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Linear operators in infinite dimensional vector spaces

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Abstract

The theory of linear operators is an extensive area. This thesis is about the linear operators in infinite dimensional vector spaces. We study elementary properties of Banach spaces, bounded operators, compact operators and spectrum of compact operators. We give an application to a two-point boundary value problem for a linear ordinary differential equation in the end.

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/Kharema

Contents

Introduction

This report contains two sections. In Section 1, we introduce basic concepts in infinite dimensional vector spaces, such as normed spaces, Banach spaces, inner product spaces and Hilbert spaces. We also study the strong and weak convergence. In Section 2, we study bounded linear operators in infinite dimensional vector spaces. In particular, we study compact operators and their spectrum. Finally, as an application we consider a two-point boundary value problem for a linear ordinary differential equation.

The results in this report are primarily taken from [1].

1 Infinite dimensional vector spaces

In this thesis we shall always assume that X is a vector space such that $\dim X = \infty$ unless otherwise stated.

1.1 Normed spaces. Banach spaces

Definition 1. A normed space is a vector space X in which a function $\| \| : X \longrightarrow R$ is defined and satisfies the following conditions:
- $\|x\| \geq 0$ and $= 0$ iff $x = 0$
- $\|\alpha x\| = |\alpha|\|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$
for all x, y in X and all scalars α. A normed space X in which every Cauchy sequence has a limit in X is said to be complete. A complete normed space is called a Banach space.

Example 1. (Space l^p). Let $1 \leq p < \infty$ be a fixed real number. By definition, each element in the space l^p is a sequence $x = (\xi_j) = (\xi_1, \xi_2, \cdots)$ of numbers such that $|\xi_1|^p + |\xi_2|^p + \cdots$ converges; thus

$$\sum_{j=1}^{\infty} |\xi_j|^p < \infty,$$

and the norm is defined by

$$\|x\| = (\sum_{j=1}^{\infty} |\xi_j|^p)^{1/p}. \tag{1}$$

Since
- $\|x\| = (\sum_{j=1}^{\infty} |\xi_j|^p)^{1/p} \geq 0$, and $= 0$ iff $x = 0$,
- $\|\lambda x\| = (\sum_{j=1}^{\infty} |\lambda \xi_j|^p)^{1/p} = (\sum_{j=1}^{\infty} |\lambda|^p |\xi_j|^p)^{1/p} = |\lambda| (\sum_{j=1}^{\infty} |\xi_j|^p)^{1/p} = |\lambda|\|x\|$,
- $\|x + y\| = (\sum_{j=1}^{\infty} |\xi_j + \eta_j|^p)^{1/p} \leq$ (by the Minkowski inequality, see (12) on p. 14 in [1]) $\leq (\sum_{j=1}^{\infty} |\xi_k|^p)^{1/p} + (\sum_{j=1}^{\infty} |\eta_k|^p)^{1/p} = \|x\| + \|y\|$,
this is indeed a norm.

(Completeness of l^p). Let (x_n) be any Cauchy sequence in the space l^p, where $x_m = (\xi_1^{(m)}, \xi_2^{(m)}, \cdots)$. Then for every $\epsilon > 0$ there is an N such that for all $m, n > N$,

$$\|x_m - x_n\| = (\sum_{j=1}^{\infty} |\xi_j^{(m)} - \xi_j^{(n)}|^p)^{1/p} < \epsilon. \tag{2}$$

It follows that

$$|\xi_j^{(m)} - \xi_j^{(n)}| < \epsilon$$

for every $j = 1, 2. \cdots$. For fixed j, $(\xi_j^{(n)})$ is a Cauchy sequence of numbers. Since the real and complex numbers are complete (see [1]), $\xi_j^{(n)} \longrightarrow \xi_j$ as $n \longrightarrow \infty$. We define $x = (\xi_1, \xi_2, \cdots)$ and show that $x \in l^p$ and $x_m \longrightarrow x$.

From (2) we have for all $m, n > N$ and a fixed k

$$\sum_{j=1}^{k} |\xi_j^{(m)} - \xi_j^{(n)}|^p < \epsilon^p.$$

Letting $n \longrightarrow \infty$, we obtain for $m > N$

$$\sum_{j=1}^{k} |\xi_j^{(m)} - \xi_j|^p \leq \epsilon^p.$$

Now let $k \longrightarrow \infty$; then for $m > N$

$$\sum_{j=1}^{\infty} |\xi_j^{(m)} - \xi_j|^p \leq \epsilon^p. \tag{3}$$

This shows that $x_m - x = (\xi_j^{(m)} - \xi_j) \in l^p$, and

$$x = x_m - (x_m - x) \in l^p.$$

The series in (3) represents $\|x_m - x\|^p$, hence $x_m \longrightarrow x$. Since (x_n) was an arbitrary Cauchy sequence in l^p, this proves the completeness of l^p.

Example 2. (Space l^∞). Every element in this space is a complex sequence $x = (\xi_j) = (\xi_1, \xi_2, \cdots)$ such that $\sup_j |\xi_j| < \infty$, and the norm is defined by

$$\|x\| = \sup_j |\xi_j|. \tag{4}$$

The first two conditions for the norm are verified similarly as in the last example. Let us show the triangle inequality.

- $\|x+y\| = \sup_j |\xi_j + \eta_j| \le \sup_j(|\xi_j| + |\eta_j|) \le \sup_j |\xi_j| + \sup_j |\eta_j| = \|x\| + \|y\|$.

(Completeness of l^∞). Let (x_m) be any Cauchy sequence in the space l^∞, where $x_m = (\xi_1^{(m)}, \xi_2^{(m)}, \cdots)$. Then for every $\epsilon > 0$ there is an N such that for all $m, n > N$

$$\|x_m - x_n\| = \sup_j |\xi_j^{(m)} - \xi_j^{(n)}| < \epsilon. \tag{5}$$

We shall show that there exists $x \in l^\infty$ which is the limit of (x_m). By (5) we have for every fixed $m, n > N$

$$|\xi_j^{(m)} - \xi_j^{(n)}| < \epsilon. \tag{6}$$

Hence for every j the sequence $\{\xi_j^{(1)}, \xi_j^{(2)}, \cdots\}$ is a Cauchy sequence of numbers. So it converges : $\xi_j^{(m)} \longrightarrow \xi_j$ as $m \longrightarrow \infty$. Let $x = (\xi_1, \xi_2, \cdots)$. We want to show that $x \in l^\infty$ and $x_m \longrightarrow x$. From (6) with $n \longrightarrow \infty$ we have for $m > N$

$$|\xi_j^{(m)} - \xi_j| \le \epsilon. \tag{7}$$

Since $x_m = (\xi_j^{(m)}) \in l^\infty$, there is a real number k_m such that $|\xi_j^{(m)}| \le k_m$ for all j. Hence by the triangle inequality

$$|\xi_j| \le |\xi_j - \xi_j^{(m)}| + |\xi_j^{(m)}| \le \epsilon + k_m.$$

This inequality holds for every j. Hence (ξ_j) is a bounded sequence of numbers. This implies that $x = (\xi_j) \in l^\infty$. From (6) we obtain

$$\|x_m - x\| = \sup_j |\xi_j^{(m)} - \xi_j| \le \epsilon$$

for $m > N$. This shows that $x_n \longrightarrow x$.

