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Some observations on Fatou sets of rational functions

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Abstract

This article is about repeated iterations of rational functions in the complex plane, Fatou sets and Julia sets. While exploring some properties of the Fatou set by computer simulations, I encountered an interesting pattern. After generating more data, I became even more convinced that there was a correlation between the Fatou set of a rational function, and its derivatives on the iterated functions. In conclusion; my research strongly suggests that in most cases, the points in the Fatou set that does not converge to infinity under iteration lie in the limit of the points where the absolute value of the derivative of the iterated function is at most one as the number of iterations grows.

Contents

Introduction

A rational function $f(z)$ is a map from $\bar{\mathbb{C}} \to \bar{\mathbb{C}}$, where $\bar{\mathbb{C}}$ is the extended complex plane, and $f(z) = \frac{P(z)}{Q(z)}$ where P, Q are polynomials. We may assume that common zeros have been cancelled out, i.e. P and Q are coprime. The *degree* of a rational function, $deg(f)$ is defined as $\max(deg(P), deg(Q))$.

Definition

Define f_k as $\underbrace{f \circ f \circ ... \circ f}_{k}$, $f_0(z) = z$, and let $z_k = f_k(z_0)$.

Obviously, z_n will for some f and z_0 diverge to infinity as n grows, ($f(z) = z^2 + 1$, $z_0 = 1$ will do the job), but the sequence behaves quite differently with different choices of z_0.

Definition of the Fatou and Julia sets

A sequence of functions $\{f_n\}$ is said to be *equicontinuous* in a set X if there for every $\epsilon > 0$ and every $z \in X$ exists a $\delta > 0$ such that for $\zeta : |\zeta - z| < \delta$ we have that $|f_n(\zeta) - f_n(z)| < \epsilon$ for all n.

The *Fatou set*, $\mathcal{F}(f)$, is defined to be the maximum open subset in $\bar{\mathbb{C}}$ where the family of functions $f_1, f_2, f_3, \ldots$ is equicontinuous. This means that f_n will preserve the proximity of points, i.e. two points near each other will behave quite similar when iterated under f.

The *Julia set* $\mathcal{J}(f)$ is defined as the complement to $\mathcal{F}(f)$. This means that the Julia set is closed and compact by definition.

Definitions

A set D is *forward invariant* if $f(D) = D$, and *backward invariant* if $f^{-1}(D) = D$. A set is *completely invariant* if it is both forward and backward invariant.

It is clear that $\mathcal{F}(f) = \mathcal{F}(f_k)$ and $\mathcal{J}(f) = \mathcal{J}(f_k)$ and one can show [1] that both these sets are completely invariant under f.

[1] p 54 in Alan F. Beardon, Iterations of Rational Functions

Definition of periodic points

A point z_0 is called a *periodic* point of f with period p if $z_0 = f_p(z_0)$ and $z_0 \neq f_k(z_0)$ for $k < p$.

Furthermore, a point z_0 is
a) *attracting* if $|f'(z_0)| < 1$
b) *indifferent* if $|f'(z_0)| = 1$
c) *repelling* if $|f'(z_0)| > 1$

A point z_0 is called *preperiodic* if $f_m(z_0)$ is periodic for some m.
It is easy to show that if z_0 is periodic, then $f(z_0)$ is also a periodic point of the same type. For rational maps, it is shown that all attracting periodic points lies in the Fatou set, and the repelling ones in the Julia set [2].

Definition of Fatou Component

A *Fatou component* is a maximum subset in the Fatou set such that there exists a path between any two points in the subset.

A Fatou component Ω is called a *limited component* of the Fatou set $\mathcal{F}(f)$ if the point at infinity is not a point in $f_n(\Omega)$ for any n. The union of all the limited components is denoted by $\mathcal{L}(f)$.

Main hypothesis

Define $A_k = \{z_0 : |f_k'(z_0)| \leq 1\}$, i.e. where the absolute value of the derivative of the iterated function in z_0 is less than or equal to one, and define A_∞ as $\{z_0 : \limsup_{k\to\infty} |f_k'(z_0)| \leq 1\}$.

The hypothesis is that *if deg*$(f) \geq 2$, *and if f is not an analytic conjugate to a Euclidean rotation of the unit disc or some annulus onto itself, then*

$$\mathcal{L}(f) \subset A_\infty$$

Computer simulations point towards this statement and an abundance of data strengthens the hypothesis. The following lemmas and sketches will show that the hypothesis seems intuitively true for some cases.

