, Göttingen and Berlin in subjects of functional theory, differential geometry and calculus of variations. He was a student of Karl Weierstrass and became a professor at the University of Berlin in 1892. He died in Berlin in 1921.

Elwin Bruno Christoffel was born in 1829 in Montjoie, now called Monschau. He lived in Cologne at a younger age and attended at the University of Berlin with Dirichlet. He received a doctorate at this University in 1856 for a thesis on the motion of electricity in homogenous bodies. A few years later he became “Privatdozent” before moving on to Gewerbeakademie in Berlin (now part of Technical University of Berlin) in 1869. In 1872 he became a professor at the University of Strasbourg. A position he retired a few years before he’s demise in Strasbourg in 1900.

Analytic preparations

The general Schwarz-Christoffel equation has the form:

$$f(z) = A\int(\xi - x_0)^{\theta 1}(\xi - x_1)^{\theta 2}(\xi - x_2)^{\theta 3}\ldots(\xi - x_{N-2})^{\theta N-1}d\xi + B \qquad \xi = [0\ z]$$

for all z members of the upper half plane **u**. The properties of this equation can be read from several introductory texts on complex analysis [2].. But for our purpose it's enough to realize that f(z) and the "constants" that appear in it has by definition the following properties:

- f(z) maps **u** conformally onto a closed polygon **P** with N corners.
- The θ_k's (k = 1, 2, 3, … N-1) are the negations of the outer angles, of **P**, divided by π. Consequently $(\sum\theta_k)+ \theta_N = -2$. (Where θ_N is set as the negation of the outer last angle divided by π.)
- **P** convex corresponds to all θ_k's < 0.
- θ_k's are always larger then -1 since otherwise f(z) would not be conformal at all.
- The complex constants A and B merely rotate resize and move the polygon **P** in the complex plane.
- $x_0<x_1<x_2<\ldots x_{N-2}$
- The x_k's (k = 0, 1, 2, 3, … N-2) are elements on the real line that are successively mapped to each of the second to last corners of **P**. Furthermore the whole real line is mapped to the contour of **P** and consequently both real infinities are mapped to the first corner.

Actually the last property could be stated that the line from $-\infty$ to x_0 is mapped to a continuous line from the first corner to the second. The line from x_0 to x_1 is mapped to a continuous line from the second to the third corner and so on until ∞ is yet again mapped to the first corner. So we only need to have N-1 arguments of **P** since the last will be decided as 2π subtracted by the sum of the others. Furthermore the mapped lengths of the lines from both infinities to the closest x_k will be automatically adjusted so as to close **P** with all the correct arguments at all corners.

Now if the x_k's where known for some **P** then we could implement this pretty straightforward into a computer program and find out the mapping of each individual z in **u**. Also if we arbitrary chose the x_k's so that each is greater then it's predecessors, then we would get a closed polygon **P**. However we want to map **u** onto a specific **P**. Now since the arguments are known we will always have a **P** with correct angles at the corners. But to obtain the correct side length's we have to do some numerical manipulation to find the correct x_k's.

For a problem like N = 4 one could solve this by inserting values until one gets the correct **P**. But for N larger then five we have to use numerical methods to solve the inverse problem to find the x_k's or "breakpoints" that will be mapped with the correct distance in between.

In order to solve all the x_k's simultaneously we want to utilise Newton's method for nonlinear multivariable systems [1]. This method utilises a start guess vector $\mathbf{v}_0$ that in each successive step is improved by subtracting $J_{\mathbf{f}}(\mathbf{v}_k)^{-1}\mathbf{f}(\mathbf{v}_k)$ from $\mathbf{v}_k$. Until **f** converges to zero, and hence gives us the final vector that satisfies the system. Here **f** is a system of n-equations and n-variables, and J is the jacobian of this system.

To accomplish this we notice that we first need to create a system of equations with as many equations as there are unknowns. Secondly we need to take the derivative of each of these equations with respect to each variable. And thirdly we need to remove eventual singularities since computers don't deal well with these.

The six corner case;

To illustrate our method in the general case we shall give here a fairly detailed explanation in the six corner case that will prove instructive. We start by noticing that we can simplify the equation somewhat. As we noticed before the constant B merely moves **P** around in the complex plane. Furthermore B can easily be adjusted after the mapping is done, so we might just as well set B = 0 from now on. One might be tempted to say the same about A = 1, since this only guards the scaling and orientation of **P**. However for convenience we want a standard base, so let's set $A = A_0$. Also the orientation and scaling of **P** can be adjusted later by multiplying a complex constant to the integral part of f(z).

Now we see that we actually have two degrees of freedom when it comes to the x_k's. Even if we had all the breakpoints at this moment the second corner could be mapped from any point on the real line (except the infinities) since we can move this sequence on the line. Remember, the first corner was mapped from both infinities. Furthermore if we had this one point fixed then we could still map the remaining corners of **P** from a number of different points on the real line. We can rescale the differences between any two points as long as all relations between all successive points remain the same. So for convenience we take $x_0 = 0$, this implies that this point will be mapped to the origin. Further we set $x_1 = b$. For the analytic part we could have chosen b = 1 but for numerical reasons we want to be able to rescale as discussed above. Most often we will set b = 100. The remaining x_k's are relabelled a_1, a_2 and a_3.

So now the problem is finding A_0 and the a_k's so that they map **u** conformally onto **P**. The first step is to find the corners of **P**. For convenience we chose the first two corners to be $c_1 = -1$ and $c_2 = 0$. Since all c_k's must be mapped from certain points of the real axis, we have a system of equations for the six cornered case;

$$A_0\int\xi^{\theta1}(\xi -b)^{\theta2}(\xi -a_1)^{\theta3}(\xi -a_2)^{\theta4}(\xi -a_3)^{\theta5}d\xi = c_3 \qquad \xi = [0\ b]$$

$$A_0\int\xi^{\theta1}(\xi -b)^{\theta2}(\xi -a_1)^{\theta3}(\xi -a_2)^{\theta4}(\xi -a_3)^{\theta5}d\xi = c_4 \qquad \xi = [0\ a_1]$$

$$A_0\int\xi^{\theta1}(\xi -b)^{\theta2}(\xi -a_1)^{\theta3}(\xi -a_2)^{\theta4}(\xi -a_3)^{\theta5}d\xi = c_5 \qquad \xi = [0\ a_2]$$

$$A_0\int\xi^{\theta1}(\xi -b)^{\theta2}(\xi -a_1)^{\theta3}(\xi -a_2)^{\theta4}(\xi -a_3)^{\theta5}d\xi = c_6 \qquad \xi = [0\ a_3]$$

that must be satisfied. We might notice now that if we subtract the right side from each equation we actually have a system of 4-variables and 4-equations just as we wanted for Newton's method. However we want to work on this a little bit more first. Now every c_k is dependent of it's predecessors since a corner in the figure has equations;

$$c_3 = l_1e^{-\pi i\theta1} \qquad c_4 = c_3 + l_2e^{-\pi i(\theta1 + \theta2)} \qquad c_5 = c_4 + l_3e^{-\pi i(\theta1 + \theta2 + \theta3)} \qquad c_6 = c_5 + l_4e^{-\pi i(\theta1 + \theta2 + \theta3 + \theta4)}$$

where the l_k's are the lengths of the sides of **P**. Now we see that in each equation the previous integral is fully contained in the subsequent equation. Furthermore the right hand side falls out as an expression where we have the previous right hand side plus a new part. If we now realize that $e^{\pi i} = -1$ and for each but the first of these "mapping equations" subtract the previous from the next one, the system becomes;

$$A_0\int\xi^{\theta 1}(\xi - b)^{\theta 2}(\xi - a_1)^{\theta 3}(\xi - a_2)^{\theta 4}(\xi - a_3)^{\theta 5}d\xi = l_1(-1)^{-\theta 1} \qquad \xi = [0\ b]$$

$$A_0\int\xi^{\theta 1}(\xi - b)^{\theta 2}(\xi - a_1)^{\theta 3}(\xi - a_2)^{\theta 4}(\xi - a_3)^{\theta 5}d\xi = l_2(-1)^{-(\theta 1+\theta 2)} \qquad \xi = [b\ a_1]$$

$$A_0\int\xi^{\theta 1}(\xi - b)^{\theta 2}(\xi - a_1)^{\theta 3}(\xi - a_2)^{\theta 4}(\xi - a_3)^{\theta 5}d\xi = l_3(-1)^{-(\theta 1+\theta 2+\theta 3)} \qquad \xi = [a_1\ a_2]$$

$$A_0\int\xi^{\theta 1}(\xi - b)^{\theta 2}(\xi - a_1)^{\theta 3}(\xi - a_2)^{\theta 4}(\xi - a_3)^{\theta 5}d\xi = l_4(-1)^{-(\theta 1+\theta 2+\theta 3+\theta 4)} \qquad \xi = [a_2\ a_3]$$

Now we want to get rid of the imaginary parts of the integrals, since we see that some of the factors inside parenthesis in the integral will be negative and we know that the sums of the θ will be less then -2. We can accomplish this by simply taking the negative of the factors that have potentiality to become complex, and we can single these out in each of the equations. We get;

$$(-1)^{(\theta 2+\theta 3+\theta 4+\theta 5)}A_0\int\xi^{\theta 1}(b-\xi)^{\theta 2}(a_1-\xi)^{\theta 3}(a_2-\xi)^{\theta 4}(a_3-\xi)^{\theta 5}d\xi = l_1(-1)^{-\theta 1} \qquad \xi = [0\ b]$$

$$(-1)^{(\theta 3+\theta 4+\theta 5)}A_0\int\xi^{\theta 1}(\xi-b)^{\theta 2}(a_1-\xi)^{\theta 3}(a_2-\xi)^{\theta 4}(a_3-\xi)^{\theta 5}d\xi = l_2(-1)^{-(\theta 1+\theta 2)} \qquad \xi = [b\ a_1]$$

$$(-1)^{(\theta 4+\theta 5)}A_0\int\xi^{\theta 1}(\xi-b)^{\theta 2}(\xi-a_1)^{\theta 3}(a_2-\xi)^{\theta 4}(a_3-\xi)^{\theta 5}d\xi = l_3(-1)^{-(\theta 1+\theta 2+\theta 3)} \qquad \xi = [a_1\ a_2]$$

$$(-1)^{\theta 5}A_0\int\xi^{\theta 1}(\xi-b)^{\theta 2}(\xi-a_1)^{\theta 3}(\xi-a_2)^{\theta 4}(a_3-\xi)^{\theta 5}d\xi = l_4(-1)^{-(\theta 1+\theta 2+\theta 3+\theta 4)} \qquad \xi = [a_2\ a_3]$$

Notice that we get the factor $-1^{(\theta 1 + \theta 2 + \theta 3 + \theta 4 + \theta 5)}$ in each of the equations, if we divide both sides of each equation with all but the length of the right side of the equation. We now put $C = (-1)^{(\theta 1 + \theta 2 + \theta 3 + \theta 4 + \theta 5)}A_0$. The system becomes;

$$C\int\xi^{\theta 1}(b-\xi)^{\theta 2}(a_1-\xi)^{\theta 3}(a_2-\xi)^{\theta 4}(a_3-\xi)^{\theta 5}d\xi = l_1 \qquad \xi = [0\ b]$$

$$C\int\xi^{\theta 1}(\xi-b)^{\theta 2}(a_1-\xi)^{\theta 3}(a_2-\xi)^{\theta 4}(a_3-\xi)^{\theta 5}d\xi = l_2 \qquad \xi = [b\ a_1]$$

$$C\int\xi^{\theta 1}(\xi-b)^{\theta 2}(\xi-a_1)^{\theta 3}(a_2-\xi)^{\theta 4}(a_3-\xi)^{\theta 5}d\xi = l_3 \qquad \xi = [a_1\ a_2]$$

$$C\int\xi^{\theta 1}(\xi-b)^{\theta 2}(\xi-a_1)^{\theta 3}(\xi-a_2)^{\theta 4}(a_3-\xi)^{\theta 5}d\xi = l_4 \qquad \xi = [a_2\ a_3]$$

Here the lengths are real positive and the integrals are always real, so C must now be a real number. Next we want to get rid of the restrictions that $b < a_1 < a_2 < a_3$. This is accomplished by replacing these by; $a_1 = b + e^{d1}$, $a_2 = b + e^{d1} + e^{d2}$ and $a_3 = b + e^{d1} + e^{d2} + e^{d3}$. Here the d_k's have only one restriction, they must be real, and hence we have a similar 4-equation 4-variable problem, as the original one. But now we are free from other restrictions except that all variables must be real. The system now looks as follows;

$$C\int\xi^{\theta 1}(b-\xi)^{\theta 2}(b+e^{d1}-\xi)^{\theta 3}(b+e^{d1}+e^{d2}-\xi)^{\theta 4}(b+e^{d1}+e^{d2}+e^{d3}-\xi)^{\theta 5}d\xi = l_1 \quad \xi = [0 \quad b]$$

$$C\int\xi^{\theta 1}(\xi-b)^{\theta 2}(b+e^{d1}-\xi)^{\theta 3}(b+e^{d1}+e^{d2}-\xi)^{\theta 4}(b+e^{d1}+e^{d2}+e^{d3}-\xi)^{\theta 5}d\xi = l_2 \quad \xi = [b \quad b+e^{d1}]$$

$$C\int\xi^{\theta 1}(\xi-b)^{\theta 2}(\xi-b+e^{d1})^{\theta 3}(b+e^{d1}+e^{d2}-\xi)^{\theta 4}(b+e^{d1}+e^{d2}+e^{d3}-\xi)^{\theta 5}d\xi = l_3 \quad \xi = [b+e^{d1} \quad b+e^{d1}+e^{d2}]$$

$$C\int\xi^{\theta 1}(\xi-b)^{\theta 2}(\xi-b+e^{d1})^{\theta 3}(\xi-b+e^{d1}+e^{d2})^{\theta 4}(b+e^{d1}+e^{d2}+e^{d3}-\xi)^{\theta 5}d\xi = l_4 \quad \xi = [b+e^{d1}+e^{d2} \quad b+e^{d1}+e^{d2}+e^{d3}]$$

One advantage with this notation is that we can now move the integrals so that they are all taken over an interval from 0 to something. We shall make the substitution ξ - b= μ in the second equation, $\xi - b - e^{d1} = \mu$ in the third and $\xi - b - e^{d1} - e^{d2} = \mu$ in the fourth.

$$C\int\xi^{\theta 1}(b-\xi)^{\theta 2}(b+e^{d1}-\xi)^{\theta 3}(b+e^{d1}+e^{d2}-\xi)^{\theta 4}(b+e^{d1}+e^{d2}+e^{d3}-\xi)^{\theta 5}d\xi = l_1 \qquad \xi = [0 \ b]$$

$$C\int(\mu+b)^{\theta 1}\mu^{\theta 2}(e^{d1}-\mu)^{\theta 3}(e^{d1}+e^{d2}-\mu)^{\theta 4}(e^{d1}+e^{d2}+e^{d3}-\mu)^{\theta 5}d\mu = l_2 \qquad \mu = [0 \ e^{d1}]$$

$$C\int(\mu+b+e^{d1})^{\theta 1}(\mu+e^{d1})^{\theta 2}\mu^{\theta 3}(e^{d2}-\mu)^{\theta 4}(e^{d2}+e^{d3}-\mu)^{\theta 5}d\mu = l_3 \qquad \mu = [0 \ e^{d2}]$$

$$C\int(\mu+b+e^{d1}+e^{d2})^{\theta 1}(\mu+e^{d1}+e^{d2})^{\theta 2}(\mu+e^{d2})^{\theta 3}\mu^{\theta 4}(e^{d3}-\mu)^{\theta 5}d\mu = l_4 \qquad \mu = [0 \ e^{d3}]$$

Now, again for convenience we make another substitution. When we start dealing with these integrals numerically it would be an advantage if the integration limits were fixed and equal therefore we want to deform the function under the integral sign so that all integrals are taken over [0 1]. So we want to do the substitution $\xi = b\rho$, $d\xi = bd\rho$ in the first equation and $\mu = e^{dk}\rho$, $d\mu=e^{dk}d\rho$ in the other equations.

$$b^{(1+\theta 1+\ldots+\theta 5)}C\int\rho^{\theta 1}(1-\rho)^{\theta 2}(1-\rho+e^{d1}/b)^{\theta 3}(1-\rho+(e^{d1}+e^{d2})/b)^{\theta 4}(1-\rho+(e^{d1}+e^{d2}+e^{d3})/b)^{\theta 5}d\rho=l_1$$

$$e^{d1(1+\theta 1+\ldots+\theta 5)}C\int(\rho+b/e^{d1})^{\theta 1}\rho^{\theta 2}(1-\rho)^{\theta 3}(1-\rho+e^{d2}/e^{d1})^{\theta 4}(1-\rho+(e^{d2}+e^{d3})/e^{d1})^{\theta 5}d\rho=l_2$$

$$e^{d2(1+\theta 1+\ldots+\theta 5)}C\int(\rho+(b+e^{d1})/e^{d2})^{\theta 1}(\rho+e^{d1}/e^{d2})^{\theta 2}\rho^{\theta 3}(1-\rho)^{\theta 4}(1-\rho+e^{d3}/e^{d2})^{\theta 5}d\rho=l_3$$

$$e^{d3(1+\theta 1+\ldots+\theta 5)}C\int(\rho+(b+e^{d1}+e^{d2})/e^{d3})^{\theta 1}(\rho+(e^{d1}+e^{d2})/e^{d3})^{\theta 2}(\rho+e^{d2}/e^{d3})^{\theta 3}\rho^{\theta 4}(1-\rho)^{\theta 5}d\rho =l_4$$

where all integrals now are taken over [0, 1]. We set $D = 1 + \theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5$. Notice that for later numerical implementation we have two problems the $\rho^{\theta k}(1-\rho)^{\theta k+1}$ factors, that will cause singularities at 0 and 1. In order to reduce this we split each of the integrals at the halfway point 0.5. Thus the left hand side in each equation is an integral taken from 0 to 0.5 which is added to an integral taken from 0.5 to 1. In the later integral we make the substitution $\rho = 1 - \rho'$ thus $d\rho = -d\rho'$ in each equation. Now all integrals are taken over 0 to 0.5 and the system becomes:

$$b^D C \int \rho^{\theta 1} (1-\rho)^{\theta 2} (1-\rho+e^{d1}/b)^{\theta 3} (1-\rho+(e^{d1}+e^{d2})/b)^{\theta 4} (1-\rho+(e^{d1}+e^{d2}+e^{d3})/b)^{\theta 5} d\rho +$$
$$+ b^D C \int (1-\rho')^{\theta 1} \rho'^{\theta 2} (\rho'+e^{d1}/b)^{\theta 3} (\rho'+(e^{d1}+e^{d2})/b)^{\theta 4} (\rho'+(e^{d1}+e^{d2}+e^{d3})/b)^{\theta 5} d\rho' = l_1$$

$$e^{Dd1} C \int (\rho+b/e^{d1})^{\theta 1} \rho^{\theta 2} (1-\rho)^{\theta 3} (1-\rho+e^{d2}/e^{d1})^{\theta 4} (1-\rho+(e^{d2}+e^{d3})/e^{d1})^{\theta 5} d\rho +$$
$$+ e^{Dd1} C \int (1-\rho'+b/e^{d1})^{\theta 1} (1-\rho')^{\theta 2} \rho'^{\theta 3} (\rho'+e^{d2}/e^{d1})^{\theta 4} (\rho'+(e^{d2}+e^{d3})/e^{d1})^{\theta 5} d\rho' = l_2$$

$$e^{Dd2} C \int (\rho+(b+e^{d1})/e^{d2})^{\theta 1} (\rho+e^{d1}/e^{d2})^{\theta 2} \rho^{\theta 3} (1-\rho)^{\theta 4} (1-\rho+e^{d3}/e^{d2})^{\theta 5} d\rho +$$
$$+ e^{Dd2} C \int (1-\rho'+(b+e^{d1})/e^{d2})^{\theta 1} (1-\rho'+e^{d1}/e^{d2})^{\theta 2} (1-\rho')^{\theta 3} \rho'^{\theta 4} (\rho'+e^{d3}/e^{d2})^{\theta 5} d\rho' = l_3$$

$$e^{Dd3} C \int (\rho+(b+e^{d1}+e^{d2})/e^{d3})^{\theta 1} (\rho+(e^{d1}+e^{d2})/e^{d3})^{\theta 2} (\rho+e^{d2}/e^{d3})^{\theta 3} \rho^{\theta 4} (1-\rho)^{\theta 5} d\rho +$$
$$+ e^{Dd3} C \int (1-\rho'+(b+e^{d1}+e^{d2})/e^{d3})^{\theta 1} (1-\rho'+(e^{d1}+e^{d2})/e^{d3})^{\theta 2} (1-\rho'+e^{d2}/e^{d3})^{\theta 3} (1-\rho')^{\theta 4} \rho'^{\theta 5} d\rho' = l_4$$

