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Proofs of certain theorems in geometry - pedagogical benefits from the use of complex numbers

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Abstract

The four plane geometrical theorems *Inscribed angle theorem, the formulae for* $\sin(x \pm y)$ *and* $\cos(x \pm y)$*, the Nine-point Circle* and *Morley's trisector theorem* are presented, as well as several proofs of each theorem. At least one proof of each theorem makes use of complex numbers. We analyse what we might benefit from this usage, and figure out what properties of complex numbers make this benefit possible.

The conclusion is that often when complex numbers are used to prove plane geometrical theorems, if the problem is arranged in a suitable way, we will not have to rely on genuine ideas in as great extent, but rather the result follows from algebraic calculations. The main reason for this is the result connecting the modulus and arguments in the product of two complex numbers.

Contents

Acknowledgement

First and foremost I would like to sincerely thank my tutor in the work on this thesis, Christian Gottlieb. He has not only provided me with the topic and some useful material, but has also shown great enthusiasm for my work from the first thought till the last written word. After our many discussions I often found myself leaping the two stairs to my room, eager to put the ideas just born into action. Further, I thank Christina Aquilonius at The Stockholm Institute of Education for her help regarding language questions. I also thank my room-mates Samuel Lundqvist and Matthias Ringqvist for patience with, and interest in, my work.

1 Introduction

This paper is a Master's thesis in mathematics, submitted to the Department of Mathematics at Stockholm university. The topic is geometry – plain plane geometry, and one striking thing about it is that at university level, geometry (or at least euclidean such) is hardly taught at all any more, courses for education students excepted. That alone is reason enough to devote some time to the subject, but to compose something on just geometry would be difficult since it is a very widespread subject, so we need some delimitation, which takes the form of usage of complex numbers in geometry. Also this would provide more material than what could be contained in a paper such as this, so we will look at a limited number of theorems chosen mainly by the interest awaken within the author at a first glance.

As the title suggests, we will interest ourselves in benefits from usage of complex numbers in the presentation of the proofs, from a pedagogical point of view. Hence the purpose of this thesis is not to present any new mathematical results, but rather to discuss different (and perhaps new) presentations of known ones. However, nor is it a thesis in pure pedagogy of mathematics. It is something in between – an analysis and discussion of some theorems in plane geometry and their proofs.

We begin by looking at the background of the two main subjects, geometry and complex numbers, in Section 2. Brief history is presented, in order to give context, and also the introduction of complex numbers is discussed. Section 3 Preliminaries is a presentation of important notation and the prerequisite theory that might not be known to the reader. The reader well conversant with the subject might want to skip these two sections. Then, in Section 4 Analysis, the four theorems are given a subsection each. Several proofs of each theorem is presented and discussed from a pedagogical point of view. The last section sums up the results in the analysis, and we draw conclusions.

2 Background

In this section we will look shortly at the historical background of geometry and complex numbers. The purpose of this is to better understand the theorems and proofs we will encounter later, as well as gaining some perspective on these subjects and the teaching of them.

2.1 Geometry

All facts in this section have been collected from [13] Stillwell (2005, chapters 1 and 2) and [18] Wikipedia - The Free Encyclopedia (2007). The word *geometry* is originally Greek, roughly meaning earth measurement. It is the branch of mathematics concerning figures (lines, points, curves, etcetera) in a space – in our case the plane – and their size, shape and relative position.

When telling the story of geometry it is almost inevitable not to begin with what may be the most famous and significant work in mathematics of all time – *Elements*[1] by Euclid. This is a collection of, by then mostly already known, lore and an introduction of axioms as a foundation of geometry. It was written about 300 BC, but was used as textbook in school mathematics until the 20th century, and has, without doubt, been a great influence on Western science over the centuries.

In *Elements* Euclid begin by stating a number of axioms (or postulates), then formulating and proving some propositions, basically saying we have access to a straightedge, used only to draw lines between points or extend existing line segments – not measuring, like we can do with a ruler – and a compass to draw (part of) a circle with given center and radius. The compass can also store the length between two points and transfer it elsewhere, and hence lengths can be added and subtracted. It is then proved that from these axioms it is possible to accomplish lots of useful results, for instance we can

[1]The reader wishing to study Euclid's *Elements* deeper is recommended to visit [17] http://aleph0.clarku.edu/~djoyce/java/elements/elements.html

construct a scale on a given line segment by dividing it into n parts of equal length, basically creating a ruler.

The creation of a ruler, along with most other results in geometry, is based on the parallel axiom, also stated by Euclid. In order to illustrate how construction with straightedge and compass are made, and since parallel lines have shown to be so vital, we will look at an example in Figure 1. The formulation is not the same Euclid would have used. We have a given line l and a given point P and intend to construct the unique line passing through P being parallel to l.

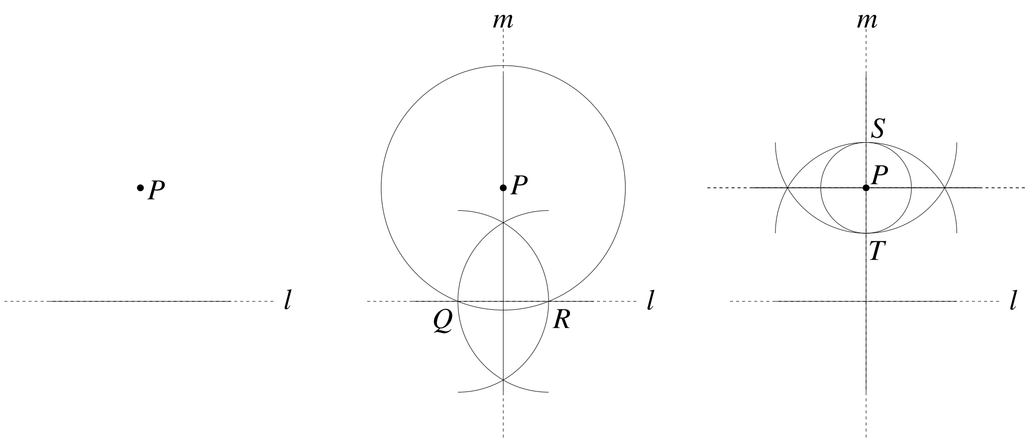

Figure 1: The construction of a parallel line through a given point

Step 1 Use the compass to draw a circle with center in P and radius large enough to intersect with l.[2] We call these points of intersection Q and R.

Step 2 Once again we use our compass, this time to draw the two circles with radii $|QR|$ and center in Q and R respectively.

Step 3 Draw the line passing through the two points of intersection of the two

[2] In fact, Euclid's axioms are not enough to show that such a point common for the circle and the line exists, but this gap was filled by David Hilbert in 1899 according to [13] Stillwell (2005, chapters 1.2 and 2.9).

circles drawn in Step 2 and call it m. This is done with the straightedge. Note that the line will pass through P and is perpendicular to l.

Step 4 Draw a new smaller circle with center in P (or use the existing one, which is more bulky though) and call the points where m intersects with this circle S and T respectively.

Step 5 Two more circles are drawn, both with radius $|ST|$ and one with center in S and the other with center in T.

Step 6 The pursued line is the one passing through the points of intersection of the two circles from Step 5.

As mentioned, with similar techniques it is easy to show many results; for example we can bisect an angle or, if we choose a length to be the unit length, multiply and divide lengths with one another. Now leaving Euclid, there are a few more selected contributions to geometry where we will make a stop.

Pappus of Alexandria, a greek mathematician born in the Egypt city Alexandria about 300 AD, developed geometry even further and his results on parallel lines, along with the results of Girard Desargues in the 17th century, became vital in the development of projective geometry in the 19th century.

The theorems of Pappus and Desargues also play an important role in David Hilbert's development of Euclid's axioms in 1899. He expands the number of axioms in order to fill in the blanks left by Euclid and also develops an arithmetic where the product of two lengths does not have to be interpreted as an area, something that would have been unacceptable in the days of Euclid. "He wants numbers to come from 'inside' geometry rather than from 'outside'. (...) It is generally easier to build geometry on numbers than the reverse".[3] ([13] Stillwell 2005, page 42 f)

[3]The Theorem of Pappus is needed to prove the commutative law for multiplication and Theorem of Desargue is needed to prove the associative law.

The milestones presented above have all taken geometry a great deal forward, but yet they are not even close to the tip of an iceberg, mere enough ice to cool a glass of water. However, this is as deep as this paper will dig. In modern mathematics, geometry has developed into a number of different fields, like algebraic geometry, complex geometry, differential geometry and topology, each with its roots in the work of Euclid. We are now leaving geometry for the moment, moving on to the other main subject of this thesis.

2.2 Complex numbers

First of all we shall clarify the purpose of this section. The intention is *not* to introduce complex numbers to the reader unfamiliar with them, but rather to present the author's view on complex numbers to the initiated reader. This is necessary because part of the objective of this entire thesis is to give the author's subjective opinion on proofs using complex numbers, and hence the reader will benefit from knowing his point of view. Despite the above, the reader who is unfamiliar with complex numbers do not have to stop the reading here, since we will develop an introduction to them.

When students first are introduced to complex numbers, it is most common ([14] Sundelin 2005, chapter 6 and [9] Ngo and Watson 1998) that the introduction is either of the form *what do we need to be able to solve equations like*

$$x^2 + 1 = 0 \tag{1}$$

or via a discussion of the known numbersystems Natural numbers ($\mathbb{N}$), Integers ($\mathbb{Z}$), Rational numbers ($\mathbb{Q}$) and Real numbers ($\mathbb{R}$), followed by an expansion of $\mathbb{R}$ into the Complex numbers ($\mathbb{C}$). Either way, the textbooks seem to give the impression that Complex numbers from the beginning were constructed in order to be able to solve equations like (1), a statement that according to [14] Sundelin (2005, chapter 5) and [8] Kleiner (1988) is incorrect.

A legitimate question about (1) is: why should we bother solving equations like this? The real story behind the origination of complex numbers ([8] Kleiner 1988) relieves us from such questions. It was in the middle of the 16th century that the Italian mathematician Cardan presented the following solution to the cubic equation $x^3 = ax + b$,

$$x = \sqrt[3]{\frac{b}{2} + \sqrt{(\frac{b}{2})^2 - (\frac{a}{3})^3}} + \sqrt[3]{\frac{b}{2} - \sqrt{(\frac{b}{2})^2 - (\frac{a}{3})^3}}.$$

As an example this yields, in the case $a = 15$ and $b = 4$, $x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}}$, which involves square roots of negative numbers but in fact is equal to the real number 4. Also the other two roots of $x^3 = 15x + 4$ are real and equal to $2 \pm \sqrt{3}$. This is the short version of the story about the birth of complex numbers which continued to evolve through the next centuries, although their relevance was questioned as late as 1825 by the great German mathematician Carl Friedrich Gauss.

Now back to the textbooks and their introduction of complex numbers. Another thing they seem to have in common is the early introduction of the *imaginary unit* (i), with the property[4] that $i^2 = -1$. This has one obvious drawback in the lack of intuition of the imaginary unit. The crucial thing, in my opinion, is that there is a one-to-one relationship between every positive real number and a distance (say the length of a stick, very substantial). No such easy-to-relate-to relationship exists for complex numbers. Altogether this effects in a way of looking at a complex number as something very abstract and probably useless when it comes to applications from the real world. Imaginary numbers give imaginary solutions – which we have seen is not true.

The way we are going to use complex numbers troughout this text is not to solve equations involving square roots of negative numbers. Therefore we may, and will, introduce complex numbers in a less mysterious way and

[4]Note that $x = i$ is a solution to (1)

never bother about the interpretation or relevance of the imaginary unit. The justification of it is its usefulness. One purpose of this text is to give examples showing that complex numbers are not useless when it comes to solving real problems, like geometrical ones, but let us begin even earlier.

When starting school, or even before that, we are introduced to natural numbers. We are taught to interpret them as *how many objects*, for example oranges. We learn how to interpret equals (=), as *just as many as* and about the operations addition (+) and multiplication (·) and how to interpret these:[5]

Let A and B be two natural numbers, we write $A, B \in \mathbb{N}$.

(+) $A+B$ equals *the total number of oranges in one pile with A oranges* and *one pile with B oranges.*

(·) $A \cdot B$ equals *the total number of oranges in A piles with B oranges in each pile.*

Note how both these operations are mappings from an ordered pair (the Cartesian product) of natural numbers into the set of natural numbers itself. We write $\mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$. From this we form subtraction

$$A, B \in \mathbb{N}, \quad (A - B) = C \iff A = B + C \quad \textit{if such a } C \in \mathbb{N} \textit{ exists}$$

and division

$$A, B \in \mathbb{N} \quad \frac{A}{B} = C \iff A = C \cdot B \quad \textit{if such a } C \in \mathbb{N} \textit{ exists}$$

and also generalise without any substantial effort to all real numbers.[6] This generalisation is worth an extra moment of thought; it can be done although we do not know how to interpret for example $-2\frac{1}{3}$ oranges or $\sqrt{\textit{two oranges}}$.

[5]This can be done more rigorously, see [3] Carlström (2006, chapter 2.2) and [2] Beachy and Blair (2006, chapter A.2)

[6]Division is not always as trivial as one might first think. One example is that in $\mathbb{Z}_5$ we get $\frac{1}{3} = 2$ since $2 \cdot 3 = 6 \equiv 1 \pmod 5$.

This is the effect of two things. First of all we are typically very young, and hence our minds suggestible to great extent, when we encounter this type of reasoning. The other reason for accepting the generalisation is the one-to-one relationship between positive real numbers and a distance, mentioned above.

The framework of real numbers is enough to make a lot of mathematics, one example being the introduction of vectors in a euclidean space. A vector, at least in $\mathbb{R}^2$ and $\mathbb{R}^3$, is quite easy to relate to, and the reason therefore is the existence of a one-to-one relationship between every vector and *a distance in a certain direction*, so the direction part is news. Here we are neglecting the problem of the 0-vector not having any distinct direction. Based upon this we claim that vectors are easy, as in not very abstract, to work with.

