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Linear preservers of hyperbolic and stable polynomials

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LINEAR PRESERVERS OF HYPERBOLIC AND STABLE POLYNOMIALS

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Abstract. In this thesis we study linear operators on the polynomial space $\mathbb{C}[z]$ that preserve the set of hyperbolic polynomials. A hyperbolic polynomial is one with all real zeros (hence an element of the Laguerre-Pólya class). We present some well known results such as the Gauss-Lucas Theorem and the Laguerre separation Theorem and we discuss their importance in view of our topic. The main purpose of this thesis is to describe all finite order linear differential operators with polynomial coefficients that are hyperbolicity preserving (HPO). Quite recently some breakthrough results regarding this have been made by Borcea, Brändén and Shapiro. This has been accomplished by using properties of the Weyl algebra and the well known example of a Hilbert space - the Fischer-Fock space. Finally experiments are made to test a conjecture that states that all HPOs also preserve the property of classical majorization. We also give some attention to similar results concerning stability preserving operators - SPOs - i.e. operators that preserve stable polynomials. A stable polynomial is one with all zeros in the left half of the complex plane. This study will be restricted to the one-variable case even if a lot of the theory that we present extends to the multivariate case.

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Contents

1. Introduction to the problem

We shall begin our exposé by giving some well known results. The motivation for reviewing the well-established theory of the geometry of polynomials is to build a solid and comprehensive structure before we initiate the reader to some recent results on so called hyperbolistic preservers due to Borcea, Brändén and Shapiro. It is crucial that we emphasize in what sense these known results that we present will be of great importance for the problems we encounter later on. To this day some very fundamental questions and problems in the area considered are still open. We now take a look at some of these, so far, open problems.

Let $S \subseteq \mathbb{C}$ denote a certain set of interest and π_n the vector space of all polynomials, $p(x)$ of at most degree n, then by $\pi_n(S)$ we mean the class of all polynomials of degree at most n whose zeros lie in S.

Problem 1. Characterize all linear operators T on $\mathbb{C}$ that preserve the set $\pi_n(S)$. Or with the notation above, characterize all linear operator T s.t.

$$T : \pi_n(S) \to \pi_n(S)$$

assuming, for simplicity, that $\deg T[p] \leq \deg p(x)$.

Despite its long history and numerous efforts Problem 1 has not yet been solved when the set containing the zeros, let us call it S, is given by important convex sets such as a sector centered in the origin or a strip. Only very recently this problem was solved in the case when S is a closed circular domain (defined in next section) or the boundary of such a domain [5]. The classical Gauss-Lucas Theorem addresses this problem in the special case where $T = \frac{d}{dx}$ and S is a convex region in $\mathbb{C}$. If S is the open upper halfplane the Hermite-Biehler Theorem provides a characterization of polynomials whose zeros lies in S and if S is the left halfplane the Hurwitz polynomials (all real polynomials whose zeros lies in the left halfplane) are of relevance. New results in this last case would be of importance for a lot of areas in applied mathematics such as for example the theory of dynamic stability. Another open problem is this:

Problem 2. Characterize all linear operators T on $\mathbb{C}$ s.t. the number of nonreal zeros of $T(P(x))$ are less then or equal to the numbers of nonreal zeros of $P(x)$ for any real polynomial $P(x)$ (i.e the Taylor coefficients are real).

When $T = D$ as above this follows from Rolle's Theorem and if $q(x)$ is a polynomial with only real zeros and $T = q(D)$ this is a consequence of the classical Hermite-Poulain-Jensen Theorem.
We could go on in this birds-eye-view manner to get to our goal faster, but as promised we shall investigate the foundations on which we shall rely on as we go further down this path. We assume the reader to be familiar with some complex analysis, so let us, without further ado, go ahead with some important basic definitions and results.

2. The Gauss-Lucas Theorem and its consequences

We begin this section by giving some preliminaries. First of all we define the *Möbius transformations*.

Definition 1. A one-to-one mapping of the extended complex plane of the form

$$\mu(z) : z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}$$

with the restriction $\alpha\delta \neq \beta\gamma$ is known as a Möbius transformation. Sometimes $f(z)$ is also referred to as a *fractional linear transformation* or a *bilinear transformation*

The Möbius transformation is one of the most important *conformal mappings*. A conformal mapping is a mapping that preserves angles and can be interpreted as compositions of translations, magnifications, rotations or inversions. The Möbius transformations are bijective and form a group and even more surprisingly it maps the set of all lines and circles on itself. Finally we mention that the inverse or composition of any Möbius transformation is again a Möbius transformation. We use the Möbius mappings to define *Circular domains*.

Definition 2. Any subset of $\mathbb{C} \cup \infty$ is called a *circular domain* if it is the image of the closed or open unit disc under a Möbius transformation.

Hence, a circular domain is either a the interior or the exterior of a disc or a halfplane so the complement of any circular domain is again a circular domain. Next we take a look at the concept of *critical points*. Consider a polynomial f of degree n and let $\zeta \in \mathbb{C}$ then by Taylor's theorem we have

$$f(z) = f(\zeta) + (z - \zeta)f'(\zeta) + g(z)$$

where g is a polynomial and $g(\zeta) = g'(\zeta) = 0$. If $f'(\zeta) \neq 0$ we can choose $r > 0$ such that for any two points z_1 and z_2 in the disc

$$\mathcal{D} = \{z \in \mathbb{C} : |z - \zeta| < r\}$$

we have

$$|g(z_1) - g(z_2)| = \left| \int_{z_1}^{z_2} g'(z)dz \right| < |f'(\zeta)| \cdot |z_1 - z_2|$$

since g' is continuous and $g'(\zeta) = 0$ and $f'(\zeta) \neq 0$. This implies that f is univalent in $\mathcal{D}$ and maps it conformally onto $f(\mathcal{D})$. If indeed $f(z_1) = f(z_2)$ for two distinct points $z_1, z_2 \in \mathcal{D}$ it would imply that $g(z_1) - g(z_2) = (z_1 - z_2)f'(\zeta)$ which would contradict the assumption that $|g(z_1) - g(z_2)| < |f'(\zeta)| \cdot |z_1 - z_2|$.
Hence the local conformity breaks down when ζ is a zero of f' and this is the reason why ζ is called a critical point.

A more formal definition of critical points is the following:

Definition 3. Let f be a polynomial (or more generally a meromorfic function), we then call the points where f' vanishes (i.e zeros of the derivative) *critical points*.
A type I critical point is *also* a zero of f and a type II critical point is *only* a zero of f', o if ζ is a type II critical point $f(\zeta) \neq 0$.

In view of the above we are ready to state the famous Gauss-Lucas theorem already mentioned in the beginning of this section.

2.1. **The Gauss-Lucas Theorem.** The Gauss-Lucas theorem is of great importance and many of the results following in this survey will be based on it.

Theorem 1 (Gauss-Lucas theorem)**.** *Every convex set containing all the zeros of a polynomial also contains all its critical points.*

Proof. We begin by showing that if the zeros of a polynomial lie in a closed half-plane $\mathcal{H}$ then so do all the critical points. Let this half-plane be

$$\mathcal{H} := \{z \in \mathbb{C} : \Re(e^{i\alpha}z) \leq b\}$$

If $z \notin \mathcal{H}$ then

$$\begin{aligned} \Re\Big(e^{-i\alpha}\frac{f'(z)}{f(z)}\Big) &= \Re\sum_{\nu=1}^{n}\frac{e^{i\alpha}}{\bar{z}-\bar{z_\nu}} = \sum_{\nu=1}^{n}\Re\frac{e^{i\alpha}(z-z_\nu)}{|z-z_\nu|^2} \\ &= \sum_{\nu=1}^{n}\Re\frac{(e^{i\alpha}z-b)-(e^{i\alpha}z_\nu-b)}{|z-z_\nu|^2} > 0. \end{aligned}$$

Hence $f'(z) \neq 0$ for $z \notin \mathcal{H}$, and so all the critical points must lie in $\mathcal{H}$. If z belongs to to the boundary of $\mathcal{H}$ and is not a zero of f but there is at least one zero of f in the interior of $\mathcal{H}$ then $f'(z) \neq 0$ as well.
Having proved that the statement applies when the convex set mentioned is a half-plane we consider the smallest convex set containing all the zeros of f. Now this set is exactly a polytope, i.e. the intersection of all half-planes which contains the zeros. Thus the statement holds. □

Recall that a set is *convex* if it contains the line segment between any two points in the set. The convex hull (of a set of point) is the smallest convex set that contains the given set. If we denote the convex hull of the zeros of f by $\mathcal{K}(f)$ we may reformulate the theorem by the following statement:

Theorem 2. *For every polynomial f, we have $\mathcal{K}(f') \subseteq \mathcal{K}(f)$.*

If we think of an arbitrary point in $\mathcal{K}(f)$ as a convex linear combination of its extreme points (i.e the zeros of f) it should be obvious that any critical point of f can be expressed as a convex linear combination of its zeros. Furthermore a critical point that is not a zero of f is an interior point of $\mathcal{K}(f)$ unless $\mathcal{K}(f)$ is a line segment.

Theorem 3. *The notion of circular domains provides the following equivalent formulation of the results above.*

(i) *Every circular domain containing all the zeros of a polynomial f, but not the point at ∞, contains all the critical points of f.*
(ii) *Let f be a polynomial of degree $n \geq 2$ and ζ a type II critical point (i.e. $f'(\zeta) = 0$ but $f(\zeta) \neq 0$). Furthermore let $\mathcal{L}$ be any straight line passing through ζ. Then the open half-planes whose boundary is $\mathcal{L}$ both contain at least one zero of f unless the zeros all lie on $\mathcal{L}$.*

In the special case when the polynomial considered has real coefficients, we may state some results similar to the Gauss-Lucas theorem. For example a zero of multiplicity $m \geq 2$ is obviously also a critical point of multiplicity $m-1$. Moreover, if f is real-valued on the real line then by Rolle's theorem

there is at least one real critical point between any two consecutive real zeros. Thus the number of non-real critical points of a polynomial with real coefficients cannot be larger then the number of non-real zeros.
We will use the fact that non-real zeros occur in conjugate pairs to derive an interesting result not covered by the Gauss-Lucas theorem. To be able to do that we define *Jensen discs*.

Definition 4 (Jensen discs)**.** Let f be a polynomial with real coefficients and let $z_1, ...z_n$ be the zeros of f which lie in the open upper halfplane. The discs

$$\mathcal{D}_\nu := \{z \in \mathbb{C} : |z - \Re(z_\nu)| \leq \Im(z_\nu)\}, \qquad \nu = 1, ..., m$$

are called the *Jensen discs* of f.

Theorem 4 (Jensen)**.** *Let f be a polynomial with real coefficients. Then the non-real critical points of f lie in the union of all Jensen discs of f.*

Proof. Let $f(z) := c\prod_{\nu=1}^{n}(z - z_\nu)$. Denoting the real and imaginary parts of z_ν by x_ν and y_ν respectively, and those of z by x and y, we find that

$$\begin{aligned}\Im\Big(\frac{f'(z)}{f(z)}\Big) &= \sum_{\Im z_\nu = 0} \Im\Big(\frac{1}{z - z_\nu}\Big) + \sum_{\Im z_\nu > 0} \Im\Big(\frac{1}{z - z_\nu} + \frac{1}{z - \bar{z_\nu}}\Big) \\ &= -y\Big(\sum_{\Im z_\nu = 0} \frac{1}{|z - z_\nu|^2} + 2\sum_{\Im z_\nu > 0} \frac{(x - x_\nu)^2 + y^2 - y_\nu^2}{|z - z_\nu|^2 \cdot |z - \bar{z_\nu}|^2}\Big).\end{aligned}$$

As such, if z is a non-real point outside all the Jensen discs of f, then

$$sgn\, \Im\Big(\frac{f'(z)}{f(z)}\Big) = -sgn\, y$$

and hence $f'(z) \neq 0$. This completes the proof of the theorem. □

A consequence that can be derived from the equations above is that a non-real critical point of the second kind lies in the interior of at least one of the Jensen discs unless it is a boundary point of each of them. In the latter case f cannot have any real zeros.
We now state some results with nice geometric interpretation. The following corollaries can be regarded as separation theorems.

Corollary 1. *Let f be a polynomial with real coefficients. Suppose that x^* is a point on the real line lying outside all of the Jensen discs of f. If $f(x^* = 0$ then, in each of the halfplanes*

$$\mathcal{H}_1 := \{z \in \mathbb{C} : \Re z < x^*\} \quad and \quad \mathcal{H}_2 := \{z \in \mathbb{C} : \Re z > x^*\}$$

the number of zeros is the same as the number of critical points.

Corollary 2. *Let f be a polynomial with real coefficients. Suppose that x^* is a point on the real line lying outside all of the Jensen discs of f. If $f(x^* \neq 0$ then, in each of the halfplanes*

$$\mathcal{H}_1 := \{z \in \mathbb{C} : \Re z < x^*\} \quad and \quad \mathcal{H}_2 := \{z \in \mathbb{C} : \Re z > x^*\}$$

the number of zeros is at least as large as the number of critical points, but can exceed it only by one.

Corollary 3. *Let f be a polynomial with real coefficients, let a and b, where $a < b$, be two points on $\mathbb{R}$ lying outside all the Jensen discs of f. Denote by m the number of zeros and m' the number of critical points in the strip $\{z \in \mathbb{C} : a < \Re z < b\}$. Then*

(i) $m' = m + 1$ *if* $f(a) = 0$ *and* $f(b) = 0$.
(ii) $m \leq m' \leq m + 1$ *if* $f(a) = 0$ *or* $f(b) = 0$.
(iii) $m - 1 \leq m' \leq m + 1$ *if* $f(a) \neq 0$ *and* $f(b) \neq 0$.

There is more to be said about this type of results. For example we mention the analogue of Jensen's theorem for finite differences due to de Brujin. However we won't examine them now but skip ahead in our survey of results necessary for our goal. In the next section we will come across a useful result due to Laguerre.

2.2. **Laguerre's separation theorem.** The Gauss-Lucas theorem and the definition of critical points motivates us to introduce the notion of the *polar derivative.*

Consider a polynomial $f(z)$ of degree $n \geq 1$. If ψ is a Möbius mapping that is not affine then $f(\psi(z))$ is no longer a polynomial. Since ψ is not an affine mapping it can be written as

$$\psi(z) = \frac{\alpha z + \beta}{z + \delta}.$$

Thus we define

$$g(z) = (z+\delta)^n f(\psi(z))$$

to avoid difficulties dealing with a meromorfic function that is not a polynomial. The derivative of g (after simplifications) is:

$$g'(z) = n(z+\delta)^{n-1}(nf(\psi(z)) - (\psi(z) - \alpha)f'(\psi(z)).$$

If ζ is a critical point of g then either $\psi(\zeta)$ is a zero of $F(\alpha, z) := nf(z) - (z-\alpha)f'(z)$ or $\zeta = -\delta$.
Hence the operator of relevance is

$$D_\alpha := n - (z-\alpha)\frac{d}{dz}.$$

Thus we make the following definition

Definition 5. $F(\alpha, z) := nf(z) - (z-\alpha)f'(z)$ is known as *Laguerre's polar derivative* of f with respect to α.

We have the following properties for $F(\alpha, z)$.