The equations become quite cumbersome so we want some kind of a functional shorthand notation to deal with these equations. First we drop the apostroph from all the second integrals since this don't really effect our calculations as long as we realize that a variable substitution was done earlier. Now all we need to achieve is to find a nice expression for the factors under the integral signs. For this we introduce two functions $g_{nf}(\ldots)$ $g_{ns}(\ldots)$ for each equation n, where n is b for the first equation and 1 to 3 for the remaining. In each equation $g_{nf}(\ldots)$ and $g_{ns}(\ldots)$ is all but the potential singular ρ^{θ} factor under the first respectively second integral sign. This might seem a bit strange but we are soon going to see one reason why we want it this way and we'll get benefit of this notation later on. Now the system is as follows;

$$b^D C \int \rho^{\theta 1} g_{bf}(\ldots) d\rho + b^D C \int \rho^{\theta 2} g_{bs}(\ldots) d\rho = l_1$$

$$e^{Dd1} C \int (\rho^{\theta 2} g_{1f}(\ldots) d\rho + e^{Dd1} C \int (\rho^{\theta 3} g_{1s}(\ldots) d\rho = l_2$$

$$e^{Dd2} C \int (\rho^{\theta 3} g_{2f}(\ldots) d\rho + e^{Dd2} C \int (\rho^{\theta 4} g_{2s}(\ldots) d\rho = l_3$$

$$e^{Dd3} C \int (\rho^{\theta 4} g_{3f}(\ldots) d\rho + e^{Dd3} C \int (\rho^{\theta 5} g_{3s}(\ldots) d\rho = l_4$$

All integrals taken over [0 0.5]

For instructive purposes and later use let's write down these g's too;

$$g_{bf}(\ldots) = (1-\rho)^{\theta 2} (1-\rho+e^{d1}/b)^{\theta 3} (1-\rho+(e^{d1}+e^{d2})/b)^{\theta 4} (1-\rho+(e^{d1}+e^{d2}+e^{d3})/b)^{\theta 5}$$
$$g_{bs}(\ldots) = (1-\rho)^{\theta 1} (\rho+e^{d1}/b)^{\theta 3} (\rho+(e^{d1}+e^{d2})/b)^{\theta 4} (\rho+(e^{d1}+e^{d2}+e^{d3})/b)^{\theta 5}$$

$$g_{1f}(\ldots) = (\rho+b/e^{d1})^{\theta 1} (1-\rho)^{\theta 3} (1-\rho+e^{d2}/e^{d1})^{\theta 4} (1-\rho+(e^{d2}+e^{d3})/e^{d1})^{\theta 5}$$
$$g_{1s}(\ldots) = (1-\rho+b/e^{d1})^{\theta 1} (1-\rho)^{\theta 2} (\rho+e^{d2}/e^{d1})^{\theta 4} (\rho+(e^{d2}+e^{d3})/e^{d1})^{\theta 5}$$

$$g_{2f}(\ldots) = (\rho+(b+e^{d1})/e^{d2})^{\theta 1} (\rho+e^{d1}/e^{d2})^{\theta 2} (1-\rho)^{\theta 4} (1-\rho+e^{d3}/e^{d2})^{\theta 5}$$
$$g_{2s}(\ldots) = (1-\rho+(b+e^{d1})/e^{d2})^{\theta 1} (1-\rho+e^{d1}/e^{d2})^{\theta 2} (1-\rho)^{\theta 3} (\rho+e^{d3}/e^{d2})^{\theta 5}$$

$$g_{3f}(\ldots) = (\rho+(b+e^{d1}+e^{d2})/e^{d3})^{\theta 1} (\rho+(e^{d1}+e^{d2})/e^{d3})^{\theta 2} (\rho+e^{d2}/e^{d3})^{\theta 3} (1-\rho)^{\theta 5}$$
$$g_{3s}(\ldots) = (1-\rho+(b+e^{d1}+e^{d2})/e^{d3})^{\theta 1} (1-\rho+(e^{d1}+e^{d2})/e^{d3})^{\theta 2} (1-\rho+e^{d2}/e^{d3})^{\theta 3} (1-\rho)^{\theta 4}$$

Now we are ready to remove the remaining singularity from the mapping equations by partial integration of each of the eight integrals, in the system. If we integrate each integral with respect to the ρ^{θ} factor and differentiate with respect to the function g, then the singularity is removed. This follows since then the argument of the ρ factor becomes 1+θ, which is always positive. Furthermore the derivatives of each g-function is well defined functions since b and all e^{d}‘s are positive. We could perform it all in one step, but again for later use and clarity in notation let’s first write down the derivatives of each g, we obtain;

$$dg_{bf}(\ldots)/d\rho=g_{bf}(\ldots)*(-\theta_2/(1-\rho)-\theta_3/(1-\rho+e^{d1}/b)-\theta_4/(1-\rho+(e^{d1}+e^{d2}/b)-\theta_5/(1-\rho+(e^{d1}+e^{d2}+e^{d3})/b))=g_{bf}(\ldots)*h_{bf}(\ldots)$$
$$dg_{bs}(\ldots)/d\rho=g_{bs}(\ldots)*(-\theta_1/(1-\rho)+\theta_3/(\rho+e^{d1}/b)+\theta_4/(\rho+(e^{d1}+e^{d2})/b)+\theta_5/(\rho+(e^{d1}+e^{d2}+e^{d3})/b))=g_{bs}(\ldots)*h_{bs}(\ldots)$$

$$dg_{1f}(\ldots)/d\rho=g_{1f}(\ldots)*(\theta_1/(\rho+b/e^{d1})-\theta_3/(1-\rho)-\theta_4/(1-\rho+e^{d2}/e^{d1})-\theta_5/(1-\rho+(e^{d2}+e^{d3})/e^{d1}))=g_{1f}(\ldots)*h_{1f}(\ldots)$$
$$dg_{1s}(\ldots)/d\rho=g_{1s}(\ldots)*(-\theta_1/(1-\rho+b/e^{d1})-\theta_2/(1-\rho)+\theta_4/(\rho+e^{d2}/e^{d1})-\theta_5/(\rho+(e^{d2}+e^{d3})/e^{d1}))=g_{1s}(\ldots)*h_{1s}(\ldots)$$

$$dg_{2f}(\ldots)/d\rho=g_{2f}(\ldots)*(\theta_1/(\rho+(b+e^{d1})/e^{d2})+\theta_2/(\rho+e^{d1}/e^{d2})-\theta_4/(1-\rho)-\theta_5/(1-\rho+e^{d3}/e^{d2}))=g_{2f}(\ldots)*h_{2f}(\ldots)$$
$$dg_{2s}(\ldots)/d\rho=g_{2s}(\ldots)*(-\theta_1/(1-\rho+(b+e^{d1})/e^{d2})-\theta_2/(1-\rho+e^{d1}/e^{d2})-\theta_3/(1-\rho)+\theta_5/(\rho+e^{d3}/e^{d2}))=g_{2s}(\ldots)*h_{2s}(\ldots)$$

$$dg_{3f}(\ldots)/d\rho=g_{3f}(\ldots)*(\theta_1/(\rho+(b+e^{d1}+e^{d2})/e^{d3})+\theta_2/(\rho+(e^{d1}+e^{d2})/e^{dN-3})+\theta_3/(\rho+e^{d2}/e^{d3})-\theta_5/(1-\rho))=g_{3f}(\ldots)*h_{3f}(\ldots)$$
$$dg_{3s}(\ldots)/d\rho=g_{3s}(\ldots)*(-\theta_1/(1-\rho+(b+e^{d1}+e^{d2)}/e^{d3})-\theta_2/(1-\rho+(e^{d1}+e^{d2})/e^{d3})-\theta_3/(1-\rho+e^{d2}/e^{d3})-\theta_4/(1-\rho))=g_{3s}(\ldots)*h_{3s}(\ldots)$$

(Remark! It’s the h’s at the far right that are the interesting parts)

So the original system now becomes;

$$b^{D}C*\ (\rho^{1+\theta 1}g_{bf}(\ldots)/(1+\theta_1)\Big| - \int\rho^{1+\theta 1}g_{bf}(\ldots)*h_{bf}(\ldots)/(1+\theta_1)d\rho + \rho^{1+\theta 2}g_{bs}(\ldots)/(1+\theta_2)\Big| - \int\rho^{1+\theta 2}g_{bs}(\ldots)*h_{bs}(\ldots)/(1+\theta_2)d\rho) = l_1$$

$$e^{Dd1}C*\ (\rho^{1+\theta 2}g_{1f}(\ldots)/(1+\theta_2)\Big| - \int\rho^{1+\theta 2}g_{1f}(\ldots)*h_{1f}(\ldots)/(1+\theta_2)d\rho + \rho^{1+\theta 3}g_{1s}(\ldots)/(1+\theta_3)\Big| - \int\rho^{1+\theta 3}g_{1s}(\ldots)*h_{1s}(\ldots)/(1+\theta_3)d\rho) = l_2$$

$$e^{Dd2}C*\ (\rho^{1+\theta 3}g_{2f}(\ldots)/(1+\theta_3)\Big| - \int\rho^{1+\theta 3}g_{2f}(\ldots)*h_{2f}(\ldots)/(1+\theta_3)d\rho + \rho^{1+\theta 4}g_{2s}(\ldots)/(1+\theta_4)\Big| - \int\rho^{1+\theta 4}g_{2s}(\ldots)*h_{2s}(\ldots)/(1+\theta_4)d\rho) = l_3$$

$$e^{Dd3}C*\ (\rho^{1+\theta 4}g_{3f}(\ldots)/(1+\theta_4)\Big| - \int\rho^{1+\theta 4}g_{3f}(\ldots)*h_{3f}(\ldots)/(1+\theta_4)d\rho + \rho^{1+\theta 5}g_{3s}(\ldots)/(1+\theta_5)\Big| - \int\rho^{1+\theta 5}g_{3s}(\ldots)*h_{3s}(\ldots)/(1+\theta_5)d\rho) = l_4$$

where all the limits are taken over 0 and 0.5. Now one could of course count the limits of the eight parts we partially integrated. The lower limit of these will always be zero, however this would still lead to a cumbersome function of the d_k’s. Also we will see that numerically we actually will get these value for free while evaluating the integral that remains. So just leave it as it is and remember that $\Big|$ means that we evaluate the function at $\rho = 0.5$ and all integrals are taken over [0 , 0.5].

Now we can reduce this system to one of 3 unknowns and 3 equations by solving the first equation for C and then setting this into all other equations:

$$l_1e^{Dd1}*(\ [\rho^{1+\theta 2}g_{1f}()/(1+\theta_2)+\rho^{1+\theta 3}g_{1s}()/(1+\theta_3)]\Big| - \int [\rho^{1+\theta 2}g_{1f}()*h_{1f}()/(1+\theta_2)+\rho^{1+\theta 3}g_{1s}()*h_{1s}\ ()/(1+\theta_3)]\ d\rho) - l_2b^{D}*([\rho^{1+\theta 1}g_{bf}()/(1+\theta_1)+\rho^{1+\theta 2}g_{bs}()/(1+\theta_2)]\Big| + \int [\rho^{1+\theta 1}g_{bf}()*h_{bf}()/(1+\theta_1)+\rho^{1+\theta 2}g_{bs}()*h_{bs}\ ()/(1+\theta_2)]\ d\rho) = 0$$

$$l_1e^{Dd2}*(\ [\rho^{1+\theta 3}g_{2f}()/(1+\theta_3)+\rho^{1+\theta 4}g_{2s}()/(1+\theta_4)]\Big| - \int [\rho^{1+\theta 3}g_{2f}()*h_{2f}()/(1+\theta_3)+\rho^{1+\theta 4}g_{2s}()*h_{2s}\ ()/(1+\theta_4)]\ d\rho) - l_3b^{D}*([\rho^{1+\theta 1}g_{bf}()/(1+\theta_1)+\rho^{1+\theta 2}g_{bs}()/(1+\theta_2)]\Big| + \int [\rho^{1+\theta 1}g_{bf}()*h_{bf}()/(1+\theta_1)+\rho^{1+\theta 2}g_{bs}()*h_{bs}\ ()/(1+\theta_2)]\ d\rho) = 0$$

$$l_1e^{Dd3}*(\ [\rho^{1+\theta 4}g_{3f}()/(1+\theta_4)+\rho^{1+\theta 5}g_{3s}()/(1+\theta_5)]\Big| - \int [\rho^{1+\theta 4}g_{3f}()*h_{3f}()/(1+\theta_4)+\rho^{1+\theta 5}g_{3s}()*h_{3s}\ ()/(1+\theta_5)]\ d\rho) - l_4b^{D}*([\rho^{1+\theta 1}g_{bf}()/(1+\theta_1)+\rho^{1+\theta 2}g_{bs}()/(1+\theta_2)]\Big| + \int [\rho^{1+\theta 1}g_{bf}()*h_{bf}()/(1+\theta_1)+\rho^{1+\theta 2}g_{bs}()*h_{bs}\ ()/(1+\theta_2)]\ d\rho) = 0$$

Notice that if we can solve this equation system for the d_k’s we can easily solve C, so we don’t bother with that for a while.

The general case;

Now let's briefly consider the general system thus far by dealing similarly step by step as in the six corner case. For every N ≥ 6 the basic system is;

$$A_0\int\xi^{\theta 1}(\xi - b)^{\theta 2}(\xi - a_1)^{\theta 3}\ldots(\xi - a_{N-3})^{\theta N-1}d\xi = c_3 \qquad \xi = [0\ b]$$

$$A_0\int\xi^{\theta 1}(\xi - b)^{\theta 2}(\xi - a_1)^{\theta 3}\ldots(\xi - a_{N-3})^{\theta N-1}d\xi = c_4 \qquad \xi = [0\ a_1]$$

$$A_0\int\xi^{\theta 1}(\xi - b)^{\theta 2}(\xi - a_1)^{\theta 3}\ldots(\xi - a_{N-3})^{\theta N-1}d\xi = c_5 \qquad \xi = [0\ a_2]$$

$$\vdots$$

$$A_0\int\xi^{\theta 1}(\xi - b)^{\theta 2}(\xi - a_1)^{\theta 3}\ldots(\xi - a_{N-3})^{\theta N-1}d\xi = c_N \qquad \xi = [0\ a_{N-2}]$$

$$c_3 = l_1 e^{-\pi i\theta 1} \quad c_4 = c_3 + l_2 e^{-\pi i(\theta 1+\theta 2)} \quad \ldots\ c_N = c_{N-1} + l_{N-2}e^{-\pi i(\theta 1+\theta 2+\ldots+\theta N-2)}$$

Removing the potentially complex parts from under the integral, setting in the corner values and setting $C = (-1)^{(\theta 2+\theta 3+\ldots+\theta N-1)}A_0$ gives;

$$C\int\xi^{\theta 1}(b-\xi)^{\theta 2}(a_1-\xi)^{\theta 3}(a_2-\xi)^{\theta 4}\ldots(a_{N-3}-\xi)^{\theta N-1}d\xi = l_1 \qquad \xi = [0\ b]$$

$$C\int\xi^{\theta 1}(\xi - b)^{\theta 2}(a_1-\xi)^{\theta 3}(a_2-\xi)^{\theta 4}\ldots(a_{N-3}-\xi)^{\theta N-1}d\xi = l_2 \qquad \xi = [b\ a_1]$$

$$C\int\xi^{\theta 1}(\xi - b)^{\theta 2}(\xi - a_1)^{\theta 3}(a_2-\xi)^{\theta 4}\ldots(a_{N-3}-\xi)^{\theta N-1}d\xi = l_3 \qquad \xi = [a_1\ a_2]$$

$$\vdots$$

$$C\int\xi^{\theta 1}(\xi - b)^{\theta 2}(\xi - a_1)^{\theta 3}\ldots(\xi - a_{N-4})^{\theta N-2}(a_{N-3} - \xi)^{\theta N-1}d\xi = l_{N-2} \qquad \xi = [a_{N-3}\ a_{N-2}].$$

Now we set $a_1 = b+e^{d1}$, $a_2 = b+e^{d1}+e^{d2}$, $a_3 = b+e^{d1}+e^{d2}+e^{d3}$ and so on. And do the same variable substitutions for the first four equations, as in the six corner case, and continue in a similar way for the rest. Thus, $\xi - b = \mu$ for the second, $\xi - b - e^{d1} = \mu$ in the third and so on. Followed by $\xi = b\rho$, $d\xi = bd\rho$ in the first equation and $\mu = e^{dk}\rho$, $d\mu = e^{dk}d\rho$ in the other equations. We get;

$$b^{(1+\theta 1+\ldots+\theta N-1)}C\int\rho^{\theta 1}(1-\rho)^{\theta 2}(1+e^{d1}/b-\rho)^{\theta 3}(1+(e^{d1}+e^{d2})/b)-\rho)^{\theta 4}\ldots(1+(e^{d1}+\ldots+e^{dN-3})/b-\rho)^{\theta N-1}d\rho = l_1$$

$$e^{d1(1+\theta 1+\ldots+\theta N-1)}C\int(\rho+b/e^{d1})^{\theta 1}\rho^{\theta 2}(1-\rho)^{\theta 3}(1+e^{d2}/e^{d1}-\rho)^{\theta 4}\ldots(1+(e^{d2}+\ldots+e^{dN-3})/e^{d1}-\rho)^{\theta N-1}d\rho = l_2$$

$$e^{d2(1+\theta 1+\ldots+\theta N-1)}C\int(\rho+(b+e^{d1})/e^{d2})^{\theta 1}(\rho+e^{d1}/e^{d2})^{\theta 2}\rho^{\theta 3}(1-\rho)^{\theta 4}\ldots(1+(e^{d3}+\ldots+e^{dN-3}/e^{d2}-\rho)^{\theta N-1}d\rho = l_3$$

$$\vdots$$

$$e^{dN-3(1+\theta 1+\ldots+\theta N-1}C\int(\rho+(b+e^{d1}+\ldots+e^{dN-4})/e^{dN-3})^{\theta 1}(\rho+(e^{d1}+\ldots+e^{dN-4})/e^{dN-3})^{\theta 2}\ldots(\rho+e^{dN-4}/e^{dN-3})^{\theta N-3}\rho^{\theta N-2}*$$

$$*(1-\rho)^{\theta N-1}d\rho = l_{N-2}$$

where all integrals now are taken over [0 1]. In order to remove one of the two singularities we proceed as before and split each integral at the 0.5 point and make the substitution $\rho = 1 - \rho'$ $d\rho = -d\rho'$ in the later integral in each equation. After setting $1+\sum\theta_n = D$ $n = [1, N-1]$ and defining the two functions $g_{nf}(\ldots)$ $g_{ns}(\ldots)$ as above we get.