[2] p 104,109 in Alan F. Beardon, Iterations of Rational Functions

Method towards a proof of the main hypothesis

Definitions

A Fatou component Ω of f is:
a) *periodic* if $f_k(\Omega) = \Omega$ for some positive integer k;
b) *eventually periodic* if $f_m(\Omega)$ is periodic for some positive integer m;
c) *wandering* if $\Omega, f(\Omega), f_2(\Omega), \ldots$ are all pairwise disjoint.

Lemma 1: No wandering domain theorem

If Ω is a Fatou component of a rational map f, where deg$(f) \geq 2$, then it is either periodic or eventually periodic.

The lemma follows directly from **The No wandering domain theorem** (Sullivan's Theorem) [3] but as the proof is very complicated it has been excluded from this article.

Lemma 2: Fixed points in Fatou components

Let Ω be a forward invariant Fatou component of the rational function f such that it contains an attracting fixed point $\alpha \neq \infty$. Then $\lim_{n\to\infty} f_n(z) = \alpha$ for all $z \in \Omega$ and $|f'(\alpha)| < 1$.

Proof:
Because there is an attracting fixed point α in Ω, $|f'(\alpha)| < 1$ and there exists an r such that $r < 1$ and a disc D centred in α where we have that $|f(z)-\alpha| = |f(z)-f(\alpha)| < r|z-\alpha|$. This shows that f_n maps D into itself, and Vitali's Theorem [4] gives the limit $f_n \to \alpha$ in Ω.

[3] p 176 in Alan F. Beardon, Iterations of Rational Functions,
p 69 in Lennart Carleson, Theodore W Gamelin, Complex Dynamics
[4] p 56 in Alan F. Beardon, Iterations of Rational Functions

The Classification theorem

One can show that a forward invariant component F_0 of $\mathcal{F}(f)$ where f is a rational function is exacly one of the following:

(a) an *attracting component* if it contains an attracting fixed point of f;
(b) a *parabolic domain* if there exists a fixed point ζ on the boundary of F_0 such that $f'(\zeta)$ is a root of unity and if $f_n \to \zeta$ on F_0.
(c) a *Siegel disc* if $f : F_0 \to F_0$ is analytically conjugate to a Euclidean rotation of the unit disc onto itself;
(c) a *Herman ring* if $f : F_0 \to F_0$ is analytically conjugate to a Euclidean rotation of some annulus onto itself.

Having this in our minds, we have come to a critical step in the proof, and the following lemma is crucial; the lemma is certainly true if f is a polynomial and the Fatou component has an attracting fixed point, but it might be possible to extend the theorem to include the parabolic domains.

The reason why this lemma feels intuitively true is that the chain rule gives that $f_k'(z_0) = f'(z_0) \cdot f'(z_1) \cdot \cdots \cdot f'(z_{k-1})$. Furthermore since each z_k lie closer and closer to the fixed point α, the derivative $f'(z_k)$ must be close to $f'(\alpha)$ which is less than one if $\alpha \in \Omega$.

Lemma 3: Convergence of the derivatives

If Ω is a forward invariant Fatou component of f which contains an attracting fixed point $\alpha \neq \infty$, where f is holomorphic in Ω, then

$$\lim_{n\to\infty} |f'(z_n)| = |f'(\alpha)| < 1 \qquad \textit{for all} \qquad z_0 \in \Omega$$

and

$$\lim_{n\to\infty} f_n'(z_0) = 0 \qquad \textit{for all} \qquad z_0 \in \Omega$$

Proof:
Let $\zeta \in \Omega$, and choose $\epsilon > 0$. Then there exists an $r > 0$ such that $|\zeta - \alpha| < r$ implies that $|f'(\zeta) - f'(\alpha)| < \epsilon$, since f is holomorphic in Ω.

But Lemma 2 gives that for each $r > 0$, there exists N, such that $n \geq N$ implies that $|f_n(z_0) - \alpha| < r$, since $f_n(z_0) \to \alpha$ as n grows.

Hence for each $\epsilon > 0$, there exists N such that $n \geq N \Rightarrow |f'(f_n(z_0)) - f'(\alpha)| < \epsilon$, and this gives that $|f'(f_n(z_0))| = |f'(z_n)|$ converges to $|f'(\alpha)| <$

1, and thus the first limit is proved.

Using the chain rule on $f_n'(z_0)$, we get $f'(z_{n-1}) \cdot f'(z_{n-2}) \cdot \dots \cdot f'(z_1) \cdot f'(z_0)$ and because of the limit we just proved only a finite number of factors have an absolute value that is greater than or equal to 1, and hence the second limit is proved.