Example 3. (Space $C[a,b]$). This space is the set of all continuous real- or complex-valued functions $x = x(t)$ defined on a given closed interval $J = [a,b]$, and the norm is defined by

$$\|x\| = \max_{t \in J} |x(t)|. \tag{8}$$

The proof that this is indeed a norm is similar as in Example 2.
(Completeness of $C[a,b]$). Let (x_m) be any Cauchy sequence in $C[a,b]$. Then for every $\epsilon > 0$ there is an N such that for all $m, n > N$,

$$\|x_m - x_n\| = \max_{t\in J} |x_m(t) - x_n(t)| < \epsilon. \tag{9}$$

Hence for any fixed $t = t_0 \in J$,

$$|x_m(t_0) - x_n(t_0)| < \epsilon, \quad m, n > N.$$

This shows that $(x_1(t_0), x_2(t_0), \cdots)$ is a Cauchy sequence of real or complex numbers, so the sequence converges, say $x_m(t_0) \longrightarrow x(t_0)$ as $m \longrightarrow \infty$. In this way we can associate with each $t \in J$ a unique number $x(t)$. This defines a function x on J, and we must show that $x \in C[a,b]$ and $x_m \longrightarrow x$.

From (9) (with $n \to \infty$) we have

$$\max |x_m(t) - x(t)| \leq \epsilon, \quad m > N.$$

Hence for every $t \in J$,

$$|x_m(t) - x(t)| \leq \epsilon, \quad m > N. \tag{10}$$

This shows that $(x_m(t))$ converges to $x(t)$ uniformly on J. Since the x_m's are continuous on J and the convergence is uniform, the limit function x is continuous on J. Hence $x \in C[a,b]$ and $x_m \longrightarrow x$. This proves the completeness of $C[a,b]$.

The following is an example of an incomplete normed space.

Example 4. Let X be the set of all continuous real-valued functions on $J = [0,1]$, and let

$$\|x\| = \int_0^1 |x(t)|dt, \tag{11}$$

then $\|x\|$ is a norm, since
$\|x+y\| = \int_0^1 |x(t)+y(t)|dt \leq \int_0^1 (|x(t)+|y(t)|)dt = \int_0^1 |x(t)|dt + \int_0^1 |y(t)|dt = \|x\| + \|y\|$, and the other two conditions are trivially satisfied.
This normed space X is not complete. Let (x_m) be defined by

$$x_m(t) = \begin{cases} 0 & \text{if } \ t \in [0, \frac{1}{2}] \\ 1 & \text{if } \ t \in [a_m, 1] \end{cases} \quad ,$$

where $a_m = \frac{1}{2} + \frac{1}{m}$ and for $\frac{1}{2} \le t \le a_m$, $(t, x_m(t))$ is the linear segment joining $(\frac{1}{2}, 0)$ and $(a_m, 1)$. For every given $\epsilon > 0$,

$$\|x_m - x_n\| < \epsilon \text{ when } m, n > \frac{1}{\epsilon}.$$

Hence (x_m) is a Cauchy sequence. For every $x \in X$,

$$\begin{aligned}\|x_m - x\| &= \int_0^1 |x_m(t) - x(t)|dt \\ &= \int_0^{\frac{1}{2}} |x(t)|dt + \int_{\frac{1}{2}}^{a_m} |x_m(t) - x(t)|dt + \int_{a_m}^1 |1 - x(t)|dt.\end{aligned}$$

Since each integrand is nonnegative, $\|x_m - x\| \longrightarrow 0$ implies that each integral approaches zero. Since x is continuous, we would have

$$x(t) = \begin{cases} 0 & \text{if } \ t \in [0, \frac{1}{2}) \\ 1 & \text{if } \ t \in (\frac{1}{2}, 1]. \end{cases}$$

But this contradicts the continuity of x, hence $x \notin X$. This proves that X is not complete.

Definition 2. Let M be a subset in a normed space X, we say M is :
- *bounded* if there is a positive number c such that $\|x\| \le c \ \forall x \in M$.
- *closed* if for any sequence (x_n) in M, $x_n \longrightarrow x$ implies that $x \in M$.
- *compact* if any sequence (x_n) in M has a convergent subsequence whose limit belongs to M.

We know that in the finite dimensional space R^N, any subset M is compact if and only if it is closed and bounded (see [1]). But in infinite dimensional normed spaces this is no longer true.

Theorem 1. A compact subset M of a normed space is closed and bounded.

Proof. The proof is the same as for R^N, see [1]. □

Remark 1. The converse of this theorem is not true in infinite dimensional spaces.

We verify our claim in the space l^2. Consider the sequence (e_n) in l^2, where $e_n = (\delta_{nj})$ has the nth term 1 and all other terms 0. This sequence is bounded since $\|e_n\| = 1$. Its terms constitute a point set which is closed because it has no point of accumulation. For the same reason, that point set is not compact. See also Theorem 2 below.

Later we shall need the following lemma:

Riesz's Lemma. Let Y and Z be subspaces of a normed space X (of any dimension), and suppose that Y is closed and is a proper subset of Z. Then for every real number θ in the interval $(0, 1)$ there is a $z \in Z$ such that

$$\|z\| = 1, \quad \|z - y\| \geq \theta \text{ for all } y \in Y.$$

Proof. Let $v \in Z - Y$ and

$$a = \inf_{y \in Y} \|v - y\|.$$

Since Y is closed, $a > 0$. We now take any $\theta \in (0, 1)$. By the definition of an infimum there is a $y_0 \in Y$ such that

$$a \leq \|v - y_0\| \leq \frac{a}{\theta}.$$

Let

$$z = c(v - y_0) \quad \text{where} \quad c = \frac{1}{\|v - y_0\|}.$$

Then $\|z\| = 1$, and we show that $\|z - y\| \geq \theta$ for every $y \in Y$. We have

$$\begin{aligned} \|z - y\| &= \|c(v - y_0) - y\| \\ &= c\|v - y_0 - c^{-1}y\| \\ &= c\|v - y_1\| \end{aligned}$$

where

$$y_1 = y_0 + c^{-1}y.$$

Since $y, y_0 \in Y$, $y_1 \in Y$. Hence by the definition of a, $\|v - y_1\| \geq a$, and

$$\|z - y\| = c\|v - y_1\| \geq ca = \frac{a}{\|v - y_0\|} \geq \frac{a}{a/\theta} = \theta.$$

Since $y \in Y$ was arbitrary, this completes the proof. □

Theorem 2. (Finite dimension). If a normed space X has the property that the closed unit ball $M = \{x | \|x\| \leq 1\}$ is compact, then X is finite dimensional.