$$b^{D}C\int\rho^{\theta1}g_{bf}(\ldots)d\rho + b^{D}C\int\rho^{\theta2}g_{bs}(\ldots)d\rho = l_1$$

$$e^{Dd1}C\int(\rho^{\theta2}g_{1f}(\ldots)d\rho + e^{Dd1}C\int(\rho^{\theta3}g_{1s}(\ldots)d\rho = l_2$$

$$e^{Dd2}C\int(\rho^{\theta3}g_{2f}(\ldots)d\rho + e^{Dd2}C\int(\rho^{\theta4}g_{2s}(\ldots)d\rho = l_3$$

$$\vdots \qquad\qquad \vdots$$

$$e^{Ddk}C\int(\rho^{\theta k+1}g_{kf}(\ldots)d\rho + e^{Ddk}C\int(\rho^{\theta k+2}g_{ks}(\ldots)d\rho = l_{k+1}$$

$$\vdots \qquad\qquad \vdots$$

$$e^{DdN-3}C\int(\rho^{\theta N-2}g_{N-3f}(\ldots)d\rho + e^{DdN-3}C\int(\rho^{\theta N-1}g_{N-3s}(\ldots)d\rho = l_{N-2}$$

All integrals taken over [0 0.5]

where;

$$g_{bf}(\ldots) = \underline{(1-\rho)^{\theta2}(1+e^{d1}/b-\rho)^{\theta3}}(1+(e^{d1}+e^{d2})/b)-\rho)^{\theta4}\ldots(1+(e^{d1}+\ldots+e^{dN-3})/b-\rho)^{\theta N-1}$$
$$g_{bs}(\ldots) = (1-\rho)^{\theta1}\underline{(\rho+e^{d1}/b)^{\theta3}(\rho+(e^{d1}+e^{d2})/b))^{\theta4}}\ldots(\rho+(e^{d1}+\ldots+e^{dN-3})/b)^{\theta N-1}$$

$$g_{1f}(\ldots) = \underline{(\rho+b/e^{d1})^{\theta1}(1-\rho)^{\theta3}}(1+e^{d2}/e^{d1}-\rho)^{\theta4}\ldots(1+(e^{d2}+\ldots+e^{dN-3})/e^{d1}-\rho)^{\theta N-1}$$
$$g_{1s}(\ldots) = (1+b/e^{d1}-\rho)^{\theta1}\underline{(1-\rho)^{\theta2}(\rho+e^{d2}/e^{d1})^{\theta4}}\ldots(\rho+(e^{d2}+\ldots+e^{dN-3})/e^{d1})^{\theta N-1}$$

$$\vdots \qquad\qquad \vdots$$

$$g_{kf}(\ldots)=(\rho+(b+e^{d1}+\ldots+e^{dk-1})/e^{dk})^{\theta1}(\rho+(e^{d1}+\ldots+e^{dk-1})/e^{dk})^{\theta2}\ldots\underline{(\rho+e^{dk-1})/e^{dk})^{\theta k}(1-\rho)^{\theta k+2}}(1+e^{dk+1}/e^{dk}-\rho)^{\theta k+3}\ldots$$
$$\ldots(1+(e^{dk+1}+\ldots+e^{dN-3})/e^{dk}-\rho)^{\theta N-1}$$

$$g_{ks}(\ldots)=(1+(b+e^{d1}+\ldots+e^{dk-1})/e^{dk}-\rho)^{\theta1}(1+(e^{d1}+\ldots+e^{dk-1})/e^{dk}-\rho)^{\theta2}\ldots(1+e^{dk-1})/e^{dk}-\rho)^{\theta k}\underline{(1-\rho)^{\theta k+1}}*$$
$$\underline{*(\rho+e^{dk+1}/e^{dk})^{\theta k+3}}\ldots(1+(e^{dk+1}+\ldots+e^{dN-3})/e^{dk}-\rho)^{\theta N-1}$$

$$\vdots \qquad\qquad \vdots$$

$$g_{N-3f}(\ldots)=(\rho+(b+e^{d1}+\ldots+e^{dN-4})/e^{dN-3})^{\theta1}(\rho+(e^{d1}+\ldots+e^{dN-4})/e^{dN-3})^{\theta2}\ldots\underline{(\rho+e^{dN-4})/e^{dN-3})^{\theta N-3}(1-\rho)^{\theta N-1}}$$
$$g_{N-3s}(\ldots)=(1+(b+e^{d1}+\ldots+e^{dN-4})/e^{dN-3}-\rho)^{\theta1}(1+(e^{d1}+\ldots+e^{dN-4})/e^{dN-3}-\rho)^{\theta2}\ldots(1+e^{dN-4})/e^{dN-3}-\rho)^{\theta N-3}\underline{(1-\rho)^{\theta N-2}}$$

(a)

Derivating a with respect to ρ we get;

$$dg_{bf}(\ldots)/d\rho=g_{bf}(\ldots)*(-\theta_2/(1-\rho)-\theta_3/(1+e^{d1}/b-\rho)-\ldots-\theta_{N-1}/(1+(e^{d1}+\ldots+e^{dN-3})/b)-\rho))=g_{bf}(\ldots)*h_{bf}(\ldots)$$
$$dg_{bs}(\ldots)/d\rho=g_{bs}(\ldots)*(-\theta_1/(1-\rho)+\theta_3/(\rho+e^{d1}/b)+\ldots+\theta_{N-1}/(\rho+(e^{d1}+\ldots+e^{dN-3})/b))=g_{bs}(\ldots)*h_{bs}(\ldots)$$

$$dg_{1f}(\ldots)/d\rho=g_{1f}(\ldots)*(\theta_1/(\rho+b/e^{d1})-\theta_3/(1-\rho)-\theta_4/(1+e^{d2}/e^{d1}-\rho)-\ldots-\theta_{N-1}/(1+(e^{d2}+\ldots+e^{dN-3})/b-\rho))=g_{1f}(\ldots)*h_{1f}(\ldots)$$
$$dg_{1s}(\ldots)/d\rho=g_{1s}(\ldots)*(-\theta_1/(1+b/e^{d1}-\rho)-\theta_2/(1-\rho)+\theta_4/(\rho+e^{d2}/e^{d1})-\ldots+\theta_{N-1}/(\rho+(e^{d2}+\ldots+e^{dN-3})/b))=g_{1s}(\ldots)*h_{1s}(\ldots)$$

$$\vdots \qquad\qquad \vdots$$

$$dg_{kf}(\ldots)/d\rho=g_{kf}(\ldots)*(\theta_1/(\rho+(b+\ldots+e^{dk-1})/e^{dk}) + \theta_2/(\rho+(e^{d1}+\ldots+e^{dk-1})/e^{dk})+\ldots+\theta_k/(\rho+e^{dk-1}/e^{dk})-\theta_{k+2}/(1-\rho)-\theta_{k+3}/(1+e^{dk+1}/e^{dk}-\rho)-\ldots$$
$$\ldots-\theta_{N-1}/(1+(e^{dk+1}+\ldots+e^{dN-3}/e^{dk}-\rho))=g_{kf}(\ldots)*h_{kf}(\ldots)$$

$$dg_{ks}(\ldots)/d\rho=g_{ks}(\ldots)*(-\theta_1/(1+(b+\ldots+e^{dk-1)}/e^{dk}-\rho)-\theta_2/(1+(e^{d1}+\ldots+e^{dk-1})/e^{dk}-\rho)-\ldots-\theta_k/(1+e^{dk-1}/e^{dk}-\rho)-\theta_{k+1}/(1-\rho)+\theta_{k+3}/(\rho+e^{dk+1}/e^{dk})+..$$
$$\ldots+\theta_{N-1}/(\rho+(e^{dk+1}+\ldots+e^{dN-3}/e^{dk}))=g_{ks}(\ldots)*h_{ks}(\ldots)$$

$$\vdots \qquad\qquad \vdots$$

$$dg_{N-3f}(\ldots)/d\rho=g_{N-3f}(\ldots)*(\theta_1/(\rho+(b+\ldots+e^{dN-4})/e^{dN-3})+\theta_2/(\rho+(e^{d1}+\ldots+e^{dN-4})/e^{dN-3})+\ldots+\theta_{N-3}/(\rho+e^{dN-4}/e^{dN-3})-\theta_{N-1}/(1-\rho)) =$$
$$=g_{N-3f}(\ldots)*h_{N-3f}(\ldots)$$

$$dg_{N-3s}(\ldots)/d\rho=g_{N-3s}(\ldots)*(-\theta_1/(1+(b+\ldots+e^{dN-4)}/e^{dN-3}-\rho)-\theta_2/(1+(e^{d1}+\ldots+e^{dN-4})/e^{dN-3}-\rho)-\ldots-\ \theta_{N-3}/(1+e^{dN-4}/e^{dN-3}-\rho)-\ -\theta_{N-2}/(1-\rho)) =$$
$$=g_{N-3s}(\ldots)*h_{N-3s}(\ldots)$$

REMARK! Notice the h's on the right are the intresting parts. **(b)**

This gives us all the functions g and h for the general case. And we get, as above after solving the first equation for C and inserting this in the rest, the system we want to solve as;

$$l_1e^{Dd1}*([\rho^{1+\theta 2}g_{1f}()/(1+\theta_2)+\rho^{1+\theta 3}g_{1s}()/(1+\theta_3)] \Big| - \int [\rho^{1+\theta 2}g_{1f}()*h_{1f}()/(1+\theta_2)+\rho^{1+\theta 3}g_{1s}()*h_{1s}()/(1+\theta_3)]\, d\rho) -$$
$$- l_2b^D*([\rho^{1+\theta 1}g_{bf}()/(1+\theta_1)+\rho^{1+\theta 2}g_{bs}()/(1+\theta_2)] \Big| + \int [\rho^{1+\theta 1}g_{bf}()*h_{bf}()/(1+\theta_1)+\rho^{1+\theta 2}g_{bs}()*h_{bs}()/(1+\theta_2)]\, d\rho) = 0$$

$$l_1e^{Dd2}*([\rho^{1+\theta 3}g_{2f}()/(1+\theta_3)+\rho^{1+\theta 4}g_{2s}()/(1+\theta_4)] \Big| - \int [\rho^{1+\theta 3}g_{2f}()*h_{2f}()/(1+\theta_3)+\rho^{1+\theta 4}g_{2s}()*h_{2s}()/(1+\theta_4)]\, d\rho) -$$
$$- l_3b^D*([\rho^{1+\theta 1}g_{bf}()/(1+\theta_1)+\rho^{1+\theta 2}g_{bs}()/(1+\theta_2)] \Big| + \int [\rho^{1+\theta 1}g_{bf}()*h_{bf}()/(1+\theta_1)+\rho^{1+\theta 2}g_{bs}()*h_{bs}()/(1+\theta_2)]\, d\rho) = 0$$

$$\vdots \qquad\qquad \vdots$$

$$l_1e^{Ddk}*([\rho^{1+\theta k+1}g_{kf}()/(1+\theta_{k+1})+\rho^{1+\theta k+2}g_{ks}()/(1+\theta_{k+2})] \Big| -\int[\rho^{1+\theta k+1}g_{kf}()*h_{kf}()/(1+\theta_{k+1})+\rho^{1+\theta k+2}g_{ks}()*h_{ks}()/(1+\theta_{k+2})]d\rho) -$$
$$- l_{k+1}b^D*([\rho^{1+\theta 1}g_{bf}()/(1+\theta_1)+\rho^{1+\theta 2}g_{bs}()/(1+\theta_2)] \Big| + \int [\rho^{1+\theta 1}g_{bf}()*h_{bf}()/(1+\theta_1)+\rho^{1+\theta 2}g_{bs}()*h_{bs}()/(1+\theta_2)]\, d\rho) = 0$$

$$\vdots \qquad\qquad \vdots$$

$$l_1e^{DdN-3}*([\rho^{1+\theta N-2}g_{N-3f}()/(1+\theta_{N-2})+\rho^{1+\theta N-1}g_{N-3s}()/(1+\theta_{N-1})] \Big| -\int[\rho^{1+\theta N-2}g_{N-3f}()*h_{N-3f}()/(1+\theta_{N-2})+\rho^{1+\theta N-1}g_{N-3s}()*h_{N-3s}()/(1+\theta_{N-1})]d\rho)-$$
$$- l_{N-2}b^D*([\rho^{1+\theta 1}g_{bf}()/(1+\theta_1)+\rho^{1+\theta 2}g_{bs}()/(1+\theta_2)] \Big| + \int [\rho^{1+\theta 1}g_{bf}()*h_{bf}()/(1+\theta_1)+\rho^{1+\theta 2}g_{bs}()*h_{bs}()/(1+\theta_2)]\, d\rho) = 0$$

All limits taken over [0 0.5] **(1)**

So this is the system of nonlinear equations we want to solve numerically. We want to use Newton's method for this. In order to do this we need to find the jacobian of the system with respect to each of the d_k's. This might appear as a formidable task but we actually get a control.

First we'll notice that all d_k's $\neq \infty, -\infty$. If this wouldn't be the case two or more breakpoints would be equal. Hence the functions under the integral signs are defined for $0 \leq \rho \leq 0.5$ and for each specific d_k if the rest are considered fixed. Furthermore the functions under the integral signs are integrable for all d_k's $\neq \infty, -\infty$. Moreover since the functions under the integral signs must converge in the Christoffel-Schwarz equations we may take the partial derivatives under the integral sign. It's enough to take the partial derivatives of the g's and h's and then combine these in different ways to get the jacobians. For example in the six corner case we get the partial derivatives;

$$\partial g_{bf}(\ldots)/\partial d_1=g_{bf}(\ldots)*(\theta_3(e^{d1}/b)/(1-\rho+e^{d1}/b)+\theta_4(e^{d1}/b)/(1-\rho+(e^{d1}+e^{d2})/b)+\theta_5(e^{d1}/b)/(1-\rho+(e^{d1}+e^{d2}+e^{d3})/b)$$
$$\partial g_{bs}(\ldots)/\partial d_1=g_{bs}(\ldots)*(\theta_3(e^{d1}/b)/(\rho+e^{d1}/b) + \theta_4(e^{d1}/b)/(\rho+(e^{d1}+e^{d2})/b)+\theta_5(e^{d1}/b)/(\rho+(e^{d1}+e^{d2}+e^{d3})/b)$$
$$\partial h_{bf}(\ldots)/\partial d_1=(\theta_3(e^{d1}/b) /(1-\rho+e^{d1}/b)^2+\theta_4(e^{d1}/b) /(1-\rho+(e^{d1}+e^{d2})/b)^2+\theta_5(e^{d1}/b)/(1-\rho+(e^{d1}+e^{d2}+e^{d3})/b)^2$$
$$\partial h_{bs}(\ldots)/\partial d_1=(-\theta_3(e^{d1}/b)/(\rho+e^{d1}/b)^2-\theta_4(e^{d1}/b)/(\rho+(e^{d1}+e^{d2})/b)^2-\theta_5(e^{d1}/b)/(\rho+(e^{d1}+e^{d2}+e^{d3})/b)^2$$

$$\partial g_{bf}(\ldots)/\partial d_2=g_{bf}(\ldots)*(\theta_4(e^{d2}/b)/(1-\rho+(e^{d1}+e^{d2})/b)+\theta_5(e^{d2}/b)/(1-\rho+(e^{d1}+e^{d2}+e^{d3})/b)$$
$$\partial g_{bs}(\ldots)/\partial d_2=g_{bs}(\ldots)*(\theta_4(e^{d2}/b)/(\rho+(e^{d1}+e^{d2})/b)+\theta_5(e^{d2}/b)/(\rho+(e^{d1}+e^{d2}+e^{d3})/b)$$
$$\partial h_{bf}(\ldots)/\partial d_2=(\theta_4(e^{d2}/b) /(1-\rho+(e^{d1}+e^{d2})/b)^2+\theta_5(e^{d2}/b)/(1-\rho+(e^{d1}+e^{d2}+e^{d3})/b)^2$$
$$\partial h_{bs}(\ldots)/\partial d_2=(-\theta_4(e^{d2}/b)/(\rho+(e^{d1}+e^{d2})/b)^2-\theta_5(e^{d2}/b)/(\rho+(e^{d1}+e^{d2}+e^{d3})/b)^2$$

$$\partial g_{bf}(\ldots)/\partial d_3=g_{bf}(\ldots)*(\theta_5(e^{d3}/b)/(1-\rho+(e^{d1}+e^{d2}+e^{d3})/b)$$
$$\partial g_{bs}(\ldots)/\partial d_3=g_{bs}(\ldots)*(\theta_5(e^{d3}/b)/(\rho+(e^{d1}+e^{d2}+e^{d3})/b)$$
$$\partial h_{bf}(\ldots)/\partial d_3=(\theta_5(e^{d3}/b)/(1-\rho+(e^{d1}+e^{d2}+e^{d3})/b)^2$$
$$\partial h_{bs}(\ldots)/\partial d_3=(-\theta_5(e^{d3}/b)/(\rho+(e^{d1}+e^{d2}+e^{d3})/b)^2$$

$$\partial g_{1f}(\ldots)/\partial d_1=g_{1f}(\ldots)*(-\theta_1(b/e^{d1})/(\rho+b/e^{d1})-\theta_4(e^{d2}/e^{d1})/(1-\rho+e^{d2}/e^{d1})-\theta_5((e^{d2}+e^{d3})/e^{d1})/(1-\rho+(e^{d2}+e^{d3})/e^{d1}))$$
$$\partial g_{1s}(\ldots)/\partial d_1=g_{1s}(\ldots)*(-\theta_1(b/e^{d1})/(1-\rho+b/e^{d1})-\theta_4(e^{d2}/e^{d1})/(\rho+e^{d2}/e^{d1})-\theta_5((e^{d2}+e^{d3})/e^{d1})/(\rho+(e^{d2}+e^{d3})/e^{d1}))$$
$$\partial h_{1f}(\ldots)/\partial d_1=(\theta_1(b/e^{d1})/(\rho+b/e^{d1})^2-\theta_4(e^{d2}/e^{d1})/(1-\rho+e^{d2}/e^{d1})^2-\theta_5((e^{d2}+e^{d3})/e^{d1})/(1-\rho+(e^{d2}+e^{d3})/e^{d1})^2)$$
$$\partial h_{1s}(\ldots)/\partial d_1=(-\theta_1(b/e^{d1})/(1-\rho+b/e^{d1})^2+\theta_4(e^{d2}/e^{d1})/(\rho+e^{d2}/e^{d1})^2+\theta_5((e^{d2}+e^{d3})/e^{d1})/(\rho+(e^{d2}+e^{d3})/e^{d1})^2)$$

$$\partial g_{1f}(\ldots)/\partial d_2=g_{1f}(\ldots)*(\theta_4(e^{d2}/e^{d1})/(1-\rho+e^{d2}/e^{d1})+\theta_5(e^{d2}/e^{d1})/(1-\rho+(e^{d2}+e^{d3})/e^{d1}))$$
$$\partial g_{1s}(\ldots)/\partial d_2=g_{1s}(\ldots)*(\theta_4(e^{d2}/e^{d1})/(\rho+e^{d2}/e^{d1})+\theta_5(e^{d2}/e^{d1})/(\rho+(e^{d2}+e^{d3})/e^{d1}))$$
$$\partial h_{1f}(\ldots)/\partial d_2=(\theta_4(e^{d2}/e^{d1})/(1-\rho+e^{d2}/e^{d1})^2+\theta_5(e^{d2}/e^{d1})/(1-\rho+(e^{d2}+e^{d3})/e^{d1})^2)$$
$$\partial h_{1s}(\ldots)/\partial d_2=(-\theta_4(e^{d2}/e^{d1})/(\rho+e^{d2}/e^{d1})^2-\theta_5(e^{d2}/e^{d1})/(\rho+(e^{d2}+e^{d3})/e^{d1})^2)$$

$$\partial g_{1f}(\ldots)/\partial d_3=g_{1f}(\ldots)*(\theta_5(e^{d3}/e^{d1})/(1-\rho+(e^{d2}+e^{d3})/e^{d1}))$$
$$\partial g_{1s}(\ldots)/\partial d_3=g_{1s}(\ldots)*(\theta_5(e^{d3}/e^{d1})/(\rho+(e^{d2}+e^{d3})/e^{d1}))$$
$$\partial h_{1f}(\ldots)/\partial d_3=(\theta_5(e^{d3}/e^{d1})/(1-\rho+(e^{d2}+e^{d3})/e^{d1})^2)$$
$$\partial h_{1s}(\ldots)/\partial d_3=(-\theta_5(e^{d3}/e^{d1})/(\rho+(e^{d2}+e^{d3})/e^{d1})^2)$$