Theorem: Convergence in components

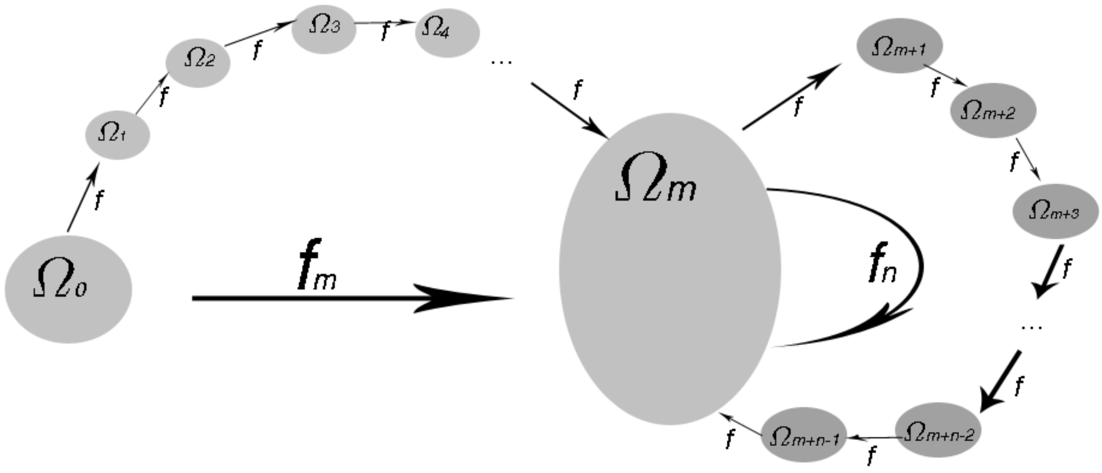
Let f be a rational function where deg$(f) \geq 2$ and let Ω be a Fatou component in $\mathcal{L}(f)$ such that $f_m(\Omega)$ contains an attracting periodic point for some m. Then $\Omega \subset A_\infty$.

Proof:
Define Ω_k as $f_k(\Omega)$, and $\Omega_0 = \Omega$. By Lemma 1, there exists a number M such that Ω_{m_1} is periodic under f with period n if $m_1 \geq M$. We also know that there exists an m_2 such that $f_{m_2}(\Omega)$ contains an attracting periodic point. Let $m = \max(m_1, m_2)$. Then Ω_m is periodic and contains an attracting periodic point with period n.

This gives us that $f_n(\Omega_m) = \Omega_m$ and it must then contain an attracting fixed point α_0 of f_n.

For each point $z \in \Omega$, $|f_k(z)| < \infty$ for all k since $\Omega \subset \mathcal{L}(f)$. Ω_m is the image of Ω under f_m so each $z \in \Omega_m$ is finite. The attracting fixed point in Ω_m must therefore also be finite and Ω_m is therefore also free from poles. We can thus be sure that f_n is holomorphic in Ω_m.

The conditions in Lemma 3 are now satisfied because Ω_m is forward invariant under f_n. Hence $\lim_{k\to\infty} f_{nk}' = 0$ in Ω_m. But if Ω_m is forward invariant for f_n, then are $\Omega_{m+1}, \Omega_{m+2}, \dots$ forward invariant as well, so in all these components, Lemma 2 and 3 are true, since they all have finite fixed points of f_n. Hence $\lim_{k\to\infty} f_{nk}' = 0$ in Ω_{m+i}.



For k large enough, we can write $k = nq + r$ where $m \leq r < m + n$. Then for an arbitrary point $z_0 \in \Omega$ we have by the chain rule that $f'_k(z_0) = f'_{nq+r}(z_0) = f'_r(z_0) \cdot f'_{nq}(z_r)$.

As $k \to \infty$, we have that $q \to \infty$ which implies that $f'_{nq}(z_r) \to 0$ since z_r lies in some Ω_{m+i} and we have already established that this limit is 0 in Ω_{m+i}.

This shows that if $z_0 \in \Omega$, $|f'_k(z_0)| \to 0 \leq 1$, and the proof is complete.

Corollary:

Let f be a rational function where $deg(f) \geq 2$. If $\mathcal{F}(f)$ is free from parabolic domains, Siegel discs and Herman rings, then $\mathcal{L}(f) \subset A_\infty$.

The Classification Theorem gives any forward invariant component $\Omega \in \mathcal{L}(f)$ must be attracting. No Wandering Domain Theorem says that Ω is eventually periodic and hence there exists m and n such that $f_m(\Omega)$ is forward invariant under f_n. We conclude that $f_m(\Omega)$ contains an attracting periodic point, and we apply the theorem about convergence in components. This implies that $\Omega \subset A_\infty$.