A reasonable question at this point would be: can we expand the known real numbers into an even wider set of numbers in a natural way? Natural here meaning that addition, subtraction, multiplication and division should work 'as usual'. One suggestion would be to introduce an *ordered pair* of real numbers.[7] Let us call these kind of numbers *New numbers*. We already have a natural interpretation of a new number as a vector (see above), and therefor also a natural interpretation of $(=)$, i.e. two new numbers are equal if they represent the same distance in the same direction, i.e. $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. The question is, can we find natural ways to interpret $(+)$ and $(\cdot)$? (Subtraction and division will follow from these.) Addition of vectors in $\mathbb{R}^2$ is already well defined by

$$(a, b) + (c, d) = (a + b,\ c + d)$$

where the '+' on the left hand side is addition of New numbers while the '+' on the right hand side is the common addition of real numbers.

What about multiplication? At least two types of products for vectors already exists, the *inner product (dot product)* and the *cross product*, each with

[7] This is not very different from the introduction of rational (non-integer) numbers, which consists of the pair of integers: numerator and denominator, see [2] Beachy and Blair (2006, chapter A.2)

its flaws. The inner product results in a scalar (real number), and is hence a mapping from the set of New numbers to the set of real numbers. The problem with the crossproduct is of similar nature. Although this product results in a vector, it is perpendicular to both factor vectors and hence is not in the same plane as these. Therefore the mapping is from $\mathbb{R}^2 \times \mathbb{R}^2$ to $\mathbb{R}^3$. The conclusion is that neither of the above will do as products for our New numbers. Inspired by the simplicity of the addition operator one suggestion might be that the product is defined by

$$(a, b) \cdot (c, d) = (a \cdot b,\ c \cdot d) \tag{2}$$

where the '·' on the left hand side is product with respect to New numbers while the '·' on the right hand side is the common product with respect to real numbers. This is obviously a mapping from the Cartesian product of New numbers into the New numbers themselves, just as we want. The remaining question is whether this gives natural results? Unfortunately the answer is no. The short answer is that with multiplication defined this way, we would not always be able to find a multiplicative inverse of every non-zero New number,[8] something that would be needed in order for the set of New numbers to be a field (see [11] Rudin 1976, chapter 1). If our set of New numbers fail of being a field, it would also fail being a natural expansion of the set of real numbers. The longer answer to the question is to diffuse to present here.

We need a better definition of multiplication of New numbers, and the suggested one is[9]

$$(a, b) \cdot (c, d) = \big((a \cdot c - b \cdot d),\ (a \cdot d + b \cdot c)\big) \tag{3}$$

[8]The multiplicative inverse of the real number r is another real number s with the property $r \cdot s = 1 = s \cdot r$. This number is usually written r^{-1}. Note that a unique such number, namely $r^{-1} = \frac{1}{r}$, exists for all real numbers except $r = 0$.

[9]The reader curious about how one can pursue such a construction in a structured way is referred to [2] Beachy and Blair (2006, chapter 4.3).

with multiplicative inverse element

$$(a,b)^{-1} := (\frac{a}{a^2+b^2},\ \frac{-b}{a^2+b^2}).$$

Note that the multiplicative inverse is well defined as long as $(a,b) \neq (0,0)$ and that we get

$$(a,b)\cdot(a,b)^{-1} = (a,b)\cdot(\frac{a}{a^2+b^2},\ \frac{-b}{a^2+b^2}) =$$

$$= (\frac{a^2}{a^2+b^2} - \frac{-b^2}{a^2+b^2},\ \frac{-b\cdot a}{a^2+b^2} + \frac{b\cdot a}{a^2+b^2}) = (\frac{a^2+b^2}{a^2+b^2},\ \frac{a\cdot b - a\cdot b}{a^2+b^2}) = (1,0)$$

and hence we define a real number r as $(r,0)$ in the new number system.[10] If we define subtraction and division in the same manner as above, we have reasoned our way to a natural expansion of the real numbers into $\mathbb{R}\times\mathbb{R}$. This expansion coincide with the complex numbers ($\mathbb{C}$) and we are ready to make following definition.

Definition 1 (Complex number)**.** *A complex number is a number* (a,b) *where* $a\in\mathbb{R}$ *and* $b\in\mathbb{R}$ *with the following rules for addition (sum)*

$$(a,b)+(c,d) = (a+b,\ c+d)$$

and multiplication (product)

$$(a,b)\cdot(c,d) = \big((a\cdot c - b\cdot d),\ (a\cdot d + b\cdot c)\big)$$

Two complex numbers (a,b) *and* (c,d) *are equal if and only if* $a=c$ *and* $b=d$*. Any real number* r *is defined as* $(r,0)$*.*

We note that $\big((r_1,0)+(r_2,0)\big)\cdot(r_3,0) = (r_1\cdot r_3 + r_2\cdot r_3, 0)$ just as we would expect, so multiplication and addition of real numbers work alright in $\mathbb{C}$. It is also easy to verify the commutative-, associative- and distributive laws. Furthermore we see that $x=(0,1)$ is a solution to (1) since $(0,1)^2+(1,0) = (0,1)\cdot(0,1)+(1,0) = (0-(1\cdot 1),0)+(1,0) = (-1,0)+(1,0) = (-1)+1 = 0$.

[10] Another suggestion that might have seemed reasonable would have been to define the real number r as (r,r).

The number $(0, 1)$ is the imaginary unit i mentioned above, and from time to time we will use the notation i for this number for the sake of convenience.

To sum up this section, what we have seen is that accepting complex numbers as actual, non-imaginary numbers need not be more abstract than accepting negative numbers or non-integers. Admittedly though, maybe we accept the latter type of numbers to easily...

3 Preliminaries

Before we buckle to the main task of this thesis we will need some more tools in our toolbox. In this section we will introduce some notation as well as results in geometry and the theory of complex numbers. These results will then be used and referred to in the analysis done in Section 4.

Geometry

We will mainly concern ourselves with triangles, and to some extent circles connected to these triangles. A triangle consists of three points in the plane, and the three line segments connecting these points two by two. The points are called the triangle's vertices and the line segments are called the sides of the triangle. Typically the vertices of a triangle will be denoted with capital letters, and if we refer to, say, the triangle ABC we will mean the (unique) triangle with vertex in A, B and C respectively. Note the difference between a line, which extends infinitely in two directions, and a line segment, which is the limited part that lies between two points of a line. We will denote the (unique) linesegment between the points A and B with either AB, $\overrightarrow{AB}$ or $B - A$. The latter two are common in vector geometry and has the *direction* from A to B, whilst the former has no direction, so $AB = BA$.[11] Hence, the directed linesegment from B to A is $\overrightarrow{BA}$ or $A - B$.

[11]Not to be misinterpreted as the *product* between two *numbers* A and B.

The principle of a regular coordinate system in the plane is considered prerequisite in this thesis, and we remind of the well known formula for the euclidean (ordinary) distance between two points $A = (a_1, a_2)$ and $B = (b_1, b_2)$, here stated without proof,

$$|AB| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}. \tag{4}$$

The distance between the two points A and B is also equal to the length of the line segment between them.

In a triangle there is an angle at each vertex, hence the name *tri-angle*. If the two sides meeting at the vertex are perpendicular, we say that the angle is right $= 90° = \frac{\pi}{2}$ radians. The angle at vertex A in triangle ABC will be denoted $\measuredangle BAC$ or equivalently $\measuredangle CAB$. Note the position of A in the middle. If there is no risk of misinterpretation, we may denote this angle by $\measuredangle A$. An angle bisector is a line dividing the angle in two equal angles, see Figure 2.

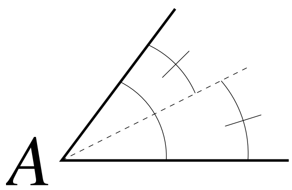

Figure 2: The angle $\measuredangle A$ and its bisector

Another statement we will not prove but nonetheless use, is that the sum of the three angles in any triangle is equal to 180°, which we will refer to as the angle sum of a triangle. There are some special types of triangles we need to know about. An equilateral triangle is one where all three sides are of equal length. One effect of this is that all three angles are equal and 60°. Another special kind of triangle is where two of the three sides are of equal length, called an isosceles triangle. The last special case of triangle we will come across is the right triangle, where one of the angles is right. These spacial cases appear in Figure 3

Two very important concepts of geometry are *congruent* and *similar* triangles. Two triangles are congruent if they are equal in both shape and

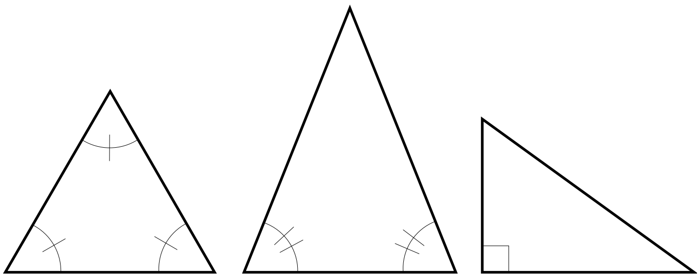

Figure 3: *(a)* an equilateral triangle, *(b)* an isosceles triangle and *(c)* a right triangle

distance, i.e. all corresponding angles are equal and all corresponding sides are equal. This makes a total of six conditions on congruence. However, three of these, if the right ones, are enough to ensure congruence ([15] Tambour 2002, chapter 2). These are

SAS *Side-Angle-Side.* If, in the two triangles ABC and DEF, the following holds, $|AB| = |DE|$, $|AC| = |DF|$ and $\measuredangle A = \measuredangle D$, then the two triangles are congruent.

SSS *Side-Side-Side.* If, in the two triangles ABC and DEF, the following holds, $|AB| = |DE|$, $|AC| = |DF|$ and $|BC| = |EF|$, then the two triangles are congruent.

ASA *Angle-Side-Angle.* If, in the two triangles ABC and DEF, the following holds, $\measuredangle A = \measuredangle D$, $\measuredangle B = \measuredangle E$ and $|AB| = |DE|$, then the two triangles are congruent.

The above conditions for congruence will not be proved. If the triangles ABC and DEF are congruent we will write $ABC \cong DEF$.

Two triangles are said to be similar if they are equal in shape but not necessarily in size, so two congruent triangles are also similar, but triangles may be similar without being congruent. The definition is that ABC is similar to

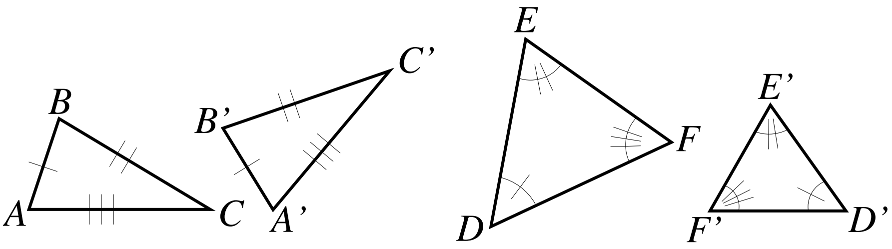

Figure 4: Two congruent triangles, $ABC \cong A'B'C'$, and two similar triangles $DEF \sim D'E'F'$

DEF if and only if

$$\frac{|AB|}{|DE|} = \frac{|AC|}{|DF|} = \frac{|BC|}{|EF|} \qquad \textit{and} \qquad \measuredangle A = \measuredangle D,\ \measuredangle B = \measuredangle E,\ \measuredangle C = \measuredangle F.$$

As in the case of congruence there exist sufficient conditions for triangles to be similar([15] Tambour 2002, chapter 3), of which we will state, but not prove, two.

SAS *Side-Angle-Side.* If, in the two triangles ABC and DEF, the following holds, $\frac{|AB|}{|DE|} = \frac{|AC|}{|DF|}$ and $\measuredangle A = \measuredangle D$, then the two triangles are similar.

SSS *Side-Side-Side.*[12] If, in the two triangles ABC and DEF, the following holds, $\frac{|AB|}{|DE|} = \frac{|AC|}{|DF|} = \frac{|BC|}{|EF|}$, then the two triangles are similar.

If the triangles ABC and DEF are similar we will write $ABC \sim DEF$.

To each triangle there are a number of interesting points we will need to investigate further. These are points of concurrence, meaning that three specific lines meet there. We formulate the concurrence as a lemma.

Lemma 1. *In any triangle the following line-triplets are concurrent, three by three. (color and line type referring to Figure 5):*

(a) The perpendicular side bisectors (yellow solid),

[12] Note that SSS, as well as SAS, has two meanings, one arguing two triangles are congruent and one arguing they are similar. The context will determine which is referred to.

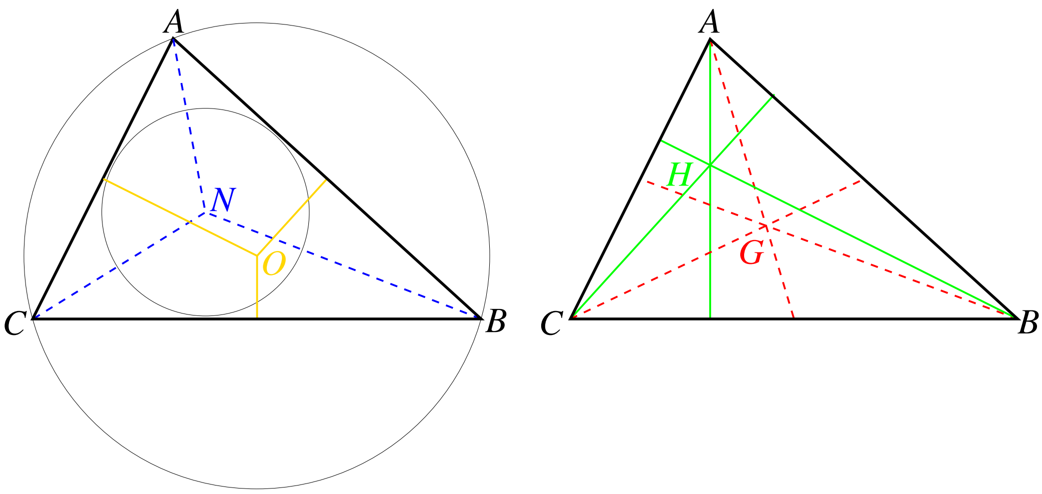

Figure 5: Left - the circumcenter (O) and incenter (N), as well as the circumcircle and incircle, of ABC. Right - the orthocenter (H) and centroid (G) of ABC

(b) *The angle bisectors (blue dashed),*

(c) *The altitudes (green solid),*

(d) *The medians (red dashed).*

The points of concurrence are called the circumcenter *(O),* incenter *(N),* orthocenter *(H) and* centroid *(G) of the triangle.*

Since we will only make use of *(a)*, *(b)* and *(c)*, these are the only ones we will prove. Elementary vector geometry is prerequisite for the proof of *(c)*.