$$\partial g_{2f}(\ldots)/\partial d_1= g_{2f}(\ldots)*(\theta_1(e^{d1}/e^{d2})/(\rho+b+e^{d1}/e^{d2})+\theta_2(e^{d1}/e^{d2})/(\rho+e^{d1}/e^{d2}))$$
$$\partial g_{2s}(\ldots)/\partial d_1= g_{2s}(\ldots)*(\theta_1(e^{d1}/e^{d2})/(1-\rho +b+e^{d1}/e^{d2})+\theta_2(e^{d1}/e^{d2})/(1-\rho+e^{d1}/e^{d2})$$
$$\partial h_{2f}(\ldots)/\partial d_1=(-\theta_1(e^{d1}/e^{d2})/(\rho+(b+e^{d1}/e^{d2})^2-\theta_2(e^{d1}/e^{d2})/(\rho+e^{d1}/e^{d2})^2)$$
$$\partial h_{2s}(\ldots)/\partial d_1=(\theta_1(e^{d1}/e^{d2})/(1-\rho +(b+e^{d1}/e^{d2})^2+\theta_2(e^{d1}/e^{d2})/(1-\rho+e^{d1}/e^{d2})^2)$$

$$\partial g_{2f}(\ldots)/\partial d_2=g_{2f}(\ldots)*(-\theta_1((b+e^{d1})/e^{d2})/(\rho+(b+e^{d1})/e^{d2})-\theta_2(e^{d1}/e^{d2})/(\rho+e^{d1}/e^{d2})-\theta_5(e^{d3}/e^{d2})/(1-\rho+e^{d3}/e^{d2}))$$
$$\partial g_{2s}(\ldots)/\partial d_2=g_{2s}(\ldots)*(-\theta_1((b+e^{d1})/e^{d2})/(1-\rho+(b+e^{d1})/e^{d2})-\theta_2(e^{d1}/e^{d2})/(1-\rho+e^{d1}/e^{d2})-\theta_5(e^{d3}/e^{d2})/(\rho+e^{d3}/e^{d1}))$$
$$\partial h_{2f}(\ldots)/\partial d_2=(\theta_1((b+e^{d1})/e^{d2})/(\rho+(b+e^{d1})/e^{d2})^2+\theta_2(e^{d1}/e^{d2})/(\rho+e^{d1}/e^{d2})^2-\theta_5(e^{d3}/e^{d2})/(1-\rho+e^{d3}/e^{d2})^2)$$
$$\partial h_{2s}(\ldots)/\partial d_2=(-\theta_1((b+e^{d1})/e^{d2})/(1-\rho+(b+e^{d1})/e^{d2})^2-\theta_2(e^{d1}/e^{d2})/(1-\rho+e^{d1}/e^{d2})^2+\theta_5(e^{d3}/e^{d2})/(\rho+e^{d3}/e^{d2})^2)$$

$$\partial g_{2f}(\ldots)/\partial d_3=g_{2f}(\ldots)*(\theta_5(e^{d3}/e^{d2})/(1-\rho+e^{d3}/e^{d2}))$$
$$\partial g_{2s}(\ldots)/\partial d_3=g_{2s}(\ldots)*(\theta_5(e^{d3}/e^{d2})/(\rho+e^{d3}/e^{d2}))$$
$$\partial h_{2f}(\ldots)/\partial d_3=(\theta_5(e^{d3}/e^{d2})/(1-\rho+e^{d3}/e^{d2})^2)$$
$$\partial h_{2s}(\ldots)/\partial d_3=(-\theta_5(e^{d3}/e^{d2})/(\rho+e^{d3}/e^{d2})^2)$$

$$\partial g_{3f}(\ldots)/\partial d_1= g_{3f}(\ldots)*(\theta_1(e^{d1}/e^{d3})/(\rho+(b+e^{d1}+e^{d2})/e^{d3})+\theta_2(e^{d1}/e^{d3})/(\rho+(e^{d1}+e^{d2})/e^{d3}))$$
$$\partial g_{3s}(\ldots)/\partial d_1= g_{3s}(\ldots)*(\theta_1(e^{d1}/e^{d3})/(1-\rho+(b+e^{d1}+e^{d2})/e^{d3})+\theta_2(e^{d1}/e^{d3})/(1-\rho+(e^{d1}+e^{d2})/e^{d3}))$$
$$\partial h_{3f}(\ldots)/\partial d_1=(-\theta_1(e^{d1}/e^{d3})/(\rho+(b+e^{d1}+e^{d2})/e^{d3})^2-\theta_2(e^{d1}/e^{d3})/(\rho+(e^{d1}+e^{d2})/e^{d3})^2)$$
$$\partial h_{3s}(\ldots)/\partial d_1=(\theta_1(e^{d1}/e^{d3})/(1-\rho+(b+e^{d1}+e^{d2})/e^{d3}-\rho)^2+\theta_2(e^{d1}/e^{d3})/(1-\rho+(e^{d1}+e^{d2})/e^{d3})^2)$$

$$\partial g_{3f}(\ldots)/\partial d_2=g_{3f}(\ldots)*(\theta_1(e^{d2}/e^{d3})/(\rho+(b+e^{d1}+e^{d2})/e^{d3})+\theta_2(e^{d2}/e^{d3})/(\rho+(e^{d1}+e^{d2})/e^{d3})+\theta_3(e^{d2}/e^{d3})/(\rho+e^{d2}/e^{d3}))$$
$$\partial g_{3s}(\ldots)/\partial d_2=g_{3s}(\ldots)*(\theta_1(e^{d2}/e^{d3})/(1-\rho+(b+e^{d1}+e^{d2})/e^{d3})+(\theta_2(e^{d2}/e^{d3})/(1-\rho+(e^{d1}+e^{d2})/e^{d3})+\theta_3(e^{d2}/e^{d3})/(1-\rho+e^{d2}/e^{d3})$$
$$\partial h_{3f}(\ldots)/\partial d_2=(-\theta_1(e^{d2}/e^{d3})/(\rho+(b+e^{d1}+e^{d2})/e^{d3})^2-\theta_2(e^{d2}/e^{d3})/(\rho+(e^{d1}+e^{d2})/e^{d3})^2-\theta_3(e^{d2}/e^{d3})/(\rho+e^{d2}/e^{d3})^2)$$
$$\partial h_{3s}(\ldots)/\partial d_2=(\theta_1(e^{d2}/e^{d3})/(1-\rho+(b+e^{d1}+e^{d2})/e^{d3})^2+\theta_2(e^{d2}/e^{d3})/(1-\rho+(e^{d1}+e^{d2})/e^{d3})^2+\theta_3(e^{d2}/e^{d3})/(1-\rho+e^{d2})/e^{d3})^2)$$

$$\partial g_{3f}(\ldots)/\partial d_3=g_{3f}(\ldots)*(-\theta_1((b+e^{d1}+e^{d2})/e^{d3})/(\rho+(b+e^{d1}+e^{d2})/e^{d3})-\theta_2((e^{d1}+e^{d2})/e^{d3})/(\rho+(e^{d1}+e^{d2})/e^{d3})-\theta_3(e^{d2}/e^{d3})/(\rho+e^{d2}/e^{d3}))$$
$$\partial g_{3s}(\ldots)/\partial d_3=g_{3s}(\ldots)*(-\theta_1((b+e^{d1}+e^{d2})/e^{d3})/(1-\rho+(b+e^{d1}+e^{d2})/e^{d3})-\theta_2((e^{d1}+e^{d2})/e^{d3})/(1-\rho+(e^{d1}+e^{d2})/e^{d3})-\theta_3(e^{d2}/e^{d3})/(1-\rho+e^{d2}/e^{d3}))$$
$$\partial h_{3f}(\ldots)/\partial d_3=(\theta_1((b+e^{d1}+e^{d2})/e^{d3})/(\rho+(b+e^{d1}+e^{d2})/e^{d3})^2+\theta_2((e^{d1}+e^{d2})/e^{d3})/(\rho+(e^{d1}+e^{d2})/e^{d3})^2+\theta_3(e^{d2}/e^{d3})/(\rho+e^{d2}/e^{d3})^2)$$
$$\partial h_{3s}(\ldots)/\partial d_3=(-\theta_1((b+e^{d1}+e^{d2})/e^{d3})/(1-\rho+(b+e^{d1}+e^{d2})/e^{d3})^2-\theta_2((e^{d1}+e^{d2})/e^{d3})/(1-\rho+(e^{d1}+e^{d2})/e^{d3})^2-\theta_3(e^{d2}/e^{d3})/(1-\rho+e^{d2}/e^{d3})^2)$$

Similarly in the general case we have the partial derivatives according to;

$$\partial g_{bf}(\ldots)/\partial d_1=g_{bf}(\ldots)*(\theta_3(e^{d1}/b)/(1+e^{d1}/b-\rho)+\theta_4(e^{d1}/b)/(1+(e^{d1}+e^{d2})/b-\rho)+\ldots +\theta_{N-1}(e^{d1}/b)/(1+(e^{d1}+\ldots+e^{dN-3})/b-\rho)$$
$$\partial g_{bs}(\ldots)/\partial d_1=g_{bs}(\ldots)*(\theta_3(e^{d1}/b)/(\rho+e^{d1}/b) + \theta_4(e^{d1}/b)/(\rho+(e^{d1}+e^{d2})/b)+\ldots +\theta_{N-1}(e^{d1}/b)/(\rho+(e^{d1}+\ldots+e^{dN-3})/b)$$
$$\partial h_{bf}(\ldots)/\partial d_1=(\theta_3(e^{d1}/b) /(1+e^{d1}/b-\rho)^2+\theta_4(e^{d1}/b) /(1+(e^{d1}+e^{d2})/b-\rho)^2+\ldots+\theta_{N-1}(e^{d1}/b)/(1+(e^{d1}+\ldots+e^{dN-3})/b-\rho)^2$$
$$\partial h_{bs}(\ldots)/\partial d_1=(-\theta_3(e^{d1}/b)/(\rho+e^{d1}/b)^2-\theta_4(e^{d1}/b)/(\rho+(e^{d1}+e^{d2})/b)^2-\ldots-\theta_{N-1}(e^{d1}/b)/(\rho+(e^{d1}+\ldots+e^{dN-3})/b)^2$$

$$\partial g_{bf}(\ldots)/\partial d_2=g_{bf}(\ldots)*(\theta_4(e^{d2}/b)/(1+(e^{d1}+e^{d2})/b-\rho)+\ldots +\theta_{N-1}(e^{d2}/b)/(1+(e^{d1}+\ldots+e^{dN-3})/b-\rho)$$
$$\partial g_{bs}(\ldots)/\partial d_2=g_{bs}(\ldots)*(\theta_4(e^{d2}/b)/(\rho+(e^{d1}+e^{d2})/b)+\ldots +\theta_{N-1}(e^{d2}/b)/(\rho+(e^{d1}+\ldots+e^{dN-3})/b)$$
$$\partial h_{bf}(\ldots)/\partial d_2=(\theta_4(e^{d2}/b) /(1+(e^{d1}+e^{d2})/b-\rho)^2+\ldots+\theta_{N-1}(e^{d2}/b)/(1+(e^{d1}+\ldots+e^{dN-3})/b-\rho)^2$$
$$\partial h_{bs}(\ldots)/\partial d_2=(-\theta_4(e^{d2}/b)/(\rho+(e^{d1}+e^{d2})/b)^2-\ldots-\theta_{N-1}(e^{d2}/b)/(\rho+(e^{d1}+\ldots+e^{dN-3})/b)^2$$

. .
. .
. .

$$\partial g_{bf}(\ldots)/\partial d_{N-3}=g_{bf}(\ldots)*(\theta_{N-1}(e^{dN-3}/b)/(1+(e^{d1}+\ldots+e^{dN-3})/b-\rho)$$
$$\partial g_{bs}(\ldots)/\partial d_{N-3}=g_{bs}(\ldots)*(\theta_{N-1}(e^{dN-3}/b)/(\rho+(e^{d1}+\ldots+e^{dN-3})/b)$$
$$\partial h_{bf}(\ldots)/\partial d_{N-3}=(\theta_{N-1}(e^{dN-3}/b)/(1+(e^{d1}+\ldots+e^{dN-3})/b-\rho)^2$$
$$\partial h_{bs}(\ldots)/\partial d_{N-3}=(-\theta_{N-1}(e^{dN-3}/b)/(\rho+(e^{d1}+\ldots+e^{dN-3})/b)^2$$

if m<k

$$\partial g_{kf}(\ldots)/\partial d_m= g_{kf}(\ldots)*(\theta_1(e^{dm}/e^{dk})/(\rho+(b+e^{d1}+\ldots+ e^{dk-1})/e^{dk})+\ldots+\theta_{m+1}(e^{dm}/e^{dk})/(\rho+(e^{dm}+\ldots+ e^{dk-1})/e^{dk}))$$
$$\partial g_{ks}(\ldots)/\partial d_m= g_{ks}(\ldots)*(\theta_1(e^{dm}/e^{dk})/(1+(b+e^{d1}+\ldots+ e^{dk-1})/e^{dk}-\rho)+\ldots+\theta_{m+1}(e^{dm}/e^{dk})/(1+(e^{dm}+\ldots+ e^{dk-1})/e^{dk}-\rho)$$
$$\partial h_{kf}(\ldots)/\partial d_m= (-\theta_1(e^{dm}/e^{dk})/(\rho+(b+e^{d1}+\ldots+ e^{dk-1})/e^{dk})^2-\ldots-\theta_{m+1}(e^{dm}/e^{dk})/(\rho+(e^{dm}+\ldots+ e^{dk-1})/e^{dk})^2)$$
$$\partial h_{ks}(\ldots)/\partial d_m= (\theta_1(e^{dm}/e^{dk})/(1+(b+e^{d1}+\ldots+ e^{dk-1})/e^{dk}-\rho)^2+\ldots+\theta_{m+1}(e^{dm}/e^{dk})/(1+(e^{dm}+\ldots+ e^{dk-1})/e^{dk}-\rho)^2)$$

if m=k

$$\partial g_{kf}(\ldots)/\partial d_k=g_{kf}(\ldots)*(-\theta_1((b+e^{d1}+\ldots+e^{dk-1})/e^{dk})/(\rho+(b+e^{d1}+\ldots+e^{dk-1})/e^{dk})-\ldots-\theta_k(e^{dk-1}/e^{dk})/(\rho+e^{dk-1}/e^{dk})-$$
$$-\theta_{k+3}(e^{dk+1}/e^{dk})/(1+e^{dk+1}/e^{dk}-\rho)-\ldots-\theta_{N-1}((e^{dk+1}+\ldots+e^{dN-3})/e^{dk})/(1+(e^{dk+1}+\ldots+e^{dN-3})/e^{dk}-\rho))$$
$$\partial g_{ks}(\ldots)/\partial d_k=g_{ks}(\ldots)*(-\theta_1((b+e^{d1}+\ldots+e^{dk-1})/e^{dk})/(1+(b+e^{d1}+\ldots+e^{dk-1})/e^{dk}-\rho)-\ldots-\theta_k(e^{dk-1}/e^{dk})/(1+e^{dk-1}/e^{dk}-\rho)-$$
$$-\theta_{k+3}(e^{dk+1}/e^{dk})/(\rho+e^{dk+1}/e^{dk})-\ldots-\theta_{N-1}((e^{dk+1}+\ldots+e^{dN-3})/e^{dk})/(\rho+(e^{dk+1}+\ldots+e^{dN-3})/e^{dk}))$$
$$\partial h_{kf}(\ldots)/\partial d_k=(\theta_1((b+e^{d1}+\ldots+e^{dk-1})/e^{dk})/(\rho+(b+e^{d1}+\ldots+e^{dk-1})/e^{dk})^2+\ldots+\theta_k(e^{dk-1}/e^{dk})/(\rho+e^{dk-1}/e^{dk})^2-$$
$$-\theta_{k+3}(e^{dk+1}/e^{dk})/(1+e^{dk+1}/e^{dk}-\rho)^2-\ldots-\theta_{N-1}((e^{dk+1}+\ldots+e^{dN-3})/e^{dk})/(1+(e^{dk+1}+\ldots+e^{dN-3})/e^{dk}-\rho)^2)$$
$$\partial h_{ks}(\ldots)/\partial d_k=(-\theta_1((b+e^{d1}+\ldots+e^{dk-1})/e^{dk})/(1+(b+e^{d1}+\ldots+e^{dk-1})/e^{dk}-\rho)^2-\ldots-\theta_k(e^{dk-1}/e^{dk})/(1+e^{dk-1}/e^{dk}-\rho)^2+$$
$$+\theta_{k+3}(e^{dk+1}/e^{dk})/(\rho+e^{dk+1}/e^{dk})^2+\ldots+\theta_{N-1}((e^{dk+1}+\ldots+e^{dN-3})/e^{dk})/(\rho+(e^{dk+1}+\ldots+e^{dN-3})/e^{dk})^2)$$

if m>k

$$\partial g_{kf}(\ldots)/\partial d_m=g_{kf}(\ldots)*(\theta_{m+2}(e^{dm}/e^{dk})/(1+(e^{dk+1}+\ldots+e^{dm})/e^{dk}-\rho)+\ldots+\theta_{N-1}(e^{dm})/e^{dk})/(1+(e^{dk+1}+\ldots+e^{dN-3})/e^{dk}-\rho))$$
$$\partial g_{ks}(\ldots)/\partial d_m=g_{ks}(\ldots)*(\theta_{m+2}(e^{dm}/e^{dk})/(\rho+(e^{dk+1}+\ldots+e^{dm})/e^{dk})+\ldots+\theta_{N-1}(e^{dm})/e^{dk})/(\rho+(e^{dk+1}+\ldots+e^{dN-3})/e^{dk}))$$
$$\partial h_{kf}(\ldots)/\partial d_m=(\theta_{m+2}(e^{dm}/e^{dk})/(1+(e^{dk+1}+\ldots+e^{dm})/e^{dk}-\rho)^2+\ldots+\theta_{N-1}(e^{dm}/e^{dk})/(1+(e^{dk+1}+\ldots+e^{dN-3})/e^{dk}-\rho)^2)$$
$$\partial h_{ks}(\ldots)/\partial d_m=(-\theta_{m+2}(e^{dm}/e^{dk})/(\rho+(e^{dk+1}+\ldots+e^{dm})/e^{dk})^2-\ldots-\theta_{N-1}(e^{dm}/e^{dk})/(\rho+(e^{dk+1}+\ldots+e^{dN-3})/e^{dk})^2)$$

(c)

where e^{d0} is considered equal to b. Now if we consider the equation;

$$l_1e^{Ddk}*([\rho^{1+\theta k+1}g_{kf}()/(1+\theta_{k+1})+\rho^{1+\theta k+2}g_{ks}()/(1+\theta_{k+2})] \,|\, -\int[\rho^{1+\theta k+1}g_{kf}()*h_{kf}()/(1+\theta_{k+1})+\rho^{1+\theta k+2}g_{ks}()*h_{ks}()/(1+\theta_{k+2})]d\rho) -$$
$$- l_{k+1}b^D*([\rho^{1+\theta 1}g_{bf}()/(1+\theta_1)+\rho^{1+\theta 2}g_{bs}()/(1+\theta_2)] \,|\, + \int [\rho^{1+\theta 1}g_{bf}()*h_{bf}()/(1+\theta_1)+\rho^{1+\theta 2}g_{bs}()*h_{bs}()/(1+\theta_2)]\, d\rho) = 0$$

which is the k:th row in the general equation system (1), here it doesn't give any more clarity to study the six corner case. Now if we in this equation take the derivative of this system with respect to d_m where $m \neq k$ we notice that the first term (upper line) obviously becomes;

$$l_1e^{Ddk}*([\rho^{1+\theta k+1}\partial g_{kf}(\ldots)/\partial d_m/(1+\theta_{k+1})+\rho^{1+\theta k+2}\partial g_{ks}(\ldots)/\partial d_m/(1+\theta_{k+2})] \,|$$
$$-\int[\rho^{1+\theta k+1}(\partial g_{kf}(\ldots)/\partial d_m*h_{kf}(\ldots)+g_{kf}(\ldots)*\partial h_{kf}(\ldots)/\partial d_m)/(1+\theta_{k+1})+$$
$$+\rho^{1+\theta k+2}(\partial g_{ks}(\ldots)/\partial d_m*h_{ks}(\ldots)+g_{ks}(\ldots)*\partial h_{ks}(\ldots)/\partial d_m)/(1+\theta_{k+2})]d\rho)$$

The second term (lower line) becomes;

$$l_{k+1}b^D*([\rho^{1+\theta 1}\partial g_{bf}(\ldots)/\partial d_m/(1+\theta_1)+\rho^{1+\theta 2}\partial g_{bs}(\ldots)/\partial d_m/(1+\theta_2)] \,|$$
$$+\int[\rho^{1+\theta 1}(\partial g_{bf}(\ldots)/\partial d_m*h_{bf}(\ldots)+g_{bf}(\ldots)*\partial h_{bf}(\ldots)/\partial d_m)/(1+\theta_1)+$$
$$+\rho^{1+\theta 2}(\partial g_{bs}(\ldots)/\partial d_m*h_{bs}(\ldots)+g_{bs}(\ldots)*\partial h_{bs}(\ldots)/\partial d_m)/(1+\theta_2)]\, d\rho)$$

Now if we consider the case when k = m; the second term must become the same as in the case $m \neq k$. However the first term has an d_k outside the parentheses so we must get an extra term to add;

$$Dl_1e^{Ddk}*([\rho^{1+\theta k+1}g_{kf}()/(1+\theta_{k+1})+\rho^{1+\theta k+2}g_{ks}()/(1+\theta_{k+2})] \,|\, -\int[\rho^{1+\theta k+1}g_{kf}()*h_{kf}()/(1+\theta_{k+1})+\rho^{1+\theta k+2}g_{ks}()*h_{ks}()/(1+\theta_{k+2})]d\rho+$$

$$+l_1e^{Ddk}*([\rho^{1+\theta k+1}\partial g_{kf}()/\partial d_m/(1+\theta_{k+1})+\rho^{1+\theta k+2}\partial g_{ks}()/\partial d_m/(1+\theta_{k+2})] \,|\, -$$

$$-\int[\rho^{1+\theta k+1}(\partial g_{kf}()/\partial d_m*h_{kf}()+g_{kf}()*\partial h_{kf}()/\partial d_m)/(1+\theta_{k+1})+\rho^{1+\theta k+2}(\partial g_{ks}()/\partial d_m*h_{ks}()+g_{ks}()*\partial h_{ks}()/\partial d_m)/(1+\theta_{k+2})]d\rho)$$

where the second and third line is exactly what the first term becomes in the case $m \neq k$ and the first line is exactly the first term in the equation system multiplied by D.