Proposition 1. *Let f be a polynomial, f of degree $n \geq 1$ and $F(\alpha, z)$ its polar derivative with respect to α as given above. Let ψ and g be as given above. Then the following statements hold:*

(i) *If z^* is a zero of g then $\psi(z^*)$ must be a zero of f.*
(ii) *If ω is a zero of f then either $\psi^{-1}(\omega)$ is a zero of g or $\omega = \alpha$ and $f(\alpha) = 0$.*
(iii) *If ζ is a critical point of g then either $\psi(z)$ is a zero of $F(\alpha, \bullet)$ or $\zeta = -\delta$ and $f^{(n-1)}(\alpha) = 0$.*
(iv) *If ω is a zero of $F(\alpha, \bullet)$ then either $\psi^{-1}(\omega)$ is a critical point of g or $\omega = \alpha$ and $f(\alpha) = 0$.*

Theorem 5 (Laguerre's separation theorem)**.** *Let f be a polynomial of degree $n \geq 2$ and $\alpha \in \mathbb{C}$*

(i) *A circular domain, $\mathcal{K}$, containing the zeros of f but not α also contains the zeros of the polar derivative, $F(\alpha, z) = nf(z) - (z-\alpha)f'(z)$*
(ii) *Let $\zeta \neq \alpha$ be a zero of $F(\alpha, z)$ such that $f(\alpha) \neq 0$. Then every circle $\mathcal{C}$ passing through α and ζ separates at least two zeros of f unless the zeros all lie on $\mathcal{C}$.*

In other words the first part of this theorem states that if the complement, $\mathcal{K}^c$ of $\mathcal{K}$ is devoid of zeros of f, and if α lies in $\mathcal{K}^c$ then $\mathcal{K}^c$ is also devoid of zeros of the polar derivative with respect to α.
We can reformulate this theorem in an equivalent form as follows:

Theorem 6 (Laguerre's theorem reformulated)**.** *If f is a polynomial of degree $n \geq 2$ and $\mathcal{K}$ is a arbitrary circular domain devoid of zeros we have that:*

$$nf(z) - (z - \alpha)f'(z) \neq 0$$

when $z, \alpha \in \mathcal{K}$

Furthermore, if $\zeta \in \mathbb{C}$ is neither a zero nor a critical point of f, then every circle $\mathcal{C}$ that passes through ζ and $\zeta - n\frac{f(\zeta)}{f'(\zeta)}$ separates at least two zeros of f unless all zeros lie on the circle $\mathcal{C}$ (the proof of the latter is omitted).

Proof. Let $\mu : z \mapsto \frac{1}{\zeta - w}$ and $\mathcal{E} = \mu(\mathbb{C} \setminus \mathcal{K})$ (i.e the image of the complement of $\mathcal{K}$ under the Möbius map μ). Now $\mathcal{E}$ is of course a circular domain and hence one of the following must be true:

(1) $\mathcal{E}$ is the interior of a disc
(2) $\mathcal{E}$ is the exterior of a disc
(3) $\mathcal{E}$ is a halfplane

The image of $\mathcal{K}$ under μ, $\mu(\mathcal{K})$, can not be limited since

$$w \to \zeta \Rightarrow \frac{1}{\zeta - w} \to \infty.$$

Furthermore μ is a one-to-one map and hence $\mu(\mathbb{C} \setminus \mathcal{K}) = \mathbb{C} \setminus \mu(\mathcal{K})$. So if we assume that (2) is true we have that $\mu(\mathcal{K})$ is the interior of a disc and this is of course a contradiction. This means that $\mathcal{E}$ is a convex set and as such it contains its arithmetic mean.
Let $z_1, ..., z_n$ denote the zeros of f. Of course the points $\frac{1}{\zeta - z_i}$ lies in $\mathcal{E}$ for $i = 1, ..., n$ and for some $w \in \mathbb{C} \setminus \mathcal{K}$ we have that

$$\frac{1}{\zeta - w} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\zeta - z_i} \in \mathcal{E}$$

. We also observe that

$$\frac{f'(z)}{f(z)} = \frac{d}{dz} \ln f(z) = \frac{d}{dz} \ln \prod_{i=1}^{n} c_i(z - z_i) = \sum_{i=1}^{n} \frac{1}{z - z_i}$$

and so

$$\exists w \in \mathbb{C} \setminus \mathcal{K} : \frac{1}{n} \frac{f'(\zeta)}{f(\zeta)} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\zeta - z_i} = \frac{1}{\zeta - w}$$

this means that for $\alpha, \zeta \in \mathcal{K}$ the statement holds when $\alpha \neq \zeta$ Since this gives that

$$\frac{1}{\zeta - w} \neq \frac{1}{\zeta - \alpha} \Rightarrow \frac{1}{n} \frac{f'(\zeta)}{f(\zeta)} \neq \frac{1}{\zeta - \alpha} \Rightarrow$$
$$nf(\zeta) - (\alpha - \zeta)f'(\zeta) \neq 0.$$

If on the other hand

$$\alpha = \zeta \Rightarrow F(\alpha, \zeta) = nf(\zeta) \neq 0$$

according to what we assumed. $\square$

We now give the proof of the original statement above (Theorem 5).

Proof. We consider the mapping $\psi : z \mapsto \frac{(\alpha z+1)}{z}$ which is a special case of a Möbius map. Define

$$g(z) := z^n f(\psi(z)).$$

Under the hypothesis of (i) it will follow that $f(\alpha) \neq 0$ and according to property (ii) in the proposition above we have that the zeros of g must lie in the circular domain $\mathcal{D} := \psi^{-1}(\mathcal{K})$. Furthermore $\mathcal{D}$ does not contain the point at infinity since α is not in $\mathcal{K}$. According to the Gauss-Lucas theorem, all the critical points of g also lie in $\mathcal{D}$ and so by the third property of the above proposition the the zeros of $F(\alpha, z)$ lie in $\psi(\mathcal{D})$, but this is exactly the set $\mathcal{K}$ and so the first part holds.
The second part of the theorem holds if α is a zero of f of order n since the zeros of f in that case all lie on the circle $\mathcal{C}$. Suppose that f has a zero $\beta \neq \alpha$ and that ζ is a zero of $F(\alpha, \bullet)$ such that $\zeta \neq \alpha$ and $f(\zeta) \neq 0$. Then by the fourth property of the above proposition $\psi^{-1}(\zeta)$ is a critical point, but not a zero, of g. But g can not be a non-zero constant since it vanishes for $\psi^{-1}(\beta)$. This means that g must be of at least degree 2. Since $\psi^{-1}(\alpha) = \infty$, a circle passing through α and ζ is mapped by ψ^{-1} onto a straight line passing through $\psi^{-1}(\zeta)$. Hence by the second part of Gauss-Lucas and the proposition above the results follows. □

The Laguerre theorem implies an alternative way of looking at the logarithmic derivative $\frac{f'(z)}{f(z)}$. Since whenever a ζ is outside the circular domain containing all the zeros of f the α defined as $\alpha := \zeta - n\frac{f(\zeta)}{f'(\zeta)}$ will satisfy the following statement.

Theorem 7 (Walsh)**.** *Let f be a polynomial of degree n with all its zeros in a circular domain $\mathcal{K}$. Then to every ζ in the extended complex plane, but not in $\mathcal{K}$ there exist an $\alpha \in \mathcal{K}$ such that:*

$$\frac{f'(\zeta)}{f(\zeta)} = \frac{n}{\zeta + \alpha}.$$

I.e the Walsh theorem states that the value of the logarithmic derivative outside the circular domain $\mathcal{K}$ are coincident at an appropriately chosen point $\alpha \in \mathcal{K}$. Thanks to apolarity, which we will define further ahead, we will be able to state a more general result called the Walsh coincidence theorem.

Proof. Let $\alpha = \zeta - n\frac{f(\zeta)}{f'(\zeta)}$. If ζ is a zero of $F(\alpha, z)$ lying in the extended complex plane, but not in $\mathcal{K}$, then, according to Laguerre, we have that if $\alpha \notin \mathcal{K}$, all the zeros of $F(\alpha, z)$ are in $\mathcal{K}$ and from our assumption if follows that ζ can not be one of them, but this is a contradiction. Hence α as above must lie in $\mathcal{K}$ and ζ does not. □

The next theorem gives us information about the function $\psi : \zeta \mapsto \frac{1}{\alpha}$ in the case where $\mathcal{K}$ is the exterior of the unit disc, $\mathcal{D}$.

Theorem 8 (Dieudonné)**.** *Let f be a polynomial of degree n without zeros in the open unit disc $\mathcal{D}$. Then*

$$\frac{f'(z)}{f(z)} = \frac{n}{z - (\phi(z))^{-1}}$$

where ϕ is analytic and $|\phi(z)| \leq 1$

Proof. According to Walsh's theorem this statement is true for $|\alpha| \geq 1|$ when $\zeta \in \mathcal{D}$ thus we obtain:

$$\phi(\zeta) := \frac{1}{\alpha} = \frac{f'(\zeta}{\zeta f'(\zeta) - nf(\zeta)}.$$

So ϕ is a rational function bounded by 1 in $\mathcal{D}$ and in particular it is analytic in $\mathcal{D}$. □

These results illustrate examples of a solutions to special cases of the open problems mention in the introduction. In this special case we consider the polar derivative as the operator T and a circular domain as the set S. Before we go on to the next section we define a recursive sequence of the polar derivative and an extension of the Laguerre theorem.

Definition 6. Let f be a polynomial of degree n, the *sequence of polar derivatives* corresponding to f is then given by:

$$f_k(z) = (n-k+1)f_{k-1}(z) + (\zeta_k - z)f'_{k-1}(z)f_0(z) = f(z).$$

With this definition $f_1(z)$ is precisely the polar derivative with $\alpha = \zeta_k$ as above. The poles ζ_k may be equal or unequal, the important thing is that the they are not in the circular domain containing the zeros of f.

Theorem 9 (Extended Laguerre)**.** *Let f be a polynomial of degree n with all its zeros in a circular domain $\mathcal{C}$. Further assume that none of the points $\zeta_1, ..., \zeta_n$ lies in $\mathcal{C}$. Then all the zeros of each one of the polar derivatives, f_k, in the sequence above also lies in $\mathcal{C}$.*

The statement obviously holds for for $k = 1$ according to the original statement of Laguerre above and so it follows that the rest of the polar derivatives will also have their zeros in $\mathcal{C}$.

2.3. **Apolarity and Grace's theorem.** In this section we will present a consequence of Laguerre's separation theorem - Grace's apolarity theorem. The importance of this theorem reaches far beyond the theory of localizing critical points and generalizations of Grace's theorem (as well as the Laguerre theorem) to abstract spaces were established by Hörmander (1954). In the next section we will also present equivalent formulations of Grace's theorem which will be of great use further ahead. We skip the proofs since they are quite technical, but details are to be found in [17].

Definition 7. Let f and g be polynomials of degree n. Then f and g are said to be apolar if:

$$\sum_{\nu=0}^{n}(-1)^{\nu}f^{(\nu)}(0)g^{n-\nu}(0)=0.$$

The apolarity condition has a geometrical interpretation in the theory of algebraic curves and surfaces and the term *apolar* was first introduced by Reyes (1874). There are a lot of useful results that follows from this definition. First of all we notice that if f and g are given by:

$$f(z)=\sum_{\nu=0}^{n}a_{\nu}z^{\nu}\Rightarrow f^{\nu}(0)=\nu!a_{\nu}$$

$$g(z)=\sum_{\nu=0}^{n}b_{\nu}z^{\nu}\Rightarrow g^{n-\nu}(0)=(n-\nu)!ba_{n-\nu}$$

then the apolarity condition is given by:

$$\sum_{\nu=0}^{n}\frac{a_{\nu}b_{n-\nu}}{\binom{n}{k}}=0.$$

Furthermore the statement below follows from the definition:

Proposition 2. *The apolarity relation has the following properties:*

(i) *It is a symmetric relation, that is the roles of f and g can be interchanged.*
(ii) *It is essentially a linear relation in the sense that if f_1 and f_2 are apolar to g and $\lambda_1,\lambda_2\in\mathbb{C}$, then $f=\lambda_1 f_1+\lambda_2 f_2$ is also apolar to g.*
(iii) *Every polynomial with odd degree is apolar to itself.*

We will not go into detail concerning further results on apolarity but simply state that apolarity is preserved under an affine transformation of the complex plane. With special conditions on f and g apolarity is also preserved under a Möbius transformation. We are now ready to move on to Grace's apolarity theorem that has to do with apolarity and the zeros of the polynomials we consider.

Theorem 10 (Grace)**.** *Let f and g be apolar polynomials. Then every circular domain containing the all the zeros of one of them contains at least one zero of the other.*

The geometric interpretation of this theorem is that two apolar polynomials cannot be separated by the boundary of a circular domain. I.e no line or circle cuts through the two circular domains containing the zeros of each of the polynomials considered if they are apolar.

For the interested reader this result should appear as somewhat surprising since the algebraic construction of apolarity is indeed a far more general formulation of the operator T (as in the introduction) than the operators in previous results.

One might raise an eyebrow since the apolarity condition as stated above takes the derivatives at the origin. But the statement that apolarity is preserved under affine transformation implies that the derivatives can actually be taken at an arbitrary point $c \in \mathbb{C}$. (For further details we refer to [21]). We end this section with a corollary to Grace's theorem that makes this statement somewhat clearer.

Corollary 4. *Let $f(z)$ and $g(z)$ be apolar polynomials and A be the convex region enclosing all the zeros of f and B the convex region enclosing all the zeros of g. Then $A \cap B \neq \emptyset$.*

Proof. If not, then there would exist a straight line separating A and B. Hence the zeros of f would lie in a circular domain, namely a halfplane which is devoid of zeros of g. $\square$

2.4. **Equivalent formulations of Grace's theorem.** We now state several equivalent forms of Graces theorem. One can either prove each of these results directly or show that they (amongst some other results) successively imply each other. The proofs however are not of interest for our purpose. We refer to [21] or [17] for further details. We now introduce the functions f, g and h that will be used through out this section.

$$f(z)=\sum_{\nu=0}^{n}\binom{n}{\nu}a_\nu z^\nu, \qquad g(z)=\sum_{\nu=0}^{n}\binom{n}{\nu}b_\nu z^\nu, \qquad h(z)=\sum_{\nu=0}^{n}\binom{n}{\nu}a_\nu b_\nu z^{nu}.$$

Theorem 11 (Walsh coincidence theorem)**.** *Let $P(z_1,...,z_n)$ be a polynomial in $z_1,...,z_n$ of total degree n symmetric in its variables, and of degree at most one in each of them. Then every circular domain containing the points $\zeta_1,...,\zeta_n$ contains at least one point ζ such that*

$$P(\zeta_1,\zeta_2,...,\zeta n)=P(\zeta,\zeta,...,\zeta).$$

Theorem 12 (Schur-Szegö composition theorem)**.** *Let $f(z)$ be a polynomial of degree n as given above , whose coefficients, a_i, $i=1,...,n$ satisfy the linear relation*

$$l_n a_0+l_{n-1}a_1+...+l_0a_n=0, \qquad l_n\neq 0.$$

Then f has at least one zero in every circular domain that contains all the zeros of

$$\psi(z):=\sum_{\nu=0}^{n}(-1)^\nu\binom{n}{\nu}l_\nu z^\nu.$$

Theorem 13 (Walsh representation theorem)**.** *Let $f(z)$ be a polynomial of degree n as given above with all its zeros in a circular domain $\mathcal{K}$, and let $\lambda_0,...,\lambda_n\in\mathbb{C}$. Then, to every $z\in\mathbb{C}$, there exists an $\alpha\in\mathcal{K}$ such that*

$$\begin{aligned}\sum_{\nu=0}^{n}\lambda_\nu f^\nu(z) &= a_n\sum_{\nu=0}^{n}\lambda_\nu\left[\frac{\partial^\nu}{\partial\omega^\nu(\omega-\alpha)^n}\right]_{\omega=z}\\ &= a_n\lambda_0(z-\alpha)^n+a_n\sum_{\nu=1}^{n}\lambda_\nu n(n-1)\cdots(n-\nu+1)(z-\alpha)^{n-\nu}.\end{aligned}$$

In the next theorem due to Schur and Szegö the polynomial h can be seen as a perturbation of the polynomial g in the case where all the coefficients $\{a_i\}$ are close to 1 which means that the zeros of f are close to -1.