Proof of Lemma 1 (a)-(c). Let ABC be an arbitrary triangle.

(a) The definition of a perpendicular side bisector is the (unique) line equidistant from the two vertices of that side, so the point where the perpendicular side bisectors of AB and AC meet, is equidistant from A and B as well as from A and C. Hence it is also equidistant from B and C, and therefore on the perpendicular bisector of the side BC. Since all vertices are equidistant from this point of concurrence, there is a circle that contains all three vertices and has the point of concurrence as its center. This circle is called the circumcircle of the triangle.

(b) An angle bisector is by definition the line being equidistant at every point to both sides of the vertex. Hence the point where the two lines bisecting A and B meet, is equidistant to AB and AC, but also equidistant to BA and BC, and therefore also equidistant to AC and BC meaning it is on the angle bisector of vertex C. The proof is done and the inscribed circle with center in the incenter is called the incircle of the triangle.

(c) Let us place ABC in $\mathbb{R}^2$ so that its circumcenter is in the origin. We then have $|\overrightarrow{A}| = |\overrightarrow{B}| = |\overrightarrow{C}| = r$, where $\overrightarrow{A}$ is short hand for the vector $\overrightarrow{OA}$ etcetera and r is the radius of the circumcircle. Define H as the point $H := \overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C}$. Note that $\overrightarrow{B} + \overrightarrow{C}$ is, by definition, perpendicular to $\overrightarrow{B} - \overrightarrow{C}$. But $\overrightarrow{H} - \overrightarrow{A} = \overrightarrow{B} + \overrightarrow{C}$, and hence H is on the altitude from A. Analogous reasoning gives that H is also on the altitudes from B and C, and the proof is complete.

$\square$

Now we are equipped with all the elementary plane geometry we will need in this thesis. The next step is to investigate what the complex numbers might contribute with.

Geometrical properties of complex numbers

From Section 2.2 we have a definition of complex numbers, but this thesis is supposed to be about geometry. What is the connection? The purpose of this section is to supply tools in theory of complex numbers, making it easier to work with geometry.

In Definition 1, complex numbers are defined as the ordered pair of two real numbers. It is easily verified that we might as well look at the complex number (a, b) as $a + bi$, where $i := (0, 1)$ is to be treated as a number with the property $i^2 = -1$. As an example

$$(1,1) \cdot (2,3) = (1+i) \cdot (2+3i) = 1 \cdot 2 + 1 \cdot 3i + i \cdot 2 + i \cdot 3i =$$

$$= 2 + 3i + 2i + 3i^2 = 2 + (-1) \cdot 3 + 5i = -1 + 5i = (-1, 5)$$

where the distributive law has been used. One might check the result using the multiplication rule presented in Definition 1. The notation $a + bi$ is usually more convenient when calculating, while it has the above mentioned drawback in the mystery of the symbol i.

Let $z = (a, b) \in \mathbb{C}$. The first coordinate, a, is called the real part of z, and we write $\Re z = a$. Likewise the second coordinate is called the imaginary part, and we write $\Im z = b$. Note that both the real and imaginary part of a complex number are real numbers! We see that it is sufficient to know the real and imaginary part of a complex number in order for it to be totally revealed. However, this description of a complex number might seem a little strange. Just as, if we are looking for something, we prefer to get directions as a distance in a certain direction instead of the longitude and latitude, it might sometimes be more convenient to have a complex number described as a distance (from the origin) in a certain direction. The description of a complex number in such a manner is called *polar form.* The distance to the origin is called the *modulus* of the complex number, while the direction is called the *argument.* If we let the modulus of $z = (a, b) \in \mathbb{C}$ be denoted by $|z|$ and the argument by $\arg z$, the polar representation is given by the simple applications of the Pythagorean theorem and the definitions of cos and sin

$$z = (a, b) = (|z| \cdot \cos(\arg z),\ |z| \cdot \sin(\arg z)), \quad \text{where}$$

$$|z| = \sqrt{a^2 + b^2} \quad \text{and } \arg z \text{ is given by the solution to}$$

$$\begin{cases} \cos(\arg z) = \frac{a}{|z|} \\ \sin(\arg z) = \frac{b}{|z|} \end{cases} \qquad 0 \leq \arg z \leq 2\pi. \tag{5}$$

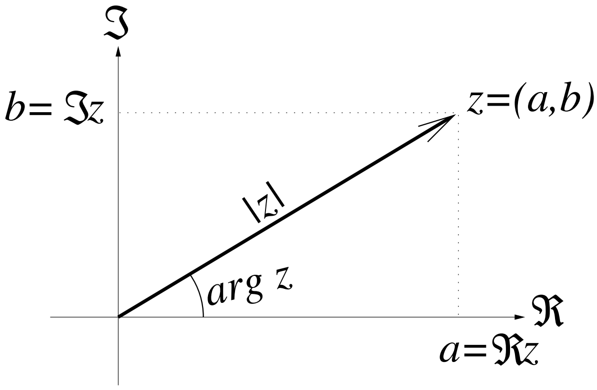

Figure 6: The complex plane and some properties of $z \in \mathbb{C}$

The concept of real and imaginary part as well as polar form is illustrated in Figure 6, where also the complex plane is introduced, which will be frequently used later on.

Next we introduce the complex conjugate of a complex number. To each complex number z there is a unique number called the complex conjugate of z, denoted $\bar{z}$. In the complex plane this number is a reflection in the real axis, and hence if $z = (a, b)$ then $\bar{z} = (a, -b)$. We now make the observations that, for any $z = (a, b) \in \mathbb{C}$,

$$z + \bar{z} = (a, b) + (a, -b) = (a + a,\ b - b) = (2a, 0)$$

$$\therefore \quad z + \bar{z} = 2\Re z \in \mathbb{R}, \tag{6}$$

$$z\bar{z} = (a, b) \cdot (a, -b) = \big(a^2 - b(-b),\ ab + a(-b)\big) = (a^2 + b^2, 0)$$

$$\therefore \quad z\bar{z} = |z|^2 \in \mathbb{R}^+, \tag{7}$$

$$z \neq (0, 0),\ \bar{z} = \frac{z}{z}\bar{z} = \frac{z\bar{z}}{z} \overset{by(7)}{=} |z|^2 \frac{1}{z}$$

$$\therefore \quad \bar{z} = |z|^2 \frac{1}{z} \quad if\ z \neq (0, 0). \tag{8}$$

We have now come to what, for our purposes, may be the most important result on complex numbers, namely how multiplication of complex numbers is to be interpreted geometrically in the complex plane. We formulate this as a lemma.

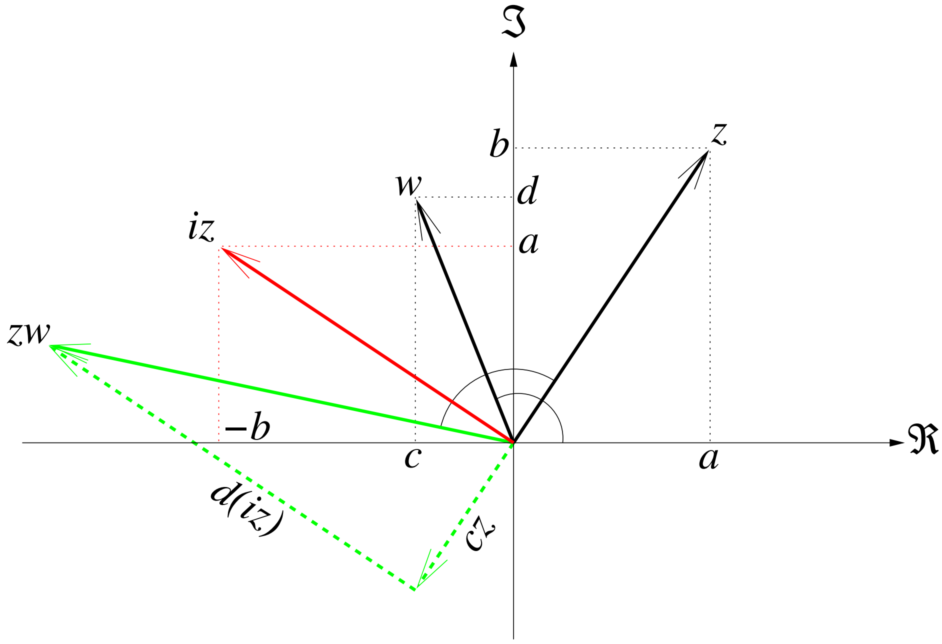

Figure 7: The product zw

Lemma 2. *If $z, w \in \mathbb{C}$, then*

(a) $\arg(z \cdot w) = \arg z + \arg w,$

(b) $|z \cdot w| = |z| \cdot |w|.$

Usually this lemma is proved using the addition formulas for sin and cos (see [10] and [12]), but this is not the way we will proceed. The reason for this is that in Section 4.2 we will do the other way around, use Lemma 2 to prove the addition formulas for sin and cos, and therefore we will need another proof of the lemma. The elegant proof presented below is from [5] Gottlieb (2002, chapter 1).

Proof. The proof relies to great extent on Figure 7, where the vectors $z = (a, b)$ and $w = (c, d)$ represent two arbitrary complex numbers with no restrictions whatsoever on neither modulus nor arguments. First note that

$$z \cdot w = z \cdot (c, d) = z\big((c, 0) + (0, d)\big) = c \cdot z + d \cdot (0, 1) \cdot z = c \cdot z + d \cdot iz.$$

Now, $iz = i(a + bi) = -b + ai = (-b, a)$ is obviously perpendicular to z. In other words z and iz makes a right angle. Further we have $|iz| = \sqrt{(-b)^2 + a^2} = |z|$ and looking at the right triangle with hypotenuse $z \cdot w$ and legs $c \cdot z$ and $d \cdot iz$ we realise that it is similar, since the legs have the same proportion, to the right triangle with one leg on the real axis and w as hypotenuse. We also note that the scaling factor of the two similar triangles is $|z|$. Since the two triangles are similar, their two angles at the origin (marked in Figure 7) are both equal to $\arg w$, and the length of their hypotenuse differ with a factor $|z|$. It is now obvious from the figure that $\arg(zw) = \arg z + \arg w$ and $|zw| = |z| \cdot |w|$.

□

One consequence of Lemma 2 is that $\arg \frac{z}{w} = \arg z - \arg w$ and $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$.

The following reasoning is inspired by different parts of chapter 2 in [6] Hahn (1994). In order not mix up with the product, we refer to the triangle with vertices in the complex numbers z_1, z_2 and z_3 as $\triangle z_1 z_2 z_3$. When looking at triangles in the complex plane we have reason to sharpen our definition of similar (and congruent) triangles. So far we have defined all angles to be positive, but the argument of a complex number, which we interpret as an angle, may well be negative. So far we have considered $\measuredangle ABC$ as being equal to $\measuredangle CBA$, but from now on we will look at it as $\measuredangle ABC = -\measuredangle CBA$ when dealing with complex numbers. This narrowing has the effect that two triangles are similar (congruent) if and only if they are in the manner defined above *and* if they have the same orientation, i.e. the order, clockwise or counterclockwise, of the corresponding vertices are the same in both triangles. If the orientation is not the same (only two options are possible) we say that the two triangles are *reversed* similar (*reversed* congruent). Returning to Figure 4 at page 16, we now see that $ABC \cong A'B'C'$, but $DEF \overset{reversed}{\sim} D'E'F'$.

Now, by SAS, we write for two triangles in the complex plane,

$$\triangle z_1z_2z_3 \sim \triangle w_1w_2w_3$$

$$\iff$$

$$\left|\frac{z_2-z_1}{z_3-z_1}\right| = \left|\frac{w_2-w_1}{w_3-w_1}\right| \qquad \textit{and} \qquad \arg\left(\frac{z_2-z_1}{z_3-z_1}\right) = \arg\left(\frac{w_2-w_1}{w_3-w_1}\right)$$

$$\iff$$

$$\frac{z_2-z_1}{z_3-z_1} = \frac{w_2-w_1}{w_3-w_1} \iff (z_2-z_1)(w_3-w_1) = (z_3-z_1)(w_2-w_1)$$

$$\iff$$

$$z_1(w_2-w_3) + z_2(w_3-w_1) + z_3(w_1-w_2) = 0 \iff$$

$$\iff \begin{vmatrix} z_1 & w_1 & 1 \\ z_2 & w_2 & 1 \\ z_3 & w_3 & 1 \end{vmatrix} = 0. \tag{9}$$

Equation (9) is very useful. We may, as an example, receive the equation of a line through two given points from it. Three points z_1, z_2 and z_3 are collinear (on the same line) if and only if $\triangle z_1z_2z_3 \sim \triangle \bar{z}_1\bar{z}_2\bar{z}_3$

$$\iff \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0,$$

which means that if two points, z_1 and z_2, are known, the line through these points are given by all $z \in \mathbb{C}$ fulfilling the equation

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0 \iff z(\bar{z}_1-\bar{z}_2) + \bar{z}(z_2-z_1) + z_1\bar{z}_2 - z_2\bar{z}_1 = 0. \tag{10}$$

This is the equation of a line through the points z_1 and z_2.

Equation (9) is also useful if we want to study equilateral triangles. The simplest equilateral triangle in the complex plane is the one with all vertices on the unit circle and one vertex in $(1,0)$. We see that if we define the complex number ω as

$$|\omega| := 1 \qquad \arg\omega := 120^\circ,$$

it follows easily that $\triangle 1\omega\omega^2$ in Figure 8 is exactly this triangle.