So each element of the jacobian **J** for row k and column m becomes;

$$
\begin{aligned}
\mathbf{J}_{km} = {} & l_1 e^{Ddk} * ([\rho^{1+\theta k+1}(E g_{kf}(\ldots)+\partial g_{kf}(\ldots)/\partial d_m)/(1+\theta_{k+1})+\rho^{1+\theta k+2}(E g_{ks}(\ldots)+\partial g_{ks}(\ldots)/\partial d_m)/(1+\theta_{k+2})] \Big| - \\
& -\int[\rho^{1+\theta k+1}(g_{kf}(\ldots)*h_{kf}(\ldots)+\partial g_{kf}(\ldots)/\partial d_m*h_{kf}(\ldots)+g_{kf}(\ldots)*\partial h_{kf}(\ldots)/\partial d_m)/(1+\theta_{k+1})+ \\
& +\rho^{1+\theta k+2}(g_{ks}(\ldots)*h_{ks}(\ldots)+\partial g_{ks}(\ldots)/\partial d_m*h_{ks}(\ldots)+g_{ks}(\ldots)*\partial h_{ks}(\ldots)/\partial d_m)/(1+\theta_{k+2})]d\rho)- \\
& - l_{k+1} b^D*([\rho^{1+\theta 1}\partial g_{bf}(\ldots)/\partial d_m/(1+\theta_1)+\rho^{1+\theta 2}\partial g_{bs}(\ldots)/\partial d_m/(1+\theta_2)] \Big| - \\
& -\int[\rho^{1+\theta 1}(\partial g_{bf}(\ldots)/\partial d_m*h_{bf}(\ldots)+g_{bf}(\ldots)*\partial h_{bf}(\ldots)/\partial d_m)/(1+\theta_1)+ \\
& +\rho^{1+\theta 2}(\partial g_{bs}(\ldots)/\partial d_m*h_{bs}(\ldots)+g_{bs}(\ldots)*\partial h_{bs}(\ldots)/\partial d_m)/(1+\theta_2)]\, d\rho)
\end{aligned}
\tag{2}
$$

where E = D if k = m and E = 0 otherwise. Now we are done with our analytic work. All these equations might seem quite overwhelming for a novice programmer but it's not hard to implement the system of equations in (1) and the jacobian (2) if we first have the equations (a), (b) and (c). And in the next section we shall see that the equations (a), (b) and (c) actually can be created through loops. Also if we can calculate (1) and (2) then this is all we need to find the roots of the original equation system. So we are now ready to start our work on the numerical implementation of these equations.

Implementing a program

Now we want to implement what we learned from our analytic preparation, into a program. We will not go into detail since our aim is only to offer a description of the main steps in the program. So let's without writing any code in any language just consider the major issues that will arise. In some instances we will consider briefly how a syntax could be created. But we won't move into dealing in any language.

Estimating an integral

Since by the Riemann definition a real one variable integral is the area between the function line and the coordinate axis of the variable, it comes quite naturally that we should try to utilize this fact to estimate an integral. All integrals we have to deal with will certainly meet this criterion. Since all integrals where taken between 0 and 0.5 we need to make a partitioning of this closed interval. Let's call this partitioning **x** and for sake of argument give it a constant step size of 0.1. In reality this step size will be considerably smaller, usually 0.0001. Then **x** is the vector [0.1 0.2 0.3 0.4 0.5].

Now if we consider the equations of (1) and (2) the estimates of the $\rho^{\theta k}$ factors at these points will all be term wise powers of the vector **x**. The θ's are all fixed and all the d's will have constant guess values each time we want to estimate an integral. Furthermore if we have two bounded functions g(x) and h(x) the estimate at a point x_0 should be the same no matter if we choose to multiply the functions and then setting $x = x_0$ or if we first set this value into the two functions and then multiply the two estimates. So now we see that if we have means to get estimates at the points of our partitioning of the functions (a), (b) and (c) then by simple term wise utilization of our four basic operators we will have estimates of everything under the integral sign in both (1) and (2).

Now let's for one second assume that we can estimate the equations of (a) (b) and (c) at all points of this partitioning. Then we simply estimate all our functions under each integral sign at each point of our partitioning and put them together according to (1) and (2). Now we can estimate the area between two successive points in our partitioning by the midpoint rule, I.e. we take the estimate at both points divide it by two and multiply with the difference between the points. Summing all these areas will finally give us the estimates of the integrals.

Now this is not quite enough though, remember we partially integrated out a part to get rid of the singularities. However now we will actually get this value for free. Now this term will include $\partial g_{kf}(\ldots)/\partial d_m$, $\partial g_{ks}(\ldots)/\partial d_m$, some powers of ρ and on the main diagonal $g_{kf}(\ldots)$ and $g_{ks}(\ldots)$. But we only want the estimates at the endpoints, that is at 0 and 0.5 (actually only 0.5 since the power of ρ will be 0 at the point 0). So now we simply take the first and last values of our vectors that are the estimates of (a) and (c) at the points of our partitioning. Now we can build our entire jacobian matrix with the four basic operators.

Of course we still don't know if we can get the estimates of the three functions. We'll get to that in one second but let's now just realize that if we can get these estimates we do have the estimates of all equations in (1) and the matrix (2). Notice that now these are functions of d-variables only, which we have as quess values and are seeking to improve. Now (1) will become a vector of numbers and (2) will become a matrix of numbers. So if we can find the inverse of the jacobian and multiply on the right side by the result vector of (1) we only need to create a loop that keeps running until we are satisfied or the loop runs out of bound. So now all that remains to be done is to find the estimates of (a), (b) and (c) at the points of our partitioning.

Estimating (a), (b) and (c)

We return to six corner case because it is instructive. Let's rewrite the equivalents of (a) and (b), but let's use a different grouping;

$$g_{bf}(\ldots) = (1-\rho)^{\theta 2}\,(1-\rho+e^{d1}/b)^{\theta 3}\,(1-\rho+(e^{d1}+e^{d2})/b)^{\theta 4}\,(1-\rho+(e^{d1}+e^{d2}+e^{d3})/b)^{\theta 5}$$
$$g_{bs}(\ldots) = (1-\rho)^{\theta 1}\,(\rho+e^{d1}/b)^{\theta 3}\,(\rho+(e^{d1}+e^{d2})/b)^{\theta 4}\,(\rho+(e^{d1}+e^{d2}+e^{d3})/b)^{\theta 5}$$
$$h_{bf}(\ldots) = (-\theta_2/(1-\rho)-\theta_3/(1-\rho+e^{d1}/b)-\theta_4/(1-\rho+(e^{d1}+e^{d2}/b)-\theta_5/(1-\rho+(e^{d1}+e^{d2}+e^{d3})/b))$$
$$h_{bs}(\ldots) = (-\theta_1/(1-\rho)+\theta_3/(\rho+e^{d1}/b)+\theta_4/(\rho+(e^{d1}+e^{d2})/b)+\theta_5/(\rho+(e^{d1}+e^{d2}+e^{d3})/b))$$

$$g_{1f}(\ldots) = (\rho+b/e^{d1})^{\theta 1}\,(1-\rho)^{\theta 3}\,(1-\rho+e^{d2}/e^{d1})^{\theta 4}\,(1-\rho+(e^{d2}+e^{d3})/e^{d1})^{\theta 5}$$
$$g_{1s}(\ldots) = (1-\rho+b/e^{d1})^{\theta 1}\,(1-\rho)^{\theta 2}\,(\rho+e^{d2}/e^{d1})^{\theta 4}\,(\rho+(e^{d2}+e^{d3})/e^{d1})^{\theta 5}$$
$$h_{1f}(\ldots) = (\theta_1/(\rho+b/e^{d1})-\theta_3/(1-\rho)-\theta_4/(1-\rho+e^{d2}/e^{d1})-\theta_5/(1-\rho+(e^{d2}+e^{d3})/e^{d1}))$$
$$h_{1s}(\ldots) = (-\theta_1/(1-\rho+b/e^{d1})-\theta_2/(1-\rho)+\theta_4/(\rho+e^{d2}/e^{d1})-\theta_5/(\rho+(e^{d2}+e^{d3})/e^{d1}))$$

$$g_{2f}(\ldots) = (\rho+(b+e^{d1})/e^{d2})^{\theta 1}\,(\rho+e^{d1}/e^{d2})^{\theta 2}\,(1-\rho)^{\theta 4}\,(1-\rho+e^{d3}/e^{d2})^{\theta 5}$$
$$g_{2s}(\ldots) = (1-\rho+(b+e^{d1})/e^{d2})^{\theta 1}\,(1-\rho+e^{d1}/e^{d2})^{\theta 2}\,(1-\rho)^{\theta 3}\,(\rho+e^{d3}/e^{d2})^{\theta 5}$$
$$h_{2f}(\ldots) = (\theta_1/(\rho+(b+e^{d1})/e^{d2})+\theta_2/(\rho+e^{d1}/e^{d2})-\theta_4/(1-\rho)-\theta_5/(1-\rho+e^{d3}/e^{d2}))$$
$$h_{2s}(\ldots) = (-\theta_1/(1-\rho+(b+e^{d1})/e^{d2})-\theta_2/(1-\rho+e^{d1}/e^{d2})-\theta_3/(1-\rho)+\theta_5/(\rho+e^{d3}/e^{d2})\ )$$

$$g_{3f}(\ldots) = (\rho+(b+e^{d1}+e^{d2})/e^{d3})^{\theta 1}\,(\rho+(e^{d1}+e^{d2})/e^{d3})^{\theta 2}\,(\rho+e^{d2}/e^{d3})^{\theta 3}\,(1-\rho)^{\theta 5}$$
$$g_{3s}(\ldots) = (1-\rho+(b+e^{d1}+e^{d2})/e^{d3})^{\theta 1}\,(1-\rho+(e^{d1}+e^{d2})/e^{d3})^{\theta 2}\,(1-\rho+e^{d2}/e^{d3})^{\theta 3}\,(1-\rho)^{\theta 4}$$
$$h_{3f}(\ldots) = (\theta_1/(\rho+(b+e^{d1}+e^{d2})/e^{d3})+\theta_2/(\rho+(e^{d1}+e^{d2})/e^{dN-3})+\theta_3/(\rho+e^{d2}/e^{d3})-\theta_5/(1-\rho))$$
$$h_{3s}(\ldots) = (-\theta_1/(1-\rho+(b+e^{d1}+e^{d2)}/e^{d3})-\theta_2/(1-\rho+(e^{d1}+e^{d2})/e^{d3})-\theta_3/(1-\rho+e^{d2}/e^{d3})-\theta_4/(1-\rho))$$

Here some of the subscripts will make more sense. First, notice that the 1-ρ term/factor moves down on the diagonal above. Thus in the first group it has powers/nominators of the first two arguments, in the second group of the second and the third arguments and so on. Furthermore these arguments never appear anywhere elsewhere in the groups. Secondly, notice that in each equation of the same group inside the parentheses of the same position, we have something plus a constant term. For example, second position of the first group 1-ρ or ρ is added to e^{d1}/b. Furthermore all parentheses in the same position and same group share the same argument that never elsewhere comes up in the particular group.

Now if we consider the first group we see an increasing term. Namely first position nothing, second position e^{d1}/b, third position $(e^{d1}+e^{d2})/b$ and fourth position $(e^{d1}+e^{d2}+e^{d3})/b$. But this doesn't seem to be right in the second grouping. Here we start with b/e^{d1} and then e^{d2}/e^{d1} followed by $(e^{d2}+e^{d3})/e^{d1}$. But here we see that after the 1-ρ term we will have an increasing term. If we look at the last group we see that we get $(b+e^{d1}+e^{d2})/e^{d3}$, $(e^{d1}+e^{d2})/e^{d3}$, e^{d2}/e^{d3}. Here it seems that we drop a term with each step, indeed the third and second groups seem to agree with this observation. Furthermore we see that the nominators before the 1-ρ term are $b+d_1+d_2+\ \ldots\ +$ up to one less then the subscript of the equations group we are considering. The denominator is of course the d with the same subscript as the equation group.

Finally we should notice that in the equations with subscript f the factors/terms before the 1-ρ factor/term contain the decreasing terms we just discussed and the term ρ. The factors after the 1-ρ factor/term contain the increasing terms and one 1-ρ term. While for the equations with subscript s this fact is reversed.

If we consider the general case in (a) and (b) we notice that these observations actually do hold. So we notice that we have regularity in these equations. Furthermore the equations in (c) are just the partial derivatives of the above groups with respect to each d, so we state that similar rules will apply to these.

This discussion can seem a bit redundant since we probably did all the same realizations while we first counted the general equations in (a) and (b). However in this compact view we get an idea how to implement these equations. If we first consider the group with subscript b, it is a decent idea to create a vector that contains the terms e^{d1}/b, $(e^{d1}+e^{d2})/b$ and $(e^{d1}+e^{d2}+e^{d3})/b$. This is easily achieved since all we have to do is a loop with three iterations that in each loop counts e^{dk}/b with k as the same number as the iteration we are running and then add what we had in the previous position to this value. In general, if we have a **P** with N corners then we have a loop of N-3 iterations and we get the correct term-vector.

Before we move on using the vector we just created let us consider how we could deal with creating similar vectors for the other groups above. I.e. consider the group with subscript 3. Here we get the opposite problem we got to remove one term in each point of the vector. This could be accomplished but would be unnecessary since we can create this vector backwards i.e. we calculate e^{dk}/e^{d3} where k is three minus the number of the iteration. Here we only loop twice and once this is done, we create the last term by adding b/e^{d3} to the second term to get the third. Now we could reverse it to get the correct vector but why bother, this only exists as a temporary step in the computer and we can just as well work with this. Also this can easily be expanded to an arbitrary problem since for an N corner problem we loop N-4 times and we start with e^{dN-4}/e^{dN-3}.

How about the other groups? Well we might want to use something in between. It might be tempting to have two vectors. If we have the group subscript as k the decreasing vector must by our discussion on the general case just discussed have d_k in the denominator, it must start with the e^{dk-1}/e^{dk} term and obviously if we loop k-1 times we will have what we wanted. This will work for any case in the general case. The growing vector is then created by starting with the e^{dk+1}/e^{dk} term and then loop from k+1 to N-3. Also this applies to the general case. So now we have vectors containing the terms that will appear in our equations. Now what do we do with it?

Well we start by reviewing the first group, i.e. the one with subscript b. We see that by the vector we created, we can make a loop that creates the estimates of this group at the points of the partitioning **x**. First we define four vectors as the term wise estimates of the factors/term only containing 1-ρ and an argument. Then we loop this N-3 times. 1-ρ or ρ is added to the element in the previously created term-vector that is contained at the same position as the number of the loop. The argument we want to use is two plus the number of the loop. Then we simply term wise multiply/add these to the estimate vectors. And hence we got the estimates for (a) and (b) for the equation group with subscript b. Note this will solve the general case and (c) is done in a similar fashion.

Now we could solve a fixed problem say the six corner case since we can solve each group individually in a similar fashion as above. Remember, having two vectors one with the terms before the 1-ρ and one with the terms after, we can first work from the first vector and then by the other. However we want to create a code that will solve the general case. But then we know that there are N-3 groups to be created so we create a loop that corresponds to each one of the groups. And it’s now a simple matter of keeping your indices right. The increasing and decreasing vectors have elements from k+1 to N-3 and 1 to k-1 respectively.

Some examples

One little extra feature

Before we start to utilize the program in the computer it would be nice to have some way to see if we are even close to the correct result. Mathematically this seems quite a task. But as we have the breakpoints we can implement these straight forward in the equation and numerically solve the corners and plot these out.

We have also quite many singularities to deal with here, but we can actually evade these. Namely by taking the integral steps up in the upper plane instead on the real line we only need to concern our self with the ones at 0 and the particular corner we are considering at the time. Thus we first partially integrate the integral with respect to the singular factor at 0. Now this equation is well defined here and we estimate the integral along the imaginary axis to some arbitrary point say from 0+0i to 0+100i. Now up here along a line parallel to the real axis the original equation must be well defined so we estimate the integral from 0+100i to x_k+100i, where x_k is the current breakpoint we consider. Then we partially integrate the equation with respect to the (ξ-x_k) factor and then count this integral from x_k+100i to x_k+0i. Finally adding these three integrals together we get an approximation of the point in the complex plane to which the point x_k is mapped.

One must notice though that this is not a formal proof but only an estimate. It's just a "self- check" to see that one has not made any mistakes in writing the code. We will also be using this to illustrate our problems. In all the following pictures the **x**'s mark the actual hit points and the lines are drawn between points that are estimated by this method.

The convex regular case

Now as mentioned above for a convex case all θ's must be less then 0 and of course for a regular **P** they must be equal. Let's start by considering a five corner case. That is **P** has all θ = -2/5 and all l = 1. As mentioned this will be mapped from two fixed points 0, 100 and two more points that converge according to;

114.3942	129.2384
143.2214	197.9051
159.3032	250.7233
161.7583	261.5371
161.8034	261.8033
161.8034	261.8034
161.8034	261.8034

A = 2.7406 - 8.4348i

We notice that difference between the first two points equals the distance between the two last, this is a motif that will be repeating it self throughout the convex regular cases. We also notice that the convergence is pretty fast.

