EXAMENSARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Algebraic properties of completely discretely stable graphs

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2008 - No 7

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Examensarbete i matematik 30 högskolepoäng, fördjupningskurs

Handledare: Jörgen Backelin

2008

Abstract

In this thesis, we investigate a class of graphs called completely discretely stable, and also a kind of graph extensions called stitches. The properties of their independence complexes and face lattices are studied. As a corollary, we calculate the f- and h-vectors of a type of combinatorially determined Gorenstein complexes.

Contents

Notation

$\binom{S}{i}$	The set of i-subsets of the finite set S
$\operatorname{Ann} M$	The annihilator of the module M, 7
$\operatorname{Ass} M$	The associated primes of the module M, 7
$\mathcal{C}^G$	The independence complex of the graph G, 38
$\tilde{\chi}\,(\Delta)$	The reduced Euler characteristic of the complex Δ, 27
$\operatorname{depth} M$	The depth of the module M, 15
$\dim R$, $\dim M$	The dimension of the ring R, the module M, 14, 14
$\operatorname{Ext}^i_R(M,N)$	The i-th Ext-module of M and N, 13
${}^*\operatorname{Ext}^i_R(M,N)$	The i-th graded Ext-module of M and N, 18
$\Delta * \Gamma$	The join of the complexes Δ and Γ, 27
$\kappa(\mathfrak{p})$	The residue class field of the localization at $\mathfrak{p}$, 6
$\operatorname{lk}_\Delta \sigma$	The link of the face σ in the complex Δ, 27
$\operatorname{cr}_n(G)$, $\operatorname{cr}_n(G\,;u_1,\dots,u_n)$	The n-stitch of the graph G, with base $u_1,\dots,u_n$, 42
$\operatorname{grade}(I,M)$	The grade of the ideal $I \subset R$ on the R-module M, 15
$\operatorname{ht} I$	The height of the ideal I, 14
$\operatorname{Hilb}_N(t)$	The Hilbert series of the graded module N,22
$L(\Delta)$	The face lattice of the complex Δ, 35
$\mathbb{N}$	The integers $0,1,2,\dots$
$P(\Delta)$	The face poset of the complex Δ, 35
$r(M)$	The type of a module M, 16
$\operatorname{Soc} M$	The socle of the module M, 7
$\operatorname{Spec} R$	The spectrum of the ring R, 4
$\operatorname{St}_\Delta \sigma$	The star of the face σ in the complex Δ, 27
$\subset$	Subset, not necessarily strict
$\subsetneq$	Subset, strict
$\operatorname{Supp} M$	The support of the module M, 7

1 Introduction

Ramsey theory originates from a paper Frank P. Ramsey published in 1930, concerning a certain problem in formal logic [9]. In general, Ramsey theory is concerned with how "big" (in some sense) a mathematical structure must be to assure the existence of certain substructures. Applied to graph theory, this can be interpreted in terms of graphs and subgraphs. No definitons will be made here, If necessary, the reader will find them in Section 3.3

Definition 1.0.1. Let $G_1, \ldots, G_p$ be graphs. Then

$$\begin{aligned} R(G_1, \ldots, G_p) \quad := \quad & \min\{n \,|\, \text{Every colouring with } p \text{ colours of } K_n \\ & \qquad \text{contains an } i\text{-coloured } G_i, \text{ for some } i.\} \end{aligned}$$

is called the *Ramsey number* of $G_i, \ldots, G_p$.

Note that

$$R(G_1, \ldots, G_i, G_{i+1}, \ldots, G_p) = R(G_1, \ldots, G_{i+1}, G_i, \ldots, G_p);$$

this follows immediately from the definition. When $G_i = K_{n_i}$, for all i, let $R(n_1, \ldots, n_p) := R(K_{n_1}, \ldots, K_{n_p})$. It is not immediate that these numbers are finite for every collection of graphs. However, if $R(G_1, \ldots, G_n)$ exists and H_i is a subgraph to G_i for every $1 \le i \le n$, clearly $R(H_1, \ldots, H_n) \le R(G_1, \ldots, G_n)$. In particular, if $|G_i| = m_i$, $R(G_1, \ldots, G_n) \le R(m_1, \ldots, m_n)$. The next theorem assures the existence of $R(G_1, \ldots, G_n)$ for every collection of graphs $G_1, \ldots, G_n$.

Theorem 1.0.2. *Assume $r_1, \ldots, r_s \ge 1$. Then $R(r_1) = r_1$, $R(r_1, r_2) \le R(r_1 - 1, r_2) + R(r_1, r_2 - 1)$, and if $s \ge 3$,*

$$R(r_1, \ldots, r_s) \le R\left(r_1, \ldots, r_{s-2}, R\left(r_{s-1}, r_s\right)\right).$$

Proof. First, obviously $R(r_1) = r_1$. Next, we show by induction that $R(r_1, r_2) \le R(r_1 - 1, r_2) + R(r_1, r_2 - 1)$. We have $R(m, 1) = R(1, m) = 1$ from definition, since every edge in K_1 is coloured in any colour. Now assume $R(r_1 - 1, r_2)$ and $R(r_1, r_2 - 1)$ satisfies the inequality, and hence are finite. Consider the complete graph on $R(r_1 - 1, r_2) + R(r_1, r_2 - 1)$ vertices, and pick a vertex v. Let M be the induced subgraph on the vertices w_i, vw_i blue, and define N in the same manner on the vertices u_i, vu_i red. Now, either $|M| \ge R(r_1 - 1, r_2)$ or $|N| \ge R(r_1, r_2 - 1)$. If $|M| \ge R(r_1 - 1, r_2)$, either M contains a red K_{r_2} or a blue $K_{r_1 - 1}$. In the latter case, the induced coloured graph on $M \cup \{v\}$ contains a blue K_{r_1}. With the same arguments, if $|N| \ge R(r_1, r_2 - 1)$, N contains either a blue K_{r_1} or ar red K_{r_2}.

Finally, by induction over s, we show that $R(r_1, \ldots, r_s)$ satisfies the inequality. We know $R(r_1)$ and $R(r_1, r_2)$ are finite. When $s \ge 3$, regard $s - 1$ and s as the same colour s'. Let $r_{s'} = R(r_{s-1}, r_s)$. We know from the induction hypothesis that $n := R(r_1, \ldots, r_{s-2}, r_{s'})$ is finite, and each colouring of K_n which contains an s'-coloured $K_{r'_s}$ contains either an $(s-1)$-coloured $K_{r_{s-1}}$, or an s-coloured K_{r_s}. $\square$

Example 1.0.3. A classical example is $R(3,3)$. First, note that from Theorem 1.0.2, we get $R(3,3) \leq 6$. Now, as seen in Figure 1, there exists an edge colouring of K_5 which neither contains a blue nor a red triangle. This colouring is in fact unique. Hence, $R(3,3) = 6$.

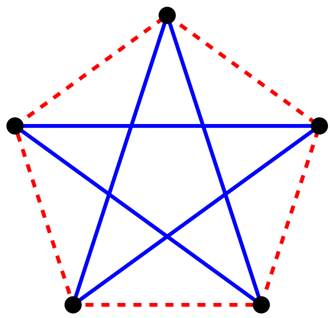


Figure 1: The unique bicolouring of K_5 with no monochromatic triangle.

Note that in the in case of two colours, the "blue" and "red" edges can be reinterpreted as edges and non-edges.

The upper bounds achieved in Theorem 1.0.2 are useful only for proving finiteness, as they are far from good estimates for larger Ramsey numbers. A great amount of work has been put into determining the Ramsey numbers, but it is a daunting task. For example, as per 2nd of May 2008, only one non-trivial Ramsey number has been determined when $s > 2$, namely $R(3,3,3) = 17$, which is shown in [5]. Table 1 gives a summary of the known exact vaules or upper and lower bounds for small Ramsey numbers, taken from [8].

One tool that has been used to determining upper and lower bounds of $R(3,j)$ are the numbers $e(i,j,n)$, the minimal number of blue edges in a K_n such that there exists no blue K_i or red K_j. Sharpening the lower bound on e-numbers may sharpen the upper bound on the Ramsey numbers $R(3,j)$, since given a least number of blue edges in a K_m, it might be possible to prove that there must exist a blue triangle. This can of course also be used to sharpen upper bounds on general Ramsey numbers $R(i,j)$.

The edge extremal graphs with respect to the $e(3,j,n)$-numbers, i.e. graphs which adhere to the conditions imposed by the definition of $e(i,j,n)$ with exactly $e(3,j,n)$ edges, are often "nice" in some sense. For example, when the ratio $\frac{j-1}{n}$ is between $\frac{1}{3}$ and $\frac{1}{2}$, all edge number extremal graphs are so-called completely discretely stable graphs. This category of graphs can be extended by dropping the condition of being triangle-free. We will study the properties of such graphs, and see which properties are invariant under certain graph extensions.

On the other hand, to every graph one can assign a commutative ring, via simplicial complexes. There is a strong interplay between properties of the

r s	3	4	5	6	7	8	9	10
3	6	9	14	18	23	28	36	40 43
4		18	25	35 41	49 61	56 84	73 115	92 149
5			43 49	58 87	80 143	101 216	125 316	143 442
6				102 165	113 298	127 495	169 780	179 1171
7					205 540	216 1031	233 1713	289 2826
8						282 1870	317 3583	6090
9							565 6588	580 12677
10								798 23556

Table 1: Known values and upper and lower bounds for small Ramsey bicoloured numbers.

ring and properties of the graph, and completely discretely stable graphs are interesting from this perspective. To understand the ring-graph correspondence, a good deal of algebra is needed. The first part of this thesis will go through the required material. The second part contains material on simplicial complexes and graphs, in particular completely discretely stable graph and a type of graph extension.

2 Algebra

The first sections will quickly go through standard material in commutative and homological algebra. Then, progressively more carefully, we will go through some dimension theory and other subjects in commutative algebra. The material is from [10] and [12].

2.1 Basic constructions

When not explicitly stated otherwise, by a ring R we will always mean a ring with a multiplicative identity $1 \neq 0$, and with associative and commutative ring multiplication. Recall that a non-empty subset $I \subset R$ of a ring R is an *ideal* of R if it is closed under addition, and $rI \subset I$ for every $r \in R$. An ideal $I \neq R$ is called a *proper ideal*. For $r_1, \ldots, r_n \in R$, let $(r_1, \ldots, r_n)$ be the ideal in R

generated by $r_1, \dots, r_n$. An ideal I is prime if for every $r, p \in R$, $rp \in I$ implies $r \in I$ or $p \in I$. The set of all prime ideals of R is called the *spectrum* of R, denoted $\operatorname{Spec}(R)$. An ideal I is *maximal* if $I \subsetneq J \subset R$ implies $J = R$. Note that all maximal ideals are prime.

Definition 2.1.1. A ring R is local if it has only one maximal ideal. A local ring is often written as $(R, \mathfrak{m})$, where $\mathfrak{m}$ is the unique maximal ideal of R. The field $R/\mathfrak{m}$ is called the *residue class field* of R.

The following theorem is called the *Prime Avoidance Theorem*.

Theorem 2.1.2. *Let R be a ring, and $I_1, \dots, I_n$ ideals of R, of which at most two are not prime. If J is another ideal in R, such that $J \subset \bigcup_{i=1}^n I_i$, $J \subset I_j$ for some $1 \le j \le n$.*

A frequent application of the above theorem is the following: If an ideal J does not lie in any finite union of primes, there is an $x \in I$ which avoids every prime; hence the name "prime avoidance".

The following definitions are mostly here for reference.

Definition 2.1.3. An *R-module* M is an abelian group together with a mapping $\varphi : R \times M \to M$ such that

(i) $r(m + n) = rm + rn$,

(ii) $(r + s)m = rs + rm$,

(iii) $(rs)m = r(sm)$,

(iv) $1m = m$,

for all $r, s \in R$ and $m, n \in M$.

A K-module, where K is a field, is called a *K-(vector) space*. A ring A which is also an R-module, with the additional condition $r(ab) = (ra)b$ for all $r \in R$ and $a, b \in A$, is called an *R-algebra*.

Let M and N be R-modules. An *(R-)module homomorphism* is a mapping $\varphi : M \to N$ such that $\varphi(r_1m_1+r_2m_2) = r_1\varphi(m_1)+r_2\varphi(m_2)$ for every $r_1, r_2 \in R$ and $m_1, m_2 \in M$. The set

$$\{n \in N \,|\, \varphi(m) = n \text{ for some } m \in M\}$$

is called the *image* of φ, denoted $\operatorname{Im}\varphi$. Similarly, the set

$$\operatorname{Ker}\varphi := \{m \in M \,|\, \varphi(m) = 0\}$$

is called the *kernel* if φ. The sets $\operatorname{Im}\varphi$ and $\operatorname{Ker}\varphi$ are submodules of M and N, respectively. A homomorphism $\varphi : M \to N$ is an *epimorphism* if $\operatorname{Im}\varphi = N$, and a *monomorphism* if $\varphi(m_1) = \varphi(m_2) \Rightarrow m_1 = m_2$. φ is an *isomorphism* if it is both an epimorphism and a monomorphism.

The set of all module homomorphisms $\varphi : M \to N$ is denoted $\mathrm{Hom}_R(M,N)$, or just $\mathrm{Hom}(M,N)$. $\mathrm{Hom}_R(M,N)$ can be given an R-module structure by letting $r\varphi$ be the mapping that takes m to $r\varphi(m)$. Note that this is not possible for non-commutative rings. Let M be an R-module.

A non-empty subset $L \subset M$ is a *submodule* of M if L is an R-module, by means of the induced operations from M. Given a submodule L of M, define a mapping from $R \times (M/L)$ to M/L by $(r, \overline{x}) \mapsto \overline{rx}$. This gives M/L an R-module structure. The module M/L is called a *quotient module* of M.

Let $M_i, i \in I$ be a family of R-modules. Let $\bigoplus_{i \in I} M_i$ be the set of all $(x_i)_{i \in I}$ with almost all $x_i = 0$. This is called the *direct sum* of $M_i, i \in I$, and is an R-module. When I is finite, there is an isomorphism from $\mathrm{Hom}_R(\bigoplus_{i\in I} M_i, N)$ to $\bigoplus_{i \in I} \mathrm{Hom}_R(M_i, N)$.

Theorem 2.1.4. *Every R-module M is a quotient module of a free R-module.*

Let M be an R-module, and $S \subset M$. If every element $m \in M$ can be written as $\sum_{s_i \in S} r_i s_i$, $r_i \in R$, with only finitely many r_i non-zero, S is said to *generate* M, and is called a *generating set for* M. If there exists a finite such $S \subset M$, M is *finitely generated*, or *finite*. A subset $T \subset M$ is *linearly independent* if $\sum_{i=1}^{n} r_i t_i = 0, r_i \in R, t_i \in T$ implies $r_i = 0$ for all i. A linearly independent generating set is called a *basis*. A module M which has a basis is a *free* R-module. When M is a finite free R-module, every basis has the same cardinality n. Note that the corresponding statement is not true for non-commutative rings. The number n is called the *rank* of M, written as $\mathrm{rank}_R M$ (or just $\mathrm{rank}\, M$). If M is a free R-module of rank n, M is isomorphic to $\underbrace{R \oplus \cdots \oplus R}_{n}$. If R is a field, every R-module has a basis, i.e. is free.

Definition 2.1.5. An R-module M is *Noetherian* if it satisfied one of the following equivalent conditions.

(i) Every submodule of M is finitely generated.

(ii) Every chain $L_1 \subsetneq L_2 \subsetneq \dots$ of submodules of M is finite.

(iii) Every non-empty family of submodules of M has a maximal member.

A ring R is Noetherian if it is Noetherian as an R-module. A useful result is the following

Proposition 2.1.6. *Let R be a Noetherian ring. Then every finitely generated R-module M is Noetherian.*

The following theorem is called *Hilbert's basis theroem.*

Theorem 2.1.7. *Let R be a Noetherian ring. Then $R[X]$ is Noetherian.*

Corollary 2.1.8. *For every field K and every $n \in \mathbb{N}$, $K[X_1, \dots, X_n]$ is Noetherian.*

Definition 2.1.9. An R-module M is *Artinian* if if it satisfies one of the following equivalent conditions

(i) Every chain $L_1 \supsetneq L_2 \supsetneq \dots$ of submodules of M is finite.

(ii) Every non-empty family of submodules of M has a minimal member.

A ring R is Artininan if it is Artinian as an R-module. Every Artinian module is Noetherian, but the converse is not true. The ring $\mathbb{Z}$ provides an example of a Noetherian, non-Artininan ring.

Definition 2.1.10. Let R be a ring. A set $S \subset R$ is *multiplicatively closed* if $1 \in S$ and $s_1, s_2 \in S$ implies $s_1 s_2 \in S$. We eill always assume $0 \notin S$.

Definition 2.1.11. Let S be a multiplicatively closed subset of a ring R. Define a relation $\sim$ on $R \times S$ as

$$(r_1, r_1) \sim (r_2, s_2) \Leftrightarrow \exists u \in S \text{ such that } u(r_1 s_2 - r_2 s_1) = 0.$$

This relation is easily seen to be an equivalence relation. For $(r, s) \in R \times S$, denote the equivalence class containing (r, s) as $\frac{r}{s}$. The set of all equivalence classes, denoted $S^{-1}R$, can be seen as a ring with the operations

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{r_1 s_2 + r_2 s_1}{s_1 s_2}, \quad \frac{r_1}{s_1}\frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2}.$$

The ring $S^{-1}R$ is called the *ring of fractions* of R with respect to S.

There is a natural ring homomorphism $f : R \to S^{-1}R$ that takes r to $\frac{r}{1}$. For an ideal $I \subset R$, let $IS^{-1}R$ be the ideal $f(I)S^{-1}R \subset S^{-1}R$. Let $r \in R$ be an element such that $r^n \neq 0, \forall n \geq 0$. Then $\{1, r, r^2, \dots\}$ is a multiplicatively closed set. Denote this set by $\langle r \rangle$. For every prime ideal $\mathfrak{p} \in R$, the set $R \setminus \mathfrak{p}$ is a multiplicatively closed set. Let $R_\mathfrak{p}$ be $(R \setminus \mathfrak{p})^{-1}R$. $R_\mathfrak{p}$ is called the *localization of* R *at* $\mathfrak{p}$. The ring $R_\mathfrak{p}$ is a local ring, with maximal ideal $\mathfrak{p}R_\mathfrak{p}$. Let $\kappa(\mathfrak{p})$ be the field $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$, called the *residue class field of* R *at* $\mathfrak{p}$.

Proposition 2.1.12. *Let S be a multiplicatively closed set in R. Then*

$$\operatorname{Spec} S^{-1}R = \{\mathfrak{p}S^{-1}R \,|\, \mathfrak{p} \in \operatorname{Spec} R, \mathfrak{p} \cap S = \varnothing\}.$$

Moreover, the mapping

$$\{\mathfrak{p}S^{-1}R \,|\, \mathfrak{p} \in \operatorname{Spec} R, \mathfrak{p} \cap S = \varnothing\} \Rightarrow \operatorname{Spec} S^{-1}R$$

is bijective and inlusion-preserving.

In particular, the above proposition gives that the prime ideals of $R_\mathfrak{p}$ are exactly the ideals $\mathfrak{q}R_\mathfrak{p}$, where $\mathfrak{q}$ is a prime of R contained in $\mathfrak{p}$.

The concept of fraction rings can be extended to modules, as follows.

Definition 2.1.13. Let S be a multiplicatively closed subset of a ring R, and let M be an R-module. Define a relation $\sim$ on $M \times S$ as

$$(m_1, s_1) \sim (m_2, s_2) \Leftrightarrow \exists u \in S \text{ such that } u(m_1 s_2 - m_2 s_1) = 0.$$

As in the case of rings, this relation is an equivalence relation. For $(m, s) \in M \times S$, denote the equivalence class containing (m, s) as $\frac{m}{s}$. The set of all equivalence classes, denoted $S^{-1}M$, can be given an $S^{-1}R$-module structure by defining

$$\frac{m_1}{s_1} + \frac{m_2}{s_2} = \frac{m_1 s_2 + m_2 s_1}{s_1 s_2}, \quad \frac{r}{s}\frac{n}{t} = \frac{rm}{st}.$$

The module $S^{-1}M$ is called the *module of fraction* of M with respect to S.

Set $M_{\mathfrak{p}} := (R \setminus \mathfrak{p})^{-1}M$ for $\mathfrak{p} \in \operatorname{Spec} R$, called the *localization of M at $\mathfrak{p}$*.

Definition 2.1.14. The *support* of an R-module M, $\operatorname{Supp} M$, is the set of prime ideals $\mathfrak{p}$ in R such that $M_{\mathfrak{p}} \neq 0$. When $M = R$, $\operatorname{Supp} M = \operatorname{Spec} M$.

The following proposition lists some properties of fraction modules.