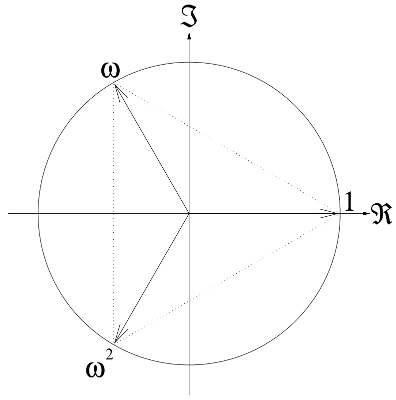

Figure 8: The equilateral triangle formed by the cubic roots of one in the complex plane

This definition gives $\omega^2 = \bar{\omega}$, since $\arg\omega^2 = 2\arg\omega = 240^\circ$, $\Re\omega^2 = \cos 240^\circ = \cos 120^\circ = \Re\omega$ and $\Im\omega^2 = \sin 240^\circ = -\sin 120^\circ = -\Im\omega$. Further we have $\omega^3 = 1 = (\omega^2)^3$, so 1, ω and ω^2 are the cubic roots of one.[13] Noting that $\Im\omega = \sin 120^\circ = -\frac{1}{2}$ and using (6) we get that ω fulfils the equation

$$\omega^2 + \omega + 1 = 0. \tag{11}$$

The conclusion is that $\triangle z_1z_2z_3$ is equilateral if and only if it is similar to $\triangle 1\omega\omega^2$ or $\triangle 1\omega^2\omega$ depending on the orientation of $\triangle z_1z_2z_3$, i.e. if and only if

$$\begin{vmatrix} z_1 & 1 & 1 \\ z_2 & \omega & 1 \\ z_3 & \omega^2 & 1 \end{vmatrix} = 0 \qquad \text{or} \qquad \begin{vmatrix} z_1 & 1 & 1 \\ z_2 & \omega^2 & 1 \\ z_3 & \omega & 1 \end{vmatrix} = 0$$

[13]According to the factorisation theorem ([2] Beachy and Blair 2006, chapter 4.2) these are the only roots.

which in turn is equivalent to

$$z_1 + z_2\omega + z_3\omega^2 = 0 \quad or \quad z_1 + z_2\omega^2 + z_3\omega = 0. \tag{12}$$

Now we are ready to face the theorems this thesis set out to deal with.

4 Analysis

In this section, the main one of this thesis, we will look at four theorems in plane geometry and several proofs of each theorem. The proofs will differ in their main idea, and at least one proof of each theorem will use theory of complex numbers. We will analyse the proofs in the sense of looking at what theory is prerequisite knowledge and discuss how easy the proof is to grasp. The arrangement of the theorems is such that the level of difficulty is increasing.

4.1 The Inscribed angle theorem

This is a theorem which students encounter early in upper secondary school mathematics. Since the theorem concerns angles only, and hence is independent of the use of a length unit, we may without loss of generality consider only circles of radius one, which might simplify the calculations a little. However, this insight might demand a higher ability of abstraction and therefore we will regard the radius as arbitrary.

Theorem 1 (The Inscribed angle theorem)**.** *Given a circle with center O and radius r, and two points A and B on the circle. The central angle $\measuredangle AOB$ is twice the inscribed (peripheral) angle $\measuredangle APB$, where P is any point on the longer arc from A to B on the circle.*

The first proof we will look at is done in two steps, first a special case is proved and the we generalise the proof to all possible cases. The special case is the one where the line AP (or BP) passes through the midpoint O.

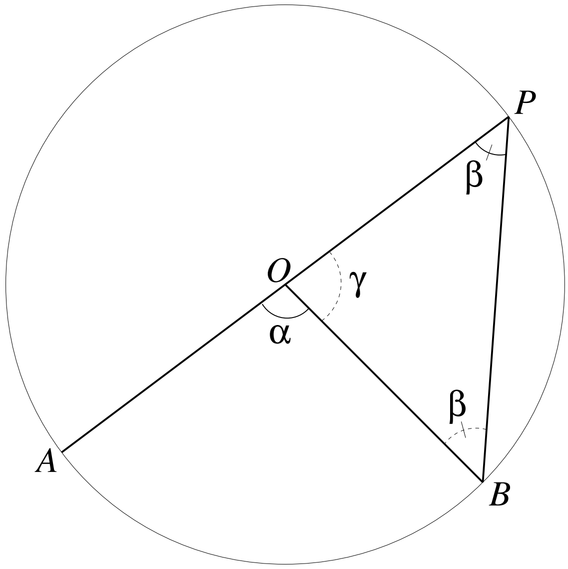

Figure 9: The central and one inscribed angle of the two points A and B

Proof (special case) by identifying angles. The situation is as in Figure 9, and we wish to show that $\alpha = 2\beta$. Since the triangle BOP is isosceles (two of the sides are of the same length as the circle's radius) we have β represented twice. Due to the angle sum of a triangle being 180°, we get $\gamma = 180° - 2\beta$, which can be rewritten as $\beta = \frac{180°-\gamma}{2}$. But we obviously have $\alpha + \gamma = 180°$, in turn equivalent to $\gamma = 180° - \alpha$, and therefor $\beta = \frac{180°-(180°-\alpha)}{2} = \frac{\alpha}{2}$ and the special case has been proved.

□

Let us now proceed to the general case presented in Figure 10, where the point P' is any point on the circle.

Generalisation of the proof. We want to show that $\measuredangle AP'B = \beta$ and know from the special case that $\measuredangle AOB = 2\beta$. With the helping line segments AB and OP' we get three more isosceles triangles AOB, AOP' and BOP' and hence δ, θ and ω are represented twice each. From the angle sum in AOB we know $2\beta + 2\delta = 180°$ which is equivalent to $\beta = 90° - \delta$. We need

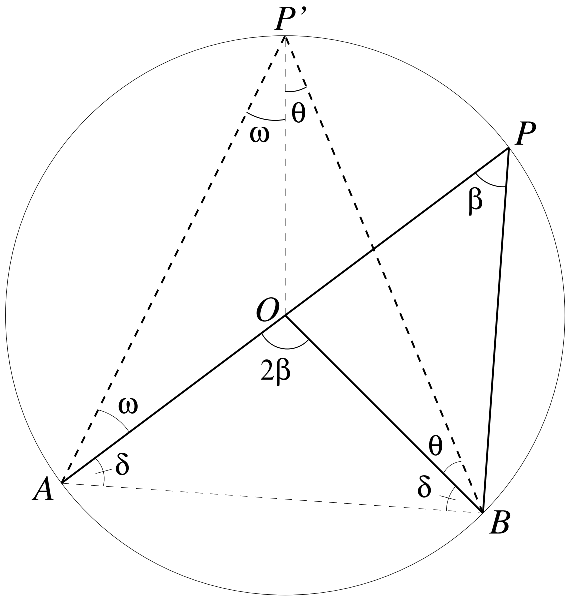

Figure 10: Two inscribed angles of the two points A and B

to show that $\omega + \theta = \beta$. The angle sum in $AP'B$ gives us $2\delta + 2\theta + 2\omega = 180° \iff \omega + \theta = 90° - \delta$, but since we know $\beta = 90° - \delta$, we get $\omega + \theta = \beta$.

$\square$

Figure 10 shows only the situation where BP' is situated between BO and BP. Strictly we have two more situations to consider, when BP' is on the other side of BP and when BP' is on the other side of BO (but still in the circle of course). These situations can be handled in a similar way as the one above. The difference is in the expression for the angle sum in $AP'B$, where some signs change. The reasoning of extracting an expression for the angle $\measuredangle AP'B$ and conclude that it is equal to β is still valid, and hence also the proof.

Theorem 1 can also be proved using complex numbers and the idea that led to the following proof was inspired by [14] Sundelin (2005), although it is not to be found there, and was simplified with great aid from Christian Gottlieb. In this case we do not have to identify any isosceles triangles or similar angles

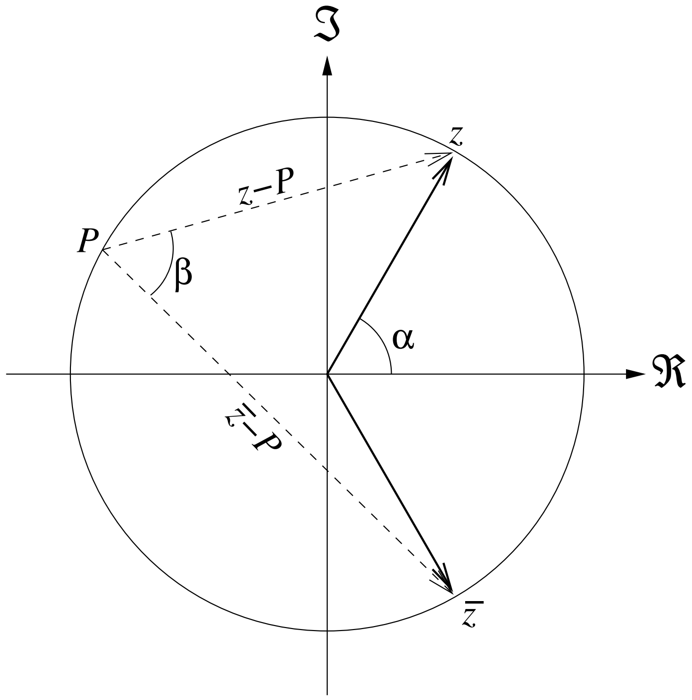

Figure 11: Circle and angles in the complex plane

using known geometry. The result follows from algebraic calculations. The key is to realise that regardless of the arrangement of A, B and P, we can always place the circle in the complex plane in such a way that the centre of the circle O is in the origin and the points A and B corresponds to the complex numbers z and its complex conjugate $\bar{z}$, see Figure 11. This gives us the situation $\arg \bar{z} = -\arg z$ and $|P| = |z| = r$.

Proof using complex numbers. Place the circle in the complex plane as described above. We see in Figure 11 that our task is to prove $\alpha = \beta$. We have

$$\alpha = \beta \iff \arg z = \arg(z - P) - \arg(\bar{z} - P)$$

and by Lemma 2 *(a)* in Section 3 we get

$$\alpha = \beta \iff \arg z = \arg(\frac{z-P}{\bar{z}-P}) \iff \frac{z-P}{\bar{z}-P} = tz$$

for some $t \in \mathbb{R}^+$. A negative t would mean $\arg z = 180° + \arg(z - P) -$

$\arg(\bar{z} - P)$ which is not what we are trying to prove. So let us look closely on $\frac{z-P}{\bar{z}-P}$ to see if we can prove it to equal tz for some $t \in \mathbb{R}$.

$$\frac{z-P}{\bar{z}-P} = \frac{(z-P)\overline{(\bar{z}-P)}}{(\bar{z}-P)\overline{(\bar{z}-P)}} = \frac{(z-P)\overline{(\bar{z}-P)}}{|\bar{z}-P|^2}$$

and we see that the denominator is real and positive, so we can focus on the numerator.

$$(z-P)\overline{(\bar{z}-P)} = (z-P)(\bar{\bar{z}}-\bar{P}) = (z-P)(z-\bar{P}) = z^2 - z(P+\bar{P}) + P\bar{P}$$

Now, since by (7) on page 20, $P\bar{P} = |P|^2 = |z|^2 = z\bar{z}$, we further get

$$(z-P)\overline{(\bar{z}-P)} = z^2 - z(P+\bar{P}) + z\bar{z} = z\big(z-(P+\bar{P})+\bar{z}\big) = z \underbrace{2(\Re z - \Re P)}_{\in\mathbb{R} \text{ by (6) on page 20.}}$$

We see that we need $\Re z > \Re P$, which can always be made sure by the choice of A or B as z. Since $t = \frac{2(\Re z - \Re P)}{|\bar{z}-P|^2} \in \mathbb{R}^+$, the proof is complete.

□

Maybe the most difficult part of the above proof is to realise we can use $P\bar{P} = z\bar{z}$. If one wishes to avoid this problem it is possible to come to the same conclusion by regarding the coordinates of each point, say $z = (x, y)$, $\bar{z} = (x, -y)$ and $P = (a, b)$ where $x^2 + y^2 = a^2 + b^2$, and do the calculations. A warning, however, this approach is more cumbersome.

Corollary 1. *The triangle with one vertex on the half circle combining the other two is a right triangle.*

Proof. In this special case z is purely imaginary and hence $\beta = \arg z = 90°$. □

Theorem 1 has now been proved twice. The first proof relies much on the idea of identifying or constructing isosceles triangles. It is not very difficult to understand that such triangles exists due to the fact that one angle in focus is the one in the midpoint of the circle. This makes sure that every triangle with the midpoint as one of its vertices (and of course the other two vertices on the circle) in fact has to be isosceles. Another important part of the proof is that the third angle in a triangle is known if the other two are so. This is basic knowledge in intermediate school mathematics and should be known to anyone facing this problem. Altogether we can conclude that the prerequisite for this proof is very basic, but some creativity is needed in order to come up with it.

The second proof, based on theory of complex numbers, has its fundamental part in the idea to place the circle in the complex plane so that A and B correspond to each others conjugates. Then the powerful results (6) and (7) can be used. The great advantage of this proof compared to the first one is that once the placement of the circle is done, almost no creativity at all is needed. We can easily see, using Lemma 2, what we are looking for, and we get there by simply 'counting on'. The identification of $P\bar{P} = z\bar{z}$ can be a problem, but as commented above this identification is not necessary – mere time saving.

So, where exactly are we using the properties of A, B and P being represented by complex numbers? First of all when we use Lemma 2 to get the simplification $\arg(z - P) - \arg(\bar{z} - P) = \arg \frac{z-P}{\bar{z}-P}$ of the angle between two vectors as the angle of one vector (relative one basic vector). There is no such tool available when dealing with vectors in $\mathbb{R}^2$ not considered as complex numbers. Also the special property (7) is unique for the complex numbers and is essential in the proof. If we, for example, would try to prove the theorem using *dot product*[14], which also deals with angles between two vectors, we would run into trouble trying to express the lengths of $\overrightarrow{PA}$ and $\overrightarrow{PB}$.

[14]Se Section 4.2 for definition.

4.2 The formulae for $\sin(x \pm y)$ and $\cos(x \pm y)$

Our next theorem, the addition formulae for the trigonometric functions sin and cos, are introduced to students taking higher mathematics courses in upper secondary school, but is also part of the first university course in calculus. Since the theorem mainly belongs to the analysis field of mathematics, radians rather than degrees will be used when dealing with angles.