Theorem 14 (Schur-Szegö convolution theorem)**.** *Let $\mathcal{K}$ be a circular domain containing all the zeros of the nth degree polynomial f as given above. Then each zero γ of h is of the form*

$$\gamma=-\alpha\beta, \qquad \alpha\in\mathcal{K}, g(\beta)=0.$$

It is remarkable how geometric results such as the Grace's theorem can be deduced from the very algebraic formulation of apolarity. The following theorem due to De Brujin (1947) was deduced from the Schur-Szegö composition theorem and provides yet another geometric interpretation of Grace's theorem.

Theorem 15. *Let f, g and h be polynomials of degree n given as above. If*
$$g(z) \neq 0, \quad g(0) = 1$$
for $|z| < 1$. Then
$$\{h(z) : |z| \leq 1\} \subseteq \{f(z) : |z| \leq 1\}.$$

2.5. **Complex analogues of Rolle's theorem.** We complete this section short note on how to locate the critical points of a complexed-valued polynomial of degree $n \geq 2$. The results presented here origins from the classical theorem of Rolle. We recall that this theorem states that if $f : [a, b] \to \mathbb{C}$ is a differentiable function and $f(a) = f(b)$ then there exists a $\varepsilon \in [a, b]$ such that $f'(\varepsilon) = 0$. This result obviously no longer holds when f is allowed to be complex-valued. Consider for example the function $f(z) = e^{iz\pi}$. For this function we have $f(-1) = f(1)$ but there is no critical point in $[-1, 1]$, in fact there is no critical point at all. If we instead consider polynomials of degree $n \geq 2$ there will be at least one critical point for these. The problem is that we can easily construct such an example where $f(a) = f(b)$ but there are no critical points in the interval $[a, b]$. The question that arise from this is: *how far away from a given interval* $[a, b]$ *can the critical points of a polynomial* f *of degree* n *lie if* $f(a) = f(b)$*?* The following theorem gives the answer.

Theorem 16 (Grace-Heawood)**.** *Let* f *be a polynomial of degree* $n \geq 2$*. If* $z_1, z_2 \in \mathbb{C}$ *are any two distinct points at which* f *takes the same value, then the disc*

$$\mathcal{D}(z_1, z_2, n) := \left\{ z \in \mathbb{C} : \left| z - \frac{z_1 + z_2}{2} \right| \leq \left| \frac{z_1 - z_2}{2} \right| \cdot \cot \frac{\pi}{n} \right\}$$

contains at least one critical point of f*.*

This theorem implies that a polynomial of degree $n \geq 3$ with no critical points in a closed disc of radius r cannot take the same value at two diametrically opposed points of a concentric disc of radius $r \tan(\frac{\pi}{n})$. Thus the determination of a concentric disc such that a polynomial cannot take on the same value at any two points is of great interest since it would lead to a sufficient condition for this polynomial to be univalent in a disc. The following theorem provides this.

Theorem 17 (Alexander-Kakeya)**.** *If a polynomial of degree* n *has no critical points in a closed disc of radius* r*, then it is univalent in the concentric closed disc of radius* $r \sin(\frac{\pi}{n})$*.*

For the proofs of these two theorems we refer to [21].
We are now familiar with some very important results concerning the special case where our operator is the differential operator. Next we will see how these results in some sense can be extended.

3. The Laguerre-Pólya class and multiplier sequences

In our efforts to try to generalize the operator T and the set S we have so far been dealing with the task of describing the zeros of f' relative to a polynomial f. In this section we extend the theory and consider a polynomial h instead of f', where h is constructed from one polynomial f or even several polynomials $f_1, ..., f_k$ by the following operations:

(i) *Linear Combinations*:

$$h(z) := \sum_{j=1}^{k} \lambda_j f_j(z).$$

(ii) *Multiplicative Compositions*:

$$f(z) := \sum_{j=1}^{k} a_j z^j, \qquad h(z) := \sum_{j=1}^{k} a_j b_j z^j, \qquad b_j \in \mathbb{C}.$$

It is especially interesting to consider the linear combination where $f_j(x) := f(z) - w_j$ and w_j are the zeros of f. For the multiplicative composition above one may think of the b_js as the coefficients of another polynomial $g(z)$ and employ the notation $h = f \star g$.

Recall the results by Grace concerning apolarity and employ the notation

$$f(z) = \sum_{\nu=0}^{n} a_\nu \frac{z^\nu}{\nu!}$$

$$g(z) = \sum_{\nu=0}^{n} b_\nu \frac{z^\nu}{\nu!}$$

where $a_\nu = f^{(\nu)}(0)$ and $b_\nu = g^{(\nu)}(0)$. Walsh's representation theorem now allows us to describe the zeros of the linear combination of the derivatives of f:

$$\lambda_0 f(z) + \lambda_1 f'(z) + ... + \lambda_n f^{(n)}(z)$$

relative to those of the polynomials f and

$$g(z) := \sum_{\nu=0}^{n} \lambda_n n(n-1)...(n-\nu+1) z^{n-\nu}$$

since the linear combinations above is equal to $h(z)/n!$ where

$$h(z) = \sum_{\nu=0}^{n} b_\nu f^{(n-\nu)}(z) = \sum_{\nu=0}^{n} a_\nu g^{(n-\nu)}(z).$$

So with the polynomials f, g and h (of degree $n \geq 1$) given as above we obtain the following result as an immediate consequence of Walsh's representation theorem:

Theorem 18. *Suppose that f has all its zeros in a circular domain $\mathcal{K}$. Then each of the zeros of h is of the form $\alpha + \beta$ where $\alpha \in \mathcal{K}$ and $g(\beta) = 0$.*

From this we can deduce results for certain linear combinations, for example this corollary:

Corollary 5. *Let f be a polynomial of positive degree n with all its zeros in a closed disc of radius ρ and center at origin denoted by $\mathcal{D}(\rho)$. Suppose that*

$$\psi(z) := \sum_{\nu=0}^{n} \binom{n}{k} \lambda_\nu z^\nu$$

has all its zeros in the half-plane $\mathcal{H} := \{z \in \mathbb{C} : |z| \leq |z-\tau|\}$ where $\tau \in \mathbb{C}\backslash 0$. Then

$$h(z) = \sum_{\nu=0}^{n} \lambda_\nu f^{(\nu)}(z) \frac{(\tau z)^\nu}{\nu!}$$

has all its zeros in $\mathcal{D}(\rho)$.

Proof. Let ζ be a zero of h and consider the polynomial

$$\chi := \sum_{\nu=0}^{n} \lambda_\nu \frac{(\tau\zeta)^\nu}{\nu!} f^{(\nu)}(z)$$

this polynomial is of the same form as h above and if we let the role of g be taken by:

$$g(z) := \frac{1}{n!} \sum_{\nu=0}^{n} \binom{n}{k} \lambda_{n-\nu} (\tau\zeta)^{n-\nu} z^\nu = \frac{z^n n!}{\psi} \left(\frac{\tau\zeta}{z}\right)$$

we get that $\zeta = \alpha + \beta$ where $\alpha \in \mathcal{D}(\rho)$ and $g(\beta) = 0$. If $\beta \neq 0$ the latter equation means that $\frac{\tau\zeta}{\beta}$ is a zero of ψ and thus $\frac{\tau\zeta}{\beta} \in \mathcal{H}$ i.e $|\zeta| \leq |\zeta - \beta|$. But $\zeta - \beta = \alpha$ and hence $|\zeta| \leq |\alpha|$ so $\zeta \in \mathcal{D}(\rho)$. $\square$

Furthermore we have by a result due to Takagi that states that $\mathcal{K}(h) \subseteq \mathcal{K}(f) + \mathcal{K}(g)$ where f, g and h again is as above. So if the roles g and f are interchanged the theorem above provides two ways of describing the location of zeros of a polynomial $\frac{h(z)}{n!}$ and the Takagi result gives a third one.

If we consider the polynomials:

$$\psi(z) := \sum_{\nu=0}^{n} g^{(\nu)}(0) z^{n-\nu} = \sum_{\nu=0}^{n} b_\nu z^{n-\nu} = b_{n-m} \prod_{\mu=1}^{m} (z - \zeta_\mu)$$

we find that it is of degree $m = n-k$, where k is the multiplicity of a possible zero of g at the origin. Furthermore it has non-vanishing zeros and we can obtain h by applying the differential operator

$$\psi\left(\frac{d}{dz}\right) = \sum_{\nu=0}^{n} b_\nu \frac{d^{n-\nu}}{dz^{n-\nu}}.$$

Using the factorization of ψ and defining

$$f_0 := f$$

$$f_\mu(z) := \left(\frac{d}{dz} - \zeta_\mu\right) f_{\mu-1} = f'_{\mu-1}(z) - \zeta f_{\mu-1}(z)$$

we find that $h(z) = b_{n-m} f_m(z)$.

3.1. **The Hermite-Poulain-Jensen Theorem.** We are now ready to approach a very central result of this paper. As we shall see in the next section this result can be made even more general. First of all we define a recursive formula and then we use three lemmas to prove the main result of this section.

Definition 8. Let f be a polynomial of degree $n \geq 1$ with real coefficient and $\psi(z) = \sum_{\nu=0}^{m} \beta_\nu z^\nu$ a polynomial of degree m with only real zeros. Further more let ζ_k, $k = 1, ..., m$ be the zeros of the polynomial ψ. Then we define the recursive formula:

$$f_0 := 0$$
$$f_k := (\frac{d}{dz} - \zeta_k) f_{k-1}(z)$$

for the sequence of polynomials $f_1, ..., f_m$.

To prove the third statement in Lemma 2 below we need an auxiliary result known as Laguerre's inequality:

Lemma 1. *Let f be a polynomial of degree $n \geq 1$ with real coefficients and only real zeros. If $x \in \mathbb{R}$ and $f^{(k)}(x) \neq 0$ for some $k \in \{0, ..., n-1\}$ then*

$$f^{(k)} f^{(k+2)}(x) - \left(f^{(k+1)}(x)\right)^2 < 0$$

Proof. Since f has only real zeros then so has $f^{(k)}$. let us denote the zeros of $f^{(k)}$ by $\xi_1, ..., \xi_{n-k}$. Then, under our hypothesis

$$\frac{f^{(k+1)}(x)}{f^{(k)}(x)} = \sum_{\nu=1}^{n-k} \frac{1}{x - \xi_\nu}.$$

Differentiation yields

$$\frac{f^{(k)} f^{(k+2)}(x) - \left(f^{(k+1)}(x)\right)^2}{\left(f^{(k)}(x)\right)^2} = -\sum_{\nu=0}^{n-k} \frac{1}{(x - \xi_\nu)^2} < 0$$

which gives the statement. □

We are now ready to state and prove the two lemmas that will give the Hermite-Poulain-Jensen theorem.

Lemma 2. *Let f be a polynomial of degree $n \geq 1$ with real coefficients and let $\epsilon \in \mathbb{R}$. Then the following holds:*

(i) *$f_1(z) := f'(z) - \epsilon f(z)$ cannot have more non-real zeros than f.*
(ii) *The non-real zeros of f_1 lie in the union of the Jensen discs of f.*
(iii) *Let f have only real zeros and let $\epsilon \neq 0$. Then $x^* \in \mathbb{R}$ is a multiple zero of f_1 of order k if and only if $k \geq 2$ and x^* is a multiple zero of f of order $k+1$.*

Proof. (i) In particular we emphasize that the zeros of f_1 are the critical points of $F(z) := e^{-\xi z} f(z)$ since $F'(z) = e^{-\xi z}(f'(z) - \xi f(z))$. Let $x_1, ..., x_k$ be the distinct real zeros of f with the multiplicities $m_1, ...m_k$ respectively. Furthermore define $m := \sum_{i=1}^{k} m_j$ We then have that each x_j is a critical point of F of multiplicity $m_j - 1$. By Rolle's theorem we also have that F has at least one real critical point in each of the intervals (x_j, x_j+1) $j = 1, .., k-1$. So when $\xi = 0$ there are a total of at least $\sum_{j=1}^{k}(m_j - 1) + k - 1 = m - k + k - 1 = m - 1$ real critical points. In the case when $\xi \neq 0$ we have that $F \to 0$ as $x \to \infty$ or $x \to -\infty$ depending on the sign of ξ. Hence there is a critical point outside the interval $[x_1, x_k]$ in addition to the $m-1$ inside the same interval. Thus there are at least m real critical points in that case. Since f_1 either has at least as many real zeros as f or at least one less than f. In the case when it has at least as many zeros it follows that it cannot have more non-real zeros than f. In the other case $\xi = 0$ and hence $f_1 = f'$ since f' has one zero less than f (i) follows in this case as well.

(ii) Consider a non-real z lying outside the union of Jensen discs of f. Then as in the proof of the Jensen Theorem (Theorem 6) we have that $\frac{\Im f'(z)}{f(z)} \neq 0$ and since ξ is real we also have that $\Im\left(\frac{f'(z)}{f(z)} - \xi\right) \neq 0$ and so $f_1(z) \neq 0$.

(iii) Suppose that x^* is a multiple zero of f_1. Then the order of x^* must be at least two since it would be simple otherwise. Let k be the order of x^*. We have that

$$f^{(\nu+1)}(x^*) - \xi f^{(\nu)}(x^*) = 0, \quad \nu = 0, ..., k-1$$

Eliminating ξ from the first two of these equations yields

$$f(x^*)f''(x^*) - \left(f'(x^*)\right)^2 = 0$$

and by lemma 1 this is only possible if $f(x^*) = 0$. So the equations above imply successively that $f^{(\nu)}(x^*) = 0$ for $\nu = 0, ..., k$. Hence the multiplicity of x^* as a zero of f is at least $k+1$. If it were higher, then the multiplicity of x^* as a zero of f_1 would be larger then k. This completes the proof.

□

For the next lemma we need to generalize the Jensen discs in Definition 4.

Definition 9 (Generalization of Jensen)**.** Let f be a polynomial with real coefficients. Suppose that $z_1, ..., z_k$ are its non-real zeros lying in the upper half-plane. Then, for $\nu \in \mathbb{N}$, the elliptical domains

$$\mathcal{D}_j^{[\nu]} := \left\{ z \in \mathbb{C} : \left(\Re(z - z_j)\right)^2 + \nu(\Im z)^2 \leq \nu(\Im z_j)^2 \right\}$$

are called the ν-th *Jensen domains* of f.