Proposition 2.1.15. *Let M and N be R-modules, L a submodule of M, and S a multiplicatively closed set in R. Then*

(a) $S^{-1}(IM) = IS^{-1}RS^{-1}M$,

(b) $S^{-1}M/S^{-1}L \cong S^{-1}(M/L)$,

(c) $S^{-1}(M \oplus N) \cong S^{-1}M \oplus S^{-1}N$,

(d) $\operatorname{Supp}_{S^{-1}M} = \{\mathfrak{p}S^{-1}R \,|\, \mathfrak{p} \in \operatorname{Supp} M, \mathfrak{p} \cap S = 0\}$.

More information on the rings and modules of fractions may be found in any standard book in commutative algebra, for example [4], [6], [10].

Definition 2.1.16. Let R be a ring, M an R-module, and $J \subset M$. The set $\{r \in R \,|\, rJ = 0\}$ is an ideal of R called the *annihilator of J*, denoted $\operatorname{Ann} J$ (or $\operatorname{Ann}_R J$, to emphasize the ring). The annihilator of $\{m\}$ is often written as $\operatorname{Ann} m$.

Lemma 2.1.17. *Let M be a finite R-module. Then*

$$\operatorname{Supp} M = \{\mathfrak{p} \in \operatorname{Spec} R \,|\, \mathfrak{p} \supset \operatorname{Ann} M\}.$$

Definition 2.1.18. Let M be an R-module.over a local ring $(R, \mathfrak{m})$, with $R/\mathfrak{m} = K$. Then

$$\operatorname{Soc} M := \operatorname{Ann} \mathfrak{m}$$

is called the *socle* of M.

Definition 2.1.19. Let R be a ring and M an R-module. An ideal $P \in \operatorname{Spec} R$ is called an *associated prime ideal of M* if $P = \operatorname{Ann} m$, for some $m \in M$. The set of all associated prime ideals of M is denoted $\operatorname{Ass}_R M$, or $\operatorname{Ass} M$ if the ring is understood.

Definition 2.1.20. Let M be an R-module. An element $0 \neq r \in R$ is a *zerodivisor* on M if there exists $0 \neq m \in M$ such that $rm = 0$. An element in R which is not a zerodivisor is called a *non-zerodivisor.* The set of zerodivisors on M is written as $\operatorname{Zdv} M$, or $\operatorname{Zdv}_R M$.

Proposition 2.1.21. *If R is a Noetherian ring and $M \neq 0$ a finite R-module,* $\operatorname{Ass} M$ *is a finite set, and*

$$\bigcup_{\mathfrak{p} \in \operatorname{Ass} M} \mathfrak{p} = \{0\} \cup \operatorname{Zdv} M.$$

The following proposition will be used later.

Proposition 2.1.22. *Let R be a ring and M an R-module. All inclusion minimal primes in* $\operatorname{Supp} M$ *lie in* $\operatorname{Ass} M$*.*

2.2 Homological Algebra

Homological methods are very powerful tools in an algebraist's toolbox. Over the last decades, homological algebra has been applied in many branches of mathematics, as well as in theoretical physics. Even the basic material presented here will provide a method useful in commutative algebra, namely the use of the Ext-*functor.* The material presented here is from [12] and [4].

2.2.1 Preliminaries

Let

$$\cdots \xrightarrow{\alpha_{i+2}} M_{i+1} \xrightarrow{\alpha_{i+1}} M_i \xrightarrow{\alpha_i} M_{i-1} \xrightarrow{\alpha_{i-1}} \cdots$$

be a sequence of R-modules and homomorphisms. The sequence is *exact at M_i* if $\operatorname{Im} \alpha_{i+1} = \operatorname{Ker} \alpha_i$. The sequence is *exact* if it is exact at every M_i where it makes sense. An exact sequence

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

is called a *short exact sequence.*

A short exact sequence of the form

$$0 \longrightarrow L \longrightarrow L \oplus N \longrightarrow N \longrightarrow 0$$

is called a *split sequence.*

Definition 2.2.1. A *covariant functor* F from R-modules to S-modules is an object that for any R-module M associates an S-module $F(M)$, and for every R-module homomorphism $\alpha : M \to N$ associates an S-module homomorphism $F(\alpha) : F(M) \to F(N)$. Moreover F must satisfy the following.

(i) If $\alpha : M \to N$ and $\beta : N \to P$ are R-module homomorphisms, $F(\beta\alpha) = F(\beta)F(\alpha)$,

(ii) $F(\mathrm{Id}_M) = \mathrm{Id}_{F(M)}$.

Examples 2.2.2. Let R be a ring and M an R-module. Let $\mathrm{Hom}(M, -)$ be the mapping that takes an R-module N and maps it to $\mathrm{Hom}(M, N)$, and $F(\alpha)$ the homomorphism that takes $\varphi \in \mathrm{Hom}(M, N)$ to $\alpha\varphi \in \mathrm{Hom}(M, P)$, for $\alpha : N \to P$. Then $\mathrm{Hom}(M, -)$ is a covariant functor from R-modules to R-modules.

Another covariant functor is the *localization functor*. For a ring R, an R-module M and a multiplicatively closed set $S \subset R$, set $F(M) = S^{-1}M$. For $\varphi : M \to N$, let $F(M)$ be the map that takes $\frac{m}{s}$ to $\frac{\varphi(m)}{s}$. This gives a functor from R-modules to S^{-1}-modules.

A *contravariant functor* F from R-modules to S-modules is defined in the same way as a covariant functor, except that for every R-module homomorphism $\alpha : M \to N$, F associates an S-module homomorphism $F(\alpha) : F(N) \to F(M)$, and 2.2.1(i) is modified accordingly.

Example 2.2.3. Let R be a ring and M an R-module. Let $\mathrm{Hom}(-, M)$ be the mapping that takes an R-module N and maps it to $\mathrm{Hom}(N, M)$, and $F(\alpha)(\varphi) = \varphi\alpha$, for $\varphi \in \mathrm{Hom}(P, M)$ and $\alpha : P \to N$. Then $\mathrm{Hom}(-, M)$ is a contravariant functor.

Definition 2.2.4. A covariant functor F is *left-exact* if it preserves exact sequences

$$0 \longrightarrow L \longrightarrow M \longrightarrow N,$$

that is,

$$0 \longrightarrow F(L) \longrightarrow F(M) \longrightarrow F(N)$$

is exact. If it preserves exact sequences

$$L \longrightarrow M \longrightarrow N \longrightarrow 0,$$

it is called *right-exact*. If a functor is both left- and right-exact, it is called *exact*.

Left- and right-exactness for contravariant functors is defined similarly, with the obvious changes made.

Examples 2.2.5. The functors $\mathrm{Hom}(M, -)$ and $\mathrm{Hom}(-, N)$ are both left-exact, but not always exact. The exact sequence

$$0 \longrightarrow 2\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0,$$

where α and β are the canonical homomorphisms, provides examples where $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, -)$ and $\mathrm{Hom}(-, 2\mathbb{Z})$ are not exact.

The localization functor is an exact functor.

Definition 2.2.6. A functor F is *additive* if $F(\alpha + \beta) = F(\alpha) + F(\beta)$, for $\alpha, \beta \in \mathrm{Hom}_R(M, N)$. It follows that F preserves direct sums of modules,thus $F(L \oplus M) = F(L) \oplus F(M)$.

An immediate consequence of the above definition is that additive functors preserve split sequences.

Example 2.2.7. $\mathrm{Hom}(M, -)$ and $\mathrm{Hom}(-, N)$ are additive functors.

Definition 2.2.8. Let R be a ring. A *chain complex* $(\mathcal{C}_\bullet, \partial)$, or just $\mathcal{C}_\bullet$, of R-modules is a sequence of modules and homomorphisms

$$\mathcal{C}_\bullet : \cdots \longrightarrow C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \longrightarrow \cdots,$$

with $\partial_i \partial_{i+1} = 0$ for every i. The *homology modules* $H_i(\mathcal{C}_\bullet)$ of $\mathcal{C}_\bullet$ are $H_i(\mathcal{C}_\bullet) := \mathrm{Ker}\,\partial_i / \mathrm{Im}\,\partial_{i+1}$. Dually, a *cochain complex* $(\mathcal{C}^\bullet, \partial)$ $(\mathcal{C}^\bullet)$ of R-modules is a sequence of modules and homomorphisms

$$\mathcal{C}^\bullet : \cdots \longleftarrow C^{i+1} \xleftarrow{\partial^i} C^i \xleftarrow{\partial^{i-1}} C^{i-1} \longleftarrow \cdots,$$

with $\partial^i \partial^{i-1} = 0$ for every i. The *cohomology modules* $H_i(\mathcal{C}^\bullet)$ of $\mathcal{C}^\bullet$ are $H^i(\mathcal{C}^\bullet) := \mathrm{Ker}\,\partial^i / \mathrm{Im}\,\partial^{i-1}$.

A (co)chain complex $\mathcal{C}_\bullet$ $(\mathcal{C}^\bullet)$ is *exact at i* if $H_i(\mathcal{C}_\bullet) = 0$ $(H^i(\mathcal{C}^\bullet) = 0)$. If $\mathcal{C}_\bullet$ $(\mathcal{C}^\bullet)$ is exact at every i, $\mathcal{C}_\bullet$ $(\mathcal{C}^\bullet)$ is *exact*.

Note that every cochain complex can be turned into a chain complex by reversing the arrows, and vice versa. However, it is convenient to have both notations. In what follows, definitions and results will be defined in terms of either chain or cochain complexes. Everything may of course be dualized.

2.2.2 Projective and Injective Modules and Resolutions

Definition 2.2.9. Let P be an R-module. If for any R-module epimorphism $\varphi : M \to N$ and homomorphism $\gamma : P \to N$, there is a homomorphism $\beta : P \to M$ such that the diagram

$$\begin{tikzcd} & P \arrow[dl, dashed, "\beta"'] \arrow[d, "\gamma"] \\ M \arrow[r, two heads, "\varphi"] & N \end{tikzcd}$$

commutes, P is called a *projective* module.

If P is a free with generators p_i, choose $q_i \in M$ such that $\varphi(q_i) = \gamma(p_i)$. Then we may let β be the map sending p_i to q_i. Thus every free module is projective.

If N is a projective module, every exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

splits.

Let M be an R-module. A chain complex

$$\mathcal{P}_\bullet : \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is called a *projective (free) resolution* of M if $\mathcal{P}_\bullet$ is exact, and all P_i are projective (free) modules. Note that since every free module is projective, every free resolution is a projective resolution. Thus, from Theorem 2.1.4, every module has a projective resolution. Sometimes the module M is removed from the resolution. In that case the chain complex need to be exact everywhere except at P_0, and $H_0 = M$.

Dual to the notion of projective modules is that of *injective* modules, although the theory is quite different; for example, injective modules are in general not finite.

Definition 2.2.10. Let I be an R-module. If for any R-module monomorphism $\varphi : M \to N$ and homomorphism $\beta : M \to I$, there is a homomorphism $\gamma : N \to I$ such that the diagram

$$\begin{tikzcd} M \arrow[r, tail, "\varphi"] \arrow[dr, "\beta"'] & N \arrow[d, dashed, "\gamma"'] \\ & I \end{tikzcd}$$

commutes, I is an *injective* module.

If L is an injective module, every exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

splits.

For every R-module M, there is an R-module I and a monomorphism $\phi : M \to I$, see for example [4]. Thus, starting with a module M, embed M in an injective module I_0. Then embed I_0/M in an injective module I_1, and so on. This gives an exact sequence

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \cdots,$$

called a *injective resolution* of M.

2.2.3 Maps, Homology and the Ext-functor

Definition 2.2.11. Let $(\mathcal{C}^\bullet, \partial)$ and $(\mathcal{D}^\bullet, d)$ be cochain complexes. A *map of cochain complexes* φ is a collection of maps $\varphi^i : C^i \to D^i$ such that the diagram

$$\begin{tikzcd} \cdots & C^{i+1} \arrow[l] \arrow[d, "\varphi^{i+1}"'] & C^i \arrow[l, "\partial^i"'] \arrow[d, "\varphi^i"] & \cdots \arrow[l] \\ \cdots & D^{i+1} \arrow[l] & D^i \arrow[l, "d^i"] & \cdots \arrow[l] \end{tikzcd}$$

commutes.

If $\varphi : \mathcal{C}^\bullet \to \mathcal{D}^\bullet$ is a map of cochain complexes, φ induces a map $\varphi : H^i(\mathcal{C}^\bullet) \to H^i(\mathcal{D}^\bullet)$, called the *induced map on cohomology.* Now, given two maps of cochain complexes, when do they induce the same map on cohomology? For a sufficient condition, we need the notion of *homotopy equivalence.*

Definition 2.2.12. Let $\varphi, \psi : (\mathcal{C}^\bullet, \partial) \to (\mathcal{D}^\bullet, d)$ be maps of cochain complexes. If there exists a collection h of module homomorphisms $h^i : C^i \to D^{i-1}$ such that $\varphi - \psi = \partial h + hd$, φ is *homotopy equivalent* to ψ.

If $\varphi, \psi : (\mathcal{C}^\bullet, \partial) \to (\mathcal{D}^\bullet, d)$ with φ homotopy equvialent to ψ, φ and ψ induce the same map on homotopy.

If F is a functor from R-modules to S-modules, and $\mathcal{C}$ is a (co)chain complex of R-modules, let FC be the result of applying F to the modules and homomorphisms of $\mathcal{C}$.

Proposition 2.2.13. *Let M be a module and $\mathcal{P}_\bullet$, $\mathcal{Q}_\bullet$ ($\mathcal{I}^\bullet$, $\mathcal{J}^\bullet$) projective (injective) resolutions of M. Then there are maps $\alpha_1 : \mathcal{P}_\bullet \to \mathcal{Q}_\bullet$ and $\alpha_2 : \mathcal{Q}_\bullet \to \mathcal{P}_\bullet$ ($\beta_1 : \mathcal{I}^\bullet \to \mathcal{J}^\bullet$ and $\beta_2 : \mathcal{J}^\bullet \to \mathcal{I}^\bullet$) such that $\alpha_1\alpha_2$ ($\beta_1\beta_2$) is homotopic to the identity map on $\mathcal{Q}_\bullet$ ($\mathcal{J}^\bullet$) and $\alpha_2\alpha_1$ ($\beta_2\beta_1$) is homotopic to the identity on $\mathcal{P}_\bullet$ ($\mathcal{I}^\bullet$). Moreover, if F is an additive functor, there is a canonical isomorphism between $H_i(F\mathcal{P})$ and $H_i(F\mathcal{Q})$ ($H^i(F\mathcal{I})$ and $H^i(F\mathcal{J})$).*

Proposition 2.2.14. *Let*

$$0 \longrightarrow \mathcal{C}' \xrightarrow{\alpha} \mathcal{C} \xrightarrow{\beta} \mathcal{C}'' \longrightarrow 0$$

be an exact sequence of cochain complexes. Then there are homomorphisms $\delta^i : H^i(\mathcal{I}'') \to H^{i+1}(\mathcal{I}')$ suh that the sequence

$$\begin{tikzcd}
 & \cdots \arrow[r, "\beta^{i-1}"] & H^{i-1}(\mathcal{I}'') \arrow[dll, "\delta^{i-1}"'] \\
H^i(\mathcal{I}') \arrow[r, "\alpha^i"] & H^i(\mathcal{I}) \arrow[r, "\beta^i"] & H^i(\mathcal{I}'') \arrow[dll, "\delta^i"'] \\
H^{i+1}(\mathcal{I}') \arrow[r, "\alpha^{i+1}"] & H^{i+1}(\mathcal{I}) \arrow[r, "\beta^{i+1}"] & H^{i+1}(\mathcal{I}'') \arrow[dll, "\delta^{i+1}"'] \\
H^{i+2}(\mathcal{I}'') \arrow[r, "\alpha^{i+2}"] & \cdots &
\end{tikzcd}$$

is exact. This is called the long exact sequence in cohomology*.*

The following proposition is equally valid for injective resolutions.

Proposition 2.2.15. *Given a short exact sequence of R-modules*

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

and projective resolutions

$$\mathcal{P} : \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow 0,$$

$$\mathcal{R} : \cdots \longrightarrow R_2 \longrightarrow R_1 \longrightarrow 0$$

of L and N, respectively. Then there exists a projective resolution

$$\mathcal{Q} : \cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow 0$$

of M and a short exact sequence

$$0 \longrightarrow \mathcal{P} \xrightarrow{\alpha'} \mathcal{Q} \xrightarrow{\beta'} \mathcal{R} \longrightarrow 0,$$

where α' and β' induce α and β.

Definition 2.2.16. Let R be a ring, M and N R-modules, and

$$\mathcal{P} : \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow 0$$

a projective resolution of M. Define $\operatorname{Ext}^i_R(M,N)$ as the i-th cohomology module of the cochain complex

$$\operatorname{Hom}_R(\mathcal{P},N) : 0 \longrightarrow \operatorname{Hom}_R(P_1,N) \longrightarrow \operatorname{Hom}_R(P_2,N) \longrightarrow \cdots .$$

That $\operatorname{Ext}^i_R(M,N)$ is independent of the choice of $\mathcal{P}$ follows from Proposition 2.2.15, since $\operatorname{Hom}(-,N)$ is additive. Actually, $\operatorname{Ext}^i_R(M,N)$ could also be defined by having an injective resolution

$$\mathcal{I} : 0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \cdots$$

of N, and letting $\operatorname{Ext}^i_R(M,N)$ be the i-th cohomology module of

$$\operatorname{Hom}_R(M,\mathcal{I}) : 0 \longrightarrow \operatorname{Hom}_R(M,I_1) \longrightarrow \operatorname{Hom}_R(M,I_2) \longrightarrow \cdots .$$

Let Λ be an R-module, and

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

an exact sequence of R-modules. Then there are long exact sequences

$$\cdots \longrightarrow \operatorname{Ext}^{i-1}_R(\Lambda,N) \longrightarrow \operatorname{Ext}^i_R(\Lambda,L) \longrightarrow \operatorname{Ext}^i_R(\Lambda,M) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \operatorname{Ext}^{i-1}_R(L,\Lambda) \longrightarrow \operatorname{Ext}^i_R(N,\Lambda) \longrightarrow \operatorname{Ext}^i_R(M,\Lambda) \longrightarrow \cdots .$$

This follows from Propositions 2.2.14 and 2.2.13.

2.3 Further topics in commutative algebra

In this section we give an introduction to the theory of dimension, grade and other measures of rings and modules, and to the notion of graded rings and modules, which may be seen as a generalization of polynomial rings and their modules. We also introduce so-called Cohen-Macaulay and Gorenstein rings and modules. Most proofs on standard material are omitted. An easily read, more complete introduction to the subject is [10]. A deeper coverage of grade and depth, Cohen-Macaulay and Gorenstein rings and modules may be found in [2] and [4], from where most material in the following sections is taken. The definition of Gorenstein ring is from [6]. Much of the theory of Gorenstein rings (and some of the theory of Cohen-Macaulay rings) is beyond the scope of this work. The interested reader may found more material in the above mentioned books [2],[4],[6].

2.3.1 Dimension Theory

Definition 2.3.1. Let R be a commutative ring, and $\mathfrak{p} \in \operatorname{Spec} R$. The *height* of $\mathfrak{p}$, $\operatorname{ht}\mathfrak{p}$, is the supremum of lengths l of chains $\mathfrak{p} = \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_l$, $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \in \operatorname{Spec} R$.. The height of an arbitrary ideal I is $\operatorname{ht} I := \inf\{\operatorname{ht}\mathfrak{p} \,|\, I \subset \mathfrak{p} \in \operatorname{Spec} R\}$.

The following theorem, *Krull's principal ideal theorem*, is a fundamental result in dimension theory.

Theorem 2.3.2. *Let R be a Noetherian ring and $I = (x_1, \ldots, x_n)$ a proper ideal. Then $\operatorname{ht}\mathfrak{p} \leq n$ for every minimal $\mathfrak{p} \in \{\mathfrak{p} \in \operatorname{Spec} R \,|\, \mathfrak{p} \supset I\}$.*

In particular, every proper ideal in a Noetherian ring has finite height.

Definition 2.3.3. The *(Krull) dimension* of a commutative ring R is $\dim R := \sup\{\operatorname{ht}\mathfrak{p} \,|\, \mathfrak{p} \in \operatorname{Spec} R\}$.

The Artinian rings are precisely the Noetherian rings of dimension zero.

Since $\operatorname{Spec} R_{\mathfrak{p}} = \{\mathfrak{q}R_{\mathfrak{p}} \,|\, \mathfrak{q} \in \operatorname{Spec} R, \mathfrak{q} \subset \mathfrak{p}\}$ for $\mathfrak{p} \in \operatorname{Spec} R$, we get that $\dim R_{\mathfrak{p}} = \operatorname{ht}\mathfrak{p}$.