We next consider the twenty corner case, which after seven loops converges to;

	0	100.0000	134.4577	152.5731	164.2040	172.6543
179.3604	185.0651	190.2113	195.1056	200.0000	205.1462	210.8509
217.5570	226.0073	237.6382	255.7536	290.2113	390.2113	

A = 91.3980 -29.6970i

The fifty corner case converges after seven loops to;

0	100.0000	133.5095	150.3974	160.6379	167.5559	172.5768
176.4134	179.4621	181.9608	184.0613	185.8648	187.4419	188.8430
190.1055	191.2576	192.3212	193.3137	194.2492	195.1391	195.9935
196.8206	197.6281	198.4229	199.2115	200.0000	200.7948	201.6023
202.4295	203.2838	204.1738	205.1092	206.1017	207.1654	208.3175
209.5799	210.9810	212.5581	214.3617	216.4621	218.9608	222.0095
225.8461	230.8670	237.7851	248.0255	264.9134	298.4229	398.4229

A = 148.6727 -18.7817i

The fifty corner cases figure will be included as figure 1 at the end of this text. As will the spread of the twenty and fifty corner cases breakpoints as figures 2 and 3. The middle points of these lists are underlined. If we work backwards from the end and forward from the beginning until this point we notice that the steps are of equal size at both ends.

Actually all of our basic breakpoint finding calculations is made by the same Matlab program. The only change made between all examples is that we define the number of corners, a vector with the arguments and a vector with the side lengths. This is not completely true but we will get to the two other changes made as we come to an appropriate point. Of which this is the first. If we try making a regular convex polygon with more corners we will be able to get to something between 60 and 75. Then Matlab will complain about a singular matrix. However this is due to that the points in the middle will be squeezed together. Notice already in the 50-corner case above the difference is less then 0.8. Now this is why we wanted an easily changeable b. If we change this value to 1000, we will be able to count over 100-corner cases. The 100-corner case will take about 15-30 minutes on a well managed average PC. Drawing the figure will however take half the night.

Note! *That in these specific examples the A's are quite irrelevant since these just guard the size and rotation. Furthermore if one wants the breakpoints for the fixed points 0 and 1 instead for 0 and 100, one could just divide all the figures by 100. If one wants the "midpoint" to be mapped from the origin one just needs to subtract this value from all in the list. This will also be valid for the other cases not just the regular convex ones.*

Non convex and non regular five corner case

Now let's consider a square with side length 1 and then step by step perturb the middle point of the upper side by pulling it down. See figure 4 at the end for reference. Now $\theta_1 = -2/4$ and $l_1 = 1$ always. But the remaining two angles and side lengths obviously differs from case to case. For this we had to create a simple program that counts the angles and lengths from the corners. This way it's a simple matter of lowering the middle point in each of the 18 runs from c_3 = - 0.5 + 0.95i to c_3 = - 0.5 + 0.10i. However this loop is so simple that it will be omitted.

A	x_3	x_4	c_3
- 0.3879i	1.4574	2.1240	-0.5000 + 0.9500i
- 0.3960i	1.5111	2.2835	-0.5000 + 0.9000i
- 0.4060i	1.5781	2.4904	-0.5000 + 0.8500i
- 0.4186i	1.6620	2.7621	-0.5000 + 0.8000i
- 0.4343i	1.7676	3.1244	-0.5000 + 0.7500i
- 0.4543i	1.9017	3.6166	-0.5000 + 0.7000i
- 0.4797i	2.0740	4.3013	-0.5000 + 0.6500i
- 0.5127i	2.2983	5.2820	-0.5000 + 0.6000i
- 0.5561i	2.5959	6.7385	-0.5000 + 0.5500i
- 0.6147i	3.0001	9.0003	-0.5000 + 0.5000i
- 0.6961i	3.5656	12.7136	-0.5000 + 0.4500i
- 0.8133i	4.3876	19.2503	-0.5000 + 0.4000i
- 0.9903i	5.6423	31.8340	-0.5000 + 0.3500i
- 1.2753i	7.6866	59.0795	-0.5000 + 0.3000i
- 1.7763i	11.3288	128.3294	-0.5000 + 0.2500i
- 2.7764i	18.7133	350.1083	-0.5000 + 0.2000i
- 5.2155i	37.0465	1371.6896	-0.5000 + 0.1500i
- 13.8177i	103.0724	10603.2503	-0.5000 + 0.1000i

NOTE! In this example we used x_0 = 0 and x_1= 1.

One might ask the question why we didn't take one more step i.e. c_3 = - 0.5 + 0.05i, well we did see figure 5. We see that the program might have failed. Now we get to the other change one would want to make. The step size in the integral should be easily accessible. Changing this to something smaller might make big changes in the results. Now this is why we chose the smaller first breakpoint for this example since the amount of steps grow so fast and with this there is at least some hope that the partitioning vector won't be too big. However in this case now matter what we do, it will either run out of bound or it will converge at 631.2031 and 397953.4697. Now remember that the drawing feature was only an estimate as well. And it seems that this is actually a part of what went wrong. Even if we in the drawing feature took steps of size 1 there would still be almost 400000 steps. Trying something smaller will cause the program to fail due to insufficient memory. Now one could of course make an attempt with a weight partitioning.

Some stars

Now we consider a five edged star with θ values of 2/5 and -4/5 altering, See figure 6. This star will have the following breakpoints after 9 loops;

0 100.0000 138.1965 161.8028 180.9007 199.9987 223.6050 261.8015 361.8015

A = 166.2938+511.7997i

Next we consider the ten edged star that will have 20 corners, and use θ values of 2/10 and -4/10 altering. See figure 7. This star will have the argument -2/10 as the sum of arguments in two consecutive corners, the same as the regular convex 20 cornered case above. The side lengths are obviously the same. We will get;

	0	100.0000	134.4577	152.5731	164.2040	172.6543
179.3604	185.0651	190.2113	195.1056	200.0000	205.1462	210.8509
217.5570	226.0073	237.6382	255.7536	290.2113	390.2113	

A = 377.5230+274.2865i

Now let's review the breakpoints from regular convex case;

	0	100.0000	134.4577	152.5731	164.2040	172.6543
179.3604	185.0651	190.2113	195.1056	200.0000	205.1462	210.8509
217.5570	226.0073	237.6382	255.7536	290.2113	390.2113	

They seem to be identical. However the numbers above are rounded. Actually if one takes the complete results in Matlab one gets a difference in the fifth or sixth decimal in some of the positions. However the question remains if the break points are truly identical in the cases star (non-convex) and the regular convex 20-corner. Figure 8 shows the spread of this star.

Slightly perturbed half unit disc

See figure 9. Our aim here is to demonstrate how one can use the program to approximate the mapping to an arbitrary **P**. Obviously this won't be exactly true, because there will be a computer limit for the number of corners. Furthermore above there was an example that might have gone haywire. But let's consider that the arc of the half unit disc can be approximated by a polygonal path and then perturb the base. Now as we said earlier the size and rotation can be adjusted later by manipulating the A value, so we will do as before. We map the base from -1 to 0 and then when the mapping is done we shrink the figure.

0	100.0000	169.2165	192.0817	202.5784	208.3560	211.9187
214.2900	215.9533	217.1606	218.0514	218.7043	219.1587	219.4204
219.4645	225.7922					

A = 4.2742 - 21.3713i

Now the half disc in this **P** has the diameter 17.7786 but if we divide A by this value, the diameter of the mapping will be 1.

Square divided to four equal squares

Now we will have to deal with some further theory to find the lines that cut the square in figure 10. But this will be relatively easy after the work we already done. We have the equation;

$$f(z) = 3.8138i \int \xi^{-0.5} (\xi - 100)^{-0.5} (\xi - 200)^{-0.5} d\xi \qquad \xi = [0\ z]$$

We know that this equation is going to map **u** onto a square in the complex plane. So we only need to deal with one unknown in this case. We start by finding out the points in which the cutting lines hit's the sides of the big square. We know that these must be some points on the real line. The middle point of the lower line must be mapped from some point < 0. The middle point of the right line must be mapped from some point strictly between 0 and 100. The upper midpoint, from some point between 100 and 200. The left side's midpoint, from a point > 200.

We will only consider how one finds the point that maps to the midpoint on the upper line. The rest will follow in a similar fashion. So the problem is solving the equation;

$$- 0.5 + i = 3.8138i \int \xi^{-0.5} (\xi -100)^{-0.5} (\xi -200)^{-0.5} d\xi \qquad \xi = [0 \; z_0]$$

but we know that f(100) = i. So the equation reduces to;

$$-0.5 = 3.8138i \int \xi^{-0.5} (\xi -100)^{-0.5} (\xi -200)^{-0.5} d\xi \qquad \xi = [100 \; z_0]$$

we know that z_0 will be between one and two hundred. So the two first factors will be real and the third imaginary so we divide the last factor with -1;

$$-0.5 = 3.8138 \int \xi^{-0.5} (\xi -100)^{-0.5} (200 - \xi)^{-0.5} d\xi \qquad \xi = [100 \; z_0]$$

now we make the substitution $\mu + 100 = \xi$;

$$-0.5 = 3.8138 \int (\mu + 100)^{-0.5} \mu^{-0.5} (\mu -100)^{-0.5} d\mu \quad \xi = [0 \quad z_0\text{-}100]$$

we set $c = z_0\text{-}100$ and make the variable substitution $\mu = c * \eta$ $d\mu = c * d\eta$;

$$-0.5 = 3.8138 \int (c\eta+100)^{-0.5} (c\eta)^{-0.5} (c\eta-100)^{-0.5} c \, d\eta \qquad \xi = [0 \quad 1]$$

we now have an singularity at 0, but we know from earlier how to deal with this. We partially integrate;

$$\int (c\eta+100)^{-0.5} (c\eta)^{-0.5} (c\eta-100)^{-0.5} c \, d\eta =$$

$$= c*c^{0.5}(c+100)^{-0.5}(c-100)^{-0.5}/0.5 - \int c(c\eta)^{0.5}(c\eta+100)^{-0.5}(c\eta-100)^{-0.5}(-0.5c/(c\eta+100) - 0.5c/(c\eta-100)$$

now it's a simple matter of utilising the bisection method to find the c. In this way we get the points that map to the side-midpoints of the square to be;

-141.4254 58.5785 141.4214 341.4254

now ones we got these points we start to examine the original equation again. For the vertical line in the middle of the square we know this line is $-0.5 + yi$ for $0 \leq y \leq 1$. From our above discussion we know that $-0.5 + i$ was mapped from $z_0 = 141.4214$. Hence we get the equation;

$$(1-y)i = 3.8138i \int \xi^{-0.5} (\xi -100)^{-0.5} (\xi -200)^{-0.5} d\xi \qquad \xi = [z_0 \; z]$$

from this we see that the integral must be real for all z that map to the line we are considering. I.e.;

$\mathrm{Im}\,(\int \xi^{-0.5}\,(\xi - 100)^{-0.5}\,(\xi - 200)^{-0.5}\,d\xi) = 0$

now all we need to do is considering the point $z_0 = 141.4214$. In **u** there must be at least one point at a distance of ε from z_0 that satisfies the criterion above for $0 \le \varepsilon \le 2*141.4214$. And furthermore any point satisfying this will be mapped to this line. In a similar fashion we can get the horizontal line. The resulting vectors that map to these cutting lines are presented in figure 11. We notice that these are actually semicircles or otherwise something really close to semicircles with radius $\sqrt{2*100^2}$ that get mapped to the cutting lines. The numbers 1, 2, 3 and 4 represents the areas in figure 11 that are mapped to the corresponding squares in figure 10.

This result deserves some extra attention. Now in order to see if these vectors truly are semicircles we simply take hundred equally spaced points on the "real" semicircular vectors and calculate the original equation for these points. (Now obviously we made some rewriting of this equation to get better estimates, but these steps are some of the above mentioned.) The results are shown in figure 10, as this figure is drawn with these estimates. Also for mathematical considerations we list each tenth estimate here;

0.00000000000000 + 0.49999889588398i
-0.25996854547006 + 0.49999889589972i
-0.43484184241095 + 0.49999889609869i
-0.55564883699446 + 0.49999889690374i
-0.64758577926513 + 0.49999889849722i
-0.72275585548992 + 0.49999890083786i
-0.78748080743678 + 0.49999890367671i
-0.84546513791567 + 0.49999890661026i
-0.89911814421131 + 0.49999890916668i
-0.95017006284764 + 0.49999891090789i

and;

-0.49999889675005 + 0.74002924934325i
-0.49999889833996 + 0.56515595280233i
-0.49999889913000 + 0.44434895822220i
-0.49999889896242 + 0.35241201565118i
-0.49999889786983 + 0.27724193887483i
-0.49999889608876 + 0.21251698617097i
-0.49999889400679 + 0.15453265477525i
-0.49999889207750 + 0.10087964744776i
-0.49999889072133 + 0.04982772770635i
-0.49999889023387 + 0.00000000000000i

Comparing results with other solutions

Now as an attempt to see how good our program is in getting the correct breakpoints we are going to make a möbius transform of the regular convex 50-corner case we had above. We are going to assume that this can be used to approximate a circle. We know that the three first corners of this **P** is 0, 0.9921 + 0.1253i and 1.9607 + 0.3740i as well we know that the three first points are mapped from 0, 100 and 133.5095. Inserting these values in we get;

a= -0.9921-0.1253i b=1.9607-0.3740i c=-0.3351 d=133.5095

$$zc/(z-d) = wa/(w-b)$$

this leads to;

$$z = d/(1-c(1-b/w)/a) \qquad \text{and} \qquad w = b/(1-a(1-d/z)/c)$$

using the first with all corners in the regular convex 50-corner case and then subtracting the breakpoints of the 50 - corner case we get;

1.0e-005 *

0	0	0.0037	0.0077	0.0112	0.0142	0.0167
0.0190	0.0209	0.0226	0.0242	0.0256	0.0269	0.0280
0.0291	0.0302	0.0312	0.0321	0.0331	0.0340	0.0349
0.0358	0.0366	0.0375	0.0384	0.0393	0.0403	0.0413
0.0423	0.0434	0.0446	0.0458	0.0472	0.0487	0.0504
0.0524	0.0546	0.0572	0.0603	0.0642	0.0691	0.0757
0.0848	0.0984	0.1211	0.1651	0.2805.		

Note! The five extra decimal points. Also remember that we used 100 as the first breakpoint.

Finally, let us compare our program with the Schwarz-Christoffel Matlab Toolbox, mentioned earlier in the introduction. Running this will give us the results;

-1.000000000000e+000	-4.980208739803e-001	-3.298102377429e-001
-2.450364529870e-001	-1.936314379788e-001	-1.589043129071e-001
-1.337004760233e-001	-1.144413573541e-001	-9.913765477643e-002
-8.659461305144e-002	-7.605077103113e-002	-6.699731833259e-002
-5.908080762888e-002	-5.204754175901e-002	-4.571018394810e-002
-3.992686449340e-002	-3.458763357308e-002	-2.960539773878e-002
-2.490968150282e-002	-2.044221715523e-002	-1.615374356997e-002
-1.200161724772e-002	-7.947973413962e-003	-3.958257962444e-003
-2.608165092422e-009	3.958252744457e-003	7.947968191199e-003
1.200161201780e-002	1.615373833112e-002	2.044221190605e-002
2.490967624186e-002	2.960539246427e-002	3.458762828344e-002
3.992685918817e-002	4.571017862888e-002	5.204753642978e-002
5.908080229539e-002	6.699731300130e-002	7.605076570828e-002
8.659460774263e-002	9.913764948687e-002	1.144413520895e-001
1.337004707914e-001	1.589043077204e-001	1.936314328617e-001
2.450364479929e-001	3.298102330069e-001	4.980208699724e-001
1.000000000000e+000		

Here the breakpoints are between -1 and 1. This is not what we had above. However by our earlier discussion this is easily modified. First we divide through our vector with half of the final element, thus getting the difference 2 between the first and last element. Next we simply subtract -1 and then we have our vector in the same format as in the toolbox. However the toolbox result is not in a format that we can immediately use, so we have to cut and paste. Removing the toolbox result from our result will finally obtain, (i.e my results – toolbox results);

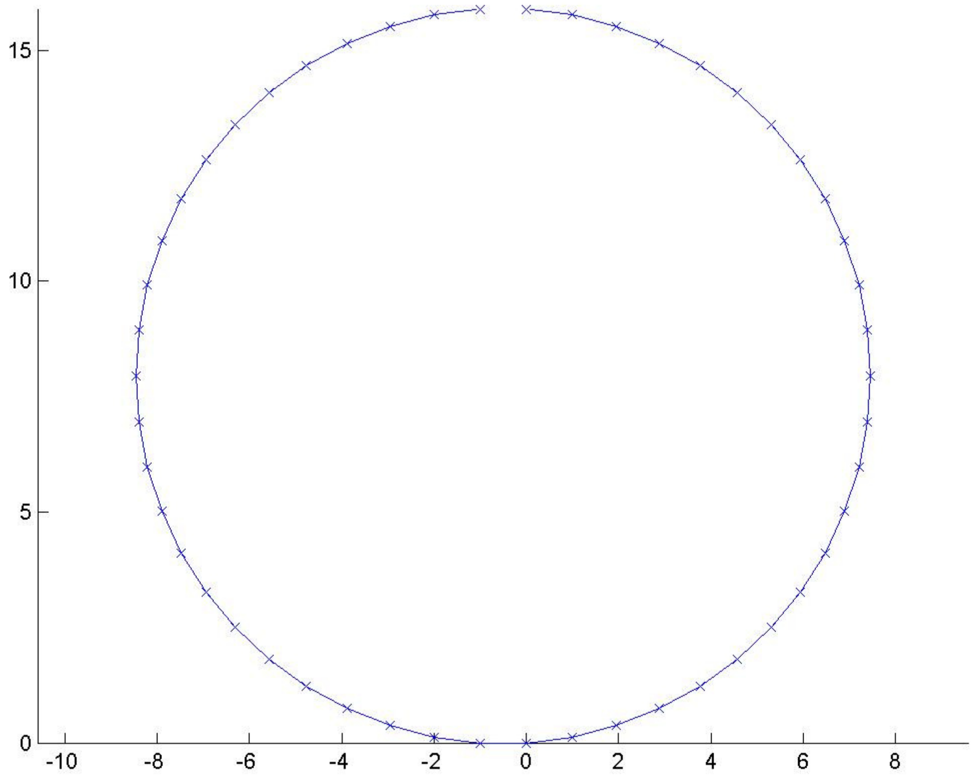
1.0e-006 *

0	0.4014	0.3156	0.2479	0.2011	0.1676	0.1425
0.1229	0.1072	0.0942	0.0832	0.0737	0.0653	0.0579
0.0512	0.0451	0.0394	0.0341	0.0291	0.0244	0.0198
0.0154	0.0111	0.0068	0.0026	-0.0016	-0.0059	-0.0102
-0.0146	-0.0191	-0.0239	-0.0288	-0.0341	-0.0398	-0.0459
-0.0526	-0.0600	-0.0683	-0.0778	-0.0888	-0.1019	-0.1176
-0.1372	-0.1624	-0.1960	-0.2429	-0.3109	-0.3974	0

Note! The six extra decimal points.