Theorem 2 (Addition and subtraction formulas)**.** *The following equalities hold*

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \tag{13}$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y \tag{14}$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y \tag{15}$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y. \tag{16}$$

Before looking at an actual proof, let us just note that if we can prove one of these formulae, the other three will follow easily from the known equalities $\cos(-x) = \cos x$, $\sin(-x) = -\sin x$, $\cos(\frac{\pi}{2}-x) = \sin x$ and $\sin(\frac{\pi}{2}-x) = \cos x$, which we do not prove but, however, are easy to accept when a picture is drawn. As an example (15) is obtained from (13) as

$$\cos(x+y) = \sin\big(\frac{\pi}{2} - (x+y)\big) = \sin\big((\frac{\pi}{2} - x) + (-y)\big) \overset{use\ (13)}{=}$$

$$= \sin(\frac{\pi}{2} - x)\cos(-y) + \cos(\frac{\pi}{2} - x)\sin(-y) = \cos x \cos y - \sin x \sin y$$

and with the exact same technique any of (13) - (16) can be obtained from any of the others.

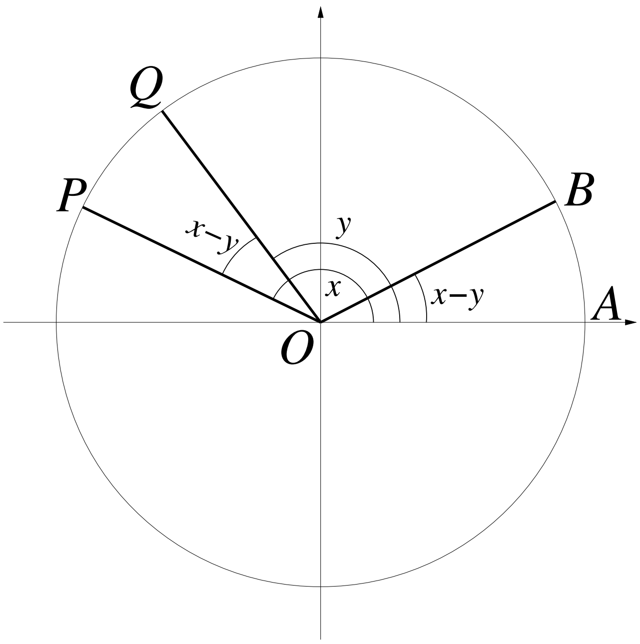

Figure 12: The unit circle

Now, let us take a look at some ways to prove one of the equalities. The first one seems to be the one most common in textbooks to a first (university) course in calculus. It can be found in [1] and [7].

Proof using coordinates on the unit circle. In Figure 12 we have $A = (1, 0)$ and, by definition of cos and sin, $P = (\cos x, \sin x)$, $Q = (\cos y, \sin y)$ and $B = \big(\cos(x - y), \sin(x - y)\big)$. From the figure we also realise that the two distances PQ and BA are equal since, by SAS, $POQ \cong BOA$, so (4) on page 14 gives

$$(\cos x - \cos y)^2 + (\sin x - \sin y)^2 = \big(1 - \cos(x - y)\big)^2 + \big(0 - \sin(x - y)\big)^2$$

and further we have the following equivalences,

$$\cos^2 x + \cos^2 y - 2\cos x \cos y + \sin^2 x + \sin^2 y - 2\sin x \sin y =$$

$$= 1^2 + \cos^2(x - y) - 2\cos(x - y) + \sin^2(x - y)$$

$$\Longleftrightarrow$$

$$\underbrace{\cos^2 x + \sin^2 x}_{=1} + \underbrace{\cos^2 y + \sin^2 y}_{=1} + 2\cos x \cos y + 2\sin x \sin y =$$

$$= 1 + \underbrace{\cos^2(x-y) + \sin^2(x-y)}_{=1} + 2\cos(x-y)$$

$$\Longleftrightarrow$$

$$2 + 2\cos x \cos y + 2\sin x \sin y = 2 + \cos(x-y)$$

$$\Longleftrightarrow$$

$$\cos x \cos y + \sin x \sin y = \cos(x-y)$$

and we are done.

$\square$

The next proof we shall look at also makes use of Figure 12, but this time the tool *dot product* from linear algebra is used to shorten the details. [16] Tengstrand (2005, chapter 4) makes the following definition.

If u and v are two vectors in the plane not equal to the zero vector, we define

$$u \cdot v := |u||v| \cos \alpha$$

where α is the angle between u and v. $u \cdot v$ is called the dot product of u and v.

Note how $u, v \in \mathbb{R}^2$ but $u \cdot v \in \mathbb{R}$. Now we easily see in Figure 12 that $\overrightarrow{OP} \cdot \overrightarrow{OQ} = |\overrightarrow{OP}||\overrightarrow{OQ}| \cos(x-y) = 1 \cdot 1 \cdot \cos(x-y)$. (From now on we write P for the vector $\overrightarrow{OP}$ and so on.) So if we can somehow get another expression for $P \cdot Q$, this might help. In [16] Tengstrand we also find the following theorem, which we here will use as a lemma.

Lemma 3. *If the vectors u and v have the coordinates (u_1, u_2) and (v_1, v_2) respectively in an orthonormal basis in the plane, then*

$$u \cdot v = u_1 v_1 + u_2 v_2.$$

Lemma 3 will not be proved here, but follows from the definition of an orthonormal basis and the following rules for vectors u, v, w and scalar λ.

(1) $u \cdot v = v \cdot u$

(2) $u \cdot (v + w) = u \cdot v + u \cdot w$

(3) $u \cdot (\lambda v) = \lambda(u \cdot v)$

(4) $u \cdot u \geq 0$ and $u \cdot u = 0 \Longrightarrow u = \mathbf{0}$

which in turn can be shown to follow from the definition of dot product. Now we are ready to prove Theorem 2.

Proof using dot product. As we have already seen the following holds

$$\cos(x - y) = P \cdot Q$$

and using Lemma 3 and remembering the coordinates for $P = (\cos x, \sin x)$ and $Q = (\cos y, \sin y)$ we get

$$\cos(x - y) = P \cdot Q = \cos x \cos y + \sin x \sin y$$

which is exactly what we were seeking.

$\square$

The next proof we will encounter, mainly inspired by [13] Stillwell (2005, chapters 3.6 and 4.7), also makes use of some linear algebra. The idea is to note that the rotation of the point $(1, 0)$ in the plane $x + y$ radians around the origin is, of course, the same as first rotating it x and then y radians, and finding out how the matrix looks for a rotation. Last we can use the result from linear algebra that a linear mapping of a point in the plane is the same as multiplying the vector representing the point with the matrix representing the mapping. Let us do this heuristically in some detail.

Proof using linear mapping. If a mapping is given by $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, then it is linear if and only if $f(\lambda_1 u_1 + \lambda_2 u_2) = \lambda_1 f(u_1) + \lambda_2 f(u_2)$ for all

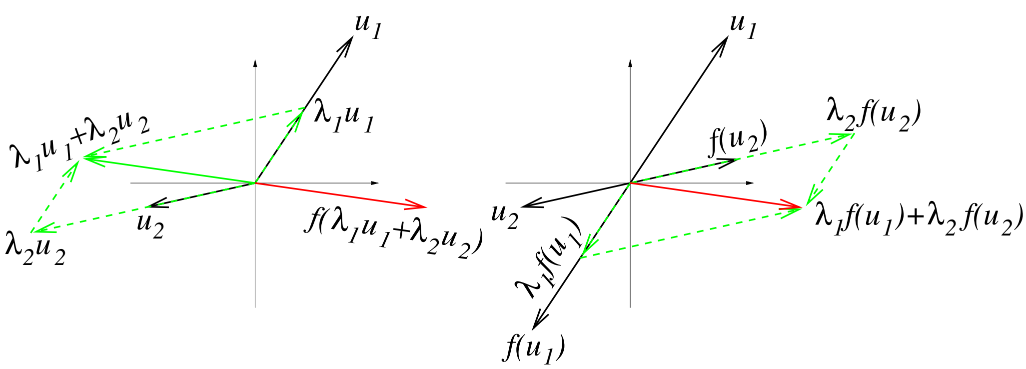

Figure 13: Rotation 180°. $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = 2$

$\lambda_1, \lambda_2 \in \mathbb{R}$ and $u_1, u_2 \in \mathbb{R}^2$. That a rotation around any point is a linear mapping is realised easily when a picture is drawn, see Figure 13 for a simplified example.

Let us find out how to construct the matrix for rotation α radians around the origin. We are looking for a function $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. If we consider the standard basis (e_1, e_2) of the plane and use the properties of a linear mapping we see that

$$\phi\big((a,b)\big) = \phi(ae_1 + be_2) = a\phi(e_1) + b\phi(e_2).$$

Now, by the definition of sin and cos, we get that $\phi(e_1) = \phi\big((1,0)\big) = (\cos\alpha, \sin\alpha)$ and, from the known equalities $\cos(\alpha + \frac{\pi}{2}) = -\sin\alpha$ and $\sin(\alpha + \frac{\pi}{2}) = \cos\alpha$, $\phi(e_2) = \phi\big((0,1)\big) = (-\sin\alpha, \cos\alpha)$, i.e.

$$\phi\big((a,b)\big) = a(\cos\alpha, \sin\alpha) + b(-\sin\alpha, \cos\alpha) = (a\cos\alpha - b\sin\alpha, a\sin\alpha + b\cos\alpha).$$

With matrix notation we have

$$\phi\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}.$$

Beginning in $(1,0)$ and first rotating x radians takes us, by definition of sin and cos, to the point $(\cos x, \sin x)$. Further rotation of this point y radians is given by

$$\begin{pmatrix} \cos y & -\sin y \\ \sin y & \cos y \end{pmatrix}\begin{pmatrix} \cos x \\ \sin x \end{pmatrix} = \begin{pmatrix} \cos y\cos x - \sin y\sin x \\ \sin y\cos x + \cos y\sin x \end{pmatrix},$$

and once again by definition of sin and cos we get

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\sin(x+y) = \cos x \sin y + \sin x \cos y$$

and the proof is complete.

$\square$

An alternative proof would be to regard the rotation of $x + y$ radians as the composite of the two linear mappings rotating x and y radians respectively, and use that the composite of two linear mappings is represented by the matrix product of the two matrices. We would get

$$\begin{pmatrix} \cos(x+y) & -\sin(x+y) \\ \sin(x+y) & \cos(x+y) \end{pmatrix} = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix} \begin{pmatrix} \cos y & -\sin y \\ \sin y & \cos y \end{pmatrix} =$$

$$= \begin{pmatrix} \cos x \cos y - \sin x \sin y & -\cos x \sin y - \sin x \cos y \\ \cos x \sin y + \sin x \cos y & \cos x \cos y - \sin x \sin y \end{pmatrix}$$

and by equating the entries in the first column of the first and last matrices we reach our goal.

The last proof we look at is the only one involving complex numbers. The main idea is to make use of Lemma 2 in Section 3.

Proof using complex numbers. Let $z = (a, b)$ and $w = (c, d)$ be complex numbers such that $|z| = |w| = 1$, $\arg z = x$ and $\arg w = y$. By (5) on page 19 we have

$$\sin x = \frac{b}{1} = b, \ \cos x = \frac{a}{1} = a, \ \sin y = \frac{d}{1} = d \text{ and } \cos y = \frac{c}{1} = c,$$

and we are interested in

$$\sin(x+y) = \sin\big(\arg(zw)\big) = \frac{\Im(zw)}{|zw|} = \frac{\Im(zw)}{|z||w|} = \frac{\Im(zw)}{1} = \Im(zw).$$

But $zw = (a, b) \cdot (c, d) =$ (see Definition 1) $= (ac - bd,\ ad + bc)$, so we get

$$\sin(x+y) = \Im(zw) = ad + bc = \cos x \sin y + \sin x \cos y$$

and analogously

$$\cos(x + y) = \Re(zw) = ac - bd = \cos x \cos y - \sin x \sin y$$

and we are done.

$\square$

We have now seen four different proofs of Theorem 2. The main idea in the proof using coordinates on the unit circle is to construct two distances we know to be equal, each involving the angle $x - y$, and plug the expressions of the two distances into an equation. The reason we are able to express the two distances in a bountiful way is that the necessary coordinates are known from the definition of sin and cos.

The prerequisite for this proof is only the definition of sin and cos, how the coordinates of a point on the unit circle is connected with these and how to calculate the euclidean distance between two points in the plane with given coordinates. All of this is basic mathematics from upper secondary school. The difficult part in grasping the proof is probably to come up with the idea of constructing the two equal distances, which might not be a natural first approach to the problem.

The second proof, the one using dot product, is only available to the reader who has knowledge of dot product, and therefor the concept of vectors. Lemma 3, which we again emphasise has not been shown, also need to be known, but is probably what most students taking a first course in linear algebra considers the definition of dot product in the plane. With these tools available the proof is rather straightforward, and the key is to identify the angle $x - y$ as the angle between two vectors with angles x and y respectively from the positive part of the horizontal axis. Note that the point B in Figure 12 is superfluous in this proof, which might be an advantage in understanding or constructing it.

In the proof based on linear mappings the reader has to be familiar with the concept and definition of matrices and matrix multiplication. Furthermore some theory about linear mappings is necessary, and the ability to identify a rotation as a linear mapping along with the knowledge of how to construct the corresponding matrix. All of this is usually included in a first course in linear algebra. A step one has to take in ones mind is that of regarding an angle as a rotation, and specifically to regard the sum of two angles as the effect of two rotations. This is the idea of the proof, and from it the result follows easily.

When complex numbers are used to prove the theorem, the thing is to consider $x + y$ as the sum of arguments of two complex numbers – something that should be natural. Then one has to think of the sum of arguments as being the argument of another complex number, namely the product of two complex numbers. This should not be a major difficulty to anyone enough familiar with complex numbers. The only difficulty left is to consider sin and cos of the arguments of the complex numbers involved. Once again we emphasise the importance of finding an independent proof of Lemma 2. As mentioned, familiarity with complex numbers is a prerequisite. This is taught the last year of some upper secondary school programmes.

Of all the proofs presented above, the one based on dot product is the shortest and probably the easiest to remember. Yet, the availability of this proof is limited since it is based on some linear algebra which is not trivial.