Proof. We argue by contradiction. We assume that M is compact but $\dim X = \infty$, and we shall show that this is impossible. Let x_1 be any vector of norm 1, and X_1 be the one dimensional subspace of X generated

by x_1, which is closed because the dimension is finite. Since $\dim X = \infty$, X_1 is a proper subspace of X.
By Riesz's Lemma there is an $x_2 \in X$ of norm 1 such that

$$\|x_2 - x_1\| \geq \frac{1}{2}.$$

Let X_2 be two dimensional proper subspace of X generated by x_1, x_2.
Again by Riesz's lemma there is an $x_3 \in X$ of norm 1 such that

$$\|x_3 - x_1\| \geq \frac{1}{2},$$

$$\|x_3 - x_2\| \geq \frac{1}{2}.$$

Continuing in this way, we obtain a sequence (x_n) of elements $x_n \in M$ such that

$$\|x_m - x_n\| \geq \frac{1}{2} \quad \text{for all } m \neq n.$$

That is, (x_n) has no convergent subsequence. But M is compact. Hence $\dim X$ must be finite. □

Definition 3. (Bounded linear functional). A linear functional f is a linear operator with domain $D(f)$ in a vector space X and range in the scalar field K, where $K = R$ or C. Thus

$$f : D(f) \longrightarrow K,$$

and we say the linear functional f is bounded if there exists a real number c such that for all $x \in D(f)$,

$$|f(x)| \leq c\|x\|.$$

Definition 4. (Dual space). Let X be a normed space. The set of all bounded linear functionals on X constitutes a normed space with the norm defined by

$$\|f\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{|f(x)|}{\|x\|} = \sup_{\substack{x \in X \\ \|x\|=1}} |f(x)|. \tag{12}$$

This set is called the dual space of X and is denoted by X^*. We also write $(f, x) = f(x)$ for $f \in X^*$ and $x \in X$.

It is easy to see from (12) that $\|f\| \geq 0$ and $\|f\| = 0$ if and only if $f = 0$, and that $\|\alpha f\| = |\alpha|\|f\|$. Moreover,

$$\|f+g\| = \sup_{\substack{x\in X\\ x\neq 0}} \frac{|f(x)+g(x)|}{\|x\|} \leq \sup_{\substack{x\in X\\ x\neq 0}} \frac{|f(x)|}{\|x\|} + \sup_{\substack{x\in X\\ x\neq 0}} \frac{|g(x)|}{\|x\|} = \|f\| + \|g\|.$$

Hence X^* is a normed space.

The following theorem shows the completeness of X^*.

Theorem 3. The dual space X^* of a Banach space X is a Banach space.

Proof. Let (f_n) be a Cauchy sequence in X^*. Then for each $\epsilon > 0$ there exists N such that

$$|(f_n, u) - (f_m, u)| = |(f_n - f_m, u)| \leq \|f_n - f_m\|\|u\| < \epsilon\|u\| \tag{13}$$

for every $n, m > N$, $u \in X$. Since (f_n, u) is a Cauchy sequence of numbers, it converges to some number c_u and we define f by setting $(f, u) = c_u$ for all u. Hence

$$\lim_{n\to\infty} (f_n, u) = (f, u).$$

We must show f is linear, bounded and $f_n \longrightarrow f$. We have

$$\lim_{n\to\infty} (f_n, \alpha_1 u_1 + \alpha_2 u_2) = (f, \alpha_1 u_1 + \alpha_2 u_2),$$

and on the other hand,

$$\begin{aligned}\lim_{n\to\infty} (f_n, \alpha_1 u_1 + \alpha_2 u_2) &= \lim_{n\to\infty} (f_n, \alpha_1 u_1) + \lim_{n\to\infty} (f_n, \alpha_2 u_2)\\ &= \alpha_1 \lim_{n\to\infty} (f_n, u_1) + \alpha_2 \lim_{n\to\infty} (f_n, u_2)\\ &= \alpha_1 (f, u_1) + \alpha_2 (f, u_2),\end{aligned}$$

thus f is linear. $\|f_n\|$ form a Cauchy sequence of positive numbers, hence $\lim \|f_n\| = M$, and

$$|(f, u)| = \lim_{n\to\infty} |(f_n, u)| \leq \lim_{n\to\infty} \|f_n\|\|u\| = M\|u\|.$$

So f is bounded. (13) gives

$$|(f_n - f, u)| = \lim_{m\to\infty} |(f_n - f_m, u)| \leq \lim_{m\to\infty} \|f_n - f_m\|\|u\| \leq \epsilon\|u\|,$$

hence

$$\|f_n - f\| = \sup_{u\neq 0} \frac{|(f_n - f, u)|}{\|u\|} \leq \epsilon \quad \text{for all } n > N.$$

Thus $f_n \longrightarrow f$ and X^* is complete. □

Example 5. Space l^1. The dual space of l^1 is l^∞.

Proof. Let $e_k = (\delta_{kj})$, then every $x \in l^1$ has a unique representation

$$x = \sum_{k=1}^{\infty} \xi_k e_k.$$

Let $f \in {l^1}^*$, since f is bounded and linear,

$$f(x) = \sum_{k=1}^{\infty} \xi_k \gamma_k, \quad \gamma_k = f(e_k),$$

γ_k's are uniquely determined by f, and

$$|\gamma_k| = |f(e_k)| \leq \|f\| \|e_k\|.$$

Since $\|e_k\| = 1$, we have

$$|\gamma_k| \leq \|f\| \quad \text{for all } k.$$

Hence

$$\sup_k |\gamma_k| \leq \|f\|. \tag{14}$$

Thus $f \in l^\infty$.