Lemma 2.3.4. *Let R be a ring, and I an ideal of R. Then*

$$\operatorname{ht} I + \dim R/I \leq \dim R.$$

Proof. Since $\operatorname{ht} I = \inf\{\operatorname{ht}\mathfrak{p} \,|\, I \subset \mathfrak{p} \in \operatorname{Spec} R\}$ and $\dim R/I = \sup\{\operatorname{ht}\mathfrak{p}/I \,|\, I \subset \mathfrak{p} \in \operatorname{Spec} R\}$, this follows. □

Let R be a ring of dimension n. If $x_1, \ldots, x_n$ are elements of R such that $\dim R/(x_1, \ldots, x_n) = 0$, $x_1, \ldots x_n$ is called a *system of parameters* of R.

Definition 2.3.5. The *dimension* of an R-module M is $\dim M := \sup\{\operatorname{ht}\mathfrak{p} \,|\, \mathfrak{p} \in \operatorname{Supp} M\}$. If M is finite, $\dim M = \dim R/\operatorname{Ann} M$. Indeed, since for a prime ideal $\mathfrak{p}$, $\mathfrak{p} \in \operatorname{Supp} M \Leftrightarrow \mathfrak{p} \supset \operatorname{Ann} M$.

Proposition 2.3.6. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring,* M *a finite* R*-module and* $x_1, \ldots, x_n \in \mathfrak{m}$*. Then*

$$\dim M/(x_1, \ldots, x_n)M \geq \dim M - n,$$

with equality if, and only if $x_1, \ldots, x_n$ *is a part of a system of parameters for* M*.*

Proposition 2.3.7. *Let* R *be a field. Then* $\dim R[X] = \dim R + 1$.

In particular, $\dim K[X_1, \ldots, X_n] = n$.

2.3.2 Grade and Depth

In the last section 2.3.1, we introduced the notion of dimension of rings and modules. In this section, we will introduce another measure, more algebraic in nature.

Definition 2.3.8. Let R be a ring and M an R-module. An element $x \in R$ is M*-regular* if it is a non-zerodivisor on M, i.e. $xm = 0 \Rightarrow m = 0$. A sequence of elements $x_1, \ldots, x_n$ is an M*-sequence* if x_i is $M/(x_1, \ldots, x_{i-1})M$-regular and $(x_1, \ldots, x_n)M \neq M$.

The following proposition follows immediately from the Prime Avoidance Theorem 2.1.2, and Proposition 2.1.21.

Proposition 2.3.9. *Let* R *be a Noetherian ring, and* M *a finite* R*-module. If an ideal* $I \subset R$ *consists of zero-divisors of* M*, there is a prime* $\mathfrak{p} \in \operatorname{Ass} M$ *with* $I \subset \mathfrak{p}$*.*

The next result shows that M-sequences are invariant under localizations.

Proposition 2.3.10. *Let* R *be a Noetherian ring, and* M *a finite* R*-module. If* $(x_1, \ldots, x_n)$ *is an* M*-sequence contained in a prime* $\mathfrak{p} \in \operatorname{Supp} M$*,* $(x_1, \ldots, x_n)$ *(as a sequence in* $R_{\mathfrak{p}}$*) is an* M_p*-sequence.*

Theorem 2.3.11 (Rees)**.** *Let* R *be a Noetherian ring,* M *a finite* R*-module and* I *an ideal in* R *such that* $IM \neq M$*. Then every inclusion maximal* M*-sequence in* I *has the same length, given by* $\min\{i \mid \operatorname{Ext}^i_R(R/I, M) \neq 0\}$.

This leads us to the following definition.

Definition 2.3.12. Let R be a Noetherian ring, and M a finite R-module, and I an ideal of R with $IM \neq M$. The *grade of* I *on* M, $\operatorname{grade}(I, M)$ is the length of the maximal M-sequence in I. Further, let $\operatorname{grade}(I, M) = \infty$ if $IM = M$.

Definition 2.3.13. When $(R, \mathfrak{m})$ is a local ring, the grade of $\mathfrak{m}$ on M is called the *depth of* M, written as $\operatorname{depth} M$.

Proposition 2.3.14. *Let* R *be Noetherian ring,* I *an ideal of* R*, and* M *a finitely generated* R*-module. Then*

(a) $\operatorname{grade}(I, M) = \inf\{\operatorname{depth} M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R, \mathfrak{p} \supset I\}$,

(b) *If* $x_1, \ldots, x_n$ *is an* M*-sequence,*

$$\begin{aligned}\operatorname{grade}(I/(x_1, \ldots, x_n), M/(x_1, \ldots, x_n)M) &= \operatorname{grade}(I, M/(x_1, \ldots, x_n)M) \\ &= \operatorname{grade}(I, M) - n.\end{aligned}$$

Proposition 2.3.15. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring and* M *a finite* R*-module,* $M \neq 0$*. Then every* M*-sequence is a part of a system of parameters of* M*. This particularly means that* $\operatorname{depth} M \leq \dim M$*.*

Proof. From Proposition 2.1.22, every minimal $\mathfrak{p} \in \operatorname{Supp} M$ lies in $\operatorname{Ass} M$. Therefore, if $x \in \mathfrak{m}$ is an M-regular element, $x \notin \mathfrak{p}$ for every minimal $\mathfrak{p} \in \operatorname{Supp} M$. (Recall that $\mathfrak{p} \in \operatorname{Ass} M$ means that $\mathfrak{p} = \operatorname{Ann} y$ for some $y \in R$.) Thus $\dim M/xM \leq \dim M - 1$. An induction then yields $\dim M/(x_1, \ldots, x_n)M \leq \dim M - n$, for any M-sequence $x_1, \ldots, x_n$, $x_i \in \mathfrak{m}$. From Proposition 2.3.6, $\dim M/(x_1, \ldots, x_n)M \geq \dim M - n$ for any $x_1, \ldots, x_n$ in $\mathfrak{m}$. We get that $\dim(M/x_1, \ldots, x_n)M = \dim M - n$, thus $x_1, \ldots, x_n$ is a part of a system of parameters for M. This holds for any M-sequence, hence $\operatorname{depth} M \leq \dim M$. □

The inequality $\operatorname{depth} M \leq \dim M$ may be sharpened, as the following proposition shows.

Proposition 2.3.16. *Let* $(R, \mathfrak{m})$ *be a local ring and* $M \neq 0$ *a finite* R*-module. Then* $\operatorname{depth} M \leq \dim R/\mathfrak{p}$ *for all* $\mathfrak{p} \in \operatorname{Ass} M$*.*

An easy consequence of Propositions 2.3.15 and 2.3.14(a) is the following

Proposition 2.3.17. *Let* R *be a Noetherian ring and* I *an ideal in* R*. Then* $\operatorname{grade} I \leq \operatorname{ht} I$*.*

Definition 2.3.18. Let $(R, \mathfrak{m})$ be a Noetherian local ring, $K := R/\mathfrak{m}$, and M a finite non-zero R-module with $\operatorname{depth} M = n$. Then define the *type of* M, $r(M)$, as $\dim_K \operatorname{Ext}^n_R(K, M)$.

Lemma 2.3.19. *Let* (R, m) *be a Noetherian local ring,* $K := R/\mathfrak{m}$*,* M *a finite* R*-module, and* $x_1, \ldots, x_n$ *a maximal* M*-sequence. Then*

$$r(M) = \dim_K \operatorname{Soc} M/(x_1, \ldots, x_n)M.$$

2.3.3 Graded Rings and Modules

Consider the ring of polynomials $K[X_1, \ldots, X_n]$. Every polynomial f therein can be written as a sum of homogeneous polynomials, i.e. polynomials in which every monomial has the same degree (with the usual $\deg X_1^{i_1} \cdots X_n^{i_n} = \sum_{j=1}^n i_j$). Moreover, although $K[X_1, \ldots, X_n]$ is not a local ring, it has a unique maximal ideal generated by homogeneous polynomials, namely $(X_1, \ldots, X_n)$. The aim of this section is to formalize these characteristics, and generalize the concept to so called graded rings and modules.

Definition 2.3.20. A ring R is a *graded ring* if $R = \bigoplus_{i\in\mathbb{Z}} R_i$ as an abelian group, with $R_iR_j \subset R_{i+j}$. An R-module M is graded if $M = \bigoplus_{i\in\mathbb{Z}} M_i$ as a $\mathbb{Z}$-module, and $R_iM_j \subset M_{i+j}$. An R-algebra S is graded if moreover, $S_iS_j \subset S_{i+j}$.

The elements $x \in M_i$ are called *homogeneous of degree i*. Denote the degree of x by $\deg x$. Every element $x \in M$ has a unique decomposition into a sum $\sum_{i\in\mathbb{Z}} x_i$ of homogeneous elements, with almost all $x_i = 0$. Note that for a graded module M, it is always possible (and often desirable) to choose a generating set for M consisting of homogeneous elements.

Definition 2.3.21. An ideal $I \in R$ is a *graded ideal* if for every $x = \sum_{i\in\mathbb{Z}} x_i \in I$, $x_i \in I$ for all i. Note that I is graded if, and only if, I is generated by homogeneous elemenents.

For an arbitrary ideal $I \subset R$, define I^* to be the (graded) ideal generated by all homogeneous elements in I.

Note that every ring is graded by letting $R_0 = R$, $R_i = 0$ for $i \neq 0$. This is called the trivial grading. The polynomial ring $K[X_1, \dots, X_n]$ is a K-algebra generated by elements of degree 1. Graded rings which as R_0-algebras are generated by elements of positive degree are called positively graded R_0-algebras. These include all rings $K[X_1, \dots, X_n]/I$, where I is a graded ideal, with the grading inherited from $K[X_1, \dots, X_n]$.

Proposition 2.3.22. *Let R be a positively graded R_0-algebra and $x_1, \dots, x_n$ homogeneous elements of positive degree. Then the following are equivalent.*

(a) *$x_1, \dots, x_n$ generate $\mathfrak{m} = \bigoplus_{i=1}^{\infty} R_i$.*

(b) *$x_1, \dots, x_n$ generate R as an R_0-algebra.*

Proof. (a) $\Rightarrow$ (b): It is enough to prove that every homogeneous element r in R can be written as a polynomial in the x_i with coefficients in R_0. We will do this by induction over $\deg r$. We know that r may be written as $\sum_{i=1}^n a_ix_i$, $a_i \in R$. Note that, since r is homogeneous, every a_i may be assumed to be homogeneous. Thus, if $\deg r = 1$, it is possible to choose the non-zero coefficients a_i to be of degree zero, since the sum in degrees other than one is 0. Thus r may be written as $\sum_{i=1}^n a_ix_i$, $a_i \in R_0$, or $a_i = 0$. Assume the claim holds when $\deg r < t$. For $\deg r = t$, write r as $\sum_{i=1}^n a_ix_i$ with a_i homogeneous, as above. Every a_i with $\deg a_ix_i \neq t$ may be chosen to be 0, as above. Thus $\deg a_i < t$ for every non-zero a_i. By induction, every non-zero a_i may be written as a polynomial in the x_i's with coefficients in R_0, whence this is also true for r.

(b) $\Rightarrow$ (a): This is obvious. □

Note that this means that a positively graded R_0-algebra R is Noetherian precisely when R_0 is Noetherian and R is finitely generated as an R_0-algebra.

A *graded homomorphism* $\varphi : M \to N$ of graded R-modules is a homomorphism of R-modules with $\varphi(M_i) \subset N_i$. The graded homomorphisms are not in general a submodule of all R-homomorphisms. This leads us to the following definition. Consider R-homomorphisms $\varphi : M \to N$ with $\varphi(M_i) \subset N_{i+j}$ for all

i. Such a homomorphism φ is called *homogeneous of degree* j. Note that graded homomorphisms are homogeneous of degree 0. Denote the set of homogeneous maps of degree j by $\mathrm{Hom}_R^j(M,N)$. Set ${}^*\mathrm{Hom}_R(M,N) = \bigoplus \mathrm{Hom}_R^j(M,N)$. For a graded R-module M, let $M(d)$ be the graded R-module with $M(d)_e = M_{d+e}$. $M(d)$ can be thought of as a shift of M in d degrees. This makes it possible to think of a homomorphism $\varphi : M \to N$ of degree j as a homomorphism $\varphi : M(-j) \to N$ of degree 0.

There is a graded version of Theorem 2.1.4, whence every graded R-module M has a projective resolution in which all modules and homomorphisms are graded. Let $\mathcal{P}_\bullet$ be a graded projective resolution of M. Let

$$ {}^*\mathrm{Ext}_R^i(M,N) = H^i({}^*\mathrm{Hom}_R(\mathcal{P}_\bullet, N)). $$

Lemma 2.3.23. *Let R be a graded ring. If M is a finite graded R-module,* ${}^*\mathrm{Hom}_R(M,N) = \mathrm{Hom}_R(M,N)$.

Proof. Every homomorphism $\varphi : M \to N$ can be written as $\sum \varphi_j$, where $\varphi_j(x)$ is the homogeneous $(i+j)$-part of $\varphi(x)$, for a homogeneous element x. Note that every φ_j is a graded homomorphism. Since M as a finite homogeneous generating set, it follows that only a finite number of graded homomorphisms are needed to describe $\varphi(M)$. Thus $\varphi \in {}^*\mathrm{Hom}_R(M,N)$, whence ${}^*\mathrm{Hom}_R(M,N) = \mathrm{Hom}_R(M,N)$ when M is finite. □

Thus also ${}^*\mathrm{Ext}_R^i(M,N) = \mathrm{Ext}_R^i(M,N)$ when R is Noetherian and M a finite R-module. The Noetherian property of R is needed since all modules P_i of the resolution of M must be finite. Otherwise, it is not certain that ${}^*\mathrm{Hom}_R(P_i,N) = \mathrm{Hom}_R(P_i,N)$ for all i.

Proposition 2.3.24. *Let R be a graded ring, and M a graded R-module. Then*

(a) $\mathfrak{p} \in \operatorname{Spec} R \Rightarrow \mathfrak{p}^* \in \operatorname{Spec} R$,

(b) $\mathfrak{p} \in \operatorname{Supp} M \Rightarrow \mathfrak{p}^* \in \operatorname{Supp} M$,

(c) $\mathfrak{p} \in \operatorname{Ass} M \Rightarrow \mathfrak{p}$ *is graded, and is the annihilator of a homogeneous element.*

Proof. (a): Let $r, s \in R$ such that $rs \in \mathfrak{p}^*$, but $r, s \notin \mathfrak{p}^*$. Write $r = \sum_i r_i$ and $s = \sum_j s_j$, as sums of graded components. There exist p, q such that $r_p \notin \mathfrak{p}^*$, but $r_i \in \mathfrak{p}^*$ for $i < p$ and $s_q \notin \mathfrak{p}^*$, but $s_j \in \mathfrak{p}^*$ for $j < q$. Consider the sum $\displaystyle\sum_{i+j=p+q} r_i s_j$, which is the $(p+q)$-th graded component of rs. Since $\mathfrak{p}^*$ is graded, $\displaystyle\sum_{i+j=p+q} r_i s_j \in \mathfrak{p}^*$. Every component except $r_p s_q$ lies in $\mathfrak{p}^*$, hence so does $r_p s_q$. This means that $r_p s_q \in \mathfrak{p}$, and since $\mathfrak{p}$ is prime, one of r_p, s_q lies in $\mathfrak{p}$. But since r_p and s_q are homogeneous, $r_p \in \mathfrak{p}^*$ or $s_q \in \mathfrak{p}^*$, a contradiction.

(b): Assume $\mathfrak{p}^* \notin \operatorname{Supp} M$, whence $M_{\mathfrak{p}^*} = 0$. Let x be a homogeneous element in M. There exists a in $R \setminus \mathfrak{p}^*$ such that $ax = 0$. With a_i the homogeneous parts of a, $a_i x = 0$ for every i. Since $a \notin \mathfrak{p}^*$, there is an i such that $a_i \notin \mathfrak{p}^*$. Since a_i is homogeneous, $a_i \notin \mathfrak{p}$. Thus $\frac{x}{1} = 0$ in $M_{\mathfrak{p}}$, for every homogeneous elements in M. Thus $\frac{x}{1} = 0$ for every element $x \in M$, whence $M_{\mathfrak{p}} = 0$.

(c): Let $x \in M$ be such that $\mathfrak{p} = \operatorname{Ann} x$. Write x as $x = x_k + \cdots + x_m$, a sum of homogeneous components x_i, $\deg x_i = i$ (so some x_i may be zero). In the same way, decompose an element $a \in \mathfrak{p}$ as $a = a_p + \cdots + a_q$. Since $ax = 0$, there are equations $\displaystyle\sum_{i+j=r} a_i x_j = 0$, for $r = k+p, \ldots, m+q$. Now, $a_p x_k = 0$. Assume by induction that $a_p^{n-1} x_{k+n-2} = 0$. Then

$$a_p^{n-1} \sum_{i+j=p+k+n-1} a_i x_j = a_p^n x_{k+n-1} = 0.$$

Thus a_p^{m-k+1} annihilates x, and since $\mathfrak{p}$ is prime, $a_p \in \mathfrak{p}$. The same can be done for $a_{p+1}, \ldots, a_q$, whence every homogeneous compononent of a lies in $\mathfrak{p}$. Hence $\mathfrak{p}$ is graded. It is left to prove that $\mathfrak{p}$ is the annihilator of a homogeneous element. Since $\mathfrak{p}$ is graded, $\mathfrak{p}$ is generated by homogeneous elements, whence $\mathfrak{p}$ annihilates every homogeneous component of x. Set $\mathfrak{p}_i = \operatorname{Ann} x_i$. Now, $\mathfrak{p}_i \supset \mathfrak{p}$ for every $k \leq i \leq m$. But it also holds that $\bigcap_{i=k}^m \mathfrak{p}_i \subset \mathfrak{p}$. Since $\mathfrak{p}$ is prime, $\mathfrak{p}_j \subset \mathfrak{p}$ for some j, whence $\mathfrak{p}_j = \mathfrak{p}$. □

Let $\mathfrak{p}$ be a prime ideal in R. The set S of all homogeneous elements outside $\mathfrak{p}$ is multiplicatively closed. Write $S^{-1}M$ as $M_{(\mathfrak{p})}$. $R_{(\mathfrak{p})}$ is a graded ring, and $M_{(\mathfrak{p})}$ is a graded $R_{(\mathfrak{p})}$-module. Note that in $R_{(\mathfrak{p})}/\mathfrak{p}^* R_{(\mathfrak{p})}$, every non-zero homogeneous element is invertible.

Lemma 2.3.25. *Let R be a graded ring. Then the following are equivalent.*

(a) *Every non-zero homogeneous element of R is invertible.*

(b) *$R_0 = K$ for a field K, and either $R = K$, or $R = K[x, x^{-1}]$ for a homogeneous element $x \in R$ which is transcendental over K.*

Proof. (a) $\Rightarrow$ (b): That $R_0 = K$ is a field is immediate. If $R \neq R_0$, there exist homogeneous elements $r \in R$ of positive degree, since $\deg r = -\deg r^{-1}$. Let x be a homogeneous element of minimal positive degree, d say. Since x is invertible, we can define a ring homomorphism $\varphi\colon K[X, X^{-1}] \to R$ by mapping K identically to R, and mapping X to x. We can define a grading on $K[X, X^{-1}]$, by setting $\deg X = d$. This makes φ a homomorphism of graded rings. We are done if φ is an isomorphism. Assume $g \in \operatorname{Ker} \varphi$, with $g = \sum a_i X^i$, $a_i \in K, i \in \mathbb{Z}$ Thus $\varphi(g) = \sum a_i x^i = 0$, whence $a_i x^i = 0$ for all i. Since x is invertible, $(a_i x^i) x^{-i} = a_i = 0$, for all i. This means that $g = 0$, thus φ is injective. To see that φ is surjective, let $a \in R$ be a homogeneous element of degree i. If $i = 0$, $a \in \operatorname{Im} \varphi$ from above,thus assume $i \neq 0$. Now, $i = jd + r$, with $0 \leq r < d$. Now, $ax^{-j} = b$ has degree r. Since d was the least positive degree, r has to be zero. Hence $a = bx^j$, whence $a = \varphi(bX^j) \in \operatorname{Im}(\varphi)$. Since every element in R is a finite sum of homogeneous elements, φ is surjective.