Remarks:
This result does not state how fast A_n converges to a set that contains $\mathcal{L}(f)$, although I will briefly touch that subject later on. Since the Fatou set can have only $0, 1, 2$ or infinitely many components [5], the hypothesis is only proved when every limited component converges to a cycle of components

[5] p 94 in Alan F. Beardon, Iterations of Rational Functions

that contains attracting periodic points and f is holomorphic. One such example is page 13 in reference [1].

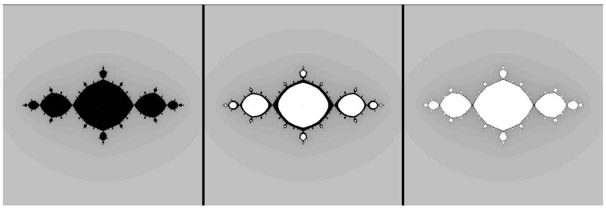


Figure 1: $f(z) = z^2 - 1$: *from left to right: The Fatou set (black), the Fatou set and* $|f'_5(z_0)| \leq 1$ *(white), the Fatou set and* $|f'_{10}(z_0)| \leq 1$ *(white).*

Another example where one can eaisly would be $f(z) = z^2 - 1/2$ which has only two Fatou components; one limited, and one unbounded. The limited component contains an attracting fixpoint $z = \frac{1-\sqrt{3}}{2}$.

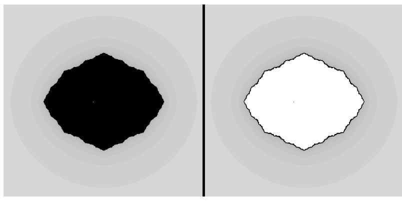


Figure 2: $f(z) = z^2 - 1/2$: *from left to right: The Fatou set (black), the Fatou set and* $|f'_{10}(z_0)| \leq 1$ *(white).*

Working with the computer

The algorithm I use to find the limited components of the Fatou set is the usual; iterate a maximum of 400 iterations and if $|f_{400}(z)| < 100$, then z is most probably in $\mathcal{L}(f)$, (or in the Julia set, but this set has almost always zero area, and can be omitted). The polynomials I have examined have zeros $a + ib : a, b \in [-1, 1]$ in the numerator and the denominator to make sure that the limited Fatou set is roughly in the center of my image. By simply calculating the pixels I compute the different areas (letting the resolution be about 800x800 pixels).

The pseudo-code for my algorithm is the following;

```
MAX_ITERATIONS := 400
FOR each point z DO

    i:=0
    WHILE abs(z)<100 AND i<MAX_ITERATIONS DO
        z := f(z)
        i := i+1

    IF i = 400 THEN point is in the limited components
    ELSE point is not in the limited components
```

Remark
This algorithm clearly gives an overestimation of the Fatou set; pixels that diverge towards infinity really slowly might be included. No pixels that should be in the Fatou set are excluded assuming all points z in the Fatou set fulfils $|z| < 100$.

Some images and examples

The leftmost image below shows the Fatou set (greyscale) and Julia set (black), and thereafter the sets A_1, A_4, A_9 (white). As we can see, A_k converges rapidly towards $\mathcal{L}(f)$, the difference between the area of $\mathcal{L}(f)$ and A_9 is only 3%.

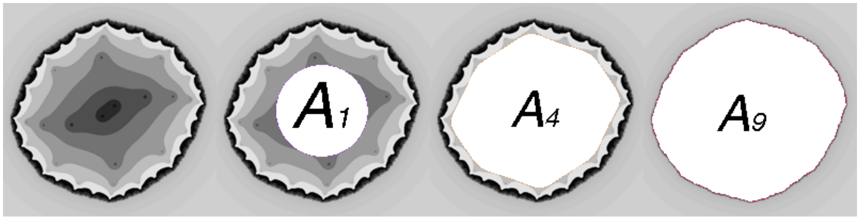


Figure 3: $f(z) = (z - 0.052 + 0.391i) \cdot (z - 0.314 - 0.332i)$

The images in Figure 4 show that even when $\mathcal{L}(f)$ is infinitely disconnected (right), A_k (left) will eventually converge to $\mathcal{L}(f)$.
In the example, the difference between $\mathcal{L}(f)$ and A_{300} is less than 0.2%.