Lemma 3. *Let f and ψ be polynomials with real coefficients. If the non-real zeros of ψ lie in the ν-th Jensens domain of f, then the Jensen discs of ψ are covered by the union of the $(\nu+1)$-th Jensen domains of f.*

Proof. A Jensen disc $\mathcal{D}$ of ψ consists of all points $u + iv \in \mathbb{C}$ with real coordinates u and v satisfying

$$(u - \xi)^2 + v^2 \leq \eta^2$$

where $\xi \pm i\eta$ is a pair of non-real conjugate zeros of ψ. By the hypothesis, there exists a ν-th Jensen domain $\mathcal{D}_j^{[\nu]}$ of f containing $\xi \pm i\eta$. The boundary of $\mathcal{D}_j^{[\nu]}$ is an ellipse given by an equation of the form

$$(x-s)^2 + \nu y^2 + \nu r^2 = 0 \quad (r, s \in \mathbb{R}, r > 0)$$

Since $\xi \pm i\eta \in \mathcal{D}_j^{[\nu]}$ we find that $|\eta| \leq \sqrt{r^2 - \frac{(\xi-s)^2}{\nu}}$. Hence the points $u + iv$ of the Jensen disc $\mathcal{D}$ given by Lemma 1 also satisfy

$$(u-\xi)^2 + v^2 \leq r^2 - \frac{(s-\xi)^2}{\nu}.$$

Multiplying this inequality by $\nu + 1$ and splitting $u - \xi$ into $(u-s)+(s-\xi)$ a short calculation gives that

$$(u-s)^2 + (\nu+1)v^2 - (\nu+1^)r^2 \leq -\nu\left((x-s) + \frac{\nu+1}{\nu}(x-\xi)\right)^2.$$

The right-hand side is certainly not positive, which means that $\xi \pm i\eta \in \mathcal{D}_j^{[\nu+1]}$ and hence that $\mathcal{D} \subset \mathcal{D}_j^{[\nu+1]}$. ☐

Theorem 19 (Hermite-Poulain-Jensen Theorem)**.** *Let f be a polynomial of degree $n \geq 1$ with real coefficients $\psi(z) = \sum_{\nu=0}^{m} \beta_\nu z^\nu$ a polynomial of degree m with only real zeros. Then the following statements hold:*

(i) *The polynomial*

$$h(z) := \sum_{\nu=0}^{m} \beta_\nu f^{(\nu)}(z)$$

cannot have more non-real zeros than f

(ii) *The non-real zeros of h lie in the union of the m-th Jensen domains of f.*

(iii) *Let f have only real zeros. Suppose that β_j is the first non-vanishing coefficient of ψ. Then $x^* \in \mathbb{R}$ is a multiple zero of h order k if and only if $k \geq 2$ and x^* is a multiple zero of $f^{(j)}$ of order $m - j + k$. In particular, if $m \geq n-1$, then h has only real simple zeros.*

Proof. If we write $\psi(z) = \beta_m \prod_{\nu=1}^{m}(z - \zeta_\nu)$ we find that

$$h(z) = \left(\sum_{\nu=0}^{m} \beta_\nu \frac{d^\nu}{dz^\nu}\right) f(z) = \beta_m \prod_{\nu=1}^{m} \left(\frac{d}{dz} - \zeta_\nu\right) f(z)$$

Using the recurrence formula above we obtain a sequence of polynomials $f_1, f_2, ..$ such that $f_m = \beta_m f^{(m)}$. Successive use of the lemmas now gives the result. ☐

If we assume that f has only real zeros then statement (i) implies that h has only real zeros as well. Letting

$$h(z) = \sum_{\nu=0}^{m} \beta_\nu \frac{d^\nu}{dz^\nu} = \sum_{\nu=0}^{m} \beta_\nu n(n-1)\cdots(n-\nu+1) z^{n.\nu}$$

then h has only real zeros since $f(z) \equiv z^n$ has only real zeros. Hence the same must be true for the polynomial

$$\left(\frac{z}{n}\right)^n h\left(\frac{n}{z}\right) = \sum_{\nu=0}^{m} \beta_\nu \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{\nu - 1}{n}\right) z^\nu.$$

Letting $n \to \infty$ and by Hurwitz Theorem (Theorem 30, Appendix A) we have that ψ also only have real zeros.

3.2. **The Laguerre-Pólya class.** In the Hermite-Poulain-Jensen theorem there is no point taking $m > n$ since h cannot have more than $n+1$ terms. But as we shall see in this section we can replace ψ with certain entire functions Ψ by letting $m \to \infty$. To do this we introduce the Laguerre-Pólya class and establish some results that go beyond the class of polynomials.

Definition 10 (The Laguerre-Pólya class)**.** An entire function Ψ belongs to the Laguerre-Pólya class if it has a representation of the form

$$\Psi(z) = cz^{\kappa} e^{-az^2+bz} \prod_{\nu=1}^{\infty} (1 - t_\nu z) e^{-\lambda t_{nu} z}$$

where $c \in \mathbb{R} \setminus \{0\}$, κ is a non-negative integer, $a, b \in \mathbb{R}$, $a \geq 0$, $\lambda \in \{0, 1\}$ and $t_\nu \in \mathbb{R}$ with $\sum_{\nu=1}^{\infty} |t_\nu|^{\lambda+1} < \infty$. Within the Laguerre-Pólya class those functions Ψ for which $\lambda = 0$, $a = 0$, $b \geq 0$ and $t_\nu \geq 0$ for $\nu \in \mathbb{N}$ are said to be of type I.

We note that the Laguerre-Pólya class includes all polynomials with real zeros. A polynomial with a κ-fold zero at the origin and non-vanishing real zeros $x_1, .., x_n$ can be represented in the form above by setting $a = b = \lambda = 0$, $t_\nu = -\frac{1}{x_\nu}$ for $\nu 1, .., n$ and $t_\nu = 0$ for $\nu > n$. An important result in this new terminology is the following:

Theorem 20. *A sequence of polynomials with real coefficients and zeros converge uniformly to a entire function not identically zero if and only if this entire function belongs to the Laguerre-Pólya class.*

We omit the proof and go ahead with the extension of the Hermite-Poulain-Jensen theorem as follows.

Theorem 21 (Pólya)**.** *Let f be a polynomial of degree $n \geq 1$ with real coefficients, and let Ψ be an entire function in the Laguerre-Pólya class. Then the following statements hold:*

(i) *The polynomial*

$$h(z) = \sum_{\nu=0}^{n} \frac{\Psi^{(\nu)}(0)}{\nu!} f^{(\nu)}(z)$$

cannot have more non-real zeros than f.

(ii) *Every strip $\mathcal{B} := z \in \mathbb{C} : |\Im z| \leq \rho$ containing all the zeros of f also contains those of h.*

(iii) *Let f have only real zeros. If Ψ has at least $n-1$ zeros, then h has only simple, real zeros.*

Proof. Let Ψ be as in definition 10 and

$$\Psi_j(z) = cz^{\kappa} \left(1 - \frac{az^2}{j}\right)^j \left(1 + \frac{\lambda_j z}{n_j}\right)^{n_j} \prod_{\nu=1}^{j} (1 + t_\nu z)$$

then the statements of theorem 21 holds with ψ replaces by Ψ_j and h replaced by

$$h_j(z) := \sum_{\nu=0}^{n} \frac{\Psi_j^{\nu}(0)}{\nu!} f^{\nu}(z).$$

(u) By the famous theorem by Weierstrass on sequences of analytic functions the convergence $\Psi_j \to \Psi$ also implies $\Psi_j^{(\nu)} \to \Psi^{(\nu)}$. Thus the first statement follows by Hurwitz's theorem and by letting $j \to \infty$.

(ii) The second statement follow since the Jensen domains of f, of any order, are subsets of $\mathcal{B}$.

(iii) Since the simple zeros of h_j might coalesce as $j \to \infty$ we use the a little trick. If the function Ψ has at least $n-1$ zeros we may write it as $\Psi(z) = \psi(z)\Phi(z)$ where ψ is a polynomial of degree $n-1$ and Φ is a function in the Laguerre-Pólya class. Now we can construct h in the following two steps:

$$g(z) := \sum_{\nu=0}^{n} \frac{\Phi^{(\nu}(0)}{\nu!} f^{(\nu)}(z), \quad h(z) := \sum_{\nu=0}^{n} \frac{\psi^{(\nu}(0)}{\nu!} g^{(\nu)}(z).$$

In the first step statement (i) ensures that g has only real zeros. Therefore in the second step statement (iii) of theorem 10 applies and therefor h has simple, real zeros.

□

Example 1. *Under the general hypothesis of Theorem 23 the polynomial*

$$\sum_{\nu=0}^{\frac{n}{2}} (-1)^{\nu} \frac{f^{(2\nu)}(z)}{\nu!}$$

cannot have more non-real zeros than f. This follows from the fact that e^{-z^2} belongs to the Laguerre-Pólya class. This does not follow directly from Theorem 21 since the polynomials

$$\psi(z) = \sum_{\mu=0}^{m} (-1)^{\mu} \frac{z^{2\mu}}{\mu!}$$

have at most two real zeros.

3.3. **Multiplier sequences.** We now introduce the concept of a multiplier sequence that was first introduced by Pólya and Schur. Let $\gamma_0, \gamma_1, ..., \gamma_n, ..,$ be an arbitrary sequence of real numbers and let T be the operator which takes the arbitrary polynomials $p(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$ to the polynomial $T[p(x)] = \gamma_0 a_0 + \gamma_1 a_1 x + \gamma_2 a_2 x^2 + ... + \gamma_n a_n x^n$. The sequence $\gamma_0, \gamma_1, ..., \gamma_n, ..,$ is called a *multiplier sequence of the first kind* if the corresponding operator takes every polynomial whose zeros are real into a polynomial of the same class. If the operator takes a polynomial whose zeros are positive into a polynomial whose zeros are real it we call it a *multiplier sequence of the second kind*.
Multiplier sequences have the following properties:

(i) If $\gamma_0, \gamma_1, ..., \gamma_n, ..,$ is a multiplier sequence of the first or second kind, then $\gamma_k, \gamma_{k+1}, ..., \gamma_{k+n}, ..,$ is of the same kind.
(ii) If a certain element in a multiplier sequence of the first kind is equal to zero, then the subsequent elements are equal to zero.

We will mainly be concerned with multiplier sequences of the first kind.

Theorem 22 (Pólya-Schur). *A sequence $(\gamma_n)_{n\in\mathbb{N}_0}$ of real numbers is a multiplier sequence of the first kind if and only if all the polynomials $\psi_n(z) := \sum_{\nu=0}^{n} \binom{n}{k} \gamma_\nu z^\nu$ of positive degrees have only real, non-negative zeros of, alternatively, only real non-positive zeros.*

A sufficient condition for a sequence $(\gamma_n)_{n\in\mathbb{N}_0}$ to be of the first kind is that those polynomials

$$\psi_n(z) := \sum_{\nu=0}^{n} \binom{n}{\nu} \gamma_\nu z^\nu \quad (n \in \mathbb{N})$$

which are of positive degree have either only real, non-negative or only real, non-positive zeros. There are some discussions left out to verify that this observation truly holds. For further details we refer to [21] on this matter. The polynomials $\psi_\nu(z)$ are often referred to as Jensen polynomials.

Proof. To characterize the multiplier sequence of the first kind it remains to prove the necessity of this condition. Since the polynomials

$$(z+1)^n = \sum_{\nu=0}^{n} \binom{n}{\nu} z^\nu,$$

for $n \in \mathbb{N}$, have only real zeros, the polynomials $\psi_n(z)$ (as above), must have only real zeros as well unless it is a constant. So it remains to show that the non-vanishing zeros are all of the same sign. Omitting trivial cases we may suppose that

$$\gamma_m \neq 0 \quad \textit{and} \quad \gamma_{m+k} \neq 0$$

for some $m \in \mathbb{N}_0$ and $k \geq 2$. The polynomials $h(z) := \gamma_m z^m - \gamma_{m+2} z^{m+2}$ has only real zeros since the polynomial $f(z) := z^m - z^{m+2}$ has only real zeros. No suppose that $\gamma_{m+2} \neq 0$, then the reality of the zeros of h implies that $\gamma_m \gamma_{m+2} > 0$ for any value of γ_{m+1}. Now assume that $\gamma_{m+1} = 0$. Then it follows by applying Laguerre's inequality (Lemma 1) to ψ_{m+k} that $\gamma_m \gamma_{m+2} < 0$. This cannot be true when $\gamma_{m+2} = 0$ which contradicts that $\gamma_m \gamma_{m+2} > 0$ for any value of γ_{m+1}. Hence $\gamma_{m+1} \neq 0$ always holds. Replacing

m with $m+1, m+2, ...$ we can successively apply the above discussion so we conclude that

$$\gamma_\nu \neq 0 \quad and \quad sgn\gamma_{\nu-1} = sgn\gamma_{n+1} \quad (\nu = m+1, ..., m+k-1)$$

whenever $\gamma_m \neq 0 \quad and \quad \gamma_{m+k} \neq 0$. Hence the polynomials $\psi_n(z)$ can be written as

$$\pm\bigg(\sum_{\nu\ even}\binom{n}{\nu}|\gamma_\nu|z^\nu + \sigma\sum_{\nu\ odd}\binom{n}{\nu}|\gamma_\nu|z^\nu\bigg) \quad (\sigma = \pm 1)$$

and if such a polynomial is not a constant, then its zeros, which are already known to be real, are non-negative when $\sigma = -1$ and non-positive if $\sigma = 1$. □

Jensen polynomials are intimately related to so called Appell polynomials associated to a given power series $\sum_{\nu=0}^{\infty} c_\nu z^\nu$. The Appell polynomials are defined as:

$$\phi_n(z) := \left(\sum_{\nu}^{\infty} c_\nu \frac{d^\nu}{dz^\nu}\right)\frac{z^n}{n!} = \sum_{\nu=0}^{n} c_\nu \frac{z^{n-\nu}}{(n-\nu)!}.$$

Theorem 23. *A power series $\sum_{\nu=0}^{\infty} c_\nu z^\nu$ which is not identically zero represents a function Ψ of type I in the Laguerre-Pólya class if and only if, for each $n \in \mathbb{N}$, the Appell polynomial $\phi_n(z)$ has only real, non-positive zeros unless it is a constant.*

The proof of Theorem 23 uses arguments similar for those in the proof of Theorem 21 (Pólya's Theorem) and we refer to [21], Theorem 5.7.3 for details.

Remark 1. Theorems 22 and 23 are both due to Pólya and Schur and their work in 1914 and they also explored multiplier sequences of the second kind. In 1913 Jensen pointed out a connection between the polynomial ϕ_n in Theorem 23 with the power series $\sum_{\nu=0}^{\infty} c_\nu z^\nu$. He also indicated certain relations between the zeros of the polynomials and those of the power series. This would motivate us to call ϕ_n the Jensen polynomials but this term is usually employed for the polynomials ψ_n in Theorem 22 and the related power series as above with $c_\nu = \frac{\gamma_\nu}{\nu!}$ while the ψ_n are referred to as the associated Appell polynomials. We adopt this notation since it is used in [21] and by Craven and Csordas for example. This however should not cause any trouble since the following identity holds:

$$\psi_n(z) = n! z^n \phi_n(\frac{1}{z})$$

so the location of the zeros is the same as far as the studies of multiplier sequences are concerned. Other remarkable properties for these polynomials is the differentiation formula:

$$\phi_n'(z) = \phi_{n-1}(z)$$

and the recovery of the power series by the limiting process:

$$\lim_{n\to\infty} \psi\left(\frac{z}{n}\right) = \sum_{\nu=0}^{\infty} \frac{\gamma_\nu}{\nu!}.$$

Jensen was aware of these properties even though he did not introduce the polynomials ψ_n explicitly.