(b) $\Rightarrow$ (a): This is obvious. $\square$

Lemma 2.3.26. *The ring $K[X, X^{-1}]$ is a principal ideal domain.*

Proof. Since $K[X, X^{-1}] = K[X]_{\langle X \rangle}$, $\langle X \rangle = \{1, X, X^2, X^3, \ldots\}$, this follows easily. $\square$

Proposition 2.3.27. *Let R be a graded ring, and M a finite graded R-module. If $\mathfrak{p} \in \operatorname{Supp} M$ is graded with $\dim M_{\mathfrak{p}} = d$, there exists a chain $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}$ of graded prime ideals, where $\mathfrak{p}_i \in \operatorname{Supp} M$ for all i. If $\mathfrak{p}$ is not graded, $\dim M_{\mathfrak{p}} = \dim M_{\mathfrak{p}^*} + 1$.*

Proof. First, $\operatorname{ht} \mathfrak{p}/\mathfrak{p}^* = 1$. Without loss of generality, replace R by $R/\mathfrak{p}^*$, whence $p^* = 0$. Since $\mathfrak{p}$ does not contain any non-zero homogeneous elements, it is safe to invert all, obtaining $R_{(0)}$. Since $\mathfrak{p}R_{(0)}$ is a non-zero prime ideal in $R_{(0)}$, $R_{(0)} \cong K[X, X^{-1}]$ by Lemma 2.3.25. Thus $\operatorname{ht} \mathfrak{p}/\mathfrak{p}^* = 1$.

Let $\mathfrak{p} \in R$ be an arbitrary ideal in $\operatorname{Supp} M$ with $\dim M_{\mathfrak{p}} = d$. If there exists a chain of prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_{d-1} \subsetneq \mathfrak{p}$ in $\operatorname{Supp} M$, $\mathfrak{p}_0, \ldots, \mathfrak{p}_{d-1}$ graded, the first claim is immediately proven. The claim that $\dim M_{\mathfrak{p}} = \dim M_{\mathfrak{p}^*} + 1$ when $\mathfrak{p}$ is not graded follows from the facts that $\mathfrak{p}_{d-1} \subset \mathfrak{p}^*$ and $\operatorname{ht} \mathfrak{p}/\mathfrak{p}^* = 1$, whence $\mathfrak{p}_{d-1} = \mathfrak{p}^*$.

Now, let $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_{d-1} \subsetneq \mathfrak{p}$ be a chain of prime ideals in $\operatorname{Supp} M$. Since $\mathfrak{p}_0$ lies in $\operatorname{Ass} M$, $\mathfrak{p}_0$ is graded by Proposition 2.3.24. Hence, if $d = 1$ we are done. Since $\operatorname{ht} \mathfrak{p}_{d-1} < \operatorname{ht} \mathfrak{p}$ we may, by induction, assume $\mathfrak{p}_0, \ldots, \mathfrak{p}_{d-2}$ graded. Assume $\mathfrak{p}$ is not graded. Then replace $\mathfrak{p}_{d-1}$ by $\mathfrak{p}^*$, which lies properly between $\mathfrak{p}_{d-2}$ and $\mathfrak{p}$, and we are done.

Else, $\mathfrak{p}$ is graded, where it contains a homogeneous element $r \notin \mathfrak{p}_{d-2}$. Let $\mathfrak{q}$ be an inclusion minimal prime such that $\mathfrak{p}_{d-2} + (r) \subset \mathfrak{q} \subset \mathfrak{p}$. Since $(\mathfrak{p}_{d-2} + (r))/\mathfrak{p}_{d-2}$ is generated by r, it follows from Theorem 2.3.2 that $\operatorname{ht}(\mathfrak{q}/\mathfrak{p}_{d-2}) = 1$. Since $\operatorname{ht}(\mathfrak{p}/\mathfrak{p}_{d-2}) = 2$, $\mathfrak{p}$ and $\mathfrak{q}$ is not equal. Moreover, since $\mathfrak{q}$ is a minimal element of $\operatorname{Supp} R/(\mathfrak{p}_{d-2} + (r))$, it is graded by Proposition 2.3.24 (c). $\square$

Lemma 2.3.28. *Every graded module M over $K[X, X^{-1}]$ is free.*

Proof. Take a minimal homogeneous generating set G for M. Assume there is a relation $\sum_{i=1}^{n} a_i g_i = 0$, $a_i \in K[X, X^{-1}]$, $g_i \in G$. Any relation between homogeneous elements of different degrees would destroy the grading on M. Thus it must hold that, with $a_{ij} g_i$ the elements of degree j, $\sum_{i=1}^{n} a_{ij} g_i = 0$ for all j. At least one of these relations must be nontrivial, in degree j say. Every homogeneous element in $K[X, X^{-1}]$ is invertible, thus $g_n = a_{nj}^{-1} \sum_{i=1}^{n-1} a_{ij} g_i$. This contradicts the minimality of G. $\square$

For the next proposition we need the following result, which is a specialization of Lemma 3.3.8 in [14],

Lemma 2.3.29. *Let R be a Noetherian ring, M and N R-modules with M finite, and $\mathfrak{p} \in \operatorname{Spec} R$. Then*

$$\operatorname{Ext}^i_R(M,N)_{\mathfrak{p}} \cong \operatorname{Ext}^i_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}).$$

Proposition 2.3.30. *Let R be a graded ring, and M a finite graded R-module. If $\mathfrak{p} \in \operatorname{Supp} M$ is not graded,* $\operatorname{depth} M_{\mathfrak{p}} = \operatorname{depth} M_{\mathfrak{p}^*} + 1$ *and* $r(M_{\mathfrak{p}}) = r(M_{\mathfrak{p}^*})$.

Proof. We may regard the modules $M_{\mathfrak{p}}$ and $M_{\mathfrak{p}^*}$ as modules over $R_{(\mathfrak{p})}$, whence $R/\mathfrak{p}^* = K[X, X^{-1}]$. Since $K[X, X^{-1}]$ is a principal ideal domain, it follows that $\mathfrak{p} = aR + \mathfrak{p}^*$ for some $a \in R$. There is an exact sequence

$$0 \longrightarrow R/\mathfrak{p}^* \xrightarrow{a} R/\mathfrak{p}^* \longrightarrow R/\mathfrak{p} \longrightarrow 0\ .$$

Thus there is an exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^i_R(R/\mathfrak{p}^*, M) \xrightarrow{a} \operatorname{Ext}^i_R(R/\mathfrak{p}^*, M) \longrightarrow \operatorname{Ext}^{i+1}(R/\mathfrak{p}, M) \longrightarrow \cdots$$

$\operatorname{Ext}^i_R(R/\mathfrak{p}^*, M) = {}^*\operatorname{Ext}^i_R(R/\mathfrak{p}^*, M)$. Since $R/\mathfrak{p}^* \cong K[X, X^{-1}]$, ${}^*\operatorname{Ext}^i_R(R/\mathfrak{p}^*, M)$ is a free $R/\mathfrak{p}^*$-module ($\mathfrak{p}^* \subset \operatorname{Ann}_R({}^*\operatorname{Ext}^i_R(R/\mathfrak{p}^*, M))$), from Lemma 2.3.28. Hence $a : {}^*\operatorname{Ext}^i_R(R/\mathfrak{p}^*, M) \to {}^*\operatorname{Ext}^i_R(R/\mathfrak{p}^*, M)$ is injective, thus there are short exact sequences

$$0 \longrightarrow a^*\operatorname{Ext}^i_R(R/\mathfrak{p}^*, M) \longrightarrow {}^*\operatorname{Ext}^i_R(R/\mathfrak{p}^*, M) \longrightarrow \operatorname{Ext}^{i+1}_R(R/\mathfrak{p}, M) \longrightarrow 0.$$

Thus

$$\operatorname{Ext}^{i+1}_R(R/\mathfrak{p}, M) \cong {}^*\operatorname{Ext}^i_R(R/\mathfrak{p}^*, M) \,/\, a^*\operatorname{Ext}^i_R(R/\mathfrak{p}^*, M).$$

Since ${}^*\operatorname{Ext}^i_R(R/\mathfrak{p}^*, M) = \bigoplus R/\mathfrak{p}^*$ and $\mathfrak{p} = aR + \mathfrak{p}^*$,

$${}^*\operatorname{Ext}^i_R(R/\mathfrak{p}^*, M) \,/\, a^*\operatorname{Ext}^i_R(R/\mathfrak{p}^*, M) = \bigoplus R/\mathfrak{p}.$$

Thus $\operatorname{Ext}^{i+1}_R(R/\mathfrak{p}, M)$ is a free $R/\mathfrak{p}$-module of the same rank as ${}^*\operatorname{Ext}^i_R(R/\mathfrak{p}^*, M)$. Hence $\operatorname{depth} M_{\mathfrak{p}} = \operatorname{depth} M_{\mathfrak{p}^*} + 1$. Localizing in $\mathfrak{p}$ respectively $\mathfrak{p}^*$, using the above Lemma 2.3.29, we get

$$\begin{aligned}
&\dim_{\kappa(\mathfrak{p})} \operatorname{Ext}^{i+1}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), M_{\mathfrak{p}}) = \operatorname{rank}_{R/\mathfrak{p}} \operatorname{Ext}^{i+1}_R(R/\mathfrak{p}, M)\\
&= \operatorname{rank}_{R/\mathfrak{p}^*} \operatorname{Ext}^i_R(R/\mathfrak{p}^*, M) = \dim_{\kappa(\mathfrak{p}^*)} \operatorname{Ext}^i_{R^*_{\mathfrak{p}}}(\kappa(\mathfrak{p}^*), M_{\mathfrak{p}}).
\end{aligned}$$

Thus, indeed $r(M_{\mathfrak{p}}) = r(M_{\mathfrak{p}^*})$.

□

Proposition 2.3.31. *Let $R = K[X_1, \ldots, X_n]$ with K an infinite field, M a finite R-module and $I \in R$ an ideal generated by elements of degree 1. Assume* $\operatorname{grade}(I, M) = p$. *Then there exist elements $x_1, \ldots, x_p$ of degree 1 in I such that $x_1, \ldots, x_p$ is an M-sequence.*

Definition 2.3.32. A graded ring with a unique graded maximal ideal is called a **local* ring.

Example 2.3.33. Let $R = \bigoplus_{i=0}^{\infty} R_i$, where R_0 is a field. Then R is a *local ring, with graded maximal ideal $\bigoplus_{i=i}^{\infty} R_i$. This ideal is called the *irrelevant ideal.*

Proposition 2.3.34. *Let $(R, \mathfrak{m})$ be a *local ring, M a finite graded R-module and I a graded ideal of R. Then*

$$\operatorname{grade}(I, M) = \operatorname{grade}(IR_{\mathfrak{m}}, M_{\mathfrak{m}}).$$

Sketch of proof. The localization functor is exact, whence the long exact * Ext-sequence is preserved. Further, one can show that if $^*\mathrm{Ext}_M^N(R, i)$ is non-zero for finite graded R-modules M and N, $^*\mathrm{Ext}_M^N(R, i)_{\mathfrak{p}}$ is non-zero for graded $\mathfrak{p} \in \operatorname{Spec} R$. This shows that $\operatorname{grade}(I, M) \geq \operatorname{grade}(IR_m, M_{\mathfrak{m}})$. That $\operatorname{grade}(I, M) \leq \operatorname{grade}(IR_{\mathfrak{m}}, M_{\mathfrak{m}})$ follows from Proposition 2.3.10. □

If $(R, \mathfrak{m})$ is a *local ring where $\mathfrak{m}$ is a maximal ideal in the non-graded sense, there is a corresponding result for dimension and height.

Proposition 2.3.35. *Let $(R, \mathfrak{m})$ be a *local ring with $\mathfrak{m}$ a maximal ideal in the non-graded sense, and let I be a graded ideal of R. Then*

(a) $\dim R = \dim R_{\mathfrak{m}}$,

(b) $\operatorname{ht}_R I = \operatorname{ht}_{R_{\mathfrak{m}}} IR_{\mathfrak{m}}$.

Proof. (a): Let $\mathfrak{p} \in \operatorname{Spec} R$, $\mathfrak{p}$ not graded. Then $\operatorname{ht} \mathfrak{p} = \operatorname{ht} \mathfrak{p}^* + 1$, from Proposition 2.3.27. But since $\mathfrak{p}^* \subset \mathfrak{m}$, $\operatorname{ht} \mathfrak{m} \geq \operatorname{ht} \mathfrak{p}^* + 1 \geq \operatorname{ht} \mathfrak{p}$.
(b): For $I \in \operatorname{Spec} R$, this follows immediately from Proposition 2.3.27. Now assume $I \notin \operatorname{Spec} R$. Since I is graded we get that for every $I \subset \mathfrak{p} \in \operatorname{Spec} R$, $I \subset \mathfrak{p}^*$. Hence

$$\begin{aligned}
\operatorname{ht}_R I &= \inf\{\operatorname{ht} \mathfrak{p} \mid I \subset \mathfrak{p} \in \operatorname{Spec} R, \mathfrak{p} \text{ graded}\} \\
&= \inf\{\operatorname{ht} \mathfrak{p}R_{\mathfrak{m}} \mid IR_{\mathfrak{m}} \subset \mathfrak{p}R_{\mathfrak{m}} \in \operatorname{Spec} R_{\mathfrak{m}}, \mathfrak{p}R_{\mathfrak{m}} \text{ graded}\} \\
&= \operatorname{ht}_{R_{\mathfrak{m}}} IR_{\mathfrak{m}}.
\end{aligned}$$

□

From Proposition 2.3.27, it also follows that if R is Noetherian and M a finite R-module, $\dim M = \dim M_{\mathfrak{m}}$.

Definition 2.3.36. Let M be a finite $\mathbb{N}$-graded $K[X_1, \ldots, X_n]$-module, K a field. Since $\dim_K M_i$ is finite for all $i \in \mathbb{N}$ (since M is Noetherian), we can define the *Hilbert series* of M as the series $\operatorname{Hilb}_M(t) = \sum_{i \in \mathbb{N}} \dim_K M_i t^i$.

Example 2.3.37. Consider the $K[X_1, X_2X_3]$-module

$$M := K[X_1.X_2, X_3]/(X_1^2, X_2X_3).$$

A K-basis for the first few M_i is as follows.

Abelian group	K-basis
M_0	1
M_1	X_1, X_2, X_3
M_2	$X_1X_2, X_1X_3, X_2^2, X_3^2$
M_3	$X_1X_2^2, X_1X_3^2, X_2^3, X_3^3$
$\vdots$	$\vdots$

Thus a K-basis for M is given by 1, X_1, X_2, $X_2^2, \ldots$, X_3, $X_3^2, \ldots$, X_1X_2, $X_1X_2^2, \ldots$, $X_1X_3, X_1X_3^2, \ldots$. Since $1 + t + t^2 + \ldots = \frac{1}{1-t}$, we get that

$$\begin{aligned}
\operatorname{Hilb}_M(t) &= 1 + t + \frac{2t}{1-t} + \frac{2t^2}{1-t} \\
&= \frac{(1+t)(1-t) + 2t + 2t^2}{1-t} \\
&= \frac{(1+t)^2}{1-t}.
\end{aligned}$$

Proposition 2.3.38. *Let M be a finite $\mathbb{N}$-graded $K[X_1, \ldots, X_n]$-module with Hilbert series $\operatorname{Hilb}_M(t)$, and let $x_1, \ldots, x_p \in R_1$ be an M-sequence. Set $N = M/(x_1, \ldots, x_p)M$. Then*

$$\operatorname{Hilb}_N(t) = (1-t)^p \operatorname{Hilb}_M(t).$$

Proof. Assume first that $p = 1$. Consider the exact sequence

$$0 \longrightarrow M(-1) \xrightarrow{x} M \longrightarrow N \longrightarrow 0.$$

Since $\operatorname{Hilb}_{M(-1)}(t) = t \operatorname{Hilb}_M(t)$,

$$\operatorname{Hilb}_N(t) = (1-t)\operatorname{Hilb}_M(t).$$

By an inductive argument, the proposition follows immediately. $\square$

2.3.4 Cohen-Macaulay and Gorenstein Rings and Modules

Definition 2.3.39. Let $(R, \mathfrak{m})$ be a Noetherian local ring. A finite R-module M is *Cohen-Macaulay* if $\operatorname{depth} M = \dim M$. For an arbitrary ring R, an R-module M is Cohen-Macaulay if $\operatorname{depth} M_{\mathfrak{m}} = \dim M_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m} \in \operatorname{Supp} M$. The ring R is a Cohen-Macaulay ring if it is Cohen-Macaulay as an R-module.

Proposition 2.3.40. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and M a Cohen-Macaulay R-module. Then*

(a) $\dim R/\mathfrak{p} = \operatorname{depth} M$, *for all* $\mathfrak{p} \in \operatorname{Ass} M$,

(b) $\operatorname{grade}(I, M) = \dim M - \dim M/IM$, *for every proper ideal* $I \subset \mathfrak{m}$.

Proof. (a): Since $\operatorname{Ass} M \subset \operatorname{Supp} M$, $\dim R/\mathfrak{p} \leq \dim M$. From Proposition 2.3.16, $\dim R/\mathfrak{p} \geq \operatorname{depth} M$. Since $\dim M = \operatorname{depth} M$, the assertion follows.

(b): If $\operatorname{grade}(I, M) = 0$, I is contained in a prime ideal $\mathfrak{p} \in \operatorname{Ass} M$, from 2.3.9. Since $\operatorname{Ann} M/\mathfrak{p}M = \mathfrak{p}$, and $\dim M/IM \geq \dim M/\mathfrak{p}M = \dim r/\mathfrak{p}$, $\dim M/IM = \dim M$, from 2.3.6. If $\operatorname{grade}(I, M) > 0$, choose an M-regular element $x \in I$. Then, from Proposition 2.3.14(b), $\operatorname{grade}(I, M/xM) = \operatorname{grade}(I, M) - 1$, $\operatorname{depth} M/xM = \operatorname{depth} M - 1$ and $\dim M/xM = \dim M - 1$. The claim follows by induction. □

Proposition 2.3.41. *Let M be a Cohen-Macaulay R-module.*

(a) *Let $x_1, \ldots, x_n$ be an M-sequence. Then $M/(x_1, \ldots, x_n)M$ is a Cohen-Macaulay module. If R is local the converse of the statement is also true, i.e. if $M/(x_1, \ldots, x_n)$ is Cohen-Macaulay, M is Cohen-Macaulay.*

(b) *For every $\mathfrak{p} \in \operatorname{Spec} R$, $M_\mathfrak{p}$ is a Cohen-Macaulay module.*

Proof. (a): Assume M is Cohen-Macaulay. Since $x_1, \ldots, x_n$ is an $M_\mathfrak{p}$-sequence for $\mathfrak{p} \in \operatorname{Supp} M$, we may assume that R is a local ring. Then

$$\operatorname{depth} M/(x_1, \ldots, x_n)M = \operatorname{depth} M - n$$

from 2.3.14(b), and

$$\dim M/(x_1, \ldots, x_n)M = \dim M - n$$

from 2.3.15. If R is local, we may use 2.3.15 to prove the converse.

(b): Let $\mathfrak{m}$ be a maximal ideal in R containing $\mathfrak{p}$. Since $M_\mathfrak{p} = (M_\mathfrak{m})_{\mathfrak{p}R_\mathfrak{p}}$, we may assume that R is local. If $M_\mathfrak{p}$ is the zero module, we are done. Otherwise, if $\operatorname{depth} M_\mathfrak{p} = 0$, $\mathfrak{p} \in \operatorname{Ass} M$ by Proposition 2.3.9. Thus $\mathfrak{p}$ is minimal in $\operatorname{Supp} M$, hence $\dim M_\mathfrak{p} = 0$, from 2.3.40(a). This also means that if $\operatorname{depth} M_\mathfrak{p} > 0$, $\mathfrak{p}$ can not be contained in some prime $\mathfrak{q} \in \operatorname{Ass} M$. Thus there exists an M-regular element $x \in \mathfrak{p}$. Since $(M/xM)_\mathfrak{p} \cong M_\mathfrak{p}/xM_\mathfrak{p}$, we get from Proposition 2.3.14(b) that $\operatorname{depth}(M/xM)_\mathfrak{p} = \operatorname{depth} M_\mathfrak{p} - 1$. We also have, from Proposition 2.3.15, that $\dim(M/xM)_\mathfrak{p} = \dim M_\mathfrak{p} - 1$. Since M/xM is Cohen-Macaulay , from 2.3.4, we may assume by induction that $(M/xM)_\mathfrak{p}$ is Cohen-Macaulay. Since R could be assumed to be local, we may use (a) to get that $M_\mathfrak{p}$ is Cohen-Macaulay. □

Proposition 2.3.42. *Let R be a Cohen-Macaulay ring, and I a proper ideal of R. Then* $\operatorname{grade}(I, M) = \operatorname{ht} I$. *Further, if R is local,* $\operatorname{ht} I + \dim R/I = \dim R$.