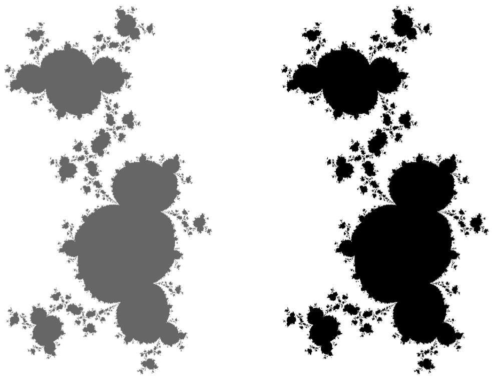


Figure 4: $f(z) = (z - 0.146 + 0.612i) \cdot (z - 0.325 - 0.993i) \cdot (z + 0.482 + 0.82i)/(z - 0.913 + 0.02i)$

The following image is the first function I found which had a really slow convergence; the difference between A_{15} and $\mathcal{L}(f)$ is usually only a few percents. (The picture shows A_1 to A_{40}, then $\mathcal{L}(f)$, and the last square is A_{400})

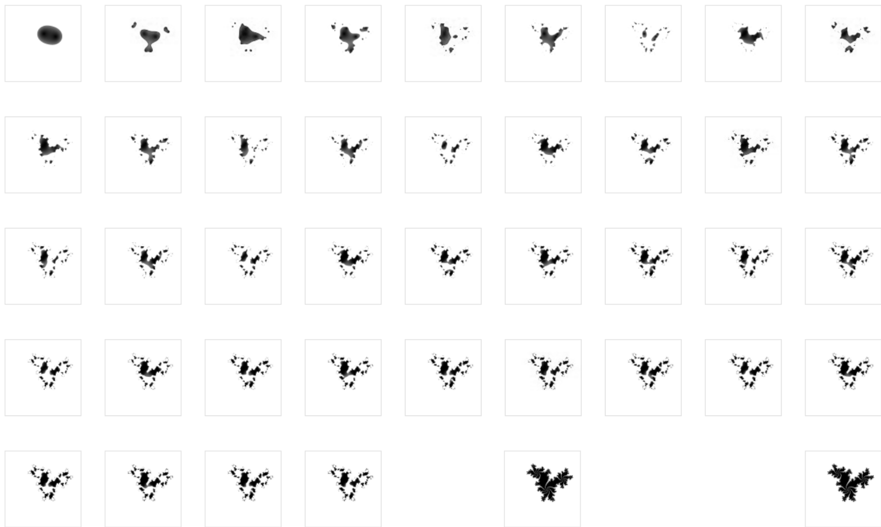


Figure 5: $f(z) = (z-0.507+0.055i)\cdot(z-0.932-0.665i)\cdot(z+0.378-0.707i)$

Practical uses

What new methods can be developed using the above results? It is only possible to approximate the Fatou set because of the infinite number of iterations needed, so instead of using the regular algorithm (which I described about approximating the Fatou set), it is sometimes more accurate to compute the derivatives. Computer simulations indicate that if the limited Fatou components of a rational map have an area close to zero and is infinitely disconnected, it is more accurate to compute the derivative instead of the absolute value of $f_n(z_0)$.

The following image shows the different results; The leftmost frame is the Fatou set, computed with 400 iterations. The second one is the same algorithm using 10 iterations. The last frame is A_{10} (white). A_{10} is an underestimate which happened in most cases, but the interesting part is that A_{10} is about 79% of the Fatou set, while the one with 10 iterations is an overestimation of about 33%.

Future work might show that an interpolation between the regular algorithm and using the derivatives gives the most accurate approximation of the Fatou set.

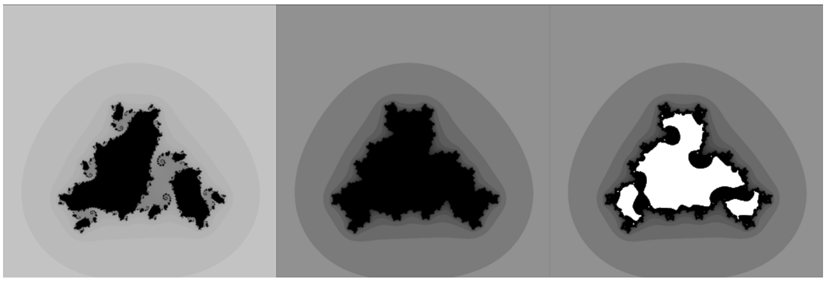


Figure 6: $f(z) = (z+0.566+0.795i)\cdot(z+0.185+0.101i)\cdot(z+0.753+0.714i)$

References

[1] Alan F. Bearden *Iterations of Rational Functions*
Springer-Verlag 1991. ISBN: 0-387-97589-6

[2] Lennart Carleson, Theodore W. Gamelin *Complex Dynamics*
Springer-Verlag 1993. ISBN: 0-387-97942-5