Figures from the examples

Figure 1: Regular convex 50-cornered case



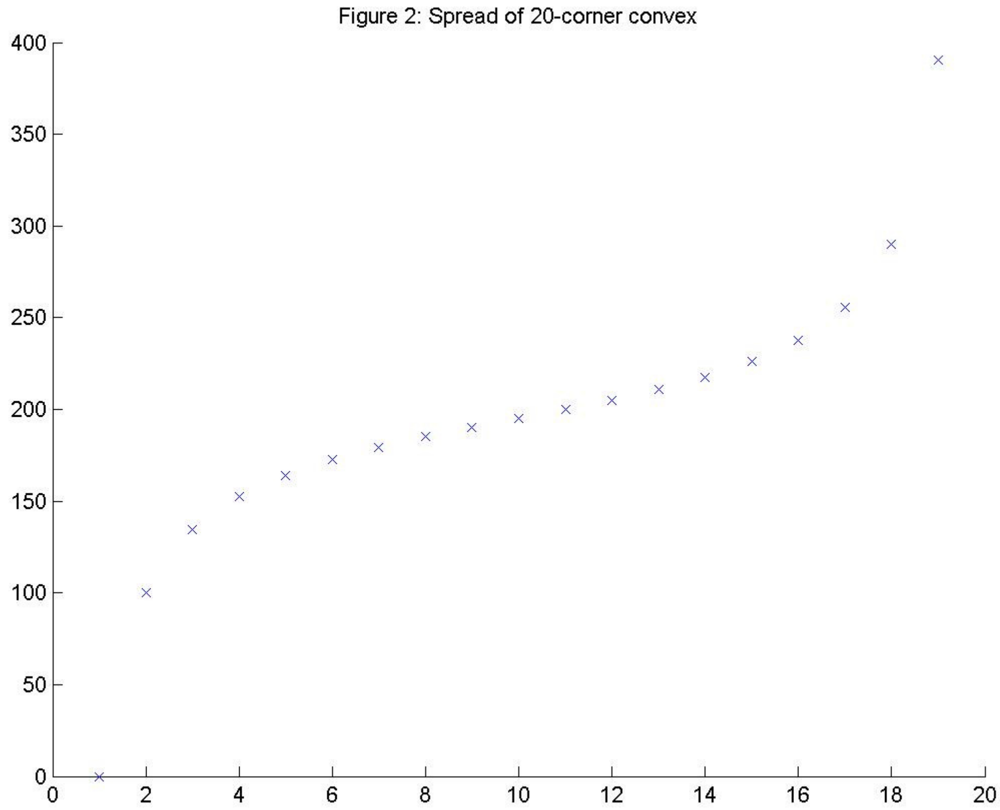
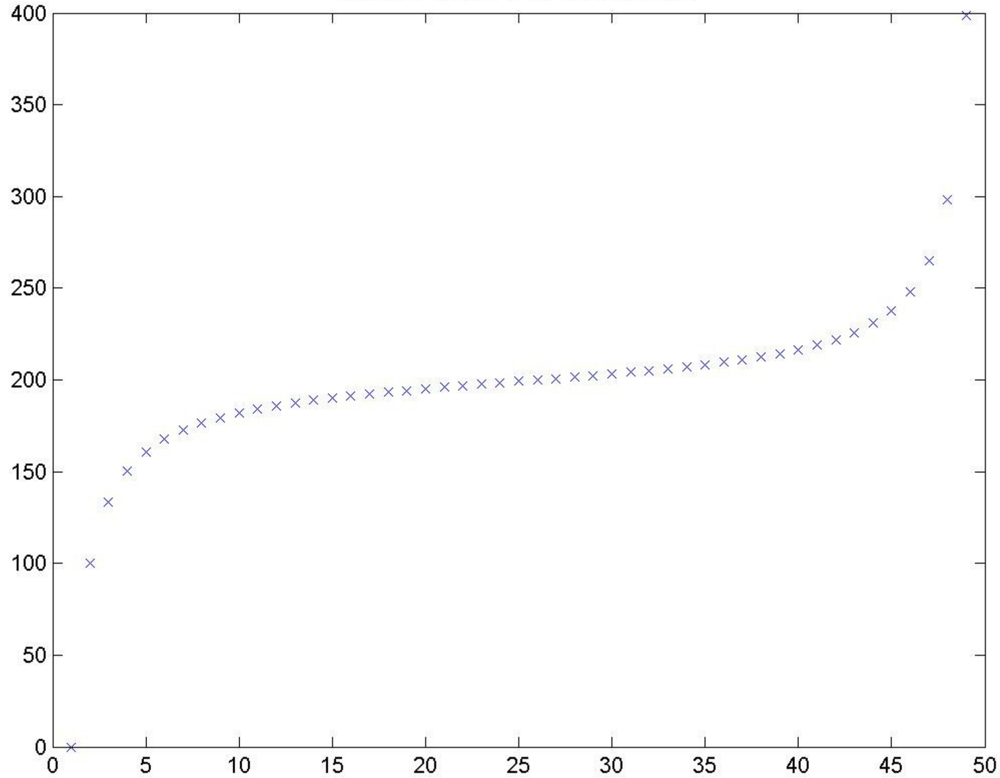


Figure 2: Spread of 20-corner convex
400
350
300
250
200
150
100
50
0
0 2 4 6 8 10 12 14 16 18 20

Figure 3: Spread of 50-corner convex



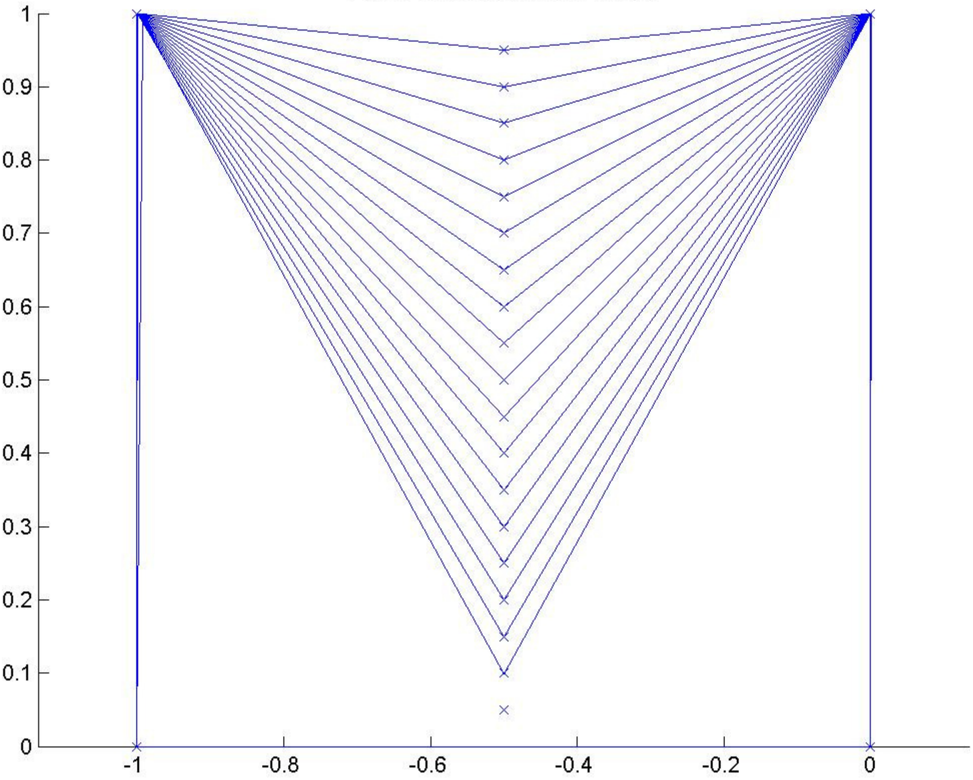


Figure 4: Nonconcex five corner

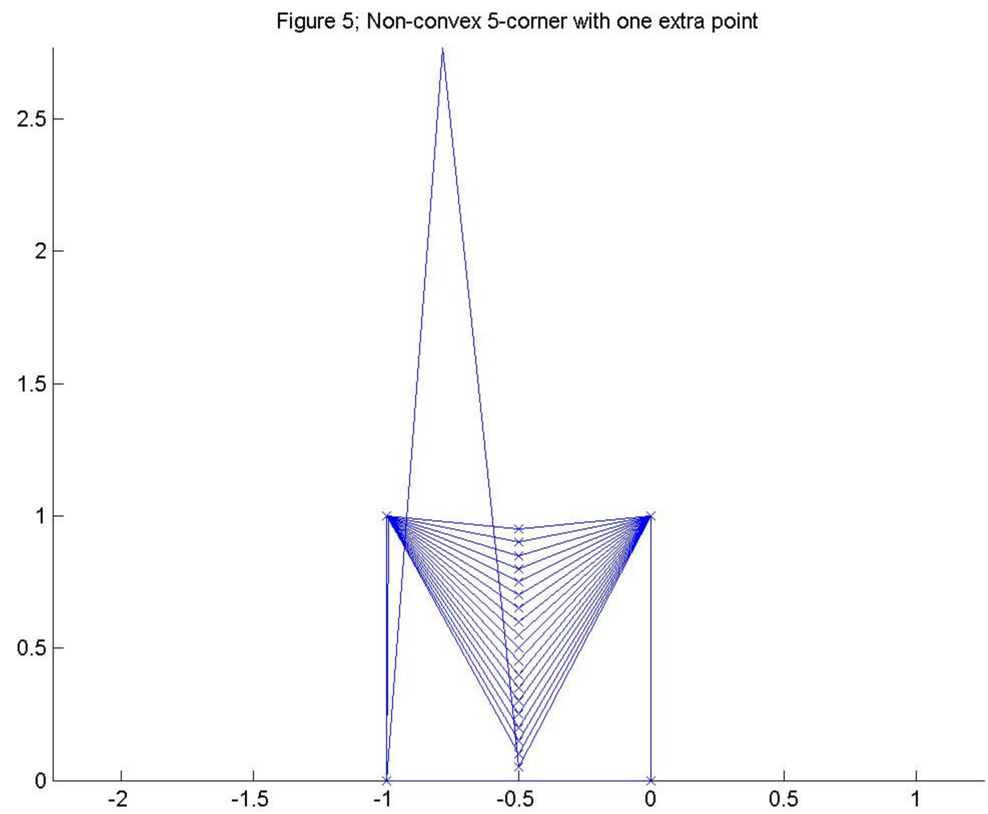


Figure 5; Non-convex 5-corner with one extra point
2.5
2
1.5
1
0.5
0
-2
-1.5
-1
-0.5
0
0.5
1