Proof. Since

$$\operatorname{ht} I = \min\{\dim R_\mathfrak{p} \mid I \subset \mathfrak{p} \in \operatorname{Spec} R\},$$
$$\operatorname{grade}(I, R) = \{\operatorname{depth} R_\mathfrak{p} \mid I \subset \mathfrak{p} \in \operatorname{Spec} R\},$$

that $\operatorname{grade}(I, R) = \operatorname{ht} I$ follows from Proposition 2.3.41(b). The second equation follows from the first and Proposition 2.3.40(b). ☐

Proposition 2.3.43. *Let R be a Noetherian ring. Then $R[X]$ is a Cohen-Macaulay ring if, and only if, R is a Cohen-Macaulay ring.*

Proof. If $R[X]$ is Cohen-Macaulay, $R[X]/(X) = R$ is Cohen-Macaulay from 2.3.41(a) since X is $R[X]$-regular. To prove the converse, let $\mathfrak{m}$ be a maximal ideal of $R[X]$, and let $\mathfrak{p} = \mathfrak{m} \cap R$. Since $R \setminus \mathfrak{p} \subset R[X] \setminus \mathfrak{m}$, we have that $R[X]_{\mathfrak{m}} = R_{\mathfrak{p}}[X]_{\mathfrak{m}}$. Thus we may assume that R is local with maximal ideal $\mathfrak{p}$. Further, $R[X]/\mathfrak{p}R[X] = (R/\mathfrak{p})[X]$ is a principal ideal domain, whence $\mathfrak{m}/\mathfrak{p} = (f)$. If $x_1, \ldots, x_n \in \mathfrak{p}$ is an R-sequence, it is also an $R[X]$-sequence. The element f is a non-zerodivisor of $R[X]/(x_1, \ldots, x_n)R[X]$, thus $x_1, \ldots, x_n, f$ is an $R[X]$-sequence in $\mathfrak{m}$. Thus $\operatorname{depth} R[X]_{\mathfrak{m}} \geq 1 + \operatorname{depth} R$. Moreover, $\operatorname{ht}_{R[X]} \mathfrak{p} = \dim R$, and by Krull's principal ideal theorem, $\operatorname{ht} \mathfrak{m} \leq 1 + \operatorname{ht} \mathfrak{p}$. Since R is Cohen-Macaulay, $\dim R = \operatorname{depth} R$, thus $\operatorname{depth} R[X]_m \geq \operatorname{ht} \mathfrak{m} = \dim R[X]_m$. Thus $\operatorname{depth} R[X]_m = \dim R[X]_m$, for every maximal ideal $\mathfrak{m} \subset R[X]$, i.e.. $R[X]$ is Cohen-Macaulay. ☐

Corollary 2.3.44. $K[X_1, \ldots, X_n]$ *is $Cohen - Macaulay$ for every field K and every n.*

Proposition 2.3.45. *Let M be finite graded module over a *local ring $(R, \mathfrak{m})$. Then M is Cohen-Macaulay if, and only if, $M_{\mathfrak{m}}$ is Cohen-Macaulay.*

Proof. Since $M_{\mathfrak{m}}$ is Cohen-Macaulay, $M_{\mathfrak{p}}$ is Cohen-Macaulay for every graded ideal $\mathfrak{p} \in \operatorname{Supp} M$. Assume $\mathfrak{p} \in \operatorname{Supp} M$ is not graded. Then, from Propositions 2.3.27 and 2.3.30, $\dim M_{\mathfrak{p}} = \dim M_{\mathfrak{p}^*} + 1 = \operatorname{depth} M_{\mathfrak{p}^*} + 1 = \operatorname{depth} M_{\mathfrak{p}}$. Hence $M_{\mathfrak{p}}$ is Cohen-Macaulay for every $\mathfrak{p} \in \operatorname{Supp} M$. The other direction is obvious. ☐

Example 2.3.46. Let $R = K[X_1, \ldots, X_n]$, and let I be a proper graded ideal. Then R/I is Cohen-Macaulay if, and only if, $R_{\mathfrak{m}}/\mathfrak{m}I_{\mathfrak{m}}$ is Cohen-Macaulay, where $\mathfrak{m} = (X_1, \ldots, X_n)$.

Cohen-Macaulay rings in many ways behave more nicely than general Noetherian rings.For example, say that a Noetherian ring R is *catenary* if every saturated chain of prime ideals from $\mathfrak{p}$ to $\mathfrak{q}$ has the same length $\operatorname{ht} \mathfrak{p}/\mathfrak{q}$, for every pair of prime ideals $\mathfrak{q} \subset \mathfrak{p}$. If every ring $R[X_1, \ldots, X_n]$ is catenary, the ring R is *universally catenary.*

Proposition 2.3.47. *Every Cohen-Macaulay ring R is universally catenary.*

Proof. Since $R[X_1, \ldots, X_n]$ is Cohen-Macaulay if R is Cohen-Macaulay, it is enough to prove that R is catenary. Let $\mathfrak{p} \subset \mathfrak{q}$ be prime ideals of R. From Proposition 2.3.41(b), $R_{\mathfrak{p}}$ is Cohen-Macaulay. Apply Proposition 2.3.42 on $R_{\mathfrak{p}}$, to get

$$\operatorname{ht} \mathfrak{q} = \dim R_{\mathfrak{q}} = \operatorname{ht} \mathfrak{p}R_{\mathfrak{q}} + \dim R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}} = \operatorname{ht} \mathfrak{p} + \operatorname{ht} \mathfrak{p}/\mathfrak{q}.$$

Now assume that there exists prime ideals $\mathfrak{p} \subset \mathfrak{q}$ of R, such that there exist maximal chains of prime ideals $\mathfrak{q} \supsetneq \mathfrak{q}_1 \supsetneq \cdots \supsetneq \mathfrak{q}_d$ and $\mathfrak{q} \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_{d-n}$, $d = \operatorname{ht} \mathfrak{q}/\mathfrak{p} - 1$, $n > 0$. Then $\operatorname{ht} \mathfrak{p}_{d-n} = \operatorname{ht} \mathfrak{p} + \operatorname{ht} \mathfrak{p}_{d-n}/\mathfrak{p} = \operatorname{ht} \mathfrak{p} + 1$, and $\operatorname{ht} \mathfrak{q}$ may be written as

$$\operatorname{ht} \mathfrak{q} = \operatorname{ht} \mathfrak{q}/\mathfrak{p}_{d-n} + \operatorname{ht} \mathfrak{p}_{d-n} = d - n + \operatorname{ht} \mathfrak{p} + 1.$$

But $\operatorname{ht} \mathfrak{q}/\mathfrak{p} = d + 1$, whence we would have

$$\operatorname{ht} \mathfrak{q} = \operatorname{ht} \mathfrak{q}/\mathfrak{p} + \operatorname{ht} \mathfrak{p} - n,$$

a contradiction. Thus a Cohen-Macaulay ring is catenary. □

Definition 2.3.48. A local ring R is *Gorenstein* if R is Cohen-Macaulay, and $r(M) = 1$. In general, a ring R is Gorenstein if every localization of R at a maximal ideal is Gorenstein.

As for Cohen-Macaulay rings, we have the following criterion for *local rings, which follows by the same results as 2.3.45.

Proposition 2.3.49. *Let $(R, \mathfrak{m})$ be a *local ring, and M a finite R-module. Then M is Gorenstein if, and only if, $M_{\mathfrak{m}}$ is Gorenstein.*

3 Combinatorics

Combinatorial objects may often be used to define algebraic structures, which then can be analyzed through their combinatorial origin. We will restrict ourself to combinatorial objects which give rise to particularly simple algebric structures. Other, more complex associations are found in [7], from which many results in this chapter are taken.

3.1 Simplicial Complexes

Definition 3.1.1. An *abstract simplicial complex* is a class of subsets Δ of a finite set V such that

$$\sigma \in \Delta \Rightarrow \tau \in \Delta, forevery \tau \subset \sigma.$$

When not explicity stated otherwise, we shall assume that every element of V lies in Δ.

The elements with cardinality $k + 1$ in Δ are called k-faces, or just faces. Thus $\{v_1\}$ is a 0-face, $\{v_1, v_3\}$ is a 1-face, and so on. The dimension of a k-face σ, $\dim \sigma$, is k. The dimension of Δ, $\dim \Delta$, is $\max\{\dim \sigma \mid \sigma \in \Delta\}$. The maximal faces of Δ are called *facets*, and the 0-faces are called *vertices*. A simplicial complex is *pure* if every facet has the same dimension. Given a simplicial complex Δ on V and $S \subset V$, call $\rangle S \langle := \{\sigma \in \Delta \mid \sigma \subset S\}$ the *induced (sub)complex on S*. Often S will be a face in Δ.

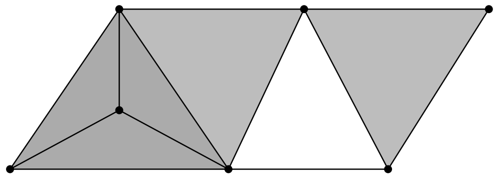


Figure 2: The simplicial complex Δ used in Example 3.1.4.

Definition 3.1.2. The *f-vector* of a simplicial complex Δ, $\dim \Delta = d$, is the vector $(f_{-1}, f_0, \ldots, f_d)$, where f_i is the number of i-dimensional faces in Δ.

Definition 3.1.3. Let Δ be a simplicial complex with f-vector $(f_{-1}, f_0, \ldots, f_d)$. Define the *$\mathcal{F}$-polynomial* of Δ as $\mathcal{F}(\Delta\,;t) := \sum_{i=-1}^{n} f_i x^{i+1}$.

Example 3.1.4. Let Δ be the simplicial complex in Figure 2. Then Δ has f-vector $(1, 7, 12, 6, 1)$.

Definition 3.1.5. A simplicial complex in which every minimal non-faces is a 2-set is called a *flag complex*.

Definitions 3.1.6. Let Δ be a simplicial complex, and $\sigma \in \Delta$. Then the *link* of σ in Δ is $\operatorname{lk}_\Delta \tau := \{\tau \in \Delta \mid \tau \cup \sigma \in \Delta, \tau \cap \sigma = \varnothing\}$, and the *star* of σ in Δ is $\operatorname{St}_\Delta \sigma := \{\tau \in \Delta \mid \tau \cup \sigma \in \Delta\}$.

Definition 3.1.7. Let Δ and Γ be simplicial complexes. The *sum* of Δ and Γ, $\Delta + \Gamma$, is the union of Δ and Γ on disjoint vertex sets.

Definition 3.1.8. The *join* of two simplicial complexes Δ and Γ is the complex $\Delta * \Gamma = \{\sigma \cup \tau \mid \sigma \in \Delta, \tau \in \Gamma\}$.

Example 3.1.9. Let Δ and $\sigma \in \Delta$ be as in Figure 3(a). The star and link of σ are depicted in Figure 3(b) respectively Figure 3(c). Let Γ be the disjoint union of two points. The join $\Delta * \Gamma$ is shown in Figure 3(d).

Definition 3.1.10. Let Δ be a simplicial complex with f-vector $(f_{-1}, f_0, \ldots, f_d)$. Define the *reduced Euler characteristic* $\tilde{\chi}(\Delta)$ as

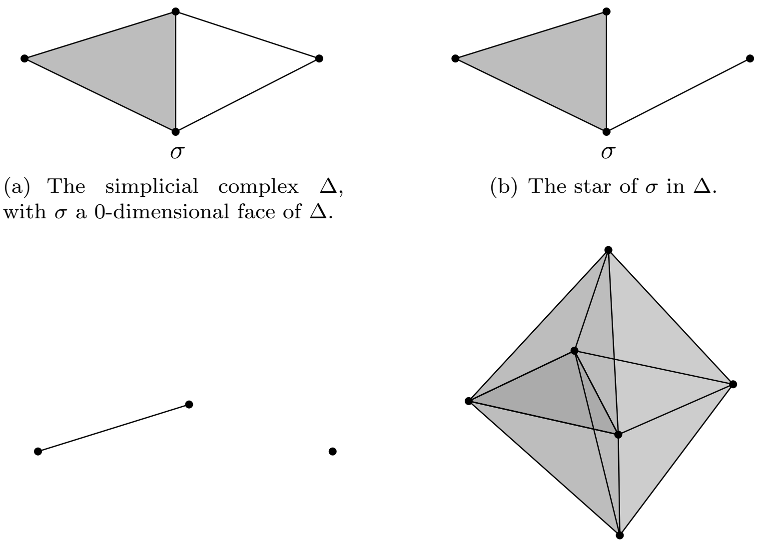
$$\tilde{\chi}(\Delta) = \sum_{i=-1}^{d} (-1)^i f_i.$$

Definition 3.1.11. A simplicial complex is *Eulerian*, or an *Euler complex*, if Δ is pure and $\tilde{\chi}(\operatorname{lk}\sigma) = (-1)^{\dim \operatorname{lk}\sigma}$, for all $\sigma \in \Delta$.

It is a well-known fact that simplicial spheres are Eulerian, see any introductory book in topology.

Proposition 3.1.12. *The h-vector of an $(d-1)$-dimensional Eulerian complex is symmetric, i.e. $h_i = h_{d-i}$ for $0 \le i \le d$.*

Thus simplicial spheres have symmetric h-vectors.



(a) The simplicial complex Δ, with σ a 0-dimensional face of Δ.

(b) The star of σ in Δ.

(c) The link of σ in Δ.

(d) The join of Δ and Γ, where Γ is as in Example 3.1.9.

Figure 3: Examples of operations on simplicial complexes.

3.1.1 The Stanley-Reisner Ring of a Simplicial Complex

Simplicial complexes can be defined to determine square-free monomial ideals, as in the following definition. For notation, given $\{v_{a_1}, \ldots, v_{a_n}\} \subset V$, let X^σ be the monomial $X^{a_1} \cdots X^{a_n}$.

Definition 3.1.13. Let Δ be a simplicial complex on the vertex set $1, \ldots, n$, and let K be a field. Define an ideal I_Δ in $K[X_1, \ldots, X_n]$ by letting $I_\Delta := (\{X^\sigma \mid \sigma \notin \Delta\})$. I_Δ is called the *Stanley-Reisner ideal* of Δ. The *Stanley-Reisner ring* of Δ is the ring $K[\Delta] := K[X_1, \ldots, X_n]/I_\Delta$.

Note that, from the results in Section 2.3.3, a Stanley-Reisner ring Rmay often be regarded as a local ring (or as a finite graded module over a local ring), in the sense that many results regarding (finite modules over) local rings may be applied to R.

There is a 1-to-1 correspondence between square-free monomial ideals in $K[X_1, \ldots, X_n]$ and simplicial complexes on $\{1, \ldots, n\}$. For $\{v_{i_1}, \ldots, v_{i_k}\} = \sigma \in \Delta$, let $\mathfrak{p}^\sigma$ be the monomial ideal $(X_{i_1}, \ldots, X_{i_k})$, and let $\overline{\sigma}$ be the complement of σ in Δ.

Theorem 3.1.14. *For Δ a simplicial complex,*

$$I_\Delta = \bigcap_{\sigma \in \Delta} \mathfrak{p}^{\overline{\sigma}},$$

and $\dim K[\Delta] = \dim \Delta + 1$.

Proof. For X^τ to lie in $\bigcap_{\sigma\in\Delta} \mathfrak{p}^{\overline{\sigma}}$, it is necessary and sufficient that τ share at least one element with every $\overline{\sigma}$. Equivalently, τ does not lie in any face $\sigma \in \Delta$, hence

$$I_\Delta = \bigcap_{\sigma\in\Delta} \mathfrak{p}^{\overline{\sigma}}.$$

Assume $\dim \Delta = d$, and the number of vertices of Δ is n. We have $\operatorname{ht} I_\Delta \leq n - \dim \Delta - 1$, since $\mathfrak{p}^{\overline{\sigma}}$, with $\dim \sigma = d$, can be generated by $n-d-1$ elements. Since

$$\operatorname{ht} I_\Delta + \dim K[\Delta] \leq \dim K[X_1, \dots, X_n] = n,$$

$\dim K[\Delta] \leq d+1$. But $\mathfrak{p}^\sigma$ is a prime ideal of $K[\Delta]$ with $\operatorname{ht} \mathfrak{p}^\sigma \geq d+1$. Hence

$$\dim K[\Delta] = \dim \Delta + 1.$$

□

A simplicial complex is Cohen-Macaulay if $K[\Delta]$ is Cohen-Macaulay. Note that this is dependent of the field K.

Proposition 3.1.15. *A Cohen-Macaulay complex is pure.*

Proof. Assume a complex Δ is not pure, and let $\dim \Delta = d$. Consider $\mathfrak{p}^{\overline{\sigma}}$, where $\sigma = \{i_1, \dots, i_p\}$ is a facet. We have $\operatorname{grade}(\mathfrak{p}^{\overline{\sigma}}, K[\Delta]) = 0$, since $X_{i_1} \cdots X_{i_p}$ is annihilated by every element in $\mathfrak{p}^{\overline{\sigma}}$. Note that $K[\Delta]/\mathfrak{p}^{\overline{\sigma}}K[\Delta] \cong K[X_{i_1}, \dots, X_{i_p}]$. Since Δ is not pure, there is a facet F with $\dim F < \dim \Delta$. But then

$$\operatorname{grade}(\mathfrak{p}^{\overline{F}}, K[\Delta] < \dim K[\Delta] - \dim K[\Delta]/\mathfrak{p}^{\overline{F}}K[\Delta],$$

which contradicts 2.3.40(b). □

By homological arguments beyond the scope of this work (see [2]), one can show that simplical spheres are Cohen-Macaulay over any field.

In Section 2.3.3, we introduced the notion of $\mathbb{N}$-graded rings and modules. For the next theorem, we need the notion of $\mathbb{N}^n$-graded rings and modules. A Stanley-Reisner ring has an obvious $\mathbb{N}^n$-grading. Let M be an $\mathbb{N}^n$-graded $K[X_1, \dots, X_n]$-module. Define the $\mathbb{N}^n$-graded (or "finely graded") Hilbert series of M as

$$\operatorname{Hilb}_M(x) = \sum_{a\in\mathbb{N}^n} \dim M_a x^a.$$

Setting $x_i = t$ for all i gives the $\mathbb{N}$-graded Hilbert series for M.

Theorem 3.1.16. *Let Δ be a simplicial complex with f-vector $(f_{-1}, f_0, \dots, f_d)$, $f_0 = n$. Then*

$$\operatorname{Hilb}_{K[\Delta]}(t) = \sum_{i=0}^{d+1} \frac{f_{i-1}t^i}{(1-t)^i}.$$

Proof. $K[\Delta]$ has naturally an $\mathbb{N}^n$-grading, thus $K[\Delta]$ can be viewed as an $\mathbb{N}^n$-graded $K[X_1, \ldots, X_n]$-module. For $a \in \mathbb{N}^n$, let the *support* of a be the set $\operatorname{supp} a = \{i \mid a_i \neq 0\}$. Thus a squarefree monimial X^a is completely determined by $\operatorname{supp} a$. An arbitrary monomial X^a lies outside I_Δ exactly when $X^{\operatorname{supp} a}$ lies outside I_Δ. Also, the non-zero monomials form a basis (as a vector space over K) for $K[\Delta]$. Thus

$$\begin{aligned}\operatorname{Hilb}_{K[\Delta]}(x) &= \sum_{\substack{a\in\mathbb{N}^n \\ \operatorname{supp} a \in \Delta}} x^a \\ &= \sum_{\sigma\in\Delta} \sum_{\substack{a\in\mathbb{N}^n \\ \operatorname{supp} a = \sigma}} x^a.\end{aligned}$$

Since

$$\sum_{\substack{a\in\mathbb{N}^n \\ \operatorname{supp} a=\sigma}} = \prod_{i\in\sigma} \frac{x_i}{1-x_i},$$

and the product over an empty index set is 1,

$$\operatorname{Hilb}_{K[\Delta]}(x) = \sum_{\sigma\in\Delta} \prod_{i\in\sigma} \frac{x_i}{1-x_i}.$$

As $K[\Delta]_i = \bigoplus_{\substack{a\in\mathbb{N}^n \\ |a|=i}} K[\Delta]_a$, the $\mathbb{N}$-graded Hilbert series of $K[\Delta]$ is

$$\begin{aligned}\operatorname{Hilb}_{K[\Delta]}(t) &= \sum_{\sigma\in\Delta} \prod_{i\in\sigma} \frac{t}{1-t} \\ &= \sum_{i=-1}^{d} \frac{f_i t^{i+1}}{(1-t)^{i+1}}.\end{aligned}$$

□

Thus $\operatorname{Hilb}_{K[\Delta]}(t) = \frac{h_0+h_1t+\cdots+h_{d+1}t^{d+1}}{(1-t)^{d+1}}$. Call the polynomal $h_0 + h_1 t + \cdots + h_{d+1}t^{d+1}$ the *$\mathcal{H}$-polynomial* of $K[\Delta]$, denoted by $\mathcal{H}(M\ ;\ t)$. The vector $(h_0, h_1, \ldots, h_{d+1})$ is the *h-vector* of Δ. The relation between the f-vector and the h-vector of a simplicial complex is given by the following proposition.