4. Recent results on HPO

So far this thesis has been dealing with historical results on operators and manipulations of polynomials that preserves certain properties of the zeros. In this section we will take it to the next level and address the problem of describing all linear operators $T : \mathbb{C}[z] \to \mathbb{C}[z]$ s.t T preserves the property of real zeros i.e T is a hyperbolicity preserving operator (HPO). We will be concerned with the one-variable case and refer to [4] for further studies of the multivariate case.

4.1. Introduction and notation regarding HPOs.

We need to formalize what we mean by hyperbolicity and provide a convenient notation.

Definition 11. The set of *hyperbolic polynomials* (i.e a polynomial with real zeros) is given by

$$\mathcal{H}(\mathbb{R}) = \{P \in \mathbb{R}[z] : P^{-1}(0) \subset \mathbb{R}\}.$$

Definition 12. A *hyperbolicity preserving operator*, HPO, is a linear operator $T : \mathbb{C}[z] \to \mathbb{C}[z]$ that satisfies the inclusion

$$T(\mathcal{H}) \in \mathcal{H} \cup \{0\}$$

And we denote the set of all HPOs by $\mathcal{A}_H$.

The stability property is very similar to the hyperbolicity property.

Definition 13. The set of *stable polynomials* is given by

$$\mathcal{H}(\mathbb{C}) = \{P \in \mathbb{C}[z] : P^{-1}(0) \subset \{z : \Im z \leq 0\}\}.$$

Definition 14. A *stability preserving operator*, SPO, is a linear operator $T : \mathbb{C}[z] \to \mathbb{C}[z]$ that satisfies the inclusion

$$T(\mathcal{H}(\mathbb{C}) \in \mathcal{H}(\mathbb{C}) \cup \{0\}$$

And we denote the set of all SPOs by $\mathcal{A}_S$.

An example of a HPO (SPO) is the following:

Example 2. *Let $\alpha, \beta \in \mathbb{R}$ and let D denote the derivative, i.e $D = \frac{d}{dz}$. Then $\alpha + \beta D \in \mathcal{A}_H$.*
(If $\alpha, \beta \in \mathbb{C}$ the operator $\alpha + \beta D$ is an example of an SPO and this is equivalent to $\Im(\beta\overline{\alpha}) \geq 0$.)

The problem of classifying all HPOs (in one variable) was pointed out by Pólya, Schur and Crelle in 1914 and this has been a open problem ever since. A more algebraic formulation of this problem is to describe the monoid $\mathcal{A}_H$ and this is the approach that lead us to a solution of this problem. We shall soon reach the breakthrough results concerning this research area due to Borcea, Brändén and Shapiro. There are some more notions and preliminaries that needs to be done.

We may also define stable polynomials the multivariate case:

Definition 15. A stable polynomial in n variables is given by

$$\mathcal{H}_n(\mathbb{C}) = \{P \in \mathbb{C}[z_1, ..., z_n] : P(z_1, ..., z_n) \neq 0 \ if \ \Im z_i > 0, 0 \leq i \leq n\}$$

and the corresponding stability-preserving operators, n-SPOs, are defined as:

Definition 16. The n-SPOs are the linear operators T on $\mathbb{C}[z_1, ..., z_n]$ that satisfy the inclusion

$$T(\mathcal{H}_n(\mathbb{C})) \subseteq \mathcal{H}_n(\mathbb{C}) \cup \{0\}.$$

The real stable polynomials (which coincides with the set of hyperbolic polynomials for $n = 1$) in n variables are given by

$$\mathcal{H}_n(\mathbb{R}) = \mathcal{H}_n(\mathbb{C}) \cap \mathbb{R}[z_1, ..., z_n]$$

and we may define real stability-preserving operators, n-RSPOs, in a similar manner as the n-SPOs. For this article it would be sufficient to define $\mathcal{H}_2(\mathbb{R})$ but since the results we are about to encounter holds in the multivariate case we might as well give the proper definition. Next we introduce the symbol curve.

Suppose that every linear operator can be uniquely represented as

$$T = \sum_{k=0}^{\infty} Q_k(z) D^k$$

where $Q_k \in \mathbb{C}[z]$ for all k and D is the differential operator. Letting our operator T act on the standard monomial basis we get the following:

$$T = \big(\sum_{k=0}^{\infty} Q_k(z) D^k\big)(1) = Q_0(z)$$

$$T = \big(\sum_{k=0}^{\infty} Q_k(z) D^k\big)(z) = Q_0(z) \cdot z + Q_1(z) \Rightarrow Q_1 = T(z) - z \cdot T(1)$$

$$T = \big(\sum_{k=0}^{\infty} Q_k(z) D^k\big)(z^2) = Q_0(z) \cdot z^2 + 2Q_1(z) \cdot z + 2Q_2(z) \Rightarrow$$

$$\Rightarrow Q_2(z) = \frac{1}{2}\Big(T(z) - zT(1) - 2zT(z) + (2z - z)T(1)\Big)$$

$$\vdots$$

Continuing in this manner we will find that Q is uniquely determined by T and vice versa. Therefor every linear operator can be uniquely represented as

$$T = \sum_{k=0}^{\infty} Q_k(z) D^k$$

as suggested. Having established this we present define the symbol curve.

Definition 17 (Symbol Curve)**.** Given an operator $T : \mathbb{C}[z] \to \mathbb{C}[z]$ with the representation as above the *symbol* of T is given by the formal power series in w with polynomial coefficients in $\Im z$. This polynomial is given by:

$$\mathcal{F}(z, w) = \sum_{k=0}^{\infty} Q_k(z) w^k \in \mathbb{C}[z, w].$$

Finally our main result requires that we introduce the Weyl Algebra denoted by $\mathcal{A}$. The Weyl Algebra is defined as

$$\mathcal{A}[\mathbb{C}] = \left\{ \sum_{k=0}^{N} Q_k(z)D^k : N \in \mathbb{N}, Q_k \in \mathbb{C}[z], 0 \leq k \leq N \right\}$$

in the case where the polynomial coefficients are allowed to be complex and

$$\mathcal{A}[\mathbb{R}] = \left\{ \sum_{k=0}^{N} Q_k(z)D^k : N \in \mathbb{N}, Q_k \in \mathbb{R}[z], 0 \leq k \leq N \right\}$$

in the case with real polynomial coefficients. For more details on the Weyl Algebra we refer the reader to the Appendix.

Now, given an operator $T \in \mathcal{A}[\mathbb{R}]$ with symbol $F_T(z,w) \in \mathbb{R}[z,w]$ of degree d consider the real algebraic (symbol) curve of degree d given by

$$\Gamma_T = \left\{ (z,w) \in \mathbb{R}^2 : F_T(z,w) = 0 \right\}.$$

This gives us a natural representation of our operators. The symbol curve and results regarding it has very nice geometrical interpretations as we shall see in the next section. As the reader may already have guessed the symbol curve is of great importance regarding this theory and the great theorem that this article evolves around.

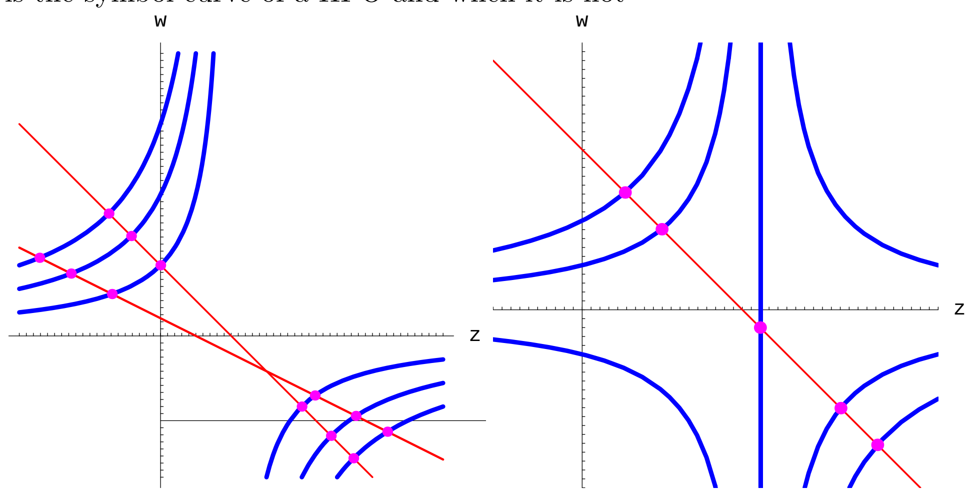
Remark 2 (Classical theory in view of HPO)**.** We have some results that are interesting to mention in terms of HPO before we reach the main results of this article. We have seen extensive theory regarding differential operators and their zero-set preserving properties such as Gauss-Lucas statement that the derivative preserve a convex set of zeros. From Gauss-Lucas to Laguerre and Grace we reach the result regarding multiplier sequences of the first kind due to Polya and Schur that characterizes all operators which are diagonal in the standard monomial basis that are HPO. Another example worth mentioning (that we have not explored above) is a result due to Carnicer and Peña that characterizes all unipotent, upper triangular linear operators that are HPO [8].

4.2. **The Borcea-Brändén-Shapiro Curve Theorem.** With all the notation as given above we are now ready to state the main result of this article. The formulation is very short and concise thanks to the the preliminaries in the previous section. And as already mentioned this is where the symbol curve plays its big part.

Theorem 24 (Borcea-Brändén-Shapiro Theorem)**.**

$$T \in \mathcal{A}_H \cap \mathcal{A}[\mathbb{R}] \Leftrightarrow F_T(z, -w) \in \mathcal{H}_2(\mathbb{R}).$$

This theorem essentially states any finite order HPO T is generated by a real stable polynomial in 2 variables via the symbol map. In theorems 1-2 in [4] this result is extended to the multivariate case (for stability preserves and real stability preservers respectively). Hence this theorem also holds in the more general case when T is a n-SPO for example. In this article however we focus on the theorem as stated above since we can visualize our operators in this case. Geometrically this theorem means that a linear operator T is an HPO if and only if each line $\mathcal{L}$ of negative slope intersects the corresponding real algebraic (symbol) curve, Γ_T, of degree d in exactly d points. Below follows two examples of how Γ_T may look in the case when it is the symbol curve of a HPO and when it is not



4.3. **Proof of Theorem 24.** Now remains the task of presenting the proof of this nice result. The proof of the general formulation of this theorem is given in [4] but we will downsize this and by that hopefully make it more comprehensive to the novice. Note that we need prove that the statement $F_T(z,-w) \in \mathcal{H}_2(\mathbb{R})$ (i.e every line of negative slope intersects the corresponding real algebraic symbol curve) is a sufficient and necessary condition for the operator T to be a HPO. The proof is done in two parts and requires some auxiliary results along the way. The ideas in this proof is deduced from [4]. We begin by proving the sufficiency condition.

4.3.1. *Sufficiency.* For $\alpha \in \mathbb{R}$, $\lambda > 0$ and $f(z_1, .., z_n) \in \mathcal{H}[\mathbb{R}(\mathbb{C})]$ the following holds:

(1) $f(\alpha + \lambda z_1, .., z_n) \in \mathcal{H}[\mathbb{R}(\mathbb{C})]$
(2) $f(\alpha, .., z_n) \in \mathcal{H}[\mathbb{R}(\mathbb{C})]$
(3) $f(z_1, .., z_n)\big|_{z_i=z_j} \in \mathcal{H}[\mathbb{R}(\mathbb{C})]$, $i \neq j$

Since $\Im z_i > 0$ for $i = 1, ..., n$, $\lambda > 0$ and $\alpha \in \mathbb{R}$ the transformation of the first variable will obviously not make any difference in (1) regarding hyperbolicity. Now (1) $\Rightarrow$ (2) if we let $\lambda \to 0$ and obviously this reduces one variable. Finally (3) holds since all z_i, $i = 1, .., n$ had $\Im z_i > 0$ and we reduce by one variable.

We now need the following result due to Lieb and Sokal and the proof of this lemma can be found in [16] (as Lemma 2.3). We somewhat modify the formulation and therefore give a version of the proof suitable for our needs.

Lemma 4 (Lieb-Sokal style)**.** *If $P_0(v), P_1(v) \in \mathbb{C}[v]$ with $P_0(v)+xP_1(v) \neq 0$ for $\Im v \geq$ and $\Im v \geq d$ then*

$$P_0(v) + \left(x - \frac{\partial}{\partial v}\right) P_1(v) \neq 0$$

for $\Im v \geq$ and $\Im v \geq d$.

The main idea in this lemma is that we can replace the variable x by $\left(x - \frac{\partial}{\partial v}\right)$. In [16], Lemma 2.3 it is proved that it is possible to do this if we replace x by $\frac{\partial}{\partial v}$. However our case follows by rotating variables.

Given $a, b \in \mathbb{C}$, $\quad 1 \leq i < j \leq n$ and

$$F(z_1, ..., z_n) = \sum a_{\alpha_1,...,\alpha_n} z_1^{\alpha_1} \cdots z_n^{\alpha_n} \in \mathbb{C}[z_1, ..., z_n]$$

let

$$F\left(z_1, ..., az_i + b\frac{\partial}{\partial z_j}, ..., z_j, ..., z_n\right)$$

denote the polynomial

$$\sum a_{\alpha_1,...,\alpha_n} z_1^{\alpha_1} \cdots \left(az_i + b\frac{\partial}{\partial z_j}\right) \cdots z_j^{\alpha_j} \cdots z_n^{\alpha_n}.$$

With this notation we are ready for the following lemma:

Lemma 5. *If $F(z_1, ..., z_n) \in \mathcal{H}_n[\mathbb{R}]$ and $1 \leq i < j \leq n$ then*

$$F\left(z_1, ..., z_i - \frac{\partial}{\partial z_j}, ..., z_j, ..., z_n\right) \in \mathcal{H}_n[\mathbb{R}]$$

Proof. Without loss of generality we can assume that $i = j - 1 = 1$. Let $c > 0$ and suppose that $\Im z_k \geq c \forall k$. Then $F(z_1, ..., z_n) \neq 0$.
Let N be the degree of F in z_1 and replace each z_1 in F by

$$\binom{N}{m}^{-1} e_m(x_1, ..., x_N)$$

where e_m is the mth elementary symmetric function (for further details on e_m we refer to appendix). This yields a polynomial in $2n - 1$ variables and we denote it by G. Now Walsh's Coincidence Theorem gives us that

$$G(x_1, ..., x_N, z_2, ...z_n) \neq 0$$

whenever all variables have imaginary part $\geq c$. Fix all variables but x_1 and z_2 in the prescribed regions. Then

$$G(x_1, ..., x_N, z_2, ...z_n) \neq 0$$

can be written as

$$P_0(z_2) + x_1 P_1(z_2) \neq 0, \quad \Im z_2 \geq c, \Im x_1 \geq c.$$

Applying the previous lemma due to Lieb-Sokal/Grace yields

$$P_0(z_2) + x_1 P_1(z_2) \neq 0 \Rightarrow$$

.

$$G\left(x_1 - \frac{\partial}{\partial z_2}, ..., x_N, z_2, ..., z_n\right) \neq 0$$

whenever all variables have imaginary part $\geq c$. Repeat this procedure with $x_1, ..., x_N$ to get

$$G\left(x_1 - \frac{\partial}{\partial z_2}, ..., x_N - \frac{\partial}{\partial z_2}, z_2, ..., z_n\right) \neq 0$$

whenever all variables have imaginary part $\geq c$. Now let $x_i = z_1$, $1 \neq i \neq N$ to get the result:

$$F\left(z_1 - \frac{\partial}{\partial z_2}, z_2, ..., z_n\right) \neq 0$$

if $\Im z_k \geq c$, $1 \neq k \neq n$, $\forall c > 0$. □

We are now ready to prove that the sufficiency condition holds. Let $T = T_F \in \mathcal{A}[\mathbb{R}]$ with symbol $F(z, w) \in \mathbb{R}$ such that $F(z, -w) \in \mathcal{H}_2(\mathbb{R})$ and let $f(v) \in \mathcal{H}(\mathbb{R})$. Then $F(z, -w)f(v) \in \mathcal{H}_3(\mathbb{R})$ from our second lemma we get that

$$F\left(z, \frac{\partial}{\partial v} - w\right) f(v) \in \mathcal{H}_3(\mathbb{R})$$

and hence for $w = 0$ we get

$$F\left(z, \frac{\partial}{\partial v}\right) f(v) \in \mathcal{H}_2(\mathbb{R})$$

and thus for any $f \in \mathcal{H}(\mathbb{R})$, this yields

$$T_F(f)(z) = F\left(z, \frac{\partial}{\partial v}\right) f(v)\Big|_{v=z} \in \mathcal{H}(\mathbb{R}).$$

This completes the part which proves that sufficiency holds.