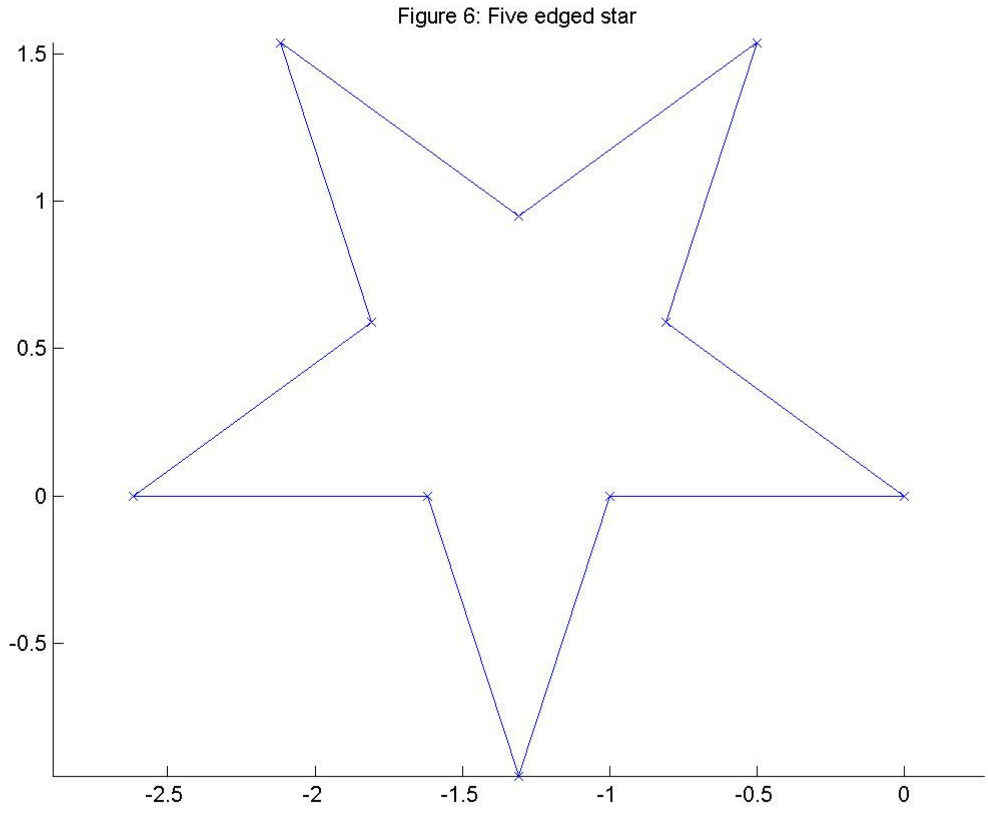


Figure 6: Five edged star
1.5
1
0.5
0
-0.5
-2.5
-2
-1.5
-1
-0.5
0

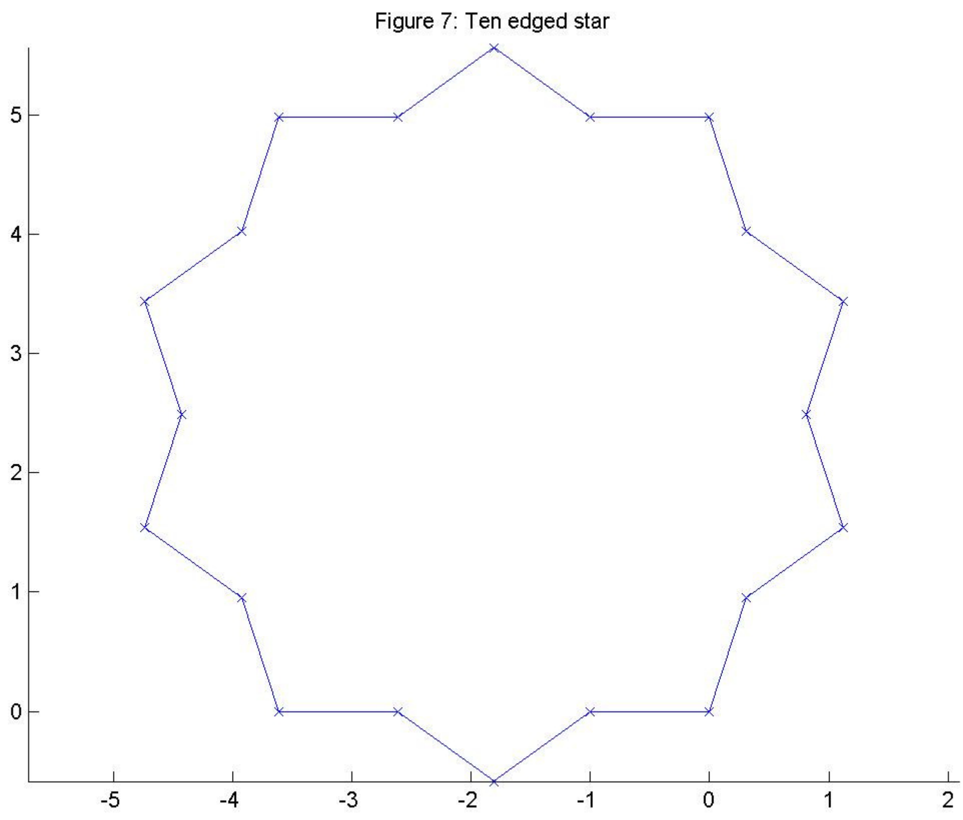


Figure 7: Ten edged star
5
4
3
2
1
0
-5
-4
-3
-2
-1
0
1
2

Figure 8: Spread ten edged star

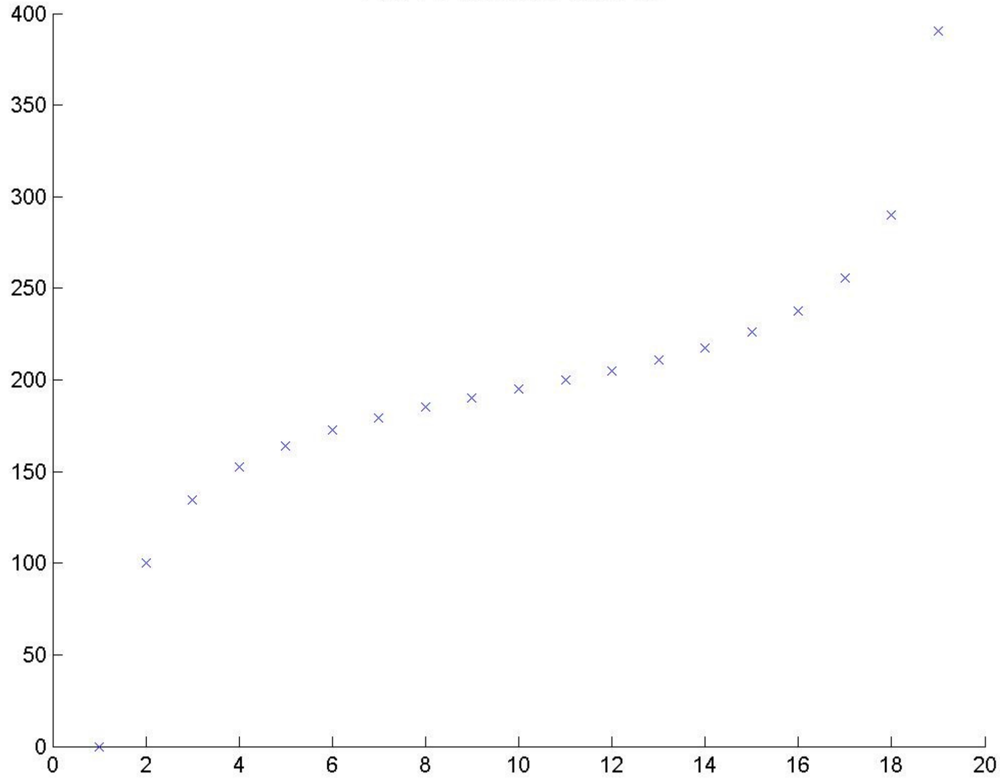


Figure 9: Slightly perturbed half unit disc

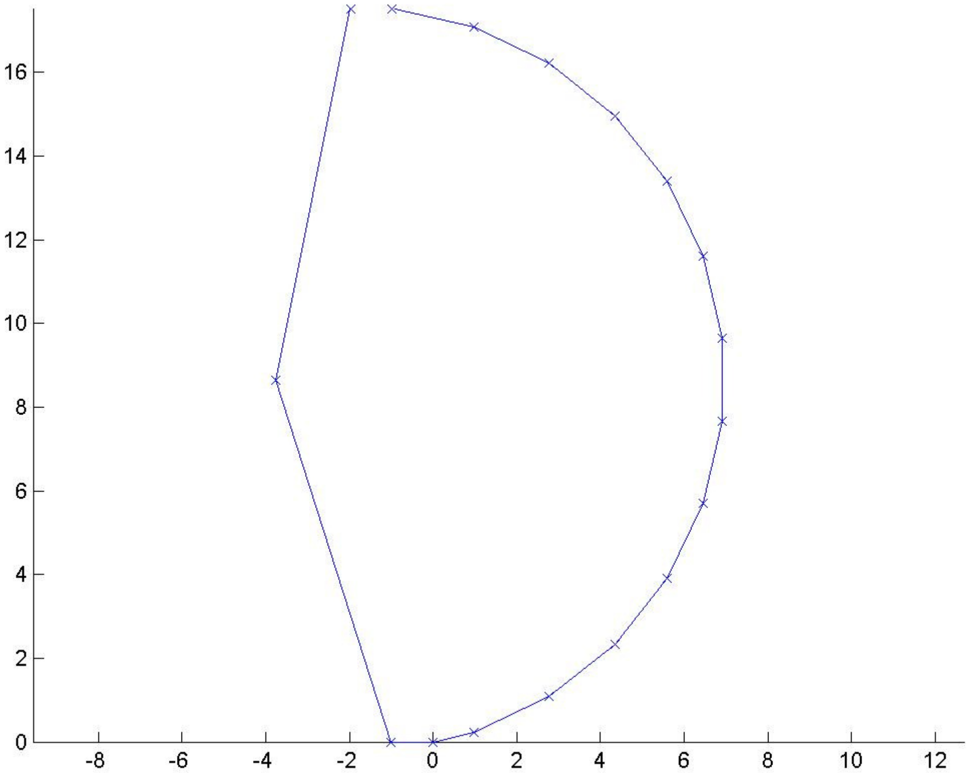


Figure 10; Square divided to four equal squares

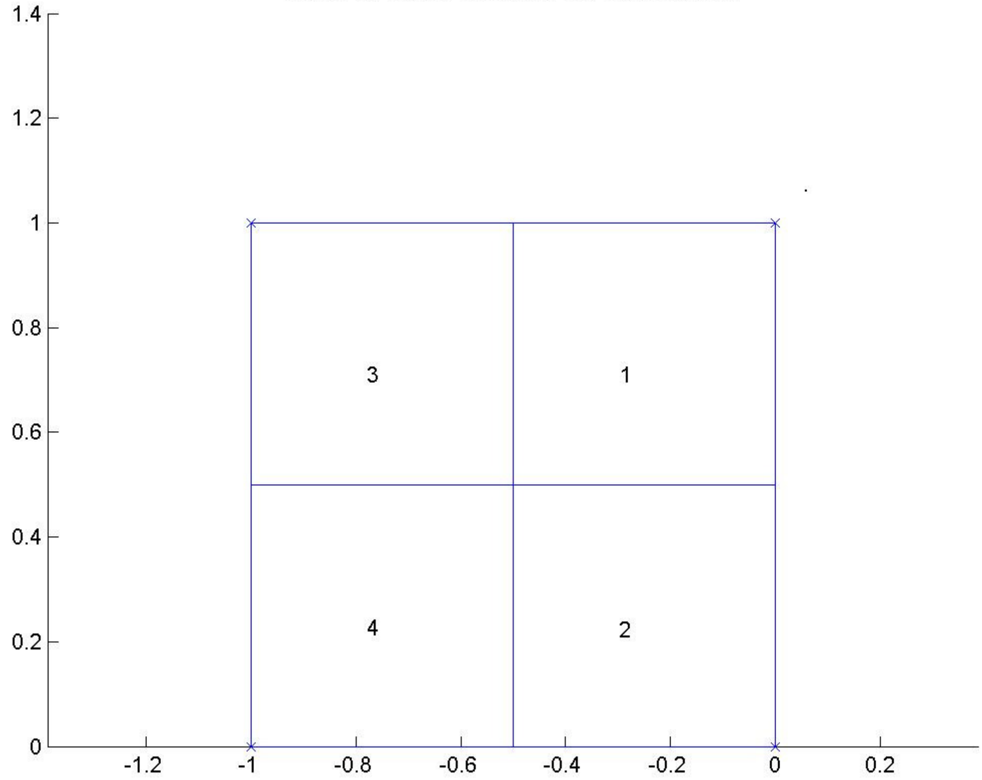
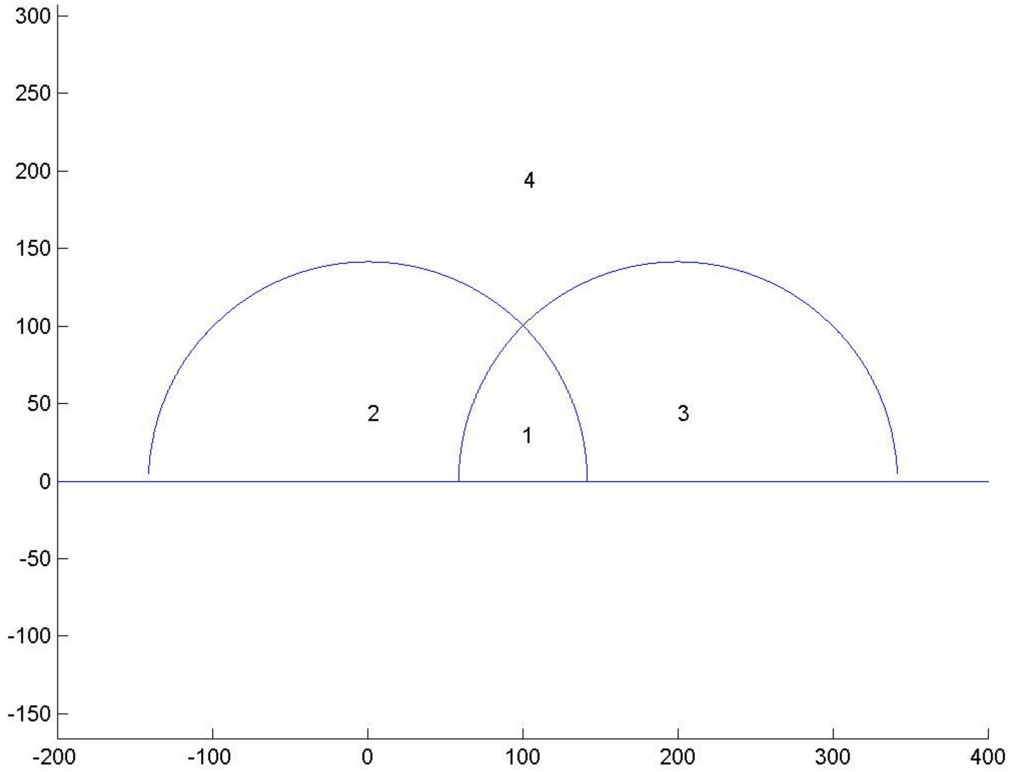


Figure 11; Vectors that cut a square



The code used in the examples

The program that finds the breakpoints;

```
clear
hold on
format compact

%%%%%%%%%%%%%%%%%%%%%%%%%%%%Program header%%%%%%%%%%%%%%%%
%   All changes betwene the examples are done here    %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%

N=50;
first=100;                    % b in the text
arg=-2/N*ones(1, N-1);                % the theta values
leng=ones(1, N-2);
dt=0.001;                     % stepsize in evaluation of integrals
vek=zeros(1, (N-3));                  % start quess vector

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%
%   For these values given, the rest is kept fixed     %
%   for all examples in this text                   %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%

x=0:dt:0.5;
change=1;
X=ones(1,N-3)'*x;
while change>0.0000000001
   ed=exp(vek);
   row=cumsum(ed/first);

   ARG=(ones(1,length(x))'*arg(3:N-1))';
   ROW=(ones(1,length(x))'*row)';
   ONE=1+ROW-X;
   TWO=ROW+X;

   gbf=arg(2)*log(1-x)+sum(ARG.*log(ONE));
   gbs=arg(1)*log(1-x)+sum(ARG.*log(TWO));
   dgbfdx=-arg(2)./(1-x)-sum(ARG./ONE);
   dgbsdx=-arg(1)./(1-x)+sum(ARG./TWO);

   gbf=(exp(gbf)).*(x.^(arg(1)+1))/(arg(1)+1);
   gbs=(exp(gbs)).*(x.^(arg(2)+1))/(arg(2)+1);

   Int=(gbf(length(gbf))-gbf(1)+gbs(length(gbs))-gbs(1))*(first^(1+sum(arg)));
   G=-(gbf.*dgbfdx)-(gbs.*dgbsdx);

   for m=1:N-3
      dgbfdm=((ARG(m:N-3,1:length(x)).*ed(m)/first)./ONE(m:N-3,1:length(x)));
      dgbsdm=((ARG(m:N-3,1:length(x)).*ed(m)/first)./TWO(m:N-3,1:length(x)));
      ddgbfdxdm=((ARG(m:N-3,1:length(x)).*ed(m)/first)./(((ONE(m:N-3,1:length(x)))).^2));
      ddgbsdxdm=-((ARG(m:N-3,1:length(x)).*ed(m)/first)./(((TWO(m:N-3,1:length(x)))).^2));
```

```
    if m<N-3
       dgbfdm=sum(dgbfdm);
       dgbsdm=sum(dgbsdm);
       ddgbfdxdm=sum(ddgbfdxdm);
       ddgbsdxdm=sum(ddgbsdxdm);
    end

    gbfid=gbf.*dgbfdm;
    gbsid=gbs.*dgbsdm;
    Int(m+1)=(gbfid(length(gbfid))-gbfid(1)+gbsid(length(gbsid))-gbsid(1))*(first^(1+sum(arg)));
    G(m+1,:)=-gbf.*ddgbfdxdm-gbfid.*dgbfdx-gbs.*ddgbsdxdm-gbsid.*dgbsdx;
end

H=G*(first^(1+sum(arg)));
Int=Int';
ARG=(ones(1,length(x))'*arg(1:N-1))';

for k=1:N-3
    clear G

    row1=cumsum(ed(k-1:-1:1)/ed(k));
    if k == 1
       term1=first/ed(k);
    else
       term1=row1(k-1)+first/ed(k);
    end
    row1=[row1 term1];
    row2=cumsum(ed(k+1:N-3)/ed(k));

    ROW1=(ones(1,length(x))'*row1)';
    ROW2=(ones(1,length(x))'*row2)';

    len=length(row1);
    if len==1
       gkf=arg(k+2)*log(1-x)+(ARG(k:-1:1,1:length(x)).*log(x+ROW1));
       gks=arg(k+1)*log(1-x)+(ARG(k:-1:1,1:length(x)).*log(1-x+ROW1));
       dgkfdx=-arg(k+2)./(1-x)+(ARG(k:-1:1,1:length(x))./(x+ROW1));
       dgksdx=-arg(k+1)./(1-x)-(ARG(k:-1:1,1:length(x))./(1-x+ROW1));

    else
       gkf=arg(k+2)*log(1-x)+sum(ARG(k:-1:1,1:length(x)).*log(X(1:k,1:length(x))+ROW1));
       gks=arg(k+1)*log(1-x)+sum(ARG(k:-1:1,1:length(x)).*log(1-X(1:k,1:length(x))+ROW1));
       dgkfdx=-arg(k+2)./(1-x)+sum(ARG(k:-1:1,1:length(x))./(X(1:k,1:length(x))+ROW1));
       dgksdx=-arg(k+1)./(1-x)-sum(ARG(k:-1:1,1:length(x))./(1-X(1:k,1:length(x))+ROW1));
    end

    len=length(row2);
    if len==1
       gkf=gkf+ARG(k+3:N-1,1:length(x)).*log(1+ROW2-x);
       gks=gks+ARG(k+3:N-1,1:length(x)).*log(x+ROW2);
       dgkfdx=dgkfdx-ARG(k+3:N-1,1:length(x))./(1+ROW2-x);
       dgksdx=dgksdx+ARG(k+3:N-1,1:length(x))./(x+ROW2);
    else
       gkf=gkf+sum(ARG(k+3:N-1,1:length(x)).*log(1+ROW2-X(1:len,1:length(x))));
       gks=gks+sum(ARG(k+3:N-1,1:length(x)).*log(ROW2+X(1:len,1:length(x))));
       dgkfdx=dgkfdx-sum(ARG(k+3:N-1,1:length(x))./(1+ROW2-X(1:len,1:length(x))));
       dgksdx=dgksdx+sum(ARG(k+3:N-1,1:length(x))./(ROW2+X(1:len,1:length(x))));
    end
```

```
        gkf=(exp(gkf)).*(x.^(arg(k+1)+1))/(arg(k+1)+1);
        gks=(exp(gks)).*(x.^(arg(k+2)+1))/(arg(k+2)+1);

        Int=[Int; (gkf(length(gkf))-gkf(1)+gks(length(gks))-gks(1))*(ed(k)^(1+sum(arg)))];
        G=(-(gkf.*dgkfdx)-(gks.*dgksdx));

        for m=1:N-3
            dgkfdm=0;
            dgksdm=0;
            ddgkfdxdm=0;
            ddgksdxdm=0;
            if m==k

                if k==1
                    dgkfdm=(-ARG(k:-1:1,1:length(x)).*ROW1./(X(1:k,1:length(x))+ROW1));
                    dgksdm=(-ARG(k:-1:1,1:length(x)).*ROW1./(1-X(1:k,1:length(x))+ROW1));
                    ddgkfdxdm=(ARG(k:-1:1,1:length(x)).*ROW1./((X(1:k,1:length(x))+ROW1).^2));
                    ddgksdxdm=(-ARG(k:-1:1,1:length(x)).*ROW1./((1-X(1:k,1:length(x))+ROW1).^2));
                else
                    dgkfdm=sum(-ARG(k:-1:1,1:length(x)).*ROW1./(X(1:k,1:length(x))+ROW1));
                    dgksdm=sum(-ARG(k:-1:1,1:length(x)).*ROW1./(1-X(1:k,1:length(x))+ROW1));
                    ddgkfdxdm=sum(ARG(k:-1:1,1:length(x)).*ROW1./((X(1:k,1:length(x))+ROW1).^2));
                    ddgksdxdm=sum(-ARG(k:-1:1,1:length(x)).*ROW1./((1-X(1:k,1:length(x))+ROW1).^2));
                end

                if k+3==N-1
                    dgkfdm=dgkfdm-(ARG(k+3:N-1,1:length(x)).*ROW2./(1-X(1:N-k-3,1:length(x))+ROW2));
                    dgksdm=dgksdm-(ARG(k+3:N-1,1:length(x)).*ROW2./(X(1:N-k-3,1:length(x))+ROW2));
                    ddgkfdxdm=ddgkfdxdm-(ARG(k+3:N-1,1:length(x)).*ROW2./((1-X(1:N-k-
3,1:length(x))+ROW2).^2));
                    ddgksdxdm=ddgksdxdm+(ARG(k+3:N-1,1:length(x)).*ROW2./((X(1:N-k-
3,1:length(x))+ROW2).^2));
                else
                    dgkfdm=dgkfdm-sum(ARG(k+3:N-1,1:length(x)).*ROW2./(1-X(1:N-k-3,1:length(x))+ROW2));
                    dgksdm=dgksdm-sum(ARG(k+3:N-1,1:length(x)).*ROW2./(X(1:N-k-3,1:length(x))+ROW2));
                    ddgkfdxdm=ddgkfdxdm-sum(ARG(k+3:N-1,1:length(x)).*ROW2./((1-X(1:N-k-
3,1:length(x))+ROW2).^2));
                    ddgksdxdm=ddgksdxdm+sum(ARG(k+3:N-1,1:length(x)).*ROW2./((X(1:N-k-
3,1:length(x))+ROW2).^2));
                end

            end

            if m<k


dgkfdm=sum((ARG(1:m+1,1:length(x))*(ed(m)./ed(k)))./(X(1:m+1,1:length(x))+ROW1(length(row1):-1:k-
m,1:length(x))));
                dgksdm=sum((ARG(1:m+1,1:length(x))*(ed(m)./ed(k)))./(1-
X(1:m+1,1:length(x))+ROW1(length(row1):-1:k-m,1:length(x))));
                ddgkfdxdm=-
sum((ARG(1:m+1,1:length(x))*(ed(m)./ed(k)))./((X(1:m+1,1:length(x))+ROW1(length(row1):-1:k-
m,1:length(x))).^2));
                ddgksdxdm=sum((ARG(1:m+1,1:length(x))*(ed(m)./ed(k)))./((1-
X(1:m+1,1:length(x))+ROW1(length(row1):-1:k-m,1:length(x))).^2));

            end

            if k<m
```

```
            if m==N-3
                dgkfdm=(ARG(m+2:N-1,1:length(x))*(ed(m)/ed(k))./(1+ROW2(m-k:N-3-k,1:length(x))-X(m:N-3,1:length(x))));
                dgksdm=(ARG(m+2:N-1,1:length(x))*(ed(m)/ed(k))./(ROW2(m-k:N-3-k,1:length(x))+X(m:N-3,1:length(x))));
                ddgkfdxdm=(ARG(m+2:N-1,1:length(x))*(ed(m)/ed(k))./((1+ROW2(m-k:N-3-k,1:length(x))-X(m:N-3,1:length(x))).^2));
                ddgksdxdm=-(ARG(m+2:N-1,1:length(x))*(ed(m)/ed(k))./((ROW2(m-k:N-3-k,1:length(x))+X(m:N-3,1:length(x))).^2));
            else
                dgkfdm=sum(ARG(m+2:N-1,1:length(x))*(ed(m)/ed(k))./(1+ROW2(m-k:N-3-k,1:length(x))-X(m:N-3,1:length(x))));
                dgksdm=sum(ARG(m+2:N-1,1:length(x))*(ed(m)/ed(k))./(ROW2(m-k:N-3-k,1:length(x))+X(m:N-3,1:length(x))));
                ddgkfdxdm=sum(ARG(m+2:N-1,1:length(x))*(ed(m)/ed(k))./((1+ROW2(m-k:N-3-k,1:length(x))-X(m:N-3,1:length(x))).^2));
                ddgksdxdm=-sum(ARG(m+2:N-1,1:length(x))*(ed(m)/ed(k))./((ROW2(m-k:N-3-k,1:length(x))+X(m:N-3,1:length(x))).^2));

            end
        end
        gkfid=gkf.*dgkfdm;
        gksid=gks.*dgksdm;
        Int=[Int; (gkfid(length(gkfid))-gkfid(1)+gksid(length(gksid))-gksid(1))*(ed(k)^(1+sum(arg)))];
        G(m+1,:)=-gkf.*ddgkfdxdm-gkfid.*dgkfdx-gks.*ddgksdxdm-gksid.*dgksdx;
      end
     H=[H; G*(ed(k)^(1+sum(arg)))];
   end


   Int=Int+dt*(sum(H')'-H(:,1)/2-H(:,length(x))/2);



%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  % Now the vector Int contains all the estimates needed for (1)   %
  % and (2) of the text                                             %

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

   F=leng(1)*Int(N-1:N-2:((N-3)*(N-1)))-leng(2:length(leng))'*Int(1);

   jac=[];
   run=N;
   for k=1:N-3
      jac(k,:)=(leng(1)*Int(run:run+N-4)-leng(k+1)'*Int(2:N-2))';
      run=run+N-2;
   end



   run=N-1;
   for k=1:N-3
        term=(1+sum(arg))*Int(run);
        jac(k,k)=jac(k,k)+leng(1)*term;
        run=run+N-2;
   end
```

```
    s=jac\(-F);

    vekpre=vek;
    vek=vek+s';

    change=norm(vekpre-vek);
    run=cumsum(exp(vek))+first ;

    disp([run])
    A=leng(1)/(Int(1)*((-1)^(sum(arg))));
end
```

And the drawing feature;

```
cor=[-1 0];
for k=3:N
    cor=[cor cor(k-1)+leng(k-2)*exp(-i*pi*sum(arg(1:(k-2))))];
end



run=[first run];
dt=0.0001;
hit=[-1 0];


x=0:dt:first;
z=x*i;

h=arg(2)*log(z-run(1));
dhdx=arg(2)./(z-run(1));
for k=2:N-2
    h=h+arg(k+1)*log(z-run(k));
    dhdx=dhdx+arg(k+1)./(z-run(k));
end
h=exp(h).*(z.^(arg(1)+1))/(arg(1)+1);
g=h.*dhdx;

int=A*h(length(h))-A*h(1);
for k=1:length(g)-1
    int=int-A*dt*i*(g(k)+g(k+1))/2;
end

int2=int;

for m=1:length(run)
    int=int2;
    dt=run(m)/10000;
    x=0:dt:run(m);
    z=x+first*i;

    h=arg(1)*log(z);
    for k=1:N-2
        h=h+arg(k+1)*log(z-run(k));
```

```
    end
    g=exp(h);

    for k=1:length(g)-1
        int=int+A*dt*(g(k)+g(k+1))/2;
    end

    dt=0.001;

    x=first:-dt:0;
    z=run(m)+x*i;

    h=arg(1)*log(z);
    dhdx=arg(1)./(z);
    for k=1:N-2
        if k==m
            h=h;
        else
            h=h+arg(k+1)*log(z-run(k));
            dhdx=dhdx+arg(k+1)./(z-run(k));
        end
    end
    h=exp(h).*((z-run(m)).^(arg(m+1)+1))/(arg(m+1)+1);
    g=h.*dhdx;
    int=int+A*h(length(h))-A*h(1);

    for k=1:length(g)-1
        int=int+A*dt*i*(g(k)+g(k+1))/2;
    end
    hit=[hit int];
end
plot(cor,'x')
hit=[hit -1];
plot(hit);

disp([A run hit(3) hit(4) hit(5)])

disp([])
```

References

Books

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At writing time 2007-05-14.