Proposition 3.1.17. *For a d-dimensional simplicial complex Δ, it holds that*

$$\begin{aligned}h_i &= \sum_{j=0}^{i} (-1)^{j-i} \binom{d+1-i}{i-j} f_{j-1}, \\ f_{j-1} &= \sum_{i=0}^{j} \binom{d+1-i}{j-i} h_i.\end{aligned}$$

Proof. Putting the expression for $\mathrm{Hilb}_{K[\Delta]}(t)$ on a common denominator gives

$$\sum_{i=0}^{d+1} h_i t^i = \sum_{j=0}^{d+1} f_{j-1} t^j (1-t)^{d+1-j}. \tag{1}$$

This gives h_i in terms the f_i, by simply summing the monomials of the same degree on the right hand side. To express f_i in terms of the h_i, make the change of variable $t = \frac{s}{1+s}$. Then (1) transforms into

$$\begin{array}{rcl} \displaystyle\sum_{i=0}^{d+1} h_i \frac{s^i}{(1+s*)^i} & = & \displaystyle\sum_{j=0}^{d+1} f_{j-1} \frac{s^j}{(1+s)^j} \frac{1}{/1+s)^{d+1-j}} \\ & \Leftrightarrow & \\ \displaystyle\sum_{i=0}^{d+1} h_i s^i (1+s)^{d+1-i} & = & \displaystyle\sum_{j=0}^{d+1} f_{j-1} s^j. \end{array}$$

From this it is easy to see how f_{j-1} can be expressed in terms of the h_i. □

Theorem 3.1.18. *Let Δ be a $(d-1)$-dimensional Cohen-Macaulay complex, with n vertices and h-vector $(h_0, \ldots, h_d)$. Then*

$$0 \leq h_i \leq \binom{n-d+i-1}{i}, \text{ for } 0 \leq i \leq d.$$

Proof. Without loss of generality we may assume K infinite, see the proof Corollary 4.1.10 in [2]. From Proposition 2.3.31, we can find a $K[\Delta]$-regular sequence $x_1, \ldots x_d$, where every x_i is of degree 1. Set $N = K[\Delta]/(x_1, \ldots, x_d)K[\Delta]$. From Proposition 2.3.38,

$$\mathrm{Hilb}_N(t) = (1-t)^d \,\mathrm{Hilb}_{K[\Delta]}(t) = h_0 + h_1 t + \ldots + h_d t^d,$$

so $h_i \geq 0$ for all $0 \leq i \leq d$. Now, N is generated over K by $n-d$ elements of degree 1. Hence $\dim_K N_i$ is bounded by $\dim_K K[X_1, \ldots, X_n]_i = \binom{n-d+i-1}{i}$. □

A simplicial complex Δ is Gorenstein over K if $K[\Delta]$ is Gorenstein.

Theorem 3.1.19. *Let Δ be a $(d-1)$-dimensional Gorenstein complex. Then $h_i = h_{d-i}$ for every $0 \leq i \leq d$.*

Thus the Gorenstein complexes satisfies the same relation on the h-vector as Eulerian complexes. In fact, a simplicial complex Δ is Gorenstein if, and only if, Δ is a Cohen-Macaulay complex and $\operatorname{core}\Delta$ is Eulerian, where $\operatorname{core}\Delta$ is the complex $\{F \in \Delta \mid \operatorname{St} v \neq \Delta, \forall v \in F\}$. Hence, if Δ is a sphere, $K[\Delta]$ is Gorenstein for any field K.

Definition 3.1.20. Let Δ be a pure simplicial complex of dimension d. A *shelling* of Δ is a total ordering $F_1, \ldots, F_n$ of the facets of Δ such that the following equivalent conditions hold.

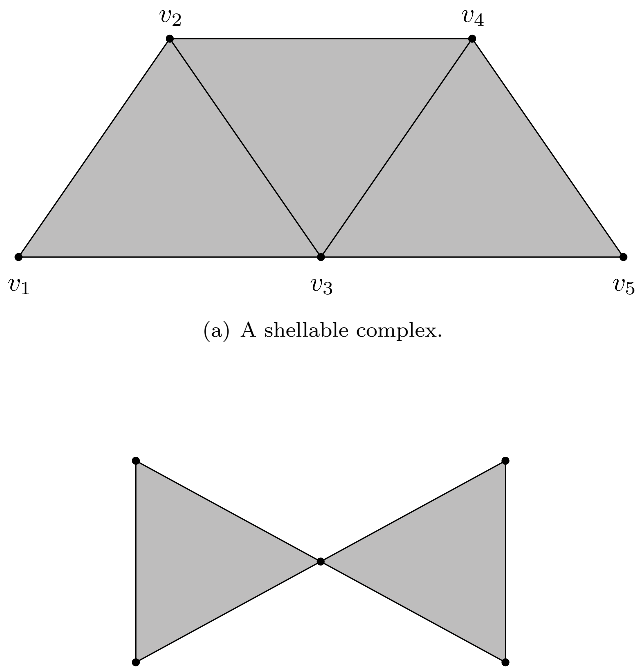
(i) For every $j < i$, there exists $k < i$ such that $F_i \setminus F_k = \{v\} \subset F_i \setminus F_j$.

(ii) For every $i > 1$, the set $\{F \in \langle F_i \rangle \,|\, F \notin \langle F_1, \dots, F_{i-1} \rangle\}$ has a unique inclusion-minimal element.

(iii) For every $i > 1$, $\langle F_i \rangle \cap \langle F_1, \dots, F_{i-1} \rangle$ is a pure, $(d-1)$-dimensional simplicial complex.

If there exists a shelling of the facets of Δ, Δ is *shellable*.

Example 3.1.21. The complex in Figure 4(a) is shellable, with a shelling given by $\{v_1, v_2, v_3\}$, $\{v_2, v_3, v_4\}$, $\{v_3, v_4, v_5\}$. The complex in Figure 4(b) is not shellable.



(a) A shellable complex.

(b) A non-shellable complex.

Figure 4: Examples of shellable and non-shellable complexes.

Lemma 3.1.22. *Let Δ be a shellable complex. Then*

(a) $\operatorname{St} \sigma$ *is shellable, for all* $\sigma \in \Delta$*,*

(b) $\operatorname{lk} \sigma$ *is shellable, for all* $\sigma \in \Delta$*.*

Further, if Γ *is another shellable complex,*

(c) $\Delta * \Gamma$ *is shellable.*

Proof. (a): Let $F_1, \dots, F_d$ be an ordering of the facets of $\operatorname{St}\sigma$, such that there exists a shelling of Δ with $F_i < F_j$ if $i < j$. From (i), for every pair F_j, F_i, $i < j$, there is a facet $F \in \Delta$, $F < F_j$ such that $F_j \setminus F = \{v\} \subset F_j \setminus F_i$. Since $\sigma \cap (F_j \setminus F_i = \varnothing$, it follows that $\sigma \subset F$. Thus $F \in \operatorname{St}\sigma$, and hence $\operatorname{St}\sigma$ is shellable.

(b): This follows easily from (a).

(c): Let $F_1, \dots F_n$ and $G_1, \dots, G_m$ be shellings of Δ and Γ, respectively.It follows easily that

$$F_1 \cup G_1, \dots, F_1 \cup G_m, F_2 \cup G_1, \dots F_n, \cup G_1, \dots F_n \cup G_m$$

is a shelling of $\Delta * \Gamma$. □

Theorem 3.1.23. *Let Δ be a shellable simplicial complex. Then $K[\Delta]$ is Cohen-Macaulay.*

Proof. We follow the proof in [7], and prove this by induction over the number of facets m in Δ. First, assume $m = 1$. Then $K[\Delta]$ is a polynomial ring, which is Cohen-Macaulay. For $m > 1$, assume $F_1, \dots, F_{m-1}$ is a shelling of the complex Δ' generated by $F_1, \dots, F_{m-1}$. By induction, $K[\Delta']$ is Cohen-Macaulay. Assume that the vertices of F_m are $x_1, \dots, x_r$, by renumbering them if necessary. By renumbering the vertices $x_1, \dots, x_r$ again, assume that $\{x_1, \dots, x_q\}$ is the unique minimal face in $\Delta \setminus \Delta'$. This means that $K[\Delta'] = K[\Delta]/(X_1 \cdots X_q)K[\Delta]$. Further, $X_k \cdot X_1 \cdots X_q = 0$ if $k > q$, whence the pincipal ideal $(X_1 \cdots X_q)$ can be seen as a free module over $K[X_1, \dots, X_n]$. This means that $(X_1 \cdots X_q)K[X_1, \dots, X_r]$ is a Cohen-Macaulay module of dimension r. Set $M = (X_1 \cdots X_q)K[X_1, \dots, X_r]$. From above, we have an exact sequence

$$0 \longrightarrow M \longrightarrow K[\Delta] \longrightarrow K[\Delta'] \longrightarrow 0,$$

where the $K[X_1, \dots, X_n]$-modules M and $K[\Delta']$ are Cohen-Macaulay modules of dimension r. Apply $\operatorname{Ext}(K, -)$ to the above sequence, to get a long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^i(K, M) \longrightarrow \operatorname{Ext}^i(K, K[\Delta]) \longrightarrow \operatorname{Ext}^i(K, K[\Delta']) \longrightarrow \cdots.$$

Since M and $K[\Delta']$ are Cohen-Macaulay of dimension r, the smallest i such that $\operatorname{Ext}(K, M)$ and $\operatorname{Ext}(K, K[\Delta'])$ are non-zero is r. Thus the same must hold for $\operatorname{Ext}^i(K, K[\Delta)]$, whence $K[\Delta]$ is Cohen-Macaulay .

□

3.2 Posets

The here presented standard material on posets is from [11].

Definition 3.2.1. A *partially ordered set* (or *poset* for short) is a pair $(P, \leq)$, where P is a set and $\leq$ is a partial ordering, i.e. is reflexive, transitive and anti-symmetric. Often $\leq$ is understood, and the poset is written simply as P. A poset $(P, \leq)$ is finite if P is a finite set.

Definitions 3.2.2. Let P be a poset. An *induced subposet* Q of P is a poset $Q \subset P$, with $x \leq y$ in Q exactly when $x \leq y$ in P. The interval $[x, y] = \{z \in P \,|\, x \leq z \leq y\}$ is an induced subposet of P. P is *locally finite* if every interval in P is finite. $x < y$ if $x \leq y$ and $x \neq y$. An element y *covers* x if $x < y$ and there is no element z with $x < z < y$.

A *chain* is a poset C in which $x \leq y$ or $y \leq x$ for all $x, y \in C$. Let C be a chain which is a subposet of P. C is saturated if $x < z < y$ and $x, y \in C$ implies $z \in C$. The *length* of a chain $C = x_0, x_1 \ldots, x_p$, $\ell(C)$, is p. P is *graded* if every maximal chain in P has the same length d. Then there exists a *rank function* $\rho : P \to [0, d]$ such that $\rho(x) = 0$ if x is minimal, and $\rho(y) = \rho(x) + 1$ if y covers x. $\rho(x)$ is called the *rank* of x. If P has a unique minimal element x, i.e. $x \leq y$ for all $y \in P$, this element is often written $\hat{0}$. In the same way, a unique maximal element is written $\hat{1}$.

Definition 3.2.3. Let $x, y \in P$. If there exists an element $z \in P$ such that $x, y \leq z$, and $z \leq w$ for any $x, y \leq w$, z is called the *join* of x and y, written $x \vee y$. In the same way, if there exists an element $z \in P$ such that $x, y \geq z$, and $z \geq$ for any $x, y \geq w$, z is called the *meet* of x and y, written $x \wedge y$. If the meet and join exist for every pair $x, y \in P$, P is a *lattice*.

There are many ways to construct new posets from old, here are but a few.

Definition 3.2.4. Let P and Q be posets on disjoint sets. Then make the following definitions.

(i) Let the *disjoint union*, or the *direct sum*, of P and Q be the poset $P + Q$ on $P \cup Q$, with $x \leq y$ in $P + Q$ if either $x, y \in P$ and $x \leq y$ in P, or $x, y \in Q$ and $x \leq y$ in Q.

(ii) The direct product of P and Q is the poset $P \times Q$ on $P \times Q$, with $(x_1, y_1) \leq (x_2, y_2)$ in $P \times Q$ if $x_1 \leq x_2$ in P and $y_1 \leq y_2$ in Q.

Definition 3.2.5. Let P be a locally finite poset. Define a function μ_P (or just μ) on the non-empty intervals of P as follows.

(i) $\mu([x, x]) = 1$,

(ii) $\mu([x, y]) = -\sum_{x \leq z < y} \mu([x, z])$ for $x \neq y$.

The function μ_P is called the *Möbius function* on P.

Example 3.2.6. Let S be a set with n elements. The set-inclusion relation is a partial ordering on 2^S, the set of all subsets of S. The poset $B_n := (2^S, \subset)$ is a lattice, with $x \vee y = x \cup y$, $x \wedge y = x \cap y$. B_n and any lattice isomorphic to B_n is called a *Boolean lattice*. B_n is ranked by the cardinality of sets.

Definition 3.2.7. Let P be a finite, graded poset, with rank n. Define a function $S_P : 2^{(\{0\}\cup[n])} \to 2^P$ with $S_P(G) = \{x \in P \mid \rho(x) \in G\}$. For simplicity, write $S_P(i)$ for $S_P(\{i\})$.

Definition 3.2.8. Let Δ be a simplicial complex. Define a lattice $L(\Delta)$, with the faces of Δ, with a $\hat{1}$ adjoined, as elements and inclusion as order relation. This gives a lattice, called the *face lattice* of Δ. Also, define $P(\Delta)$, the *face poset* of Δ, as the poset consisting of all non-empty faces of Δ. Note that if Δ is pure, $L(\Delta)$ is ranked by the cardinality of faces.

Example 3.2.9. Let Δ be as in Figure 5(a). Then $L(\Delta)$ is the lattice depicted in Figure 5(b). Note that $[x, y]$ is a Boolean lattice for every pair of faces $x, y \in \Delta$ with $x \subsetneq y$. As Figure 5(b) shows, $L(\Delta)$ need not even be ranked. However, $L(\Delta)$ is ranked whenever Δ is pure.

Lemma 3.2.10. *Let Δ be a simplicial complex. For every $\sigma \in \Delta$, $L(\operatorname{lk}\sigma)$ is isomorphic to the induced sublattice $Q := \{\tau \mid \tau \geq \sigma\} \subseteq L(\Delta)$.*

Proof. There is an obvious isomorphism φ between $L(\operatorname{lk}\sigma)$ and $[\sigma, \hat{1}]$, which sends $\tau \in L(\operatorname{lk}\sigma) \setminus \{\hat{1}\}$ to $\tau \cup \sigma \in Q$, and sends $\hat{1}$ to $\hat{1}$. $\square$

Corollary 3.2.11. *If moreover Δ is pure, the isomorphism φ preserves rank differences of the elements, i.e.*

$$\rho_{L(\operatorname{lk}\sigma)}(y) - \rho_{L(\operatorname{lk}\sigma)}(x) = k \Leftrightarrow \rho_Q(\varphi(y)) - \rho_Q(x) = k.$$

Definition 3.2.12. A graded poset P is *Eulerian* if $\mu([x, y]) = (-1)^{\rho(y)-\rho(x)}$ for every $x < y$, or equivalently, if every such interval $[x, y]$, contains the same number of even- as odd-ranked elements.

Example 3.2.13. Let $P = B_n$. For $x, y \in P$ with $x < y$ and $|y| - |x| = j$, $[x, y]$ is isomorphic to B_j. Since $\sum_{i=0}^{j}(-1)^i\binom{j}{i} = 0$, $[x, y]$ has the same number of odd- as even-ranked elements. Hence B_n is Eulerian.

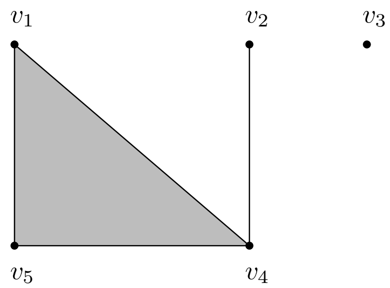
Lemma 3.2.14. *Let Δ be a pure simplicial complex. Then the following are equivalent.*

(a) *Δ is an Euler complex.*

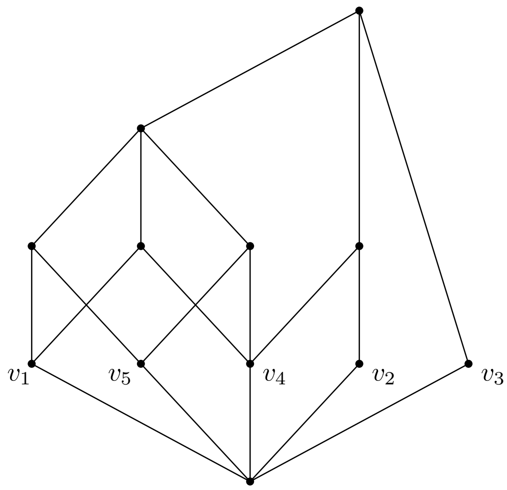
(b) *$L(\Delta)$ is Eulerian.*

Proof. (a) $\Rightarrow$ (b): Since every interval $[x, y] \subset L(\Delta), y \neq \hat{1}$, is isomorphic to a Boolean lattice, it is enough to show that $[x, \hat{1}]$ has the same number of odd- as even-ranked elemets, for all $x \in L(\Delta)$, $x \neq \hat{1}$. From 3.2.10, $[x, \hat{1}]$ is isomorphic to the face lattice $L(\operatorname{lk}\sigma)$, where σ is the face of Δ corresponding to x. Since $\tilde{\chi}(\operatorname{lk}\sigma) = (-1)^{\dim\operatorname{lk}\sigma}$, and $\dim\operatorname{lk}\sigma = \rho(x) - 1$, it follows that the number of odd-ranked elements in $[x, \hat{1}]$ is equal to the number of even-ranked elements. Hence $L(\Delta)$ is Eulerian.

(b) $\Rightarrow$ (a) follows similarly. $\square$



(a) The simplicial complex Δ.



(b) The face lattice of Δ.

Figure 5: A simplicial complex and its face lattice.

3.3 Graphs

The non-standard material is from [1], including the notion of (completely) discretely stable graphs and stitches. It is indicated where results are taken therefrom.

Let $\binom{S}{i}$ be the set of all i-subsets of S.

Definition 3.3.1. A *simple, loop-free graph* is a pair (V, E), where V is a finite set and $E \subset \binom{V}{2}$. An element $\{u, v\} \in E$ will often be abbreviated as uv. The set V is often called the *vertices* of G, and the set E the *edges* of G.

A graph will always mean a simple, loop-free graph.

Example 3.3.2. The *path* P_n, $n \geq 1$ is the graph with $V_{P_n} = \{v_1 \ldots, v_n\}$, $E_{P_n} = \{v_i v_{i+1} \,|\, 1 \leq i \leq n-1\}$. The *cycle* C_n, $n \geq 3$, is the graph with

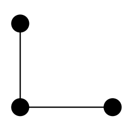
$V_{C_n} = V_{P_n}$, $E_{C_n} = E_{P_n} \cup \{v_1 v_n\}$. The *complete graph* K_n is the graph with $V_{K_n} = \{v_1, \dots, v_n\}$, $E_{K_n} = \binom{V_{K_n}}{2}$.

Definition 3.3.3. Let $G = (V, E)$ be a graph. A *subgraph* of G is a pair (V', E'), where $V' \subset V$ and $E' \subset \binom{V'}{2} \cap E$. A subgraph of G is an *induced subgraph* if $E' = \binom{V'}{2} \cap E$. (V', E') is said to be the subgraph induced by V'.

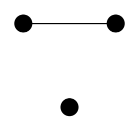
A graph $G = (V, E)$ will often be identified with its vertices. For example, if $S \subset V$, $G \setminus S$ may either denote $V \setminus S$ or the induced subgraph on $G \setminus S$. This will not cause any confusion.

Definition 3.3.4. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be graphs on disjoint vertex sets. Define the *sum* of G and H as $G + H := (V_G \cup V_H, E_G \cup E_H)$, and $G \uplus H := (V_G \cup V_H, V_G \cup V_H \cup \{uv \mid u \in V_G, v \in V_H\})$.

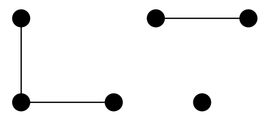
Example 3.3.5. Let G be the graph in Figure 6(a), and H the graph in Figure 6(b). Then $G + H$ is the graph in Figure 6(c), and $G \uplus H$ is the graph in Figure 6(d).



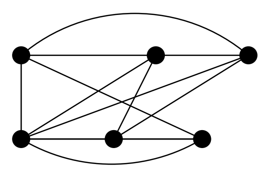
(a) The graph G.



(b) The graph H.



(c) The graph $G + H$.



(d) The graph $G \uplus H$.

Figure 6: Graphs and their different sums.