Remark 3. Yet another way to obtain results similar to the Lieb-Sokal lemma is by using the property of polynomials that are in *proper position* (see Appendix A) This approach also gives us an extension of the Hermite-Biehler theorem and a nice set property for polynomials in proper position that will come in handy during the process of establishing necessity. So let us take a closer look at these results.

Theorem 25. *Let $f, g \in \mathbb{R}[z_1, ..., z_n]$ and z_{n+1} be a new indeterminate. Then $f \ll g$ if and only if $g + z_{n+1}f \in \mathcal{H}_{n+1}(\mathbb{R})$. Moreover, if $f \in \mathcal{H}_n(\mathbb{R})$ then $f \ll g$ if and only if*

$$\Im\left(\frac{g(z)}{f(z)}\right) \geq 0$$

whenever $\Im(z) > 0$.

The proof of this can be found on p. 7 in [4].

Theorem 26. *Let $f \in \mathcal{H}_n(\mathbb{R})$. Then the sets*

$$\{g \in \mathcal{H}_n(\mathbb{R}) : f \ll g\}$$

and

$$\{g \in \mathcal{H}_n(\mathbb{R}) : f \gg g\}$$

are nonnegative cones, i.e., they are closed under nonnegative linear combinations.

Proof. Let $f \in \mathcal{H}_n(\mathbb{R})$ and suppose that $f \ll g$ and $f \ll h$. Then by our previous result we have that

$$\frac{\Im(g(z))}{\Im(f(z))} \geq 0 \quad \frac{\Im(h(z))}{\Im(f(z))} \geq 0$$

whenever $\Im(z) > 0$. Hence if $\mu, \lambda \geq 0$ we have

$$\Im(\frac{(\lambda g(z) + \mu h(z))}{f(z)}) \geq 0$$

whenever $\Im(z) > 0$. Again by the previous result we have that $f \ll \lambda g + \mu h$. The other part follows in a similar manner. □

The Lieb-Sokal lemma and results similar to it can now be obtained in the following manner. Let

$$R(v, w) = Q_0(w) + vQ_1(w)$$

and suppose that we want to show that if $R(v, w) \neq 0$ the same goes for this expression where we replace v with $\frac{\partial}{\partial w}$ when $\Im v > 0$, $\Im w > 0$. So let us assume that $R(v, w) \neq 0$ when $\Im v > 0$, $\Im w > 0$. This is equivalent to $R(v, w) \in \mathcal{H}_2(\mathbb{R})$ and according to Theorem 25 this means $Q_1(w) \ll Q_0(w)$. On the other hand $Q_1(w) \ll -Q_1'(w)$ assume namely that $-Q_1'(w)+iQ(w) = 0$ whenever $\Im(w_0) > 0$. That is $\frac{Q_1'(w_0)}{Q_1(w_0)} = i$ whenever $\Im(w_0) > 0$ and Q_1 has real zeros $\alpha_1, ..., \alpha_n$. Then we get that

$$\sum_{j=1}^{n} \frac{1}{w_0 - \alpha_j} = i \Leftrightarrow \sum_{j=1}^{n} \frac{\bar{w_0} - \alpha_j}{|w_0 - \alpha_j|^2} = i \Rightarrow \Im\left(\sum_{j=1}^{n} \frac{\bar{w_0} - \alpha_j}{|w_0 - \alpha_j|^2}\right) = 1.$$

But on the other hand we assumed that $0 > -\Im(w_0)$ so we must have that

$$0 > -\Im(w_0)\Im\left(\sum_{j=1}^{n} \frac{\bar{w_0}}{|w_0-\alpha_j|^2}\right) = \Im(\bar{w_0})\Im\left(\sum_{j=1}^{n} \frac{1}{w_0-\alpha_j}\right)$$

which contradicts the above and hence $-Q_1'(w) + iQ_1(w) \neq 0$ and by Theorems 25-26 this means that $Q_1(w) \ll Q_0(w) - Q_1'(w) \Rightarrow Q_0(w) - Q_1'(w) \in \mathcal{H}_1(\mathbb{R})$ and so the statement follows.

4.3.2. *Necessity.* Among some auxiliary results the following Lemma is needed to prove necessity.

Lemma 6. *If $F(z,w) \in \mathbb{R}[z,w]$ is a symbol curve for an HPO, T, then so is $F(z,\lambda w)$ for any $w \in [0,1]$.*

Assuming the above lemma the proof goes as follows.

Proof. Let the operator T be given by

$$T = \sum_{k=0}^{N} Q_k(z)D^k \in \mathcal{A}_{\mathcal{H}} \cap \mathcal{A}_1[\mathbb{R}]$$

and $F_T(z,w) = \sum_{k=0}^{N} Q_k(z)w^k \in \mathbb{R}[z,w]$ be its symbol. Given $\lambda \in (0,1)$ let T_λ be the operator with symbol $F(z,\lambda w)$. That is

$$T_\lambda = \sum_{k=0}^{N} Q_k(z)\lambda^k D^k$$

Then Lemma 6 asserts that $T_\lambda \in \mathcal{A}_{\mathcal{H}}$. Hence

$$z^{-n}T_\lambda(z) = \sum_{k=0}^{N} Q_k(z)n\cdots(n-k+1)\lambda^k z^{-k}$$

has all real zeros $\forall n \in \mathbb{N}$. Let $\lambda = (n\mu)^{-1}$ where $\mu > 0$ is arbitrarily fixed and $n \to \infty$ (so $\lambda \in (0,1)$). Since also

$$\left(z+\frac{\alpha}{\mu}\right)^{-n} T_{(n\mu)^{-1}}\left(\left(z+\frac{\alpha}{\mu}\right)^{n}\right) =$$

$$= \sum_{k=0}^{N} Q_k(z)n\cdots(n-k+1)n^{-k}(\mu z+\alpha)^{-k}$$

has all real zeros $\forall n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ we get that

$$F\left(z(\mu z+\alpha)^{-1}\right) = \sum_{k=0}^{N} Q_k(z)(\mu z+\alpha)^{-k}$$

has all real zeros for all $\mu > 0$ and $\alpha \in \mathbb{R}$ (or is $\equiv 0$).

If $\Im z > 0$ and $\Im w < 0$, then $w = (\mu z+\alpha)^{-1}$ for some $\mu > 0$ and $\alpha \in \mathbb{R}$. Since $F(z,(\mu z+\alpha)^{-1})$ has all real zeros (and coefficients) it follows that

$$F(z,(\mu z+\alpha)^{-1}) \neq 0$$

whenever $\Im z > 0$, and this implies that:

$$F(z,-w) \neq 0$$

if $\Im z > 0, \Im w > 0$, which proves the necessity of Theorem 24. □

It remains to prove Lemma 6 and in fact this result can be extended to a more general homotopical property for HPO symbols:

Theorem 27. *If $F(z,w) \in \mathbb{R}[z,w]$ is an HPO symbol then so is $F(\mu z, \lambda w)$ for any $(\mu, \lambda) \in [0,1]^2$.*

Lemma 6 follows from another Lemma which follows from a Proposition. This Proposition, in turn, follows from this Lemma:

Lemma 7. *Let $T \in \mathcal{A}[\mathbb{R}], \delta \geq 0$ and define a linear operator $\mathcal{R}_\delta T$ on $\mathbb{C}[z]$ by setting*

$$\mathcal{R}_\delta T(f) = T(F) + \delta z T(Df), \quad f \in \mathbb{C}[z].$$

If $T \in \mathcal{A}_H$ then $\mathcal{R}_\delta T \in \mathcal{A}_H$.

Proof. We will show that

$$T \in \mathcal{A}_H \cap \mathcal{A}[\mathbb{R}] \Rightarrow \mathcal{R}_\delta T(f) := T(F) + \delta z T(Df) \in \mathcal{H}(\mathbb{R}) \cup \{0\}.$$

We have that $T(Df) \equiv 0 \Rightarrow \mathcal{R}_\delta T(f) = T(f)$ and clearly $T(f) \in \mathcal{H}(\mathbb{R}) \cup \{0\}$ so let us assume that $T(Df) \not\equiv 0$. By Laguerre's inequality (Lemma 1) we have that $Df \ll f$ and Theorem 34 (in Appendix A) now yields that

$$T(Df) \ll T(f).$$

Obviously it is also true that

$$T(Df) \ll zT(Df)$$

and these two facts together with Theorem 26 asserts that

$$T(Df) \ll T(f) + \delta z T(Df) \quad \forall \delta \geq 0.$$

In particular this means that

$$\mathcal{R}_\delta T(f) := T(F) + \delta z T(Df) \in \mathcal{H}(\mathbb{R}) \cup \{0\}$$

and this proves Lemma 7 □

The following proposition can now be obtained from the preceding:

Proposition 3. *Let $T \in \mathcal{A}[\mathbb{R}], \delta \geq 0$ and define a linear operator $\varepsilon^\delta T$ on $\mathbb{C}[z]$ by*

$$\varepsilon^\delta T(f) = \sum_{n=0}^{\infty} \frac{\delta^n z^n T(D^n f)}{n!}, \quad f \in \mathbb{C}[z].$$

If $T \in \mathcal{A}_H$ then $\varepsilon^\delta T \in \mathcal{A}_H$.

Proof. Let $n \geq 1$. Now apply $\mathcal{R}_{\frac{\delta}{n}}$ to T n times. This yields

$$\mathcal{R}^n_{\frac{\delta}{n}} T(f) = \sum_{k=0}^{n} \binom{n}{k} n^{-k} \delta^k z^k T(D^k f), \quad f \in \mathbb{C}[z]$$

and by repeated use of the previous lemma we know that $\mathcal{R}^n_{\frac{\delta}{n}} T \in \mathcal{A}_H \forall n \geq 1$. So for any $f \in \mathcal{H}(\mathbb{R})$ one has that

$$\sum_{k=0}^{n} (1 - \frac{1}{n}) \cdots (1 - \frac{(k-1)}{n}) \frac{\delta^k z^k T(D^k f)}{k!} =$$

$$\mathcal{R}^n_{\frac{\delta}{n}} T(f) \in \mathcal{H}(\mathbb{R})$$

so $\mathcal{R}^n_{\frac{\delta}{n}} T(f)$ tends uniformly to $\varepsilon^{\delta} T(f)$ on compacts sets and by Hurwitz theorem (Theorem 30, Appendix A) this means that $\varepsilon^{\delta} T(f) \in \mathcal{H}(\mathbb{R}) \cup \{0\}$ and thus $\varepsilon^{\delta} T \in \mathcal{A}_H$. □

The proof of Lemma 6 now follows as below:

Proof. If $F(z,w)$ is the symbol of T and T_δ denotes the operator with symbol $F(z,(1+\delta)^{-1}w)$ then a calculation shows that

$$T_\delta(f(z)) = \varepsilon^{\delta} T(f((1+\delta)z)), \quad f \in \mathbb{C}[z].$$

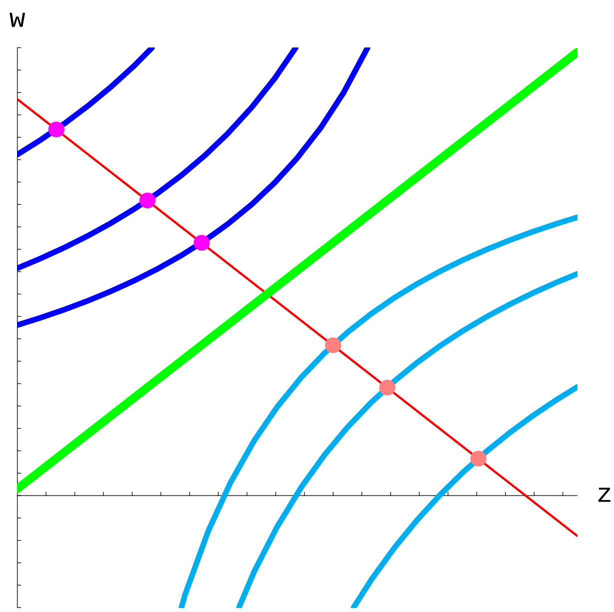
Thus if $f \in \mathcal{H}(\mathbb{R})$ then $T_\delta(f) \in \mathcal{H}(\mathbb{R})$ so that $T_\delta \in \mathcal{A}_H$. If we set $\lambda = (1+\delta)^{-1}$ the result follows from the previous proposition. □

4.4. **The HPO Dual Operator Theorem.** The second great result of this thesis is also due to Borcea-Brändén-Shapiro and we will refer to it as the Dual Operator Theorem. For this result we need to be able to define a dual operator, T^*, for every HPO operator T. As can be seen in the Appendix the Fischer-Fock space, denoted by $\mathcal{F}$, is just what we need for this matter since we are able to define a so called Fischer-Fock dual to any operator $T \in \mathcal{A}$. Hence the dual T^* satisfy $\langle T(f), g\rangle = \langle f, T^*(g)\rangle \forall f, g \in \mathcal{F}$. Geometrically this means that the algebraic symbol curve, Γ_T, as mentioned earlier is reflected by the corresponding dual symbol curve, Γ_{T^*} along the line of identity in the complex plane.

Theorem 28 (HPO Dual Operator Theorem)**.** *Let $T \in \mathcal{A}_H$ then*

$$T \in \mathcal{A}_H \cup \mathcal{A}(\mathbb{C}) \Leftrightarrow T^* \in \mathcal{A}_H \cup \mathcal{A}(\mathbb{C}).$$

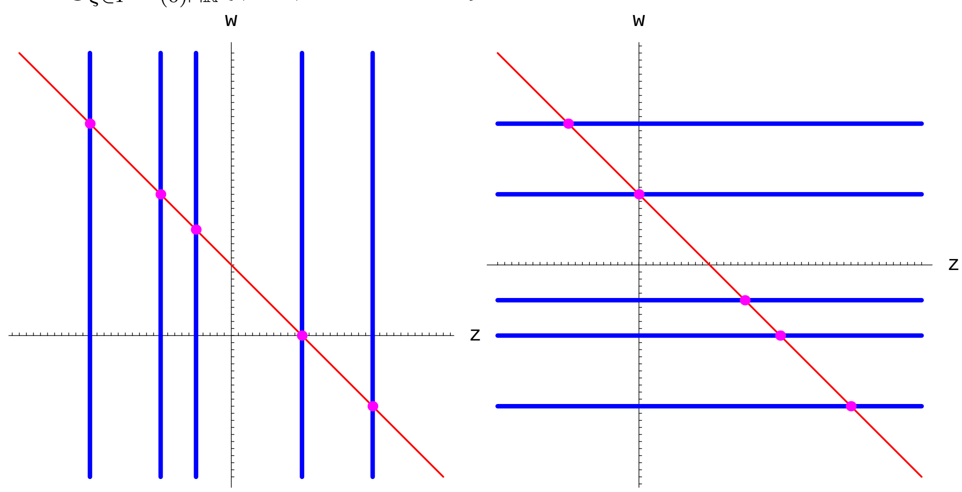
So a linear operator T is a HPO if and only if its dual operator, T^*, is a HPO. This result can be established geometrically. As we already know each line of negative slope must intersect the algebraic symbol curve of a HPO in exactly d points where d is the degree and vice versa. This property is of course preserved when we reflect Γ_T along the line of identity and hence the same intersection property is true for the dual T^*.