4.3 The Nine-point Circle

This theorem states that in any triangle, nine specific points are cocyclic, i.e. on the same circle (or line, but that is not the case here). The points appear to be unrelated at a first sight. In triangle ABC in Figure 14 we denote the midpoints of the sides by A', B' and C'; the feet of the altitudes by D, E and F; the orthocenter by H and the midpoints of the line segments from H to the vertices by K, L and M.

Theorem 3 (The Nine-point Circle)**.** *In any triangle, the following points*

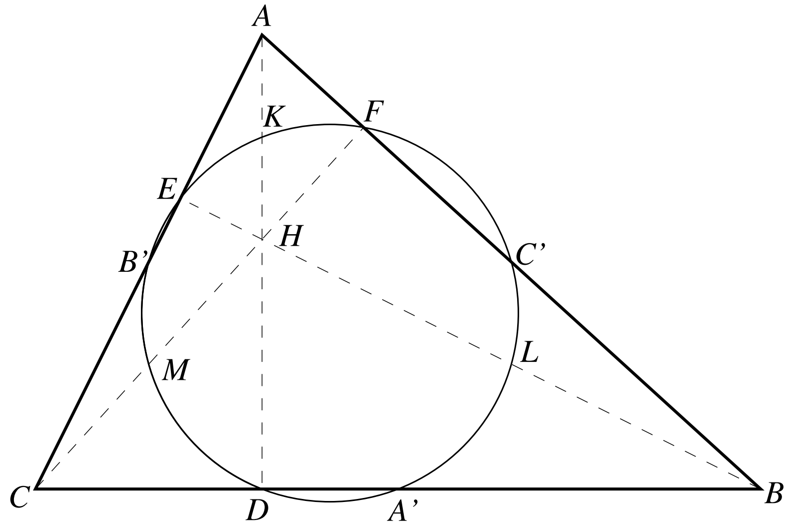

Figure 14: The nine points of the Nine-point Circle

are on the same circle with half the radius of the circumcircle:

- *The midpoints of the sides.*
- *The feet of the altitudes.*
- *The midpoints of the segments joining each vertex with the orthocenter.*

This circle is called the Nine-point Circle and has its center in the middle of the segment joining the orthocenter with the circumcenter of the triangle.[15]

Note how the orthocenter is constructed in Lemma 1 in Section 3. We will look at two proofs of the theorem, of which the first, inspired by [4] Coxeter and Greitzer (1967, chapter 1.8) and by [15] Tambour (2002, chapter 16), uses classic geometry only whilst the second, found in [6] Hahn (1994, chapter 2.4), uses complex numbers. Before looking at the first one we remember that all four vertices of a rectangle lie on the same circle, with diameter equal to the length of any of the diagonals of the rectangle and center on the midpoint of

[15]This line segment is called the *Euler line*, and also contains the centroid.

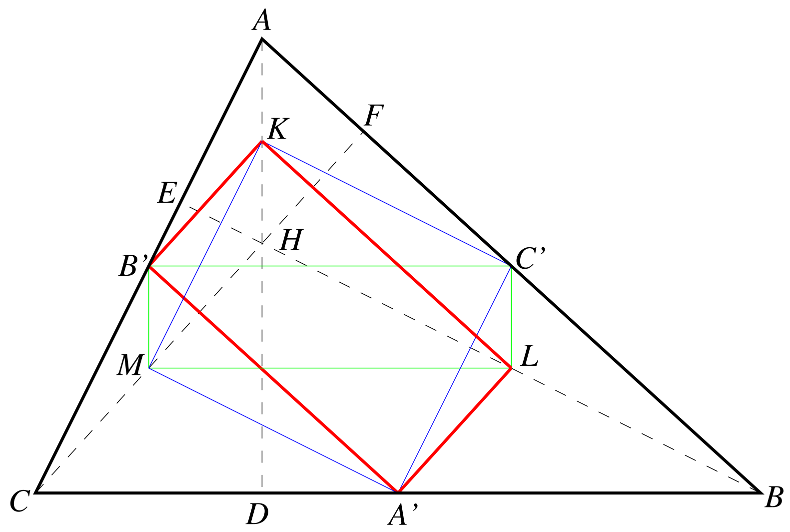

Figure 15: Three quadrangles in ABC

any of the diagonals of the rectangle (the lengths are equal and the midpoints coincide).

Proof using classic geometry. First we wish to prove that the quadrangles $B'A'LK$, $MLC'B'$ and $A'C'KM$ (see Figure 15) are rectangles. If this is the case then they all have the same circumcircle since they have one diagonal in common with each of the others.

To prove that $B'A'LK$ is a rectangle we start by noting that, by SAS, $B'CA' \sim ACB$, since the two triangles have $\measuredangle C$ in common and B' and C' bisects the sides CA and CB respectively. Hence $B'A'$ is parallel to AB and half as long. By exactly the same reasoning we conclude that KL is parallel to AB and half as long, this time considering the similar triangles KHL and AHB. We now know that $B'A'$ and KL are parallel and of the same length, and hence $B'A'LK$ is a parallelogram. It remains to show that $B'K$ (and $A'L$) are perpendicular to $B'A'$ (and KL).

The same reasoning as above gives us that $CAH \sim B'AK$ and $CBH \sim A'BL$. This shows that $B'K$ as well as $A'L$ is parallel to CH, but by definition CH is perpendicular to AB. We have proved that the parallelogram $B'A'LK$

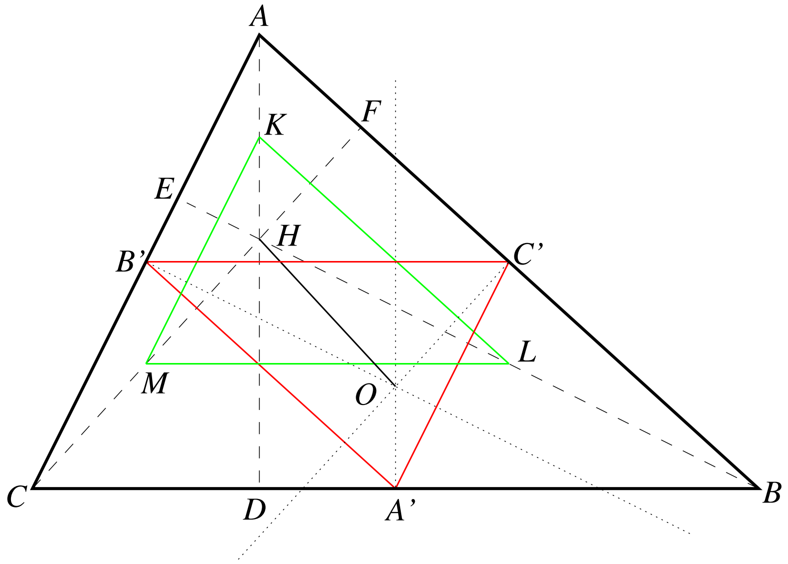

Figure 16: Two triangles in ABC

is in fact a rectangle. The same reasoning holds for $MLC'B'$ and $A'C'KM$ as well, and we know that the points A', L, C', K, B' and M are on the same circle. We also know that $A'K$, $B'L$ and $C'M$ are three diameters of this circle.

The next step is to show that the three remaining points, D, E and F, are also on the same circle. Since $A'K$ is a diameter of the circle and $\measuredangle A'DK$ by definition is right, it follows immediately from Corollary 1 in Section 4.1 (or actually from its converse, which can also be shown is true) that D is on the circle. Likewise it follows that E and F are on the circle from the fact that $B'L$ and $C'M$ are diameters of the circle and $\measuredangle B'EL$ and $\measuredangle C'FM$ are right angles.

We have now proved the existence of the Nine-point Circle, but still remaining is its radius and center. Concider the two triangles $A'B'C'$ and KLM in Figure 16. According to SSS, $A'B'C' \sim ABC$ and $KLM \sim ABC$ (in length scale 1:2), since every side in each is exactly half the length of a corresponding side in ABC, and hence $A'B'C' \cong KLM$. We have proved above that they both have the Nine-point Circle of ABC as their circumcircle. We realise that one is derived from the other by a rotation through 180° about the center of the circle, since the sides $B'A'$ and KL are parallel. But this rotation through 180° must interchange the orthocenters of the two triangles, so the center of the Nine-point Circle must be situated in the midpoint between the two orthocenters. From Figure 16 it should be obvious that the orthocenter of KLM is H – the same as the orthocenter of ABC, whilst the orthocenter of $A'B'C'$ is O – the same as the *circumcenter* of ABC.

From this we conclude that the center of the Nine-point Circle is the midpoint of the segment joining the orthocenter with the circumcenter of the triangle, i.e. the Euler line. Further, since the Nine-point Circle is also the circumcircle of a triangle with sides half as long as the original triangle, the radius of the Nine-point Circle is half that of the circumcircle, and the proof is complete.

$\square$

Our next goal is to prove the theorem using complex numbers.

Proof using complex numbers. Let ABC be any triangle and place it in the complex plane such that its circumcenter is in the origin. Now all possible points concerning the triangle correspond to a certain complex number. As before we have $|A| = |B| = |C| = r \in \mathbb{R}$, where r equals the radius of the circumcircle. We define $H := A + B + C$ (which we know from the proof of Lemma 1 *(c)* in Section 3 to be the orthocenter of ABC) and by simple (real) vector geometry we get the following definitions, if we let the points A', B', C', D, E, F, K, L and M have the same meaning as above.

$$A' := \frac{B+C}{2} \quad \text{(and analogous for } B' \text{ and } C'\text{)}$$

$$K := \frac{A+H}{2} \quad \text{(and analogous for } L \text{ and } M\text{).}$$

We wish to prove that the distance from $\frac{1}{2}H$ to each of these points are $\frac{1}{2}r$, but note that D, E and F lacks obvious definitions. We begin with the distance between $\frac{1}{2}H$ and A', B' and C' respectively.

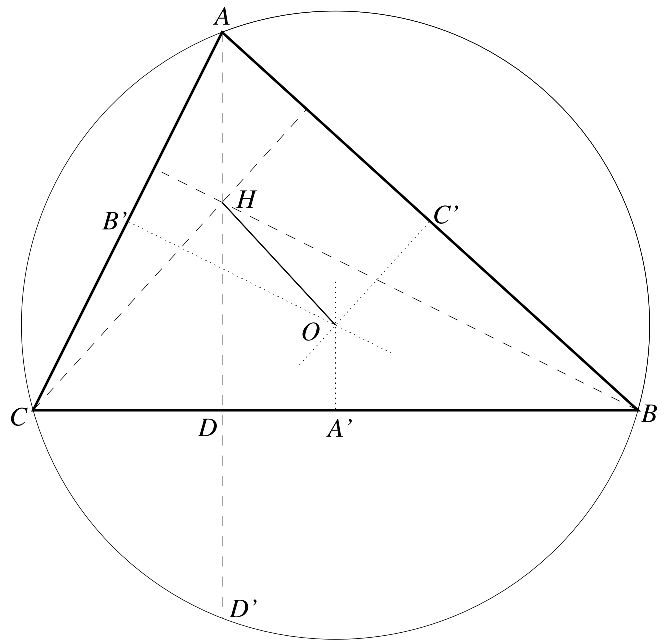

Figure 17: The triangle ABC and its circumcircle in the complex plane

First of all, note that the three cases are completely analogous, so it is enough to consider one of them. Let us choose the distance from $\frac{1}{2}H$ to A'.

$$\left|A' - \frac{1}{2}H\right| = \left|\frac{B+C}{2} - \frac{A+B+C}{2}\right| = \left|\frac{-A}{2}\right| = \frac{1}{2}r$$

and this part of the proof is done.

Our next goal is to show that the distance from $\frac{1}{2}H$ to each of K, L and M is $\frac{1}{2}r$. As above the three cases are analogous, and we may consider only one. Let us choose the distance $\left|K - \frac{1}{2}H\right|$.

$$\left|K - \frac{1}{2}H\right| = \left|\frac{A+H}{2} - \frac{H}{2}\right| = \left|\frac{A}{2}\right| = \frac{1}{2}r$$

and also this part of the proof is done with only very simple calculations so far. Yet to come though, is the tricky part – to show that also the distance from $\frac{1}{2}H$ to D, E and F equals $\frac{1}{2}r$.

Of course also these three cases are completely analogous, so we only need to consider one, say $\left|D - \frac{1}{2}H\right| = \frac{1}{2}r$. The tricky part is that we have no easy definition of D. Before we consider D further, let us look at another complex number, namely the one where the extension of the altitude from vertex A meets the circumcircle again. Let us call this number D' (see Figure 17). About D' we know that $\overrightarrow{AD'} \perp \overrightarrow{BC}$ and that $|D'| = r$. As a consequence of first Lemma 2 *(a)* in Section 3 and then (8) on page 20

$$\overrightarrow{AD'} \perp \overrightarrow{BC} \iff \frac{D'-A}{C-B} \text{ purely imaginary } \iff \frac{D'-A}{C-B}+\frac{\bar{D}'-\bar{A}}{\bar{C}-\bar{B}} = 0 \iff$$

$$\iff \frac{D'-A}{C-B} + \frac{r^2}{r^2}\cdot\frac{\frac{1}{D'}-\frac{1}{A}}{\frac{1}{C}-\frac{1}{B}} = 0 \iff \frac{D'-A}{C-B} + \frac{C}{D'}\cdot\frac{B}{A}\cdot\frac{A-D'}{B-C} = 0 \iff$$

$$\iff \frac{D'-A}{C-B}(1+\frac{C}{D'}\cdot\frac{B}{A}) = 0 \iff \frac{C}{D'}\cdot\frac{B}{A} = -1 \iff$$

$$D' = -\frac{C\cdot B}{A}. \tag{17}$$

Now we note the interesting fact that D' and H are equidistant from B (and C) since

$$|B-D'| = \left|B+\frac{C\cdot B}{A}\right| = \left|\frac{B}{A}(C+A)\right| = \frac{1}{1}\left|C+A\right| = |C+A|$$

and

$$|B-H| = |B-(A+B+C)| = |A+C|$$

and therefore the triangle HBD' is isosceles. Since DB by definition is perpendicular to HD', we know that D is the midpoint of the segment joining H with D'. Hence

$$D = \frac{1}{2}(H+D') = \frac{1}{2}(H-\frac{C\cdot B}{A})$$

and

$$\left|D-\frac{1}{2}H\right| = \left|\frac{1}{2}(H-\frac{C\cdot B}{A})-\frac{1}{2}H\right| = \frac{1}{2}\left|-\frac{C\cdot B}{A}\right| = \frac{1}{2}\cdot\frac{r^2}{r} = \frac{1}{2}r.$$

We have shown that all nine points are at the distance $\frac{1}{2}r$ from the midpoint of the segment joining the orthocenter with the circumcenter, hence the theorem is proved.