Definition 3.3.6. Let G be a graph. A set $S \subset G$ is called an *independent set* if the subgraph induced by S has no edges. A set $T \subset G$ is called a *clique* if the graph induced by T is isomorphic to $K_{|T|}$. The *independence number* of G is defined as $\alpha(G) := \max\{|S| \mid S \text{ independent in } G\}$. The *clique number* of G is $\omega(G) := \max\{|T| \mid T \text{ clique in } G\}$. If all inclusion-maximal independent sets of G have the same cardinality, G is *unmixed*.

Definition 3.3.7. Let G be a graph and v and w two vertices of G. The *distance* between v and w, $\mathrm{d}(v,w)$, is the smallest n such that P_{n+1} is a subgraph of G with v and w as endpoints, if such n exists. Otherwise $\mathrm{d}(v,w) := \infty$. For $v \in V$, let $N(v)$ denote the set of neighbours of v, i.e. $N(v) = \{w \mid \mathrm{d}(v,w) = 1\}$. More generally, if $S \subset G$, $N(S)$ denotes the set of vertices v not in S such that $d(s,v) = 1$ for some $s \in S$.

Definition 3.3.8. Let G be a graph and S an independent set in G. The graph $G_S := G \setminus (S \cup N(S))$ is called a *standard subgraph* of G.

Definition 3.3.9. Let $G = (V_G, E_G)$ be a graph. Define an ideal $I(G)$ of $K[X_1, \ldots, X_{|V_G|}]$ by identifying the variables X_i with the vertices $v_i \in V_G$, and letting $X_i X_j \in I(G)$ if $v_i v_j \in E_G$. The ideal $I(G)$ is called the graph ideal of G, and was introduced by Villareal in [13]. He then studied the rings $K[X_1, \ldots, X_{|V_G|}]/I(G)$.

The ring $K[X_1, \ldots, X_{|V_G|}]/I(G)$ may also be obtained by from the graph G by determining a simplicial complex and taking the Stanley-Reisner ring of that complex, as follows.

Definition 3.3.10. Let $G = (V, E)$ be a graph. Define a simplicial complex $\mathcal{C}^G$ on V by letting E be the minimal non-faces of $\mathcal{C}^G$. If G is the empty graph, let $\mathcal{C}^G = \{\varnothing\}$. To simplify notation, let $K[G]$ be $K[\mathcal{C}^G]$, $P(G)$ be $P(\mathcal{C}^G)$, and $L(G)$ be $L(\mathcal{C}^G)$.

Note that $\mathcal{C}^G$ is a flag complex, and that there is a 1-to-1 correspondence between flag complexes and graphs. Further, $\mathcal{C}^G$ is pure exactly when G is unmixed. The Stanley-Reisner ring of $\mathcal{C}^G$ is the ring $K[X_1, \ldots, X_{|V_G|}]/I(G)$.

Example 3.3.11. Let G be the graph in Figure 7(a). Then

$$\begin{aligned} \mathcal{C}^G \quad = \quad & \{\varnothing, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_1, v_4\}, \{v_1, v_5\}, \\ & \{v_2, v_5\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_1, v_4, v_5\}\}, \end{aligned}$$

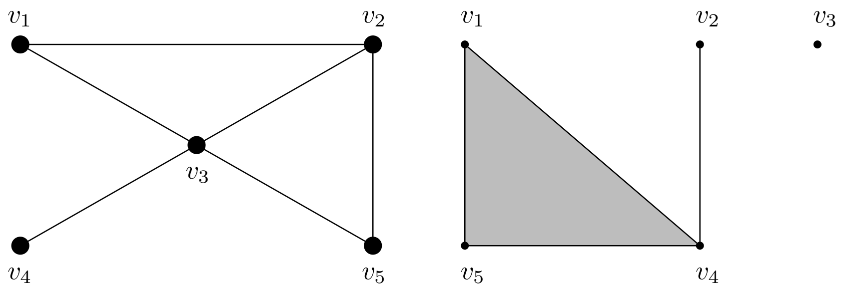
depicted in Figure 7(b). The Stanley-Reisner ring $K[G]$ is

$$K[X_1, X_2, X_3, X_4, X_5]/(X_1X_2, X_1X_3, X_2X_3, X_2X_5, X_3X_4, X_3X_5).$$

Let G be a graph. To G there is an associated simplicial complex $\mathcal{C}^G$, and also an associated lattice $L(G)$. Now, take for example a standard subgraph of G, say G_u. What is then the operation d on $\mathcal{C}^G$, such that $d(\mathcal{C}^G) = \mathcal{C}^{G_u}$? The following proposition collects the correspondences beween a few of the most basic operations on graphs and operations on $\mathcal{C}^G$ and $L(G)$.

Proposition 3.3.12. *Let G be a graph.*

(a) *Standard subgraphs of G corresponds to links in $\mathcal{C}^G$ and upper intervals in $L(G)$, in the sense that $\mathcal{C}^{G_{u_1,\ldots,u_n}} \cong \mathrm{lk}_\Delta\{u_1, \ldots, u_n\}$ and $L(G_{u_1,\ldots,u_n}) \cong [\{u_1, \ldots, u_n\}, \hat{1}] \subset L(G)$.*



(a) The graph G used in Example 3.3.11. (b) The independence complex of G.

Figure 7: A graph and its independence complex.

Let H be another graph. Then

(b) $\mathcal{C}^{G+H} \cong \mathcal{C}^G * \mathcal{C}^H$ *and* $L(G+H) \cong \big((P(G) \cup \{\hat{0}\}) \times (P(H) \cup \{\hat{0}\})\big) \cup \{\hat{1}\}$,

(c) $\mathcal{C}^{G \uplus H} \cong \mathcal{C}^G + \mathcal{C}^H$ *and* $L(G \uplus H) \cong \big(P(G) + P(H)\big) \cup \{\hat{0}, \hat{1}\}$.

Proof. (a): A set $S \subset G$ is an independent set of $G_{u_1,\ldots,u_n}$ precisely when S is an independent set of G such that $S \cap \{u_1, \ldots, u_n\} = \varnothing$ and $S \cup \{u_1, \ldots, u_n\}$ is an independent set. Thus $\mathcal{C}^{G_{u_1,\ldots,u_n}} \cong \mathrm{lk}_\Delta\{u_1, \ldots, u_n\}$ follows immediately. The isomorphism of lattices is the content of Lemma 3.2.10.

(b): The isomorphism $\mathcal{C}^{G+H} \cong \mathcal{C}^G * \mathcal{C}^H$ follows at once, since every independent set of $G+H$ is a union of an independent set of G and an independent set of H. Further, the set of independent sets S of $G+H$ can be seen as the cartesian product of the sets of independent sets S_1' and S_2' of G resp H. Since $s_1 \subset s_2$ in S exactly when $s_1 \cap S_1' \subset s_2 \cap S_1'$ and $s_1 \cap S_2' \subset s_2 \cap S_2'$, the lattice isomorphism follows.

(c): This is immediate. □

3.3.1 Completely Discretely Stable Graphs

Definition 3.3.13. Let G be a graph. Then

(i) G *discretely stable* if $\alpha(G \setminus S) = \alpha(G)$ for every independent set $S \subset G$,

(ii) G is *completely discretely stable* if every standard subgraph of G is discretely stable.

Note that it is enough to consider inclusion-maximal independent sets S.

It follows easily that if G and H are completely discretely stable, both $G+H$ and $G \uplus H$ are completely discretely stable.

Lemma 3.3.14 ([1])**.** *Let G be a graph, and $\{v, w\}$ an edge in G. If $\alpha(G_v) = \alpha(G_w) = \alpha(G) - 1$, and G_v and G_w are discretely stable, so is G.*

Proof. Let $S \subset G$ be an independent set. Not both of v and w lie is S. Assume $v \notin S$. $S \cap G_v$ is independent in G_v, and since G_v is discretely stable, there is an independent set $T \in G_v \setminus S$, with $|T| = \alpha(G_v)$. Thus $T \cup \{v\}$ is an independent set of size $\alpha(G_v) + 1 = \alpha(G)$ in $G \setminus S$. □

Proposition 3.3.15. *Let G be a graph, with $\alpha(G) = k$. Then the following are equivalent.*

(a) *G is completely discretely stable.*

(b) *For every standard subgraph G' of G with $\alpha(G') = 1$, $G' \cong K_m$ for some $m > 1$.*

(c) *There are no intervals in $L(G)$ with 3 elements.*

(d) *Every proper standard subgraph of G is discretely stable, and $G \cong K_m$, $m > 1$, if $k = 1$.*

Proof. That the conditions are equivalent when $k = 0$ is trivial. Assume $k \geq 1$. (a) $\Rightarrow$ (b): This is immediate.

(b) $\Rightarrow$ (c): Note that, since $L(P)$ is a face lattice of a simplicial complex, the only possibility for $[x, y]$ to be a three element interval is for y to be $\hat{1}$. In view of Lemma 3.2.10, the implication follows immediately.

(c) $\Rightarrow$ (b): This follows from Lemma 3.2.10.

(b) $\Rightarrow$ (d): Since (b) holds for every standard subgraph of G, and all conditions are equivalent by induction, (d) follows.

(d) $\Rightarrow$ (a): Let S be a k-independent set, and let $u \in S$. $S \cap V_{G_u}$ is a $(k-1)$-independent set in G_u. Since G_u is discretely stable, there is at least one $(k-1)$-independent set T in G_u disjoint from S. Then $T \cup \{u\}$ is a k-independent set in V_G. There is a $w \in T$ which is not in S, and such that $\alpha(G_w) = k - 1$. Moreover, $w \in \operatorname{lk} v$ for some $v \in S$, otherwise $S \cup \{w\}$ would be a $(k+1)$-independent set in G. Apply Lemma 3.3.14 with v and w as adjacent vertices.

□

Since 3.3.15(b) is satisfied for every Eulerian poset, graphs corresponding to Eulerian complexes are completely discretely stable. In particular, every flag sphere comes from a completely discretely stable graph.

Proposition 3.3.16. *Let G and H be two completely discretely stable graphs, with independence complexes $\mathcal{C}^G$ and $\mathcal{C}^H$. Let $v_1, \ldots, v_p$ be vertices of $\mathcal{C}^G$ and $w_1, \ldots, w_p$ vertices of $\mathcal{C}^H$, such that there is a bijection between $\{v_1, \ldots, v_p\}$ and $\{w_1, \ldots, w_p\}$. If Δ, constructed by identifying every pair $(v_i,\, w_i)$ in $\mathcal{C}^G + \mathcal{C}^H$, is a flag simplicial complex, the graph determined by Δ is completely discretely stable.*

Note that if the induced subcomplexes on $\{v_1, \ldots, v_p\}$ resp. $\{w_1, \ldots, w_p\}$ are isomorphic, the result will always be a flag complex.

Proof. From Proposition 3.3.15 it is enough to verify that every 0-dimensional link in Δ contains at least two vertices. But this is obvious, since any 0-dimensional link $\mathrm{lk}_{\Delta}\,\tau$ contains at least as many vertices as the 0-dimensional link of τ, in the complex(es) where τ is a member. $\square$

Example 3.3.17. Let G be the graph shown in Figure 8, and let Δ be the complex contructed by identifying faces of $\mathcal{C}^G + \mathcal{C}^G$ as in Figure 9. Then the graph H determined by Δ is completely discretely stable. The graph H is displayed in Figure 10.

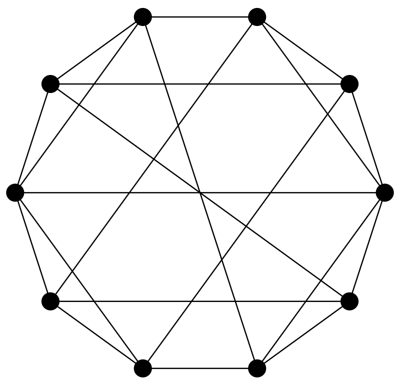


Figure 8: The graph G.

In [1], Backelin and Torsten Ekedahl prove that the triangle-free completely discretely stable graphs are precisely those graphs with spheres as independence complexes. This is not true in the general case, since the independence complex of a completely discretely stable graph need not even be pure. However, an interesting question is if every completely discretely stable graph can be constructed by the method of iterating Proposition 3.3.16, with spheres as "base objects" of the iteration.

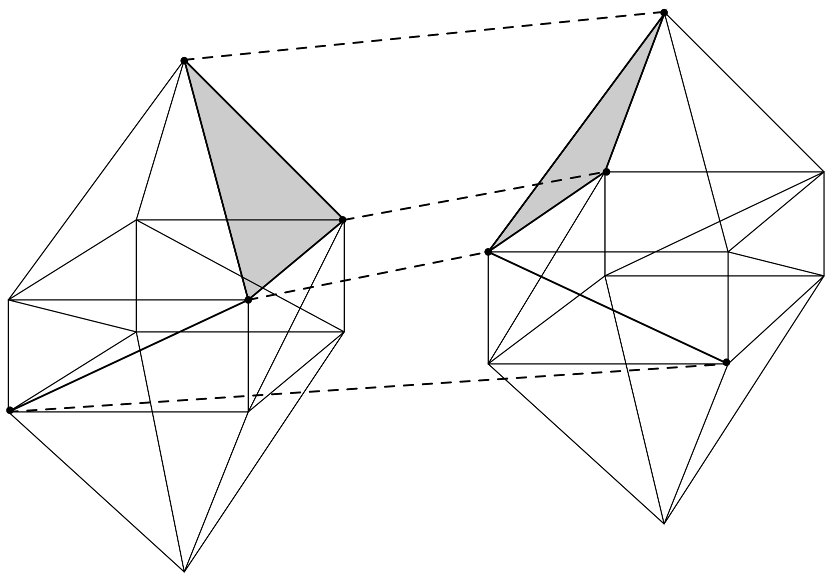


Figure 9: The complex Δ, constructed by identifying the indicated faces of $\mathcal{C}^G + \mathcal{C}^G$.

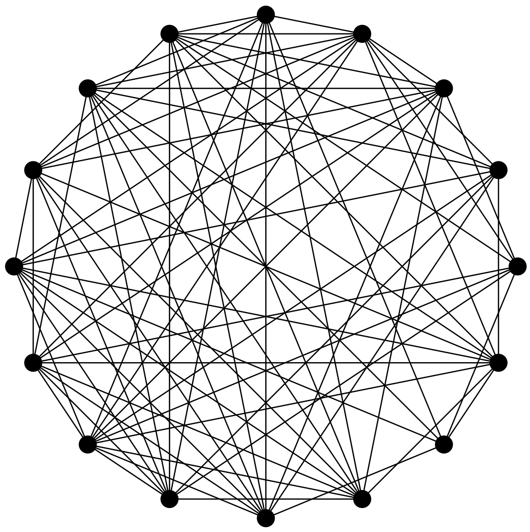


Figure 10: The graph H with independence complex Δ as in Figure 9.

3.4 Extensions of Graphs

Definition 3.4.1. Let G be a graph, $u_1, \ldots, u_{n-1}$ be vertices of G with pairwise distance at least 3, and $n \geq 1$. Define the *(n-)stitch*, based at $u_1, \ldots, u_{n-1}$, denoted $\mathrm{cr}_n(G\ ;\ u_1,\dots,u_{n-1})$, as follows. First, add n vertices $v_0,\dots v_{n-1}$. Add edges from u_j to v_i for $i+1 \le j \le n-1$, and from every $w \in N(u_i)$ to v_i, for $1 \le i \le n-1$. Finally add a vertex u', and add edges from every v_i to u'. The vertices $u_1,\dots,u_{n-1}$ is the *base* of the stitch, and u' is the *node*. When the base is understood, the stitch is written as $\mathrm{cr}_n(G)$.

Note that when $n=1$, $\mathrm{cr}_n(G) = G + P_2$. See Figure 11 for an example of a 2-stitch.

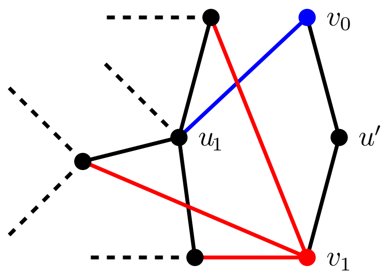


Figure 11: A 2-stitch.

Lemma 3.4.2 ([1])**.** *Let G be a graph. With the notation above,*

(a) $\mathrm{cr}_n(G)_{v_0} \cong \mathrm{cr}_n(G)_{u'} = G$,

(b) $\mathrm{cr}_n(G)_{v_i,u_i} \cong G_{u_i}$ *for* $1 \le i \le n-1$.

Proof. If not stated otherwise, $N(v)$ will denote the neighbourhood of v in $\mathrm{cr}_n(G)$.

(a): For each vertex $u_1,\dots,u_{n-1}$, $N(u_i) = N(v_i) \setminus \{u'\}$. Since $u' \in N(v_0)$, (a) follows.

(b): Since $N(v_j) \setminus \{u'\} = N(u_j)$, it follows that $\mathrm{cr}_n(G)_{v_i}$ is isomorphic to the graph $G' := \mathrm{cr}_{i-1}(G) \setminus (N_G(u_i) \cup u')$, where the $(i-1)$-stitch has the base $u_1,\dots,u_{i-1}$. Since $N_{G'}(u_i) = \{v_0,\dots,v_{i-1}\}$, (b) follows. □

Many graph properties are preserved by stitches, or can easily be expressed in terms of the original graph.

Proposition 3.4.3 ([1])**.** *Let $G' = \mathrm{cr}_n(G)$. Then*

(a) $\omega(G') = \max\{2, \omega(G)\}$,

(b) $\alpha(G') = \alpha(G) + 1$,

(c) *if G is unmixed, so is G',*

(d) *if G is discretely stable, so is G',*

(e) *G' is completely discretely stable if, and only if, G is completely discretely stable.*

Proof. (a): If G contains no edges, $\omega(G') = 2$. Assume $\omega(G) \geq 2$. An $(\omega(G)+1)$-clique S in G' must contain exactly one vertex in $G' \setminus G$, and neither u' nor v_0 may be a part of such S. Assume $v_k \in S$. Further, $S \setminus v_k \subset N(u_k)$, since $d(u_i, u_j) \geq 3$ for $i \neq j$. But then $N(u_k)$ contains an $\omega(G)$-clique T, and $T \cup \{u_k\}$ is an $(\omega(G) + 1)$-clique in G, a contradiction.

(b): This follows from Lemma 3.4.2.

(c): Given an inclusion maximal independent set S, if $v_k \in S$ with $k \geq 1$, $u_k \in S$. Thus also (c) follows from Lemma 3.4.2.

(d): Let S be an independent set in G'. Then there exists an independent $\alpha(G)$-set T in $G' \setminus S$. If $u' \notin S$, $T \cup \{u'\}$ will be an independent $(\alpha(G) + 1)$-set in $G' \setminus S$. If $u' \in S$, $v_k \notin S$ for $0 \leq k \leq n - 1$. If $u_i \notin T$ for all $1 \leq i \leq n - 1$, $S \cup \{v_0\}$ is an independent $(\alpha(G) + 1)$-set in G'. Otherwise, there is a minimal i such that $u_i \in T$. Then $S \cup \{v_i\}$ is an independent $(\alpha(G) + 1)$-set in G'.

(e): If G' is completely discretely stable, so is $G'_{u'} = G$. For the converse, assume G completely discretely stable. If $n = 1$, $G' = G + P_2$, which is completely discretely stable. Thus we may make an induction over the vertices of G', and assume that $n \geq 2$. From (d), we know that G' is discretely stable. It remains to show that G'_v is completely discretely stable, for every $v \in G'$. If $v \in G$ and $v \notin (N_{v_i})$ for $0 \leq i \leq n - 1$, $G'_v = \mathrm{cr}_n(G_v)$, which, by induction, is completely discretely stable. If $v \in N_G(u_i)$, $G'_v = \mathrm{cr}_{n-1}(G_v\ ;\ u_1, \ldots, u_{i-1}, u_{i+1}, \ldots u_{n-1})$, which is completely discretely stable by induction. If $v = u_i$, $G' = \mathrm{cr}_{n-i}(G_v\ ;\ u_{i+1}, \ldots, u_{n-1})$. If $v \in \{u', v_0\}$, $G'_v \cong G$. Lastly, if $v = v_i$ with $i \geq 1$, $G'_v \cong \mathrm{cr}_i(G_{u_i}\ ;\ u_1, \ldots, u_{i-1})$, with u_i the node of the stitch. □

Before the next proposition, let us recapitulate the principle of inclusion-exlusion. We will only need a simple version of it, see for example [11] for a more general formulation.