Conversely, let $b = (\beta_k) \in l^\infty$, then we can define the action of g on l^1 by

$$g(x) = \sum_{k=1}^{\infty} \xi_k \beta_k,$$

where $x = (\xi_k) \in l^1$. Clearly, g is linear. It remains to show that g is bounded. Note that

$$\begin{aligned} |g(x)| &\leq \sum_{k=1}^{\infty} |\xi_k \beta_k| \\ &\leq \sup_j |\beta_j| \sum_{k=1}^{\infty} |\xi_k| \\ &= \|x\| \sup_j |\beta_j|, \end{aligned}$$

hence g is bounded. Therefore g belongs to ${l^1}^*$. Finally we show that $\|f\|_{l^{1*}} = \|f\|_{l^\infty}$. We have

$$|f(x)| = |\sum_{k=1}^{\infty} \xi_k \gamma_k| \leq \sup_j |\gamma_j| \sum_{k=1}^{\infty} |\xi_k| = \|x\| \sup_j |\gamma_j|.$$

Since $\|f\| = \sup\limits_{\|x\|=1} \dfrac{|f(x)|}{\|x\|}$, then

$$\|f\| \le \sup_j |\gamma_j|.$$

From this and (14),

$$\|f\| = \sup_j |\gamma_j|.$$

This is the norm on l^∞. So we have shown that the l^{1^*} norm of f is the norm on l^∞. $\square$

Example 6. Space $l^p,\ 1 < p < \infty$. The dual space of l^p is l^q, where q is the conjugate of p, that is, $1/p + 1/q = 1$.

Proof. Let $x \in l^p$ and $e_k = (\delta_{kj})$. Then every $x \in l^p$ has a unique representation

$$x = \sum_{k=1}^{\infty} \xi_k e_k.$$

Let $f \in l^{p^*}$, since f is linear and bounded,

$$f(x) = \sum_{k=1}^{\infty} \xi_k f(e_k). \tag{15}$$

Let q be the conjugate of p and consider $x_n = (\xi_k^{(n)})$ with

$$\xi_k^{(n)} = \begin{cases} |\gamma_k|^q/\gamma_k & \text{if } k \le n \text{ and } \gamma_k \ne 0 \\ 0 & \text{if } k > n \text{ or } \gamma_k = 0, \end{cases} \tag{16}$$

where $\gamma_k = f(e_k)$. By substituting this into (15) we obtain

$$f(x_n) = \sum_{k=1}^{\infty} \xi_k^{(n)} \gamma_k = \sum_{k=1}^{n} |\gamma_k|^q.$$

By (16) and $(q-1)p = q$, we have

$$\begin{aligned} f(x_n) \le \|f\| \|x_n\| &= \|f\| (\sum_{k=1}^{n} |\xi_k^{(n)}|^p)^{1/p} \\ &= \|f\| (\sum_{k=1}^{n} |\gamma_k|^{(q-1)p})^{1/p} \\ &= \|f\| (\sum_{k=1}^{n} |\gamma_k|^q)^{1/p}. \end{aligned}$$

Then

$$f(x_n) = \sum_{k=1}^{n} |\gamma_k|^q \leq \|f\|(\sum_{k=1}^{n} |\gamma_k|^q)^{1/p}.$$

Dividing by the last factor, we get

$$(\sum_{k=1}^{n} |\gamma_k|^q)^{1-1/p} = (\sum_{k=1}^{n} |\gamma_k|^q)^{1/q} \leq \|f\|.$$

Letting $n \longrightarrow \infty$, we obtain

$$(\sum_{k=1}^{\infty} |\gamma_k|^q)^{1/q} \leq \|f\|. \tag{17}$$

This implies that $(\gamma_k) \in l^q$.
On other hand, from (15) and the Hölder inequality (see (10) on p. 14 in [1]), we have

$$\begin{aligned} |f(x)| = |\sum_{k=1}^{\infty} \xi_k \gamma_k| &\leq (\sum_{k=1}^{\infty} |\xi_k|^p)^{1/p} (\sum_{k=1}^{\infty} |\gamma_k|^q)^{1/q} \\ &= \|x\|(\sum_{k=1}^{\infty} |\gamma_k|^q)^{1/q} \end{aligned}$$

and

$$\|f\| \leq (\sum_{k=1}^{\infty} |\gamma_k|^q)^{1/q} \tag{18}$$

since $\|f\| = \sup_{x \neq 0} \dfrac{|f(x)|}{\|x\|}$. We have, by (17) and (18),

$$\|f\| = (\sum_{k=1}^{\infty} |\gamma_k|^q)^{1/q}. \tag{19}$$

So $\|f\| = \|\gamma\|_q$, where $\gamma = (\gamma_k) \in l^q$ and $\gamma_k = f(e_k)$. The mapping of $(l^p)^*$ to l^q given by $f \longmapsto \gamma$ is linear and injective. And by (19) it is norm preserving, so it is an isomorphism. Thus we have shown that l^{p^*} is isometrically embedded in l^q. We complete the proof by showing that $f \longmapsto \gamma$ is bijective, i. e., for each element of l^q there is a corresponding element of l^{p^*}. Let $(\beta_k) \in l^q$, then we can define g on l^p by

$$g(x) = \sum_{k=1}^{\infty} \xi_k \beta_k,$$

where $x = (\xi_k) \in l^p$. Then g is linear and by the Hölder inequality, we have

$$|g(x)| = |\sum_{k=1}^{\infty} \xi_k \beta_k| \leq (\sum_{k=1}^{\infty} |\xi_k|^p)^{1/p} (\sum_{k=1}^{\infty} |\beta_k|^q)^{1/q} = \|\beta\| \|x\|.$$

Hence g is bounded and thus g belongs to l^{p^*}. □

1.2 Inner product spaces. Hilbert spaces

Definition 5. An inner product space is a vector space X with an inner product $(\ , \ )$ defined on X. A Hilbert space is a complete inner product space.

Remark 2. The inner product has been defined in [3].

An inner product on X defines a norm on X given by

$$\|x\| = (x, x)^{\frac{1}{2}}. \tag{20}$$

A norm on an inner product space satisfies :
- Parallelogram equality:

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2). \tag{21}$$

- Polarization identity:

$$\begin{aligned} \mathrm{Re}(x, y) &= \tfrac{1}{4}(\|x + y\|^2 - \|x - y\|^2) \\ \mathrm{Im}(x, y) &= \tfrac{1}{4}(\|x + iy\|^2 - \|x - iy\|^2). \end{aligned} \tag{22}$$

(22) holds only in complex inner product spaces. By adding

$$\|x + y\|^2 = (x + y, x + y) = \|x\|^2 + (x, y) + (y, x) + \|y\|^2,$$

and

$$\begin{aligned} \|x - y\|^2 = (x - y, x - y) &= \|x\|^2 + (x, -y) + (-y, x) + \|y\|^2 \\ &= \|x\|^2 - (x, y) - (y, x) + \|y\|^2 \end{aligned}$$

we obtain the parallelogram equality.