□

If we choose to place the triangle so that $\Im B = \Im C$, which implies $C = -\bar{B}$ and $D' = \bar{A}$, we can simplify the proof further. Note that this is possible without loss of generality. In this case we get the following proof of HBD' being isosceles.

$$HBD' \text{ isosceles} \iff \measuredangle HBC = \measuredangle D'BC \iff$$
$$\iff \arg\left(\frac{-\bar{B}-B}{H-B}\right) = -\arg\left(\frac{-\bar{B}-B}{\bar{A}-B}\right) \Longleftarrow \left(\frac{-\bar{B}-B}{H-B}\right) = \overline{\left(\frac{-\bar{B}-B}{\bar{A}-B}\right)} \iff$$
$$H - B = A - \bar{B} \iff A + C = A + C$$

which is obviously the case.

So we have seen two proofs of Theorem 3. One based on similar triangles, where the proof is two-parted. The first part is the proof of all nine points being on the same circle, a part where the main thought is to identify three quadrangles connecting six of the nine points as vertices in rectangles. This is done by, through similar triangles, identifying the sides of the quadrangles as being parallel to either a side or an altitude in the triangle.

The second part of the proof is the one identifying the center and radius of the circle. This part is based upon finding two triangles partly being similar to (and in length scale 1:2 of) the original triangle, partly having all vertices on the Nine-point Circle and partly being rotated 180° from each other. Then we must realise what this means for the center of the Nine-point Circle.

In the other proof, based on complex numbers, the approach is quite different. Instead of first finding the circle and then its radius and center, we suppose the center is known (or can be intelligently guessed) and try to show that all nine points are equidistant to the center. If we do not know it from the beginning, how do we come up with the idea of this point being an important one in the triangle? This is one of the tricky parts of this proofs. The idea of placing the circumcenter in the origin of the complex plane is not equally tricky, partly since that is already done in the proof of Lemma 1 and partly since this gives the property of all vertices in the triangle having the same modulus.

Once this is accomplished, the only obstacle left is really the crucial identification of the isosceles triangle. This is the only place we use properties of complex numbers, when we make use of the fact that the ratio of two perpendicular vectors is purely imaginary, and hence can be expressed as an equation using the complex conjugate. This way any of the endpoints of the two vectors can be expressed from the other three. This step would be very difficult not looking at the points as complex numbers.

Note how, in the first proof, the points A', B', C', K, L and M seem to be the tricky ones to handle, while D, E and F are easy to show the property for when the others are known. In the second proof it is the other way around. Could we combine the two proofs in a way to simplify them both? The answer is yes, and the crucial thing is to identify the diameters known in the first proof also in the second one. This combining proof is not to be found in any of the references.

Proof by combining the other two. Place any triangle ABC in $\mathbb{R}^2$ with its circumcenter in the origin. Note the following:

$$|A| = |B| = |C| = r,$$

$$H := A + B + C,$$
$$A' := \frac{B + C}{2} \quad \text{(and analogous for } B' \text{ and } C'\text{)},$$
$$K := \frac{A + H}{2} \quad \text{(and analogous for } L \text{ and } M\text{)}.$$

Now we investigate distances from $\frac{1}{2}H$.

$$\left|A' - \frac{1}{2}H\right| = \left|\frac{B + C}{2} - \frac{A + B + C}{2}\right| = \left|\frac{-A}{2}\right| = \frac{1}{2}r,$$

$$\left|K - \frac{1}{2}H\right| = \left|\frac{A + H}{2} - \frac{H}{2}\right| = \left|\frac{A}{2}\right| = \frac{1}{2}r.$$

$$\therefore \quad A',\ B',\ C',\ K,\ L \text{ and } M \text{ are on the pursued circle.}$$

Further note that $A'K$ (and $B'L$ and $C'M$) are diagonals of the circle since

$$|A' - K| = \left| \frac{B+C}{2} - \frac{A+H}{2} \right| = \frac{1}{2} |-2A| = |A| = r,$$

and hence D (and E and F) are on the circle by the reversion of Corollary 1 since $\measuredangle A'DK$ is right (as well as $\measuredangle B'EL$ and $\measuredangle C'FM$).

□

This last proof requires neither identification of similar or isosceles triangles nor any knowledge of complex numbers. It is very straightforward when Lemma 1 and Corollary 1 are used. Still it requires an intelligent guessing of where the center of the Nine-point Circle should be and some very basic knowledge of vector geometry.

4.4 The Morley trisector theorem

This theorem, the last one we will encounter in this thesis, was first discovered in the early 20th century[16] by Frank Morley (1860-1934), an English mathematician active at Cambridge university at the time. It was not published immediately after the discovery, but several years later. One interesting thing about this theorem is that it involves the trisection of an arbitrary angle, a construction that has been proved to be impossible using only straightedge and compass ([2] Beachy and Blair, 2006 chapter 6.3). So the equilateral triangle mentioned in the theorem can not be constructed, but nevertheless the theorem holds.

Theorem 4 (Morley's theorem)**.** *In any triangle, the intersections of the adjacent pairs of angle trisectors are vertices in an equilateral triangle.*

As with most theorems, there are several proofs of this one. We will only look at two of these, of which none is Morley's original one. The first is found

[16]Late 19th century, according to some sources.

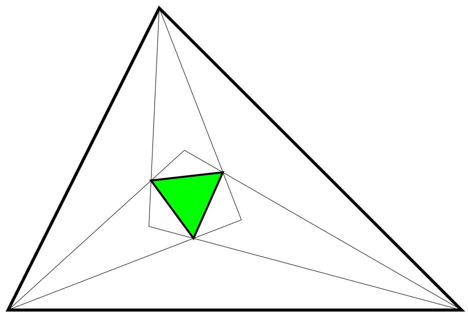

Figure 18: The filled equilateral triangle in Morley's theorem

in [4] Coxeter and Greitzer (1967, chapter 2.9) but was discovered by M. T. Naraniengar in 1909. Before we can look at the actual proof, however, we need a lemma.

Lemma 4. *If four points A, B, C and D satisfy*

$$|AB| = |BC| = |CD| \tag{18}$$

and

$$\measuredangle ABC = \measuredangle BCD > 60^\circ \tag{19}$$

then they are cocyclic. Let us denote $\measuredangle ABC$ $(= \measuredangle BCD)$ by $180^\circ - 2\alpha$ (this seems strange, but will eventually facilitate things). If another point, P, satisfies

$$\measuredangle APD = 3\alpha$$

and is on the opposite side of AD as B and C, then P is also on the same circle as A, B, C and D.

Before proving the lemma, let us do some heuristic reasoning as to why it is true. Concider the two middle points B and C. There are infinitely many circles passing through these points, but all of them are symmetric with respect to the perpendicular bisector of the segment BC. If we choose a point on one side of this perpendicular bisector we can always find a circle passing through it. Due to the construction of the uttermost points A and D, these are symmetric with respect to the perpendicular bisector mentioned,

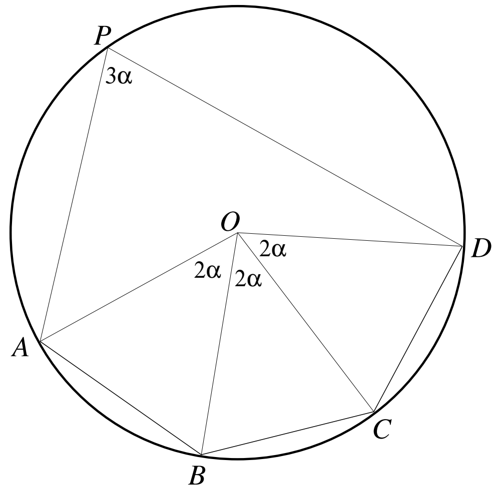

Figure 19: Five cocyclic points

and hence both are on the constructed circle. To find the fifth point P, we only need to choose one so that the angle $\measuredangle APD$ is adequate.

Proof of Lemma 4. Let the points A, B, C and D satisfy (18) and (19), and let the angle bisectors of $\measuredangle ABC$ and $\measuredangle BCD$ meet at O. Draw the segments OA and OD. The triangles AOB, BOC and COD are congruent by SAS. Furthermore they are isosceles since BOC obviously is so. The angles at O are $180° - 2(90° - \alpha) = 2\alpha$. Since $|OA| = |OB| = |OC| = |OD|$ the first part of the lemma is proved, and the second part follows from the reversion of the Inscribed angle theorem (Theorem 1 in Section 4.1).

□

In fact we will also need the Inscribed angle theorem itself as a lemma when proving Theorem 4. The structure of the proof is to first construct an equilateral triangle with one vertex in the intersection of two trisectors, and then prove that the other two vertices are on the trisectors of the third angle.

Proof by Naraniengar. Let ABC be any triangle and let the intersections of the trisectors of A and B be called P and Q (Q being the the intersection

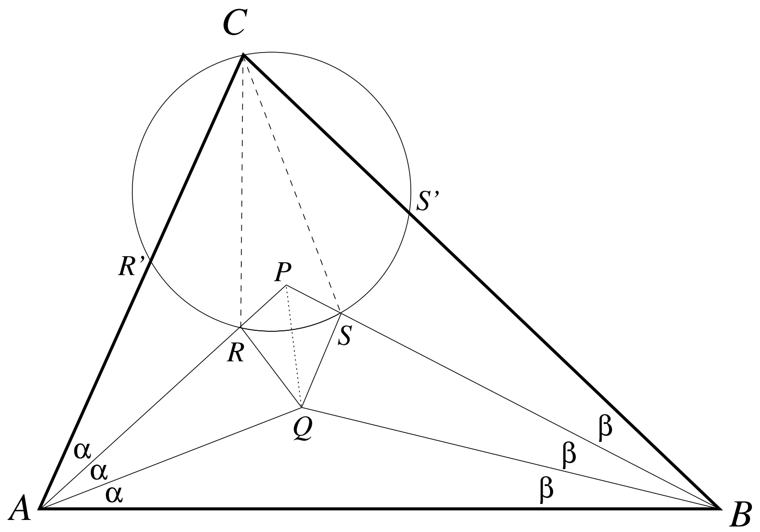

Figure 20: Construction of the equilateral triangle in Morley's theorem

of the adjacent trisectors). Note how Q is on the bisectors of $\measuredangle PAB$ and $\measuredangle PBA$, and hence is the incenter of ABP. Lemma 1 *(b)* in Section 3 now gives that PQ is the angle bisector of $\measuredangle APB$. Now construct the points R and S, on the line segments AP and BP respectively, so that $\measuredangle PQR = \measuredangle PQS = 30°$.[17] By ASA we have that $PQR \cong PQS$, and hence QRS is an equilateral triangle since $\measuredangle RQS = 60°$.

Another result is that RPS is isosceles, and by regarding the angle sum in APB and RPS (and noting that $\measuredangle RPS = \measuredangle APB$) we get $\measuredangle SRP = \measuredangle RSP = \alpha + \beta$. We can define the angle C to be 3γ, and then get, from the angle sum in ABC, $3\alpha + 3\beta + 3\gamma = 180°$, or equivalently $\alpha + \beta = 60° - \gamma$. So we know $\measuredangle SRP = 60° - \gamma$ and therefore

$$\measuredangle QRP = \measuredangle QSP = 120° - \gamma.$$

Next we define the points R' and S', on AC and BC respectively, so that

[17]Note how this is possible if and only if $\measuredangle AQP > 30°$ and $\measuredangle BQP > 30°$. This is the case since $\measuredangle AQP = 180° - \measuredangle QPA - \alpha$, and $\measuredangle QPA < 90°$ (since P is vertex in a triangle and PQ is the angle bisector of that vertex) and $3\alpha < 180°$ so $\alpha < 60°$, and analogously for $\measuredangle BQP$.

$|AR'| = |AQ|$ and $|BS'| = |BQ|$. By SAS we get that $ARQ \cong ARR'$ and $BSQ \cong BSS'$, and hence $|R'R| = |RS| = |SS'|$. What we further need, in order to use Lemma 4, is an evaluation of the angles $\measuredangle R'RS$ and $\measuredangle RSS'$. Since $ARQ \cong ARR'$, and P is on the extension of AR, also $APQ \cong APR'$, and then obviously also $PRQ \cong PRR'$. Hence $\measuredangle SRR' = \measuredangle PRR' + \measuredangle SRP = \measuredangle PRQ + \measuredangle SRP = (120° - \gamma) + (60° - \gamma) = 180° - 2\gamma$, and by the same reasoning $\measuredangle RSS' = 180° - 2\gamma$. Now Lemma 4 gives that the points R', R, S, S' and C are cocyclic, and further Theorem 1 gives that $\measuredangle R'CR = \measuredangle RCS = \measuredangle SCS' = \gamma$, since all the chords $R'R$, RS and SS' are of the same length. The proof is complete.

□

Next we will prove the theorem with aid from complex numbers as done in [6] Hahn (1994, chapter 2.9). The calculations will be longer than in the proof above, but we will not let this discourage us since they are also quite straight ahead. In fact the resolving ideas in this proof is simpler than the ones in Naraniengar's proof above. First we need a lemma.