The corresponding theorem holds for SPOs in the multivariate case.

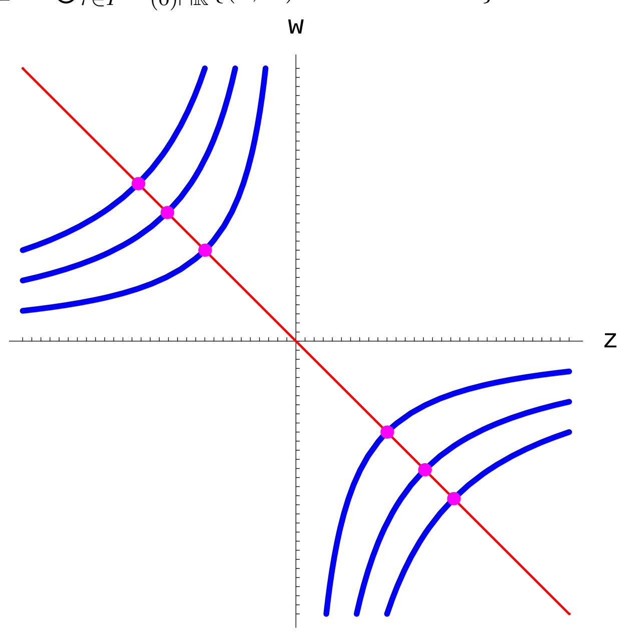
A very nice example of how this result applies is the following:

Example 3. *Let* $P \in \mathbb{R}[z]$, $degP = d$ *and* $T =$ *multiplication by* $P(z)$. *Then* $F_T(z,w) = P(z)$ *and hence* $\mathbf{\Gamma_T} = \bigcup_{\zeta \in P^{-1}(0)\cap\mathbb{R}} \{(z,w) \in \mathbb{R}^2 : z = \zeta\}$. *Dualizing provides the Hermite-Poulain Theorem and this yields* $T^* = P(D)$ *where* $D = \frac{d}{dz}$ *and* $F_{T^*}(z,w) = P(w) \Rightarrow$
$\mathbf{\Gamma_{T^*}} = \bigcup_{\xi \in P^{-1}(0)\cap\mathbb{R}} \{(z,w) \in \mathbb{R}^2 : z = \xi\}$.



This Theorem provides a criterion for how we can establish whether a finite-order multiplier sequence is HPO.

Example 4. *Simple criterion for finite-order multiplier sequences:*
$T = \sum_{k=0}^{N} a_k z^k D^k$ *is an HPO if and only if* $P(t) := \sum_{k=0}^{N} a_k t^k$ *has all non-positive zeros.*
Indeed, $F_T(z,w) = \sum_{k=0}^{N} a_k z^k w^k$ *so that*
$\mathbf{\Gamma_T} = \bigcup_{\tau \in P^{-1}(0)\cap\mathbb{R}} \{(z,w) \in \mathbb{R}^2 : zw = \tau\}$.



5. Testing the spectral order conjecture

The example in the last section encourage us to investigate if HPOs also satisfy another property. In this section we will present results on some laboratory experiments to test the following conjecture due to Borcea, Brändén and Shapiro, see e.g. [6]. Let us define an involution $\iota : \mathbb{C}[z] \mapsto \mathbb{C}[z]$ by $\iota f(z) = f(-z)$.

Conjecture 1 (Spectral order conjecture). *Let T be any HPO. Then either T or $T \circ \iota$ preserves the classical majorization property (or the spectral order).*

First of all let us learn something about what the classical majorization property means.

5.1. A short introduction to Classical Majorization Theory.

We only present a minimum of theory concerning classical majorization theory. The reader will hopefully find it quite sufficient for the contents of the rest of this section. We refer to [18] for further details on Majorization Theory.

Let x and y be n-tuples of elements in $\mathbb{R}$, i.e

$$x = (x_1, ..., x_n), \quad y = (y_1, ..., y_n), \quad x_i, y_i \in \mathbb{R}.$$

Assuming that $x_1 \geq ... \geq x_n$ and $y_1 \geq ... \geq y_n$ we say that x *is majorized by* y and write $x \prec y$ if the following n equalities is satisfied:

$$\begin{array}{rcl} y_1 & \geq & x_1 \\ & \vdots & \\ \sum_{i=1}^{n-1} y_i & \geq & \sum_{i=1}^{n-1} x_i \\ \sum_{i=1}^{n} y_i & = & \sum_{i=1}^{n} x_i \end{array}$$

Another condition for majorization is that $x = yP$ for a doubly stochastic matrix P. There are several equivalent conditions for the classical majorization property but the two mentioned above are the ones used in this experiment.

5.2. **Description of the experiment.** The first thing we want to do is generate two random polynomials $P(x)$ and $Q(x)$ with arbitrary real zeros $Z(P)$ and $Z(Q)$ in a given interval such that $Z(P) \prec Z(Q)$. To do this we begin by letting Mathematica randomly give us the tuple $Z(P)$ and then multiply it with a doubly stochastic matrix in order to obtain the zeros of the polynomials P and Q with $Z(P) \prec Z(Q)$. According to the results of section 5 every operator is hyperbolicity preserving if and only if its symbol polynomial, $\sum_{k=0}^{n} a_k t^k$, has non-positive zeros so we generate such a polynomial in the same manner as we generated P and Q to obtain our HPO, let us call it T. The only thing left to do now is to let the operator T act on our polynomials P and Q and check if the zero set of $T(P)$ - $Z[T(P)]$ - is majorized by the zeros of $T(Q)$. Let us take a closer look on how this is done in the Mathematica Software. In the following code extract our polynomials are of degree 25 and the zeros lie in the interval $[-100, 100]$ this is of course easily adjusted as preferred. We define the zeros of P and Q in the following manner:

```
y=Sort[Table[Random[Real,{-100,100}],{25}],Greater]


RandomPermutationMatrix[n_]:= Module[{P,M,j,k},P={1};
      Do[j=Random[Integer,{1,k}];P=Insert[P,k,j],{k,2,n}];
      M=Table[0,{n},{n}];
      Do[M[[j,P[[j]]]]=1,{j,1,n}];M];

RandomDoublyStochasticMatrix[n_]:=Module[{p=0.1,s=0,M,t},
      M=Table[0,{n},{n}];
      While[Random[Real,{0,1}]>p,t=Random[Real,{0,1-s}];s=s+t;
        M=M+t*RandomPermutationMatrix[n]];
      M+(1-s)*RandomPermutationMatrix[n]];

x=Sort[Table[y.RandomDoublyStochasticMatrix[Length[y]]],Greater]
```

Our polynomials are then composed like this:

```
Q=Expand[Product[t-y[[j]],{j,1,Length[y]}]]

t/.NSolve[Q\[Equal]0,t]

P=Expand[Product[s-x[[j]],{j,1,Length[x]}]]

s/.NSolve[P\[Equal]0,s]
```

To create a hyperbolic operator T recall that we can apply the symbol curve to obtain the coefficients of T:

```
k=Sort[Table[Random[Real,{-100,0}],{Random[Integer,{0,100}]}]]

fT=Expand[Product[r-k[[j]],{j,1,Length[k]}]]

a=CoefficientList[fT,r]
```

Next we let our operator act on P and Q and pick out the zeros of the new polynomials $T(P)$ and $T(Q)$.

```
TP=Expand[Simplify[Sum[a[[i]]s^(i-1)*D[P,{s,(i-1)}],
   {i,1,Length[a]}]]]

zP=Sort[s/.NSolve[TP\[Equal]0,s],Greater]

TQ=Expand[Simplify[Sum[a[[i]]t^(i-1)*D[Q,{t,(i-1)}],
   {i,1,Length[a]}]]]

zQ=Sort[t/.NSolve[TQ\[Equal]0,t],Greater]
```

Finally let us check if the spectral order conjecture holds numerically:

```
Table[Sum[zQ[[i]]-zP[[i]],{i,1,j}],{j,1,Length[zQ]}]

Table[Sum[y[[i]]-x[[i]],{i,1,j}],{j,1,Length[y]}]
```

5.3. **Results from the experiment.** Let us take a look at a numerical example to visualize this assertion. Degrees, range of zeros and our operator may of course vary infinitely. In this example the polynomials P and Q are both of degree 5 and the zeros range from -100 to 100. First of all we generate the zeros of Q:

```
y=Sort[Table[Random[Real,{-100,100}],{5}],Greater]
```

This generates the set:

```
{68.0254,54.0586,-11.2007,-24.9176,-74.124}
```

and from this we get our Q like this:

```
Q=Expand[Product[t-y[[j]],{j,1,Length[y]}]]
```

and this yields

```
Q(t) = 76075703486.99 + 8345849.81*t + 65167.63*t^2
- 6825.13*t^3 - 11.84*t^4 + t^5
```

Using the doubly stochastic matrix command to generate P gives us the set of P's zeros

```
{66.296,55.2719,-22.2928,-24.6853,-62.748}
```

And P is therefore

```
P(s) = 126530645.53 + 8620267.36*s + 11345.25*s^2
- 6176.78*s^3 - 11.84*s^4 + s^5
```

Our operator T is obtained by the following computations:

```
k=Sort[Table[Random[Real,{-100,0}],{Random[Integer,{0,100}]}]]=
{-99.726,-93.2003,-89.5969,-84.686,-84.2998,-81.4396,-74.4853,-74.003,
-68.2823,-57.6073,-54.4361,-53.0949,-48.7852,-48.4883,-48.0321,-47.1922,
-46.9346,-46.2657,-43.6218,-38.8106,-33.9966,-31.8379,-31.205,-29.9167,
-29.4717,-28.1798,-27.7739,-25.8896,-23.1169,-22.6722,-22.2253,-19.5669,
-16.8949,-16.1378,-12.2691,-11.6361,-5.90262,-4.41032,-2.67206}
```

The symbol polynomial F is now obtain by this command

```
fT=Expand[Product[r-k[[j]],{j,1,Length[k]}]]
```

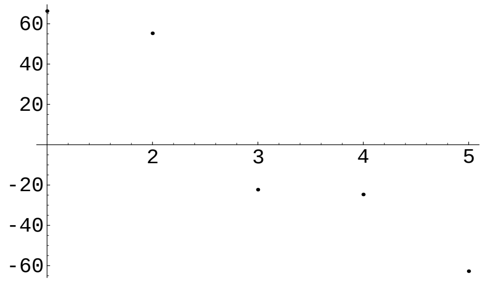
this and the following computations of $T(P)$ and $T(Q)$ yields very tedious expressions so let us skip ahead and take a look at the zeros of $T(P)$ and $T(Q)$:

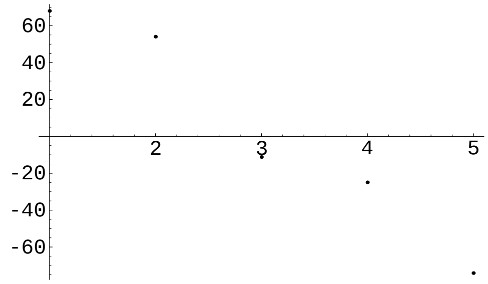
```
z(T(P)) = {27.6656,18.8548,-6.93476,-10.1045,-24.8218}
z(T(Q))= {28.116,18.6973,-3.84931,-9.29134,-29.0133}
```

And the difference between these zeros are given by this set:

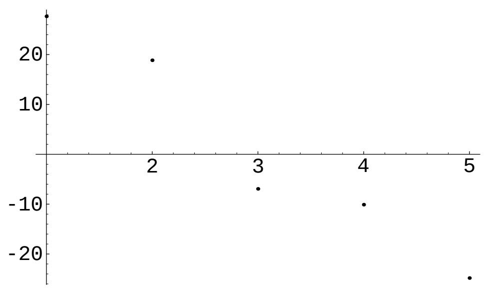
```
{0.45, 0.29, 3.38, 4.19, 3.55*10^(-15)}
```

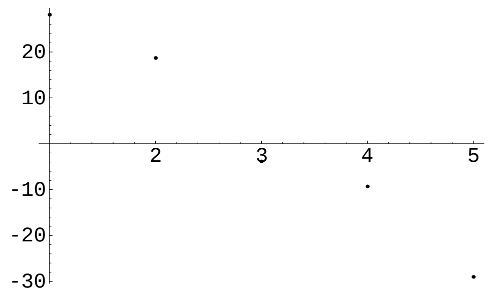
If we plot the zeros of P and Q respectively it is clear that P majorized by Q





The same goes for $z(T(P))$ and $z(T(Q))$ which is what the conjecture suggests





Remark 4. We rarely get 0 in the final position in the set of differences between $z(T(Q))$and $z(T(P))$. This is because the computations are based on approximations and round offs are made here and there.

Remark 5. Running through a numerous examples of polynomials and operators has provided convincing evidence in favor of the conjecture - the latter was actually proved in full generality quite recently by Julius Borcea and Petter Brändén (unpublished).

Appendix A. Mixed results

There should be no problem for the undergraduate Mathematic student to follow the theory presented in chapters 1-3. However in chapter 4 several well known theorems and notions are mentioned without any further explanation. It seems motivated to provide an appendix including some definitions, explanations and formulation of some of these results and notions. We give them without any particular order.

Theorem 29 (The Continuity theorem)**.** *Let $f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu = \prod_{j=1}^{k}(z-z_j)^{m_j}$, where $m_1 + ... + m_k = m$, be a monic polynomial of degree n with distinct zeros $z_1, ..., z_k$ of multiplicities $m_1, ..., m_k$. Then, given a positive $\varepsilon < min_{1\leq i<j\leq k}\frac{|z_i-z_j|}{2}$ there exists a $\delta > 0$ so that any monic polynomial $g(z) = \sum_{\nu=0}^{n} b_\nu z^\nu$ whose coefficients satisfy $|b_\nu - a_\nu| < \delta$ for $\nu = 1, ..., n-1$, has exactly m_j zeros in the disc $\mathcal{D}(z_j, \varepsilon)$ (with radius ε and center in z_j) $j = 1, .., k$.*

Theorem 30 (Hurwitz)**.** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of analytic functions defined in a region $\Omega \subset \mathbb{C}$. Suppose that this sequence converges to a function $f \not\equiv 0$, uniformly on every compact subset of Ω. Then $\zeta \in \Omega$ is a zero of f of multiplicity m if and only if there exists a neighbourhood $\mathcal{V} \subset \Omega$ of ζ such that, in every disc $\mathcal{D}(\zeta, \varepsilon) \subset \mathcal{V}$, each polynomial f_n whose index exceeds some bound $n(\varepsilon)$ has exactly m zeros, counted according to their multiplicities.*

Hurwitz theorem follows from the theorem due to *Rouché* which says that $f + g$ has the same numbers of zeros as f *inside* a Jordan curve if $|g(z]| < |f(z)|$ *on* the Jordan curve. We also remind the reader of results such as the *maximum modulus principle*, *principle of the argument*. Finally we mention that every polynomial of positive degree n may be represented as a product of $(z - z_j)$ where z_j is a zero if we multiply this product with the coefficient of the term of highest degree. If in a addition the polynomial is symmetric we can equate the coefficients by the *formula of Viète*.