To show (22), by (20), the definition of the inner product and since $(x, y) + (y, x) = (x, y) + \overline{(x, y)} = 2\mathrm{Re}(x, y)$, we have

$$\begin{aligned}\frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) &= \frac{1}{4}[(x+y, x+y) - (x-y, x-y)] \\ &= \frac{1}{4}[(x,x) + 2\mathrm{Re}(x,y) + (y,y) - (x,x) + 2\mathrm{Re}(x,y) - (y,y)] \\ &= \mathrm{Re}(x,y).\end{aligned}$$

Similarly, we can prove the other equality. Since $\mathrm{Im}(x, y) = \mathrm{Re}(x, iy)$, we get

$$\begin{aligned}\frac{1}{4}(\|x+iy\|^2 - \|x-iy\|^2) &= \frac{1}{4}[\|x\|^2 + (x, iy) + (iy, x) + \|y\|^2 - \\ &\quad (\|x\|^2 - (x, iy) - (iy, x) + \|y\|^2)] \\ &= \frac{1}{4}[2(x, iy) + 2(iy, x)] \\ &= \frac{1}{2}[(x, iy) + \overline{(x, iy)}] \\ &= \mathrm{Re}(x, iy) \\ &= \mathrm{Im}(x, y).\end{aligned}$$

Example 7. Hilbert sequence space l^2. The inner product is defined by

$$(x, y) = \sum_{j=1}^{\infty} \xi_j \bar{\eta_j},$$

and the norm is defined by

$$\|x\|^2 = (x, x) = (\sum |\xi_j|^2)^{\frac{1}{2}}.$$

We have shown the completeness of l^2 in Example 1.

The inner product spaces are normed spaces and the Hilbert spaces are Banach spaces, but the converse is not always true. The following examples show that.

Example 8. The space l^p with $p \neq 2$ is not an inner product space, hence not a Hilbert space.

Proof. We prove this by showing that the norm does not satisfy the parallelogram equality. Let $x = (1, 1, 0, 0, \cdots) \in l^p$ and $y = (1, -1, 0, 0, \cdots) \in l^p$, so

$$\|x\| = \|y\| = 2^{1/p}, \quad \|x + y\| = \|x - y\| = 2,$$

hence

$$8 = \|x + y\|^2 + \|x - y\|^2 \neq 2(\|x\|^2 + \|y\|^2) = 4 \cdot 2^{2/p}$$

if $p \neq 2$. □

Example 9. The space $C[a, b]$ is not an inner product space, hence not a Hilbert space.

Proof. We show that the norm defined by

$$\|x\| = \max_{t \in J} |x(t)|, \; J = [a, b]$$

cannot be obtained from an inner product since this norm does not satisfy the parallelogram equality. Indeed, if we take $x(t) = 1$ and $y(t) = \frac{t-a}{b-a}$, we have $\|x\| = 1, \|y\| = 1$ and

$$x(t) + y(t) = 1 + \frac{t - a}{b - a}$$

$$x(t) - y(t) = 1 - \frac{t - a}{b - a}.$$

Hence $\|x + y\| = 2, \|x - y\| = 1$ and

$$\|x + y\|^2 + \|x - y\|^2 = 5 \quad \text{but} \quad 2(\|x\|^2 + \|y\|^2) = 4.$$

This completes the proof. □

Definition 6. Let M be a subset of an inner product space X. We say M is an orthogonal set if $(x, y) = 0 \; \forall x, y \in M, x \neq y$ and orthonormal if

$$(x, y) = \begin{cases} 0 & \text{if} \;\; x \neq y \\ 1 & \text{if} \;\; x = y. \end{cases}$$

Later we shall need the following theorem:

Riesz's Representation Theorem (Functionals on Hilbert spaces). Every bounded linear functional f on a Hilbert space H can be represented in terms of the inner product, namely,

$$f(x) = (x, z)$$

where z depends on f, is uniquely determined by f and has norm

$$\|z\| = \|f\|.$$

For a proof, see 3.8-1 in [1].

1.3 Strong and weak convergence

Definition 7. Let (x_n) be a sequence in a normed space X. We say that:
- (x_n) is strongly convergent if there is an $x \in X$ such that $\lim\limits_{n\to\infty} \|x_n - x\| = 0$. This is written $\lim\limits_{n\to\infty} x_n = x$ or $x_n \longrightarrow x$.
- (x_n) is weakly convergent if there is an $x \in X$ such that for every $f \in X^*$, $\lim\limits_{n\to\infty} f(x_n) = f(x)$. This is written $x_n \xrightarrow{\mathrm{w}} x$ or $x_n \rightharpoonup x$.

Example 10. Let $e_n = (1, 0, 0, \cdots), (0, 1, 0, \cdots), \cdots$ in $H = l^2$. Then (e_n) converges weakly to 0, but not strongly. In fact, every $f \in H^*$ has a Riesz representation $z \in H$. Hence $f(e_n) = (e_n, z)$. Now by the Bessel inequality (see 3.4-6 in [1]),

$$\sum_{n=1}^{\infty} |(e_n, z)|^2 \leq \|z\|^2.$$

Hence the series on the left converges, so that its terms must approach zero as $n \longrightarrow \infty$. This implies

$$f(e_n) = (e_n, z) \longrightarrow 0.$$

Since $f \in H^*$ was arbitrary, we see that $e_n \xrightarrow{\mathrm{w}} 0$. However, (e_n) does not converge strongly because

$$\|e_m - e_n\|^2 = (e_m - e_n, e_m - e_n) = 2, \quad m \neq n.$$

Lemma 1. Let (x_n) be a weakly convergent sequence in a normed space X, say, $x_n \xrightarrow{\mathrm{w}} x$. Then:

1. The weak limit x of (x_n) is unique.

2. Every subsequence of (x_n) converges weakly to x.

Proof. 1. If $x_n \xrightarrow{\text{w}} x, x_n \xrightarrow{\text{w}} y$, then $f(x_n) \longrightarrow f(x), f(x_n) \longrightarrow f(y)$ for $f \in X^*$. Since $(f(x_n))$ is a sequence of numbers, its limit is unique. Hence $f(x) = f(y)$, that is

$$f(x) - f(y) = f(x - y) = 0$$

for every $f \in X^*$. This implies $x - y = 0$ by Corollary 4.3-4 in [1].

2. $(f(x_n))$ is a convergent sequence of numbers, and every subsequence of $(f(x_n))$ converges and has the same limit as the sequence.

□

A subset S of X is said to be fundamental if the closed span of S is X, i. e., if the set of all finite linear combinations of elements of S is dense in X.

Example 11. Let $u_n \in X$ be a bounded sequence. In order that u_n converge weakly to u, it suffices that (f, u_n) converge to (f, u) for all f in a fundamental subset S^* of X^*.