Equation (10) on page 23 gives us the equation of the line through z_1 and z_2. If z_1 and z_2 further are on the unit circle, we may use that, by (8) on page 20, $\bar{z}_i = \frac{1}{z_i}$, $i = 1, 2$, which reduces (10) to

$$z + z_1 z_2 \bar{z} = z_1 + z_2.$$

So, if the four points z_1, z_2, z_3 and z_4 are on the unit circle, the point of intersection of the lines (extension of the chords) through z_1 and z_2 and through z_3 and z_4 is the solution to the system

$$\begin{cases} z + z_1 z_2 \bar{z} = z_1 + z_2 \\ z + z_3 z_4 \bar{z} = z_3 + z_4 \end{cases}.$$

Subtracting the second equation from the first gives

$$\bar{z}(z_1 z_2 - z_3 z_4) = z_1 + z_2 - z_3 - z_4 \iff \bar{z} = \frac{z_1 + z_2 - z_3 - z_4}{z_1 z_2 - z_3 z_4},$$

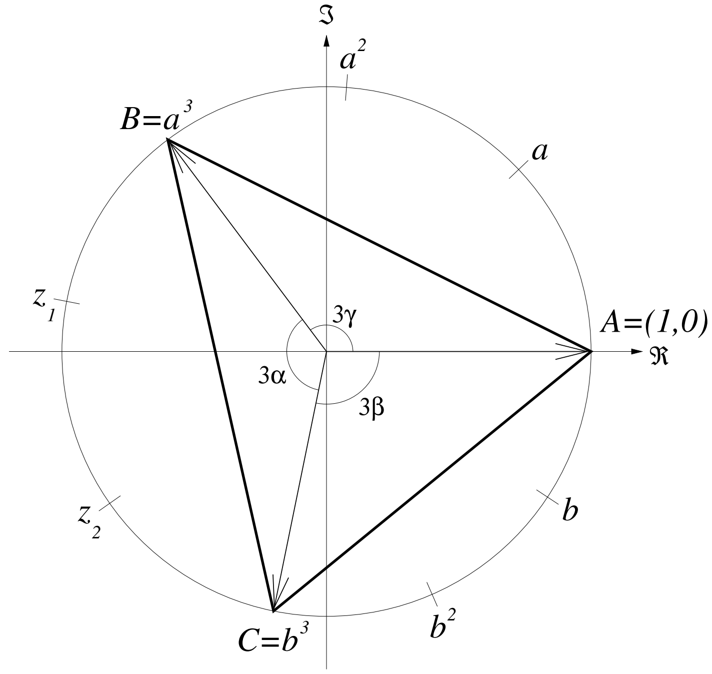

Figure 21: An arbitrary triangle in the complex plane

and by taking the complex conjugate of each side we have come to the following lemma.

Lemma 5. *If* $|z_1| = |z_2| = |z_3| = |z_4| = 1$*, then the extension of the chords joining* z_1*,* z_2 *and* z_3*,* z_4 *respectively meet at*

$$z = \frac{\bar{z}_1 + \bar{z}_2 - \bar{z}_3 - \bar{z}_4}{\bar{z}_1\bar{z}_2 - \bar{z}_3\bar{z}_4}.$$

Now we are ready to prove the theorem.

Proof of Theorem 4 using complex numbers. Without loss of generality, we may place every vertex of the arbitrary triangle ABC on the unit circle in the complex plane, and we choose to place vertex A in the point $(1, 0)$. Further the notation

$$\measuredangle AOB = 3\gamma, \qquad \measuredangle AOC = 3\beta, \qquad \measuredangle AOB = 3\alpha = 360^\circ - (3\gamma + 3\beta)$$

will be used. Note how α can be expressed in terms of β and γ as

$$\alpha = 120^\circ - \gamma - \beta. \tag{20}$$

We are interested in the trisection of the angles at A, B and C, which is the vector from the vertex to the points trisecting the 'opposite arc' (another consequence of Theorem 1). Therefore we interest ourselves in the points trisecting the arcs $\overset{\frown}{AB}$ (not containing C), $\overset{\frown}{AC}$ (not containing B) and $\overset{\frown}{BC}$ (not containing A). Let a be the trisecting point nearest to A on $\overset{\frown}{AB}$. Then the other trisecting point is a^2, due to $|a| = 1$ and Lemma 2 in Section 3. Analogously the points trisecting $\overset{\frown}{AC}$ are b and b^2, and note how $\arg b = -\beta$. Note further how $B = a^3$ and $C = b^3$. However, the points trisecting $\overset{\frown}{BC}$, referred to as z_1 and z_2 in Figure 21, are not as easily expressed. We have $|z_1| = |z_2| = 1$ and

$$\arg z_1 = 3\gamma + \alpha = 2\gamma - \beta + 120° = 2\arg a + \arg b + \arg \omega = \arg(a^2 b\omega),$$

$$\arg z_2 = -3\beta - \alpha = -2\beta + \gamma + 240° = 2\arg b + \arg a + \arg \omega^2 = \arg(ab^2\omega^2),$$

where we have used (20), Lemma 2 and (11) on page 24. Remember the definition of ω from Section 3, i.e. $1 + \omega + \omega^2 = 0$. Now, since equal in both modulus and argument, we have $z_1 = a^2 b\omega$ and $z_2 = ab^2\omega^2$, and all that is left is to use Lemma 5 to calculate the interesting points of intersection and then use (12) on page 25 to show that this triangle is equilateral.

Let D, E and F be the intersections of the adjacent trisectors of B, C; A, B and A, C respectively. Then, by Lemma 5, we get

$$D = \frac{\bar{a^2} + \bar{b^3} - \bar{b^2} - \bar{a^3}}{\bar{a^2}\bar{b^3} - \bar{b^2}\bar{a^3}} = \frac{a^{-2} + b^{-3} - b^{-2} - a^{-3}}{a^{-2}b^{-3} - b^{-2}a^{-}3} =$$

$$= \frac{b^{-3}a^{-3}(ab^3 + a^3 - a^3 b - b^3)}{b^{-3}a^{-3}(a-b)} = \frac{ab^3 + a^3 - a^3 b - b^3}{a-b} =$$

$$= \frac{\overbrace{(a-b)(a^2 + ab + b^2)}^{a^3 - b^3} + \overbrace{ab(b^2 - a^2)}^{ab^3 - a^3 b}}{a-b} =$$

$$= \frac{(a-b)\big[(a^2 + ab + b^2) - ab(a+b)\big]}{a-b} = (a^2 + ab + b^2) - ab(a+b),$$

$$E = \frac{\bar{1} + \overline{a^2 b\omega} - \bar{a^3} - \bar{b}}{\overline{a^2 b\omega} - \bar{a^3}\bar{b}} = \frac{1 + a^{-2}b^{-1}\omega^{-1} - a^{-3} - b^{-1}}{a^{-2}b^{-1}\omega^{-1} - a^{-3}b^{-1}} =$$

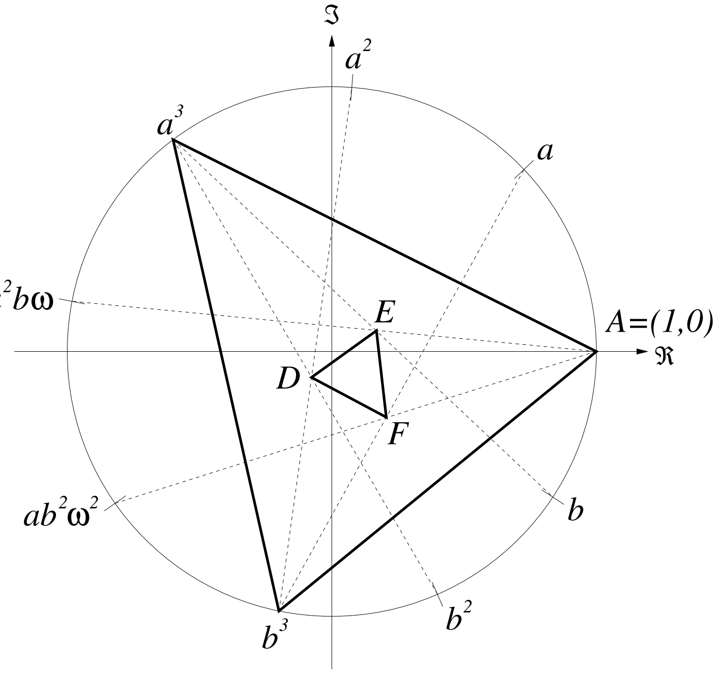

Figure 22: The angle trisectors of an arbitrary triangle in the complex plane

$$
\begin{aligned}
&= \frac{a^3b + a\omega^2 - b - a^3}{a\omega^2 - 1} = \frac{b(a^3 - \omega^3) + a(\omega^2 - a^2)}{\omega^2(a - \omega)} \\
&= \frac{b(a^2 + a\omega + \omega^2) - a(a + \omega)}{\omega^2} = \omega\big[b(a^2 + a\omega + \omega^2) - a(a + \omega)\big] \\
F &= \ldots \text{similar calculations} \ldots = \omega^2\big[a(b^2 + b\omega^2 + \omega) - b(b + \omega^2)\big].
\end{aligned}
$$

According to (12), and considering the orientation, $\triangle DEF$ is equilateral if and only if $D + F\omega + E\omega^2 = 0$, in turn equivalent to

$$a^2+ab+b^2-ab(a+b)+a(b^2+b\omega^2+\omega)-b(b+\omega^2)+b(a^2+a\omega+\omega^2)-a(a+\omega) = 0,$$

which is what we need to show. Multiplying into the parenthesis and rearranging the left hand side we get

$$a^2 - a^2 + b^2 - b^2 + ab(1 + \omega^2 + \omega) + a^2b - a^2b + ab^2 - ab^2 + a\omega - a\omega + b\omega^2 - b\omega^2$$

which is obviously equal to zero, and the proof is complete.

$\square$

The last theorem we will make acquaintance with has now been proved twice. The proof by Naraniengar is based upon a construction of an equilateral triangle symmetric about the line passing through the intersections of the adjacent and remote trisectors of two vertices respectively. Once this construction is at place the proof uses the theory of similar triangles (again by contruction of new points) several times and the angle sum in a triangle, as well as – the quite intuitive – Lemma 4. One severe pedagogical obstacle is the lack of motivation for the contructions made. For instance, it is far from obvious that the point P in Figure 18 should be relevant.

The motivation is far more persuasive in the second proof, with complex numbers. The reasoning might be something like 'we know the angle bisectors in any triangle are interesting since they are concurrent. Is there anything interesting about the angle trisectors? Let us place the triangle in the complex plane (here we have a limited number of rational choices), extract expressions for these vectors and use our tool for calculating points of intersection to see what wee get', and this is basically the entire proof. The "tool" mentioned is of course Lemma 5. Note how easy it would be to generalise the reasoning to the angle n-section of an arbitrary triangle, something not possible in the proof by Naraniengar.

So, why will it not do with ordinary vector representation of the vertices in the triangle, why is it important to use complex numbers in the last proof? We find three striking things, first the simple expressions of the trisection of the arcs $\overset{\frown}{AB}$ and $\overset{\frown}{AC}$ in Figure 21, then the possibility of extracting the expressions for z_1 and z_2 and also the easy way presented in (12) of determining whether given points are vertices in an equilateral triangle. The first two are variations of Lemma 2. Further there is no counterpart to Lemma 5 in the non complex case.

We have now discussed different proofs of four plane geometric theorems, and especially the use of complex numbers. What is left is a summation of our results to see what conclusions can be drawn.

5 Summing up and Conclusions

The four theorems presented in the previous section have two things in common, they are of plane geometric nature and they can be proved by use of complex numbers. We will try to generalise the solvent thought in the different proofs, looking for patterns.

Starting with the proofs based on classic geometry, one thing is striking – the need for constructions of new objects, may they be points, lines, triangles or other. In the Inscribed angle theorem we start by constructing a special case by placing a point where we need it to be. In the first proof of the formulae for sin and cos, we construct congruent triangles with vertices in the origin and on the unit circle. In the Nine-point Circle we construct three rectangles by identifying similar triangles in the first part, and then two congruent triangles are constructed in the second part. Last, in Morley's theorem, there are several constructions of triangles and points in the proof by Naraniengar.

A second striking thing these proofs have in common is the identification of similar and/or congruent triangles, either they exist in the first place or are the results of constructions. Such identifications are present in the proofs of the last three theorems. Also the angle sum of a triangle seems to be reappearing.

When it comes to the proofs using some other branch of mathematics, like linear algebra or complex numbers, the beginning is to realise we are dealing with something familiar from that branch. Two examples are to connect the angle between two lines of arbitrary length with the definition of dot product of vectors in the formulae for sin and cos, and to look at the trisection of an angle as the argument of a complex number raised to the power three as done in the proof of Morley's theorem. When it comes to the proofs using complex numbers we note that Lemma 2 in Section 3, connecting the product of two complex numbers with the sum of their arguments and product of their modulus, seems to be very applicable, since it is represented in proofs of all four theorems. Further the results (6) and (7), page 20, on addition and multiplication of a complex number with its conjugate, and thereby also the concept of complex conjugate, are important tools.

When using complex numbers, it also seems to be fruitful to place as many interesting points as possible in the complex plane such that they have modulus 1. If the interesting points are the vertices of a triangle, this means that the circumcenter of the triangle is in the origin and the circumcircle is in fact the unit circle. In doing this, we have infinitely many options of which basically three are smarter, each represented in a proof in the previous section. The first one, represented in the proof of the Inscribed angle theorem, is to place two of the points such that they are the conjugates of each other, i.e. if the points are z and w then $\Re z = \Re w$, giving $\arg z = -\arg w$. The other alternative, found in the proof of the Nine-point Circle, is to place two of the points, say z and w, such that $\Im z = \Im w$, resulting in $z = -\bar{w}$. The last smart positioning of the interesting points is to simply let one point be the simplest possible, namely $(1, 0)$. This gives simple expressions for the arguments of the other two interesting points, something which is used in the proof of Morley's theorem. Which of these three alternatives is best, is determined by the situation, but at least it leaves us with a very limited number of options to work our way through.

Once we have identified the tool we wish to use and have placed the problem in the complex plane in a suitable way, the rest of the proof is often just algebraic calculations. This is definitely the case in the proof of the Inscribed angle theorem and Morley's theorem. One exception is the proof of the Nine-point Circle, where some creativity is needed in the construction of yet another point. However, we have seen how this theorem can be proved in an even simpler way using only real vector algebra.

We conclude that when proving a plane geometric theorem, we often have two choices. We can find a way to construct geometrical objects so that the theorem follows from congruence or similarity, or is obvious; or we can identify something in the problem which we know we have a fitting tool for, like for instance a linear mapping, dot product or product of complex numbers. The former often claims an original idea and results in a beautiful proof which is easy to follow but difficult to come up with or reproduce. In the latter, complex numbers are often very powerful – mostly due to the result presented in Lemma 2, and the proof is often based upon straightforward calculations which might be difficult to follow but relatively easy, yet time-consuming, to perform.

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