Theorem 31 (Implicit function theorem)**.** *Let $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable in the neighbourhood of the fix point $(a, b) \in \mathbb{R}^m \times \mathbb{R}^n$ and assume that the derivative of $\mathbb{R}^n \ni y \mapsto f(a, y) \in \mathbb{R}^n$ is invertible in b. $\Rightarrow \exists$ a continuously differentiable function φ defined in a neighbourhood of a s.t $b = \varphi(a)$ and $f(x, \varphi(x)) = f(a, b)$ for x close to a.*

The implicit function theorem so to speak convert complicated relations to functions by representing the relation as the graph of a function. This allows us to implicitly differentiate such relations.

Definition 18. The *elementary symmetric function* in n variables is defined as

$$\begin{aligned} e_0(x_1,...,x_n) &= 1\\ e_1(x_1,...,x_n) &= \sum_{1\le j\le n} x_j\\ e_2(x_1,...,x_n) &= \sum_{1\le j<k\le n} x_j x_k\\ e_3(x_1,...,x_n) &= \sum_{1\le j<k<l\le n} x_j x_k x_l\\ &\vdots\\ e_n(x_1,...,x_n) &= x_1\cdots x_n \end{aligned}$$

Definition 19. The so called Wronskian is given by $W[f,g] := f'g - fg'$.

Definition 20 (Interlacing zeros)**.** Let $\alpha_1 \le \alpha_2 \le ... \le \alpha_n$ and $\beta_1 \le \beta_2 \le ... \le \beta_m$ be the zeros of two given polynomials $f, g \in \mathcal{H}_1(\mathbb{R})$. We say that the zeros *interlace* if they can be ordered so that either $\alpha_1 \le \beta_1 \le \alpha_2 \le \beta_2 \le ...$ or $\beta_1 \le \alpha_1 \le \beta_2 \le \alpha_2 \le ....$

If the zeros of f and g interlace then the Wronskian is either nonnegative or nonpositive on the whole real axis $\mathbb{R}$. In the case when $W[f,g] \le 0$ we say that f and g are in *proper position* and denote this by $f \ll g$. If $f \ll g$ and $g \ll f$ this implies that f and g are constant multiples of each other and the Wronskian is equal to zero in this case. For technical reasons we define the zeros of polynomial $h \equiv 0$ to interlace the zeros of any (nonzero) hyperbolic polynomial. We even write $0 \ll f$ or $f \ll 0$. We now give the classical Hermite-Biehler theorem in terms of proper position.

Theorem 32 (Hermite-Biehler)**.** *Let $h := f + ig \in \mathbb{C}[z]$, where $f, g \in \mathbb{R}[z]$. Then $h \in \mathcal{H}_1(\mathbb{C})$ if and only if $g \ll f$.*

This in turn gives us a generalization of the proper position property in higher dimensions:

Definition 21. Two polynomials $f, g \in \mathbb{R}[z_1, ..., z_n]$ are in proper position if $g + if \in \mathcal{H}_n(\mathbb{C})$.

Theorem 33 (Obreschkoff's Theorem)**.** *Let $f, g \in \mathbb{R}[z]$. Then $\alpha f + \beta g \in \mathcal{H}(\mathbb{R}) \cup \{0\}$ for all $\alpha, \beta \in \mathbb{R}$ if and only if either $f \ll g$ or $f = g \equiv 0$.*

We now introduce the concept of an operator being *sign monotone*.

Definition 22. A linear operator T on $\mathbb{C}[z]$ is *monotone* if there exists $M \in \mathbb{N}$ and $d \in \mathbb{Z}$ such that

$$T(z^n) = 0, \quad n < M, \quad degT(z^n) = n + d, \quad n \ge M$$

where d is the degree shift of T.

Such a T is sign monotone if either the leading coefficient of $T(z^n)$ is $\ge 0 \forall n \in \mathbb{N}$ or the leading coefficient of $T(z^n)$ is $\le 0 \forall n \in \mathbb{N}$. In the first case we say that T is positive monotone and in the second one that it is negative monotone. Obviously T is positive monotone if and only if $-T$ is negative monotone and vice versa.

Theorem 34. *If* $T \in \mathcal{A}_H \cap \mathcal{A}[\mathbb{R}]$ *and* $f, g \in \mathcal{H}(\mathbb{R})$ *with* $f \ll g$ *then* $T(f) \ll T(g)$ *or* $T(f) = T(g) = 0$*.*

Remark 6. This result is actually equivalent to

$$\mathcal{A}_H \cap \mathcal{A}[\mathbb{R}] \subset \mathcal{A}_S \cap \mathcal{A}[\mathbb{R}]$$

i.e the set of HPOs (hyperbolicity preservers) is a subset of the set of SPOs (stability preservers). It is of course very nice that the inclusion of the sets containing hyperbolic and stable polynomials applies also to the set containing hyperbolicity and stability preservers.

Appendix B. The Fischer-Fock space and Weyl Algebra

Here we give some important basic notions regarding the Hilbert space and measure theory are mentioned in this section in order to give a proper introduction of the Weyl Algebra and Fischer-Fock space. Not in any way do we claim this to be a full account on these matters since there are extensive literature for the reader to consult should he or she feel the need. It is crucial that the reader is familiar with these two concepts in order to understand the main results given in chapter 4.

B.1. **The Fischer-Fock space.**

The Hilbert space is a complete linear vector space that has a norm and an inner product. We remind the reader of the formal definition of a vector space and give the definitions a norm, an inner product and explain the concept of completeness.

Definition 23. In order for V to be a linear vector space the following must hold for all elements $x, y, z \in V$ and scalars $\lambda, \mu \in \mathbb{C}$

(i) Vector addition is commutative, $x + y = y + x$.
(ii) Vector addition is associative, $(x + y) + z = x + (y + z)$.
(iii) There is an additive identity element, 0, s.t. $0 + x = x + 0 = x$.
(iv) For every x there is an element $-x$ s.t. $x + (-x) = 0$.
(v) Scalar multiplication is associative, $\lambda(\mu x) = (\lambda\mu)x$.
(vi) Scalar addition is distributive, $(\lambda + \mu)x = \lambda x + \mu x$.
(vii) Vector addition is distributive, $\lambda(x + y) = \lambda x + \lambda y$.
(viii) There is a scalar multiplication identity element, 1, s.t. $1x = x$.

Definition 24. A norm $\|\cdot\|$ satisfy:

(i) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$
(ii) $\|\lambda x\| = |\lambda|\|x\|$
(iii) $\|x + y\| \leq \|x\| + \|y\|$.

Definition 25. And inner product, $\langle \cdot, \cdot \rangle$ satisfy:

(i) $\langle x, y \rangle = \overline{\langle y, x \rangle}$
(ii) $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$
(iii) $\langle x, \lambda y \rangle = \overline{\lambda} \langle x, y \rangle$
(iv) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
(v) $\langle x, x \rangle \geq 0$ with equality iff $x = 0$.

Finally a vector space is said to be *complete* if every Cauchy sequence (in that space) converges to a point in the space. (A Cauchy sequence $\{x_n\}$ satisfy that $\|x_n - x_m\| \to 0$ as $m, n \to \infty$). The definition of the Hilbert space is now very short and concise!

Definition 26. A Hilbert space is a complete linear vector space that has a norm and an inner product.

The Fischer-Fock space is a well known example of a Hilbert space with an inner product. More precisely the Fischer-Fock space is given by the following definition:

Definition 27. The Fischer-Fock space $\mathcal{F}$ is the space of all entire functions in $\mathbb{C}$ which are square-integrable with respect to Gaussian measure.

$$\mathcal{F} = \left\{ f \text{ analytic in } \mathbb{C} \text{ s.t} \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} d\mu < \infty \right\}$$

where μ is a Lebesgue measure on $\mathbb{C}$.

The inner product is given by:

$$\langle f, g \rangle = \frac{1}{\pi} \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-|z|^2} d\mu$$

and we may also express it as

$$\langle f, g \rangle = f(D) g^*(z) \Big|_{z=0}$$

where $D = \frac{d}{dz}$ and $g^*(z) = \sum_k \overline{a_k} z^k$ if $g^*(z) = \sum_k a_k$ F has an reproducing kernel: $K_w(z) = e^{z\overline{w}}$ that is $\langle f(z), K_w(z) \rangle = f(w)$ when $f \in \mathcal{F}$.
Using the fact than $\overline{\langle f, g \rangle} = \langle f, g \rangle$ the expression $\langle f, g \rangle = f(D) g^*(z) \Big|_{z=0}$ the following relations may be deduced:

$$\langle z^k f, g \rangle = \langle f, D^k g \rangle$$
$$\langle D^k f, g \rangle = \langle f, z^k g \rangle$$

for $f, g \in \mathcal{F}$. Since there is a natural way to introduce the notion of a dual in any Hilbert space we define the Fischer-Fock dual as the following:

Definition 28. Given the linear operator T on $\mathcal{F}$ its dual, T^*, is given by the unique linear operator on that satisfy $\langle T(f), g \rangle = \langle f, T * (g) \rangle \forall f, g \in \mathcal{F}$.

Remark 7. The above account on the Fischer-Fock space is of importance in subsection 4.4 and the HPO Dual Operator Theorem since it allows us to define the Fischer-Fock dual.

B.2. **The Weyl Algebra.** Since some of the following theory assumes that we know something about measures we will briefly try to explain this even though the term "measure" or "measurable," have very precise technical definitions that can make them appear difficult and perhaps a bit tedious to understand. To jump ahead we give the definition of a measure space:

Definition 29. A *measure space* is a triple (X, Σ, μ) where X is a space, Σ is a σ-algebra and μ is a measure with domain Σ.

This definition requires that we know what a σ-algebra and a measure is. A σ-ring is a class $\mathcal{R}$ of sets such that

- $\emptyset \in \mathcal{R}$
- $A, B \in \mathcal{R} \Rightarrow A - B \in \mathcal{R}$
- $A_n \in \mathcal{R}$ for $n = 1, 2... \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$.

If in addition the whole space is in $\mathcal{R}$, i.e $X \in \mathcal{R}$, we say that $\mathcal{R}$ is a σ-algebra.
A measure is an extended real-valued set function having the following properties:

- The domain Σ of μ is a σ-algebra
- μ is non-negative on Σ
- μ is completely additive on Σ
- $\mu(\emptyset) = 0$

A set function is a function defined on a class of sets, it is extended-real-valued if its values are extended real numbers. This set is called the domain of μ. To emphasize the importance of measure theory note that every definition of an integral is based on a particular measure: the Riemann integral is based on Jordan measure and the Lebesgue integral is based on Lebesgue measure.
The Weyl Algebra is an important object in representation theory, quantum mechanics etc (see [3]) and it is a famous example of a simple algebra which is not Artinian. In algebraic terms the Weyl Algebra is the subalgebra of all linear endomorphisms of the polynomial algebra $\mathbb{C}[z]$ generated by z and $D = \frac{d}{dz}$ where the canonical commutator relation is satisfied. But it is the following definition that will be of relevance for our purpose:

Definition 30. The Weyl Algebra is the set of all operators T on $\mathbb{C}[z]$ on the form

$$T = \sum_{k=M}^{N} Q_k(z) D^k$$

where $Q_k \in \mathbb{C}[z]$ for all k and $M \leq N$.

I.e The Weyl Algebra is the space of all finite order linear ordinary differential operators with polynomial coefficients,

Remark 8. The Weyl Algebra is of importance through out section 4 and both the formulation and proof of the Borcea-Brändén-Shapiro Theorem relies on theory regarding the Weyl Algebra.

References

[1] V. Bargman, *On Hilbert space of analytic functions and an associated integral transform*, Comm. Pure Appl. Math 14 (1955), 293-328.

[2] L. Bieberbach, *Conformal Mapping*, Chelsea Publishing Company, 1953, New York.

[3] J-E. Björk, *Rings of differential operators*, North-Holland Publishing Co., Amsterdam-New York (1979) xvii-374 pp.

[4] J. Borcea, P. Brändén and B. Shapiro, *Classification of Hyperbolicity and Stability Preservers: the Weyl Algebra case* math.CA/0606360

[5] J. Borcea, P. Brändén and B. Shapiro, *Polya-Schur master theorems for circular domains and their boundaries* math.CA/0607416

[6] J. Borcea *Spectral order and isotonic differential operators of Laguerre-Pólya type* math.CA/0404336

[7] P. Brändén, *On linear transformations preserving the Pólya frequency property*, Trans. Amer. Math. Soc. 358 (2006), 3697-3716.

[8] J.M. Carnicer, J.M. Peña, A. Pinkus, *On some zero-increasing operator*, Acta Math. Hungar. 94, (2004), 173-190.

[9] T. Craven, G. Csordas, *Composition Theorems, Multiplier Sequences and complex zeros decreasing sequences*, in "Value Distribution Theory and Its Related Topics", ed. G. Barsegian, I.Laine and C.C Yang, pp. 131-166, Kluver Press, 2004.

[10] T. Craven, G. Csordas, *The Gauss-Lucas Theorem and Jensen Polynomials*, Trans. Amer. Math. Soc 278 (1983), 415-429.

[11] T. Craven, G. Csordas, *Multiplier sequences for fields*, Illinois J. Math. 21 (1977), 801-817.

[12] G.H. Hardy, J.E. Littlewood, G.Pólya, *Inequalities*, Cambridge Univ. Press. 2nd ed. 1988

[13] O. Holtz, *Hermite-Biehler, Routh-Hurwitz and total positivity*, Linear Algebra and its Applications 372 (2003) 105-110, Berlin.

[14] K. Kodaira, *Introduction to Complex Analysis*, Cambridge University Press, 1984 Cambridge.

[15] B. Levin, *Distribution of zeros of entire functions*. Translated from Russian by R.P.Boas, J.M.Danskin, F.M.Goodspeed, J.Korevaar, A.L.Shields and H.P.Thielman. Transl. Math. Monogr. Vol. 5. Amer. Math. Soc., Providence, R.I., 1980. xii+523 pp.

[16] E.H. Lieb and A.D. Sokal, *A General Lee-Yang Theorem for One-Component and Multicomponent Ferromagnets*, Commun. Math. Phys. 80, 153-179 (1981)

[17] M. Marden, *The Geometry of the Zeros*, Mathematical Surveys nr III, American Math. Soc., New York, 1949.

[18] A.W. Marshall, I. Olkin, *Inequalities: Theory of Majorization and Its Applications*, Academic Press, New York, NY, 1979.

[19] D.J. Newman, H.S. Shapiro, *Certain Hilbert spaces of entire functions*, Bull. Amer. Math. Soc 72 (1966), 971-977

[20] G. Pólya and J. Schur, *Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen*, J.Reine Angew. Math. **144**(1914), 89-113

[21] Q. I. Rahman, G. Schmeisser, *Analytic theory of polynomials*, London Math. Soc. Monogr. (N. S.) Vol. **26**, Oxford Univ. Press, New York, NY, 2002.

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