Proof. Let D^* be the span of S^*; D^* is dense in X^*. Since f is a finite linear combination of elements of S^*, (f, u_n) converges to (f, u) for all $f \in D^*$. Let $\epsilon > 0$ and $g \in X^*$ There exists $f \in D^*$ such that

$$\|g - f\| < \epsilon.$$

Since (f, u_n) converges, there is an N such that

$$|(f, u_n - u_m)| < \epsilon \quad \text{for } n, m > N.$$

Thus

$$\begin{aligned} |(g, u_n - u_m)| &= |(g - f, u_n) + (f, u_n - u_m) + (f - g, u_m)| \\ &\leq |(g - f, u_n)| + |(f, u_n - u_m)| + |(f - g, u_m)| \\ &\leq M\epsilon + \epsilon + M\epsilon = (2M + 1)\epsilon \quad \text{for } n, m > N, \end{aligned}$$

where $M = \sup \|u_n\|$. This shows that (g, u_n) converges for all $g \in X^*$. □

The relationship between strong and weak convergence is given by

Theorem 4. A sequence $(u_n) \subset X$ converges strongly if and only if (f, u_n) converges uniformly for $\|f\| \leq 1, \ f \in X^*$.

Proof. The "only if" part follows from

$$|(f, u_n) - (f, u_m)| \leq \|u_n - u_m\| \|f\| \leq \|u_n - u_m\|$$

because for each $\epsilon > 0$ there exists N such that $\|u_n - u_m\| < \epsilon$ for $n, m > N$. To prove the "if" part, suppose that (f, u_n) converges uniformly for $\|f\| \leq 1$. This implies that for any $\epsilon > 0$, there exists an N such that

$$|(f, u_n - u_m)| \leq \epsilon \ \text{ if } n, m > N \ \text{ and } \|f\| \leq 1.$$

Hence

$$\|u_n - u_m\| = \sup_{\|f\| \leq 1} |(f, u_n - u_m)| \leq \epsilon \ \text{ for } n, m > N$$

by (12).

$\square$

2 Linear operators

2.1 The domain and range

Let X, Y be normed spaces, and let T be a linear operator defined on a subspace $D(T)$ of X and taking values in Y. $D(T)$ is called the domain of T. The range $R(T)$ of T is defined as the set of all vectors of the form Tu with $u \in D(T)$. We write $T : D(T) \longrightarrow Y$.

Example 12. A finite real matrix $(\tau_{jk}), \ j = 1, \cdots, r, \ k = 1, \cdots, n$ defines a linear operator $T : R^n \longrightarrow R^r$ by

$$y = \tau x,$$

where $x = (\xi_1, \cdots, \xi_n)$ and $y = (\eta_1, \cdots, \eta_r)$. In matrix form we write

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \cdot \\ \cdot \\ \cdot \\ \eta_r \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ t_{r1} & t_{r2} & \cdots & t_{rn} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \\ \vdots \\ \\ \xi_n \end{bmatrix}.$$

Example 13. Let X be the vector space of all polynomials on $[a, b]$. We may define a linear operator T on X by setting

$$Tx(t) = x'(t)$$

for every $x \in X$, where the prime denotes differentiation with respect to t. This operator T maps X onto itself.

Example 14. A linear operator T from $C[a, b]$ into itself can be defined by

$$Tx(t) = \int_a^t x(\tau)d\tau, \quad t \in [a, b].$$

Another linear operator from $C[a, b]$ into itself is defined by

$$Tx(t) = tx(t).$$

In these examples we can easily verify that the dimension of the range of T is not exceeding the dimension of the domain of T. Let us prove this in the following theorem.

Theorem 5. Let T be a linear operator. If $\dim D(T) = n < \infty$, then $\dim R(T) \leq n$.

Proof. If $y_1, \ldots, y_{n+1}$ are elements in $R(T)$, then there are $x_1, \cdots, x_{n+1}$ in $D(T)$ such that $y_1 = Tx_1, \cdots, y_{n+1} = Tx_{n+1}$. Since $\dim D(T) = n$, this set $\{x_1, \cdots, x_{n+1}\}$ must be linearly dependent. Hence

$$\alpha_1 x_1 + \cdots + \alpha_{n+1} x_{n+1} = 0$$

for some scalars $\alpha_1, \cdots, \alpha_{n+1}$, not all zero, and

$$T(\alpha_1 x_1 + \cdots +_{n+1} x_{n+1}) = \alpha_1 y_1 + \cdots + \alpha_{n+1} y_{n+1} = 0.$$

This shows that $\{y_1, \cdots, y_{n+1}\}$ is a linearly dependent set. Hence dim $R(T) \leq n$. $\square$

2.2 Bounded and continuous operators

Definition 8. (Bounded linear operator). Let X and Y be normed spaces and $T : D(T) \to Y$ a linear operator, where $D(T) \subset X$. The operator T is said to be bounded if there is a real number c such that for all $x \in D(T)$

$$\|Tx\| \leq c\|x\|.$$

Example 15. Let X and Y be normed spaces. $T : X \longrightarrow Y$ is bounded if and only if T maps bounded sets in X into bounded sets in Y.

Proof. Suppose T is bounded, then there is a number c such that $\|Tx\| \leq c\|x\|$. Let A be a bounded subset, and $c_1 = \max_{x\in A} \|x\|$, then $\|Tx\| \leq cc_1 \ \forall x \in A$, thus the image of any bounded set in X is bounded. Conversely, suppose we have $\|Tx\| \leq M \ \forall \ \|x\| \leq 1$. Then $\|Tx\| \leq M\|x\| \ \forall x$, so T is a bounded operator. $\square$

Example 16. The operator $T : l^\infty \longrightarrow l^\infty$ defined by $y = (\eta_j) = Tx, \ \eta_j = \xi_j/j, \ x = (\xi_j)$, is bounded.

Proof. Let A be any bounded set, then there is c such that $\|x\| \leq c$ for all $x \in A$. Then

$$\|Tx\| = \max_j |\frac{\xi_j}{j}| \leq \max_j |\xi_j| = \|x\|,$$

hence T is bounded. $\square$

Example 17. Let T be a bounded linear operator from a normed space X onto a normed space Y. If there is a positive b such that

$$\|Tx\| \geq b\|x\| \quad \text{for all } x \in X,$$

then $T^{-1} : Y \longrightarrow X$ exists and is bounded.