Theorem 3.4.4. *Let A be a set, and $A_1, \ldots, A_n$ subsets of A. Then*

$$|\bigcup_{i=1}^{n} A_i| = \sum_{\substack{S \subset [n] \\ S \neq \varnothing}} (-1)^{|S|-1} |\bigcap_{s \in S} A_i|.$$

Proposition 3.4.5. *Let G be a graph for which $\mathcal{C}^G$ is Eulerian. Then $\Delta := \mathcal{C}^{\mathrm{cr}_{k+1}(G)}$, where the stitch has base $u_1, \ldots, u_k$, is Eulerian.*

Proof. We will show that $L(\Delta)$ is Eulerian. Then, from Lemma 3.2.14, we get that Δ is Eulerian. To see that $L(\Delta)$ is Eulerian, we only need to prove that every interval $[\sigma, \hat{1}]$ has the same number of odd- and even-ranked elements. First, the interval $[u', \hat{1}]$ is isomorphic to $L(\mathcal{C}^G)$. Thus, if σ contains u', the assertion follows.

Next we will show by induction over l that if σ contains v_l, $[\sigma, \hat{1}]$ satisfies the assertion. Since $[v_0, \hat{1}]$ is isomorphic to $L(\mathcal{C}^G)$, it is true for v_0. Let $\sigma \geq v_l$. Now, if σ contains v_i, $0 \le i < l$, we are done. Also note that $[\{v_l, u_l\}, \hat{1}]$ is isomorphic to $L(\operatorname{lk} u_l)$. Hence, we can suppose that σ neither contains v_i, $0 \le i < l$, nor u_l. Now there are integers $0 \le m_0 < \cdots < m_j \le l-1$, depending on σ, such that $\sigma \cup \{v_s\} \in L(\Delta) \Leftrightarrow s = m_i$. We want to show that

$$\begin{gathered} |\{\tau \in L(\Delta) \mid \tau \ge \sigma, \rho(\tau) \text{ odd}\}| \\ = \\ |\{\tau \in L(\Delta) \mid \tau \ge \sigma, \rho(\tau) \text{ even}\}|. \end{gathered} \tag{2}$$

Define the following induced subposets of $L(\Delta) \setminus \{\hat{1}\}$: $P_1 := \{\tau \mid \tau \supseteq \sigma \cup \{u_l\}\}$, $Q_i := \{\tau \mid \tau \supseteq \sigma \cup \{v_{m_i}\}\}$, $P_2 := \displaystyle\bigcup_{i=0}^{j} Q_i$ and $P_3 := \{\tau \mid \tau \supseteq \sigma\} \setminus (P_1 \cup P_2)$. Note that $P_1 \cap P_2 = \varnothing$, whence P_1, P_2 and P_3 are pairwise disjoint. Both P_1 and P_3 are isomorphic to $P\{\tau \mid \tau \supseteq \sigma \setminus \{v_l\} \cup \{u_l\}\}$ as subposets of $L(\Delta) \setminus \{\hat{1}\}$. This follows from Lemma 3.4.2 and the fact that

$$L(\operatorname{cr}_{k+1}(G)_{v_l} \setminus \{v_1, \ldots, v_{l-1}\}) \cong L(G_{u_l}).$$

Since $\rho(\sigma \cup \{u_l\}) = \rho(\sigma) + 1$,

$$|\{\tau \in P_1 \cup P_3 \mid \rho(\tau) - \rho(\sigma) \text{ odd}\}| = |\{\tau \in P_1 \cup P_3 \mid \rho(\tau) - \rho(\sigma) \text{ even}\}|.$$

In what follows, ρ will denote the rank function of P_2. We have that $\bigcap_{i \in I} Q_i$ is graded for every $\varnothing \ne I \subset [j]$. Moreover, if an element $x \in \bigcap_{i \in I} Q_i$ has rank n in $\bigcap_{i \in I} Q_i$, the rank of x in P_2 is $k + |I| - 1$. From the principle of inclusion-exclusion, we thus have

$$|S_{P_2}(k)| = \sum_{\substack{T \subset [j] \\ T \ne \varnothing}} (-1)^{|T|+1} |S_{\bigcap_{i \in T} Q_i (k - |T| + 1)}|.$$

This gives a sum

$$\Sigma_T := \sum_{i=0}^{p} (-1)^{|T|+1} |S_{\cap_{t \in T} Q_t}(i)|,$$

where $p = \rho(\bigcap_{t \in T} Q_t)$, for every non-empty $T \in [j]$. Note that

$$\sum_{i=0}^{\rho(P_2)} |S_{P_2}(k)| = \sum_{T} \Sigma_T.$$

Now, consider

$$\sum_{i=0}^{\rho(P_2)} (-1)^i |S_{P_2}(i)|. \tag{3}$$

The rank of $\bigcap_{t \in T} Q_t$ is $\rho(P) - |T| + 1$, thus (3) is equal to

$$\sum_{\substack{T \subset [j] \\ T \ne \varnothing}} \sum_{i=0}^{\rho(P) - |T| + 1} (-1)^i |S_{\bigcap_{t \in T} Q_t}(i)|.$$

Every Q_i is Eulerian by induction, thus for every non-empty $T \subset [j]$ it holds that

$$\sum_{i=0}^{\rho(P_2)-|T|+1} (-1)^i |S_{\bigcap_{t\in T} Q_t}(i)| = (-1)^{\rho(P_2)-|T|+1}.$$

Since $\sum_{\substack{T\in[j]\\ T\neq\varnothing}} (-1)^{|T|} = -1$ and $\rho(P_2) = \rho([\sigma, \hat{1}]) - 1$, it follows that

$$\sum_{i=0}^{\rho(P_2)} (-1)^i |S_{P_2}(i)| = (-1)^{\rho(P_2)+1}.$$

Thus the subposet $[\sigma, \hat{1}] \supset P_2' := P_2 \cup \{\hat{1}\}$ contains the same number of odd-ranked as equal-ranked elements, since the rank of $\hat{1}$ in P_2' is even if the rank of P_2 is odd and vice versa and any other element in P_2' has the same rank in P_2' as in P_2. Summing up, (2) holds. Thus the assertion follows when σ contains some v_i. The only case left to consider is when σ does not contain any v_i nor u'. Define P_1, P_2 and P_3 as above, but use u' instead of u_l. Thus P_1 is $\{\tau \mid \tau \supset \sigma \cup u'\}$ and P_3 is simply $\{\tau \mid \tau \supset \sigma\}$ as a subposet of $L(\Delta) \setminus \{\hat{1}\}$. □

Proposition 3.4.6. *Let G be a shellable graph. Then $G' := \mathrm{cr}_{r+1}(G)$ is shellable.*

Proof. The subcomplex of $\mathcal{C}^{G'}$ generated by all facets containing u' (the base of the stitch, recall the notation in Definition 3.4.1) is just $\{u'\} * \mathcal{C}^G$, which is shellable by Lemma 3.1.22. Let $G_1, \ldots, G_p$ be a shelling of $\{u'\} * \mathcal{C}^{G'}$. Next, all facets containing the vertex v_r, and no other v_i, also contain the vertex u_r. From Lemma 3.4.2(b)$\mathcal{C}^{(G_{v_r,u_r}} \cong \mathcal{C}^{G'_{u_r}}$. The complex $\mathcal{C}^{G_{u_r}}$ is shellable, again by Lemma 3.1.22, whence $\mathcal{C}^{G'_{v_r,u_r}}$ is shellable too. Let $H_1, \ldots, H_n$ be a shelling of $\mathcal{C}^{G_{u_r}}$. From Definition 3.1.20(ii), there exists a unique minimal face σ_i in

$$\{F \in \langle H_i \rangle \mid F \notin \langle H_1, \ldots, H_{i-1} \rangle\},$$

for each $1 \le i \le n$. The face $\sigma_i \cup \{v_r\}$ is then a unique minimal face in

$$\{F \in \langle H_i \cup \{v_r, u_r\} \rangle \mid F \notin \langle G_1, \ldots, G_p, H_1 \cup \{v_r, u_r\}, \ldots, H_{i-1} \cup \{v_r, u_r\} \rangle\}.$$

Indeed, from the fact that every face therein contains v_r, if there were two minimal faces τ_1 and τ_2, $\tau_1 \setminus \{v_r\}$ and $\tau_2 \setminus \{v_r\}$ would be two minimal faces in

$$\{F \in \langle H_i \rangle \mid F \notin \langle H_1, \ldots, H_{i-1} \rangle\}.$$

Thus $G_1, \ldots, G_p, H_1 \cup \{v_r, u_r\}, \ldots, H_n \cup \{v_r, u_r\}$ is a partial shelling of $\mathcal{C}^{G'}$. Continuing with the facets containing v_s and no v_i for $i \le s-1$, for $s = r-1, \ldots, 1$, the same argument as for the facets containing v_r and no other v_i shows that there exists a partial shelling of $\mathcal{C}^{G'}$ consisting of all facets not containing v_0, say $F_1, \ldots, F_j$. Since $\mathcal{C}^{G'_{v_0}}$ is isomorphic to $\mathcal{C}^G$, the complex generated by the facets $I_1, \dots, I_m$ containing $\{v_0\}$ is $\{v_0\} * \mathcal{C}^{G'}$, hence shellable. If κ_i is a minimal face in

$$\{F \in \langle I_i \rangle \mid F \notin \langle I_1, \dots, I_{i-1} \rangle\},$$

$\kappa_i \cup \{v_0\}$ is a minimal face in

$$\{F \in \langle I_i \rangle \mid F \notin \langle F_1, \dots, F_j, I_1 \cup \{v_0\}, \dots, I_{i-1} \cup \{v_0\} \rangle\}.$$

Hence $\mathcal{C}^{G'}$ is shellable. □

The Propositions 3.4.5 and 3.4.6 can be used in conjunction with the following consequence of Propositions 1.2 and 1.3 in [3].

Theorem 3.4.7. *Let Δ be a simplicial complex of dimension d, in which every $(d-1)$-face lies in exactly 2 facets. Then Δ is (homeomorphic to) a d-sphere.*

Since an Eulerian simplicial complex Δ of dimension d has the property that every $(d-1)$-face is contained in exactly 2 facets, every shellable Eulerian complex is a shellable sphere. Thus, if $\mathcal{C}^G$ is a shellable sphere, and hence Eulerian, $\mathcal{C}^{\mathrm{cr}_k G}$ is a shellable sphere as well.

Proposition 3.4.8. *Let G be a graph and $u_1, \dots, u_{r-1} \in G$ with $\mathrm{d}(u_i, u_j) \geq 3$. Then*

$$\begin{aligned}\mathcal{F}(\mathcal{C}^{\mathrm{cr}_r(G)}\,;x) =& \mathcal{F}(\mathcal{C}^G\,;x) + 2x\mathcal{F}(\mathcal{C}^G\,;x) + \\ &+ \sum_{i=1}^{r-1} \left(x\mathcal{F}(\mathcal{C}^{G_{u_i}}\,;x) + x^2\mathcal{F}(\mathcal{C}^{G_{u_i}}\,;x) \right). \end{aligned} \tag{4}$$

Proof. For each independent k-set $T \in \mathrm{cr}_r(G)$, exactly one of the following holds: $T \subset G$, $u' \in T$, or there is a minimal $i \in \mathbb{N}$ such that $v_i \in T$. If $u' \in T$, $T \setminus \{u'\} \subset G$, and we get $x\mathcal{F}(\mathcal{C}^G\,;x)$. Since $\mathrm{cr}_k(G)_{v_0} \cong G$, if $\{v_0\} \in T$ we get $x\mathcal{F}(\mathcal{C}^G\,;x)$. Let v_i, $i \geq 1$, be the minimal v_i in T. We have $\mathrm{cr}_k(G)_{v_i} \setminus \bigcup_{j=0}^{i-1} \{v_j\} \cong G_{u_i} + P_1$, and we get $xF(\mathcal{C}^{G_{u_i}}\,;x) + x^2F(\mathcal{C}^{G_{u_i}}\,;x)$. Summing up, we get (4). □

Corollary 3.4.9. *Let G and $u_1, \dots, u_{r-1} \in V$ be as above. Then*

$$\mathcal{H}(K[\mathrm{cr}_r(G)]\,;x) = \mathcal{H}(K[G]\,;x) + x\mathcal{H}(K[G]\,;x) + \sum_{i=1}^{r-1} x\mathcal{H}(K[G_{u_i}]\,;x).$$

This will follow from the following lemma. Although not standard by any means, set $f_j = 0$ for $j < -1$ or $j > d$. This is to simplify the formulæ.

Lemma 3.4.10. *With the notation as in Proposition 3.4.8,*

(a) $h_{i-1}(\mathcal{C}^G) = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{i-j} f_{j-2}(\mathcal{C}^G)$,

(b) $h_i(\mathcal{C}^G) = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{i-j} (f_{j-1}(\mathcal{C}^G) + f_{j-2}(\mathcal{C}^G))$,

(c) $h_{i-1}(\mathcal{C}^{G_{u_i}})\sum_{j=0}^{i}(-1)^{i-j}\binom{d-j}{i-j}(f_{j-2}(\mathcal{C}^{G_{u_i}})+f_{j-3}(\mathcal{C}^{G_{u_i}}))$.

Proof. Recall that $f_i(\Delta)$ is defined to be zero for $i<-1$.

(a): The change of variable $l=j-1$ transforms the sum in j to

$$\sum_{l=0}^{i-1}(-1)^{i-1-l}\binom{d-1-l}{i-1-l}f_{l-1}(\mathcal{C}^G)=h_{i-1}(\mathcal{C}^G).$$

(b): Since $\binom{m}{n}=\binom{m-1}{n}+\binom{m-1}{n-1}$,

$$\begin{aligned}\sum_{j=0}^{i}(-1)^{i-j}\binom{d-j}{i-j}(f_{j-1}(\mathcal{C}^G) &= \sum_{j=0}^{i}(-1)^{i-j}\binom{d-1-j}{i-j}(f_{j-1}(\mathcal{C}^G)+\\ &\quad \sum_{j=0}^{i-1}(-1)^{i-j}\binom{d-1-j}{i-1-j}(f_{j-1}(\mathcal{C}^G).\end{aligned}$$

The first term is equal to $h_i(\mathcal{C}^G)$, and the second is equal to $-h_{i-1}(\mathcal{C}^G)$, from 3.4. Thus

$$\begin{aligned}&\sum_{j=0}^{i}(-1)^{i-j}\binom{d-j}{i-j}(f_{j-1}(\mathcal{C}^G)+\\ &\quad\sum_{j=0}^{i}(-1)^{i-j}\binom{d-j}{i-j}f_{j-2}(\mathcal{C}^G)) = h_i(\mathcal{C}^G)-h_{i-1}(\mathcal{C}^G)+h_{i-1}(\mathcal{C}^G)\\ &\qquad = h_i(\mathcal{C}^G).\end{aligned}$$

(c): This follows immediately from (b) and the change of variable $l=j-2$. □

Proof of Corollary 3.4.9. From Proposition 3.4.8, it follows that

$$f_j(\mathcal{C}^{\mathrm{cr}_r(G)})=f_j(\mathcal{C}^G)+2f_{j-1}(\mathcal{C}^G)+\sum_{k=1}^{r-1}\left(f_{j-1}(\mathcal{C}^{G_{u_k}})+f_{j-2}(\mathcal{C}^{G_{u_k}})\right).$$

Hence

$$\begin{aligned}h_i(\mathcal{C}^{\mathrm{cr}_r(G)}) &= \sum_{j=0}^{i}(-1)^{i-j}\binom{d-j}{i-j}f_{j-1}(\mathcal{C}^{\mathrm{cr}_r(G)})\\ &= \sum_{j=0}^{i}(-1)^{i-j}\binom{d-j}{i-j}(f_{j-1}(\mathcal{C}^G)+2f_{j-2}(\mathcal{C}^G)+\\ &\quad \sum_{k=1}^{r-1}\left(f_{j-2}(\mathcal{C}^{G_{u_i}})+f_{j-3}(\mathcal{C}^{G_{u_i}})\right)\\ &= h_i(\mathcal{C}^G)+h_{i-1}(\mathcal{C}^G)+\sum_{k=0}^{r-1}h_{i-1}(\mathcal{C}^{G_{u_k}}).\end{aligned}$$

Interpreting this in terms of the $\mathcal{H}$-polynomial gives

$$\mathcal{H}(K[\mathrm{cr}_r(G)]\ ;x) = \mathcal{H}(K[G]\ ;x) + x\mathcal{H}(K[G]\ ;x) + \sum_{i=1}^{r-1} x\mathcal{H}(K[G_{u_i}]\ ;x).$$

$\square$

3.4.1 The graph $\mathrm{cr}_2^n(P_2)$.

Definition 3.4.11. Begin with the graph P_2. Then successively add 2-stitches to the graph, where the base of each stitch is the node of the previous. This is called a *simple wale*, or in the introduced notation $\mathrm{cr}_2^n(P_2)$, where n is the number of stitches of the wale.

Note that, since $\mathcal{C}^{P_2}$ is a (or the) 0-dimensional shellable sphere, any simple wale is a shellable sphere. In particular, $K[\mathrm{cr}_2^n(P_2)]$ is Gorenstein.

As an application of Proposition 3.4.8 and its corollary, the f- and h-vector of $\mathcal{C}^{\mathrm{cr}_2^n(P_2)}$ is determined. This will thus give the Hilbert series of the Artinian, Gorenstein rings $K[\mathrm{cr}_2^n(P_2)]/(x_1,\ldots,x_{n+1})K[\mathrm{cr}_2^n(P_2)]$ (where $x_1,\ldots,x_n$ is a maximal M-sequence, $\deg x_i = 1$). As a corollary, a closed form solution to a complicated (well) recurrence equation is somewhat serendipitously found.

Proposition 3.4.12. *Let $G = \mathrm{cr}_2^n(P_2)$, $n \geq 0$. Also, let G be the empty graph for $n = -1$. Then*

(a) $h_i(\mathcal{C}^G) = \displaystyle\sum_{j=0}^{i} \binom{i}{j}\binom{n+1-j}{i}$,

(b) $f_i(\mathcal{C}^G) = \displaystyle\sum_{j=0}^{i+1} \left[\binom{n+1-j}{i+1-j} \sum_{l=0}^{j} \binom{j}{l}\binom{n+1-l}{j}\right]$.

Proof. For convenience, set $h_i^n = h_i(\mathcal{C}^{\mathrm{cr}_2^n(P_2)})$.

(a): For $n = -1$ and $n = 0$, G has h-vector (1) resp $(1,1)$. By the change of variables

$$\begin{aligned} i &= i, \\ m &= n - i + 1, \end{aligned}$$

the recursion $h_i^n = h_i^{n-1} + h_{i-1}^{n-1} + h_{i-1}^{n-2}$ transforms to $h_i^m = h_i^{m-1} + h_{i-1}^m + h_{i-1}^{m-1}$, $i \geq 0, m \geq 1$, with initial conditions $h_1^0 = h_0^1 = 1$. It is possible to extend this recursion to $m \geq 0$ by setting $h_0^0 = 1$, and this is consistent with the h-vector of G for $n = -1$. This renders the inital condidions $h_1^0 = h_0^1 = 1$ superfluous. What is left is the recursion

$$h_i^m = h_i^{m-1} + h_{i-1}^m + h_{i-1}^{m-1}, h_0^0 = 1, i, m \geq 0,$$

and this is the recursion for the *Delannoy numbers*. The number h_i^m can be interpreted as the number of ways to move a king from $(0,0)$ to (i,m) on a chessboard if only steps in one quadrant are allowed (i.e. the steps $(1,0)$, $(0,1)$ and $(1,1)$). The recursion may be solved by first choosing j rows of i rows, which may be done in $\binom{i}{j}$ ways, corresponding to rows with a $(1,1)$-step. Then choose $m-j$ $(0,1)$-steps, which may be done in $\binom{m+i-j}{m-j} = \binom{m+i-j}{i}$ ways, since it is not possible to place a $(0,1)$-step in a row with a $(1,1)$-step. The walk is completely determined by the placements of the $(1,0)$- and $(1,1)$-steps, whence $h_i^m = \sum_{j=0}^{i} \binom{i}{j}\binom{m+i-j}{i}$. Changing back to the variables i and n gives

$$h_i^n = \sum_{j=0}^{i} \binom{i}{j}\binom{n+1-j}{i}.$$

(b): This follows immediately from (a) and Proposition 3.1.17. □

The formula for $f_i(\mathcal{C}^G)$ in (b) gives a solution to the recurrence equation

$$a_i^n = a_i^{n-1} + 2a_{i-1}^n + a_{i-1}^{n-1} + a_{i-2}^n, a_0^0 = 1, a_i^j = 0 \text{ for } i < 0 \text{ or } j < 0,$$

namely

$$a_i^n = \sum_{j=0}^{i} \left[\binom{n+i-j}{i-j} \sum_{l=0}^{j} \binom{j}{l}\binom{n+i-l}{j} \right].$$

Note that the f-vector of $\mathcal{C}^{\mathrm{cr}_2^n(P_2)}$ may be obtained from this equation by taking diagonal slices from a_0^{n+1} to a_{n+1}^0. Thus $f_i(\mathcal{C}^{\mathrm{cr}_2^n(P_2)}) = a_{i+1}^{n-i}$.

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