Proof. T is bounded, hence there is $c > 0$ such that $\|Tx\| \leq c\|x\|$. So

$$b\|x\| \leq \|Tx\| \leq c\|x\|.$$

Then $Tx = 0$ iff $x = 0$, that is, T^{-1} exists. Thus $\forall x \ \exists y$ such that the inequality becomes

$$b \leq \frac{\|y\|}{\|T^{-1}y\|} \leq c.$$

In particular,
$$\|T^{-1}y\| \leq \frac{1}{b}\|y\|.$$
Hence T^{-1} is bounded. □

Example 18. The inverse $T^{-1} : R(X) \longrightarrow X$ of a bounded linear operator $T : X \longrightarrow Y$ need not be bounded.
The operator defined in Example 16 is bounded and T^{-1} exists. It is given by the formula
$$T^{-1}(y) = (j\eta_j) = (\eta_1, 2\eta_2, 3\eta_3, \cdots)$$
and is not bounded.

Theorem 6. (Finite dimension). If a normed space X is finite dimensional, then every linear operator on X is bounded.

Proof. Let $\dim X = n$ and let $\{e_1, \cdots, e_n\}$ be a basis for X. We take any $x = \sum \xi_j e_j$ and consider any linear operator T on X. We have
$$\|Tx\| = \|\sum_{j=1}^{n} \xi_j Te_j\| \leq \sum_{j=1}^{n} |\xi_j| \|Te_j\| \leq \max_k \|Te_k\| \sum_{j=1}^{n} |\xi_j|.$$
To the last sum we apply Lemma 2.4-1 in [1] which states that
$$\sum |\xi_j| \leq \frac{1}{c}\|\sum \xi_j e_j\| = \frac{1}{c}\|x\|$$
for some $c > 0$. So
$$\|Tx\| \leq \max \|Te_k\| \sum |\xi_j| \leq \frac{1}{c} \max \|Te_k\| \|x\|.$$
Let $\gamma = \frac{1}{c}\max\limits_{k}\|Te_k\|$, then
$$\|Tx\| \leq \gamma\|x\|$$
and T is bounded. □

Definition 9. (Continuous linear operator). Let X, Y be normed spaces, $D(T) \subset X$, and let $T : D(T) \longrightarrow Y$ be a linear operator. T is continuous at $x_0 \in D(T)$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that
$$\|Tx - Tx_0\| < \epsilon \text{ for all } x \in D(T) \text{ satisfying } \|x - x_0\| < \delta.$$

In fact there is an immediate relation between bounded and continuous operators. We show this in next theorem.

Theorem 7. T is continuous if and only if T is bounded.

Proof. For $T = 0$ the statement is trivial. Let $T \neq 0$ be bounded, we consider any $x_0 \in D(T)$. Let $\epsilon > 0$, then for every $x \in D(T)$ such that

$$\|x - x_0\| < \delta \quad \text{where} \quad \delta = \frac{\epsilon}{\|T\|}$$

we obtain

$$\|Tx - Tx_0\| = \|T(x - x_0)\| \leq \|T\|\|x - x_0\| < \|T\|\delta = \epsilon.$$

Since $x_0 \in D(T)$ was arbitrary, this shows that T is continuous.

Conversely, if T is continuous at an arbitrary $x_0 \in D(T)$, then for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\|Tx - Tx_0\| \leq \epsilon \quad \text{for all } x \in D(T) \quad \text{satisfying } \|x - x_0\| \leq \delta.$$

We now take any $y \neq 0$ in $D(T)$ and set

$$x = x_0 + \frac{\delta}{\|y\|} y.$$

Hence $\|x - x_0\| = \delta$ and

$$\|Tx - Tx_0\| = \|T(x - x_0)\| = \|T(\frac{\delta}{\|y\|} y)\| = \frac{\delta}{\|y\|}\|Ty\| \leq \epsilon.$$

Thus

$$\|Ty\| \leq \frac{\epsilon}{\delta}\|y\|.$$

Hence T is bounded. □

Example 19. Let T be a bounded linear operator. Then $x_n \longrightarrow x$ implies $Tx_n \longrightarrow Tx$ where $x_n, x \in D(T)$. By the last theorem, as $n \longrightarrow \infty$ we have

$$\|Tx_n - Tx\| = \|T(x_n - x)\| \leq \|T\|\|x_n - x\| \longrightarrow 0.$$

Let X, Y be Banach spaces. We denote by $B(X, Y)$ the set of all bounded operators from X to Y.

Lemma 2. $B(X,Y)$ is a Banach space under the norm $\|T\| = \sup_{x\neq 0} \dfrac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\|$.

Proof. That $\|T\|$ is a norm can be seen in the same way as for $\|f\|$ in (12). Let (T_n) be a Cauchy sequence of elements of $B(X,Y)$. Then $(T_n u)$ is a Cauchy sequence in Y for all $u \in X$, hence for each $\epsilon > 0$ there exists N such that

$$\|T_n u - T_m u\| \leq \|T_n - T_m\| \|u\| < \epsilon \|u\|. \tag{23}$$

Since Y is complete, there is $v \in Y$ such that

$$T_n u \longrightarrow v$$

and we can define $v = Tu$. As in the proof of Theorem 3, we see that T is linear. We show that T is bounded and $T_n \longrightarrow T$. Since $\|T_n\|$ form a Cauchy sequence of positive numbers, then $\|T_n\| \leq M$ for some M and all n, and

$$\|Tu\| = \lim_{n\to\infty} \|T_n u\| \leq \lim_{n\to\infty} \|T_n\| \|u\| \leq M\|u\| \text{ for all } u,$$

so T is bounded. Now (23) gives

$$\|(T_n - T)u\| = \lim_{m\to\infty} \|(T_n - T_m)u\| \leq \lim_{m\to\infty} \|T_n - T_m\| \|u\| \leq \epsilon \|u\|.$$

Since this holds for all $u \in X$,

$$\lim_{n\to\infty} \|T_n - T\| = 0.$$

□

Definition 10. (Convergence of sequences of operators). Let X and Y be normed spaces. A sequence (T_n) of operators $T_n \in B(X,Y)$ is said to be:
- *uniformly operator convergent* if (T_n) converges to some T in the norm on $B(X,Y)$, i.e.,

$$\|T_n - T\| \longrightarrow 0.$$

We use the notation $T_n \longrightarrow T$.
- *strongly operator convergent* to T if $(T_n x)$ converges strongly in Y for every $x \in X$, i. e.,

$$\|T_n x - Tx\| \longrightarrow 0.$$

This is denoted by $T_n \xrightarrow{\text{s}} T$.
- *weakly operator convergent* to T if $(T_n x)$ converges weakly in Y for every $x \in X$, i. e.,

$$|f(T_n x) - f(Tx)| \longrightarrow 0 \quad \text{for all } f \in Y^*.$$

This is denoted by $T_n \xrightarrow{\text{w}} T$. T is called the uniform, strong and weak operator limit of (T_n), respectively.

Example 20. Let $u_n \in X$ and $T_n \in B(X)$. If $u_n \xrightarrow{\text{s}} u$ and $T_n \xrightarrow{\text{s}} T$, then $T_n u_n \xrightarrow{\text{s}} Tu$. If $u_n \xrightarrow{\text{s}} u$ and $T_n \xrightarrow{\text{w}} T$, then $T_n u_n \xrightarrow{\text{w}} Tu$.

Proof. If $u_n \xrightarrow{\text{s}} u$ and $T_n \xrightarrow{\text{s}} T$ then,

$$\begin{aligned}
\|T_n u_n - Tu\| &= \|T_n u_n - Tu_n + Tu_n - Tu\| \\
&\leq \|T_n u_n - Tu_n\| + \|Tu_n - Tu\| \\
&= \|(T_n - T)u_n\| + \|T(u_n - u)\| \\
&\leq \|T_n - T\|\|u_n\| + \|T\|\|u_n - u\| \longrightarrow 0.
\end{aligned}$$

If $u_n \xrightarrow{\text{s}} u$ and $T_n \xrightarrow{\text{w}} T$, then for all $f \in X^*$ we have

$$\begin{aligned}
|f(T_n u_n) - f(Tu)| &= |f(T_n u_n) - f(T_n u) + f(T_n u) - f(Tu)| \\
&\leq |f(T_n u_n) - f(T_n u)| + |f(T_n u) - f(Tu)| \\
&\leq \|f\|\|T_n\|\|u_n - u\| + |f(T_n u) - f(Tu)| \longrightarrow 0.
\end{aligned}$$

□

2.3 Compact operators

Definition 11. Let X and Y be normed spaces. A linear operator $T : X \longrightarrow Y$ is called a compact linear operator if for every bounded subset M of X, the set $\overline{T(M)}$ is compact.

Lemma 3. Let X and Y be normed spaces. Then:

1. Every compact linear operator $T : X \longrightarrow Y$ is bounded, hence continuous.

2. If $\dim X = \infty$, the identity operator $I : X \longrightarrow X$ is not compact.

Proof. 1. The unit sphere $U = \{x \in X | \, \|x\| = 1\}$ is bounded. Since T is compact, $\overline{T(U)}$ is compact, and is bounded, so that

$$\sup_{\|x\|=1} \|Tx\| < \infty.$$

Hence T is bounded and Theorem 7 shows that it is continuous.

2. The closed unit ball $M = \{x \in X | \, \|x\| \leq 1\}$ is bounded. If M is compact, then $\dim X < \infty$ by Theorem 2, which contradicts that X is of infinite dimension. Thus M cannot be compact, showing that $\overline{I(M)} = \overline{M} = M$ is not compact, i.e. I is not compact.

□

From this lemma it follows that the identity operator in a normed space is compact if and only if X is finite dimensional.

Theorem 8. Let X and Y be normed spaces and $T : X \longrightarrow Y$ a linear operator. Then T is compact if and only if it maps every bounded sequence (x_n) in X onto a sequence (Tx_n) in Y which has a convergent subsequence.

Proof. If T is compact and (x_n) is bounded, then $(\overline{Tx_n})$ is compact and Definition 2 shows that (Tx_n) contains a convergent subsequence.

Conversely: Let B be any bounded subset in X, and let (y_n) be any sequence in $T(B)$. Then $y_n = Tx_n$ for some $x_n \in B$, and (x_n) is bounded since B is bounded. By assumption, (Tx_n) contains a convergent subsequence. Hence $\overline{T(B)}$ is compact, this shows that T is compact. □

Example 21. If T_1 and T_2 are compact linear operators from a normed space X into a normed space Y, then $T_1 + T_2$ is a compact linear operator.

Proof. Let (x_n) be any bounded sequence in X. Since T_1 is compact, by Theorem 8 (T_1x_n) has a convergenct subsequence $(T_1x_{n_1})$ in Y and similarly, $(T_2x_{n_1})$ has a convergent subsequence $(T_2x_{n_2})$ in Y. Then $((T_1 + T_2)x_{n_2})$ is a convergent subsequence of $((T_1 + T_2)x_n)$. Hence $T_1 + T_2$ is a compact operator. □

Theorem 9. (Finite dimensional domain or range). Let X and Y be normed spaces and $T : X \longrightarrow Y$ a linear operator. Then:

1. If T is bounded and $\dim \; T(X) < \infty$, the operator T is compact.

2. If $\dim X < \infty$, T is compact.

Proof. 1. Let (x_n) be any bounded sequence in X. Then the inequality $\|Tx_n\| \leq \|T\|\|x_n\|$ shows that (Tx_n) is bounded. Hence $\overline{(Tx_n)}$ is compact since $\dim T(X) < \infty$. It follows that (Tx_n) has a convergent subsequence. Since (x_n) was an arbitrary bounded sequence in X, the operator T is compact by Theorem 8.

2. By Theorem 6 T is bounded, by Theorem 5 $\dim T(X) \leq \dim X$. Hence by 1. T is compact.

□

Theorem 10. (Sequence of compact linear operators). Let (T_n) be a sequence of compact linear operators from a normed space X into a Banach space Y. If (T_n) is uniformly operator convergent, say, $\|T_n - T\| \longrightarrow 0$, then the limit operator T is compact.

Proof. Let (x_m) be any bounded sequence in X, then it has subsequences $(x_{1,m}), (x_{2,m}), \cdots$ such that $(T_1 x_{1,m}), (T_2 x_{2,m}), \cdots$ are Cauchy. Here $(x_{1,m})$ is a subsequence of (x_m), $(x_{2,m})$ is a subsequence of $(x_{1,m})$ etc. Let $(y_m) = (x_{m,m})$. This is a subsequence of (x_m) such that for every fixed n the sequence $(T_n y_m)$ is Cauchy. Since $\|x_m\| \leq c$ for some $c > 0$, also $\|y_m\| \leq c$ for all m. Since $\|T_n - T\| \longrightarrow 0$, $\exists p$ such that $\|T - T_p\| < \frac{\epsilon}{3c}$. Since $(T_p y_m)$ is Cauchy, $\exists N$ such that

$$\|T_p y_j - T_p y_k\| < \frac{\epsilon}{3} \quad \text{for all } j, k > N.$$

Hence for $j, k > N$

$$\begin{aligned}
\|Ty_j - Ty_k\| &\leq \|Ty_j - T_p y_j\| + \|T_p y_j - T_p y_k\| + \|T_p y_k - Ty_k\| \\
&\leq \|T - T_p\|\|y_j\| + \frac{\epsilon}{3} + \|T_p - T\|\|y_k\| \\
&< \frac{\epsilon}{3c}c + \frac{\epsilon}{3} + \frac{\epsilon}{3c}c = \epsilon.
\end{aligned}$$

Hence (Ty_m) is Cauchy and therefore convergent, that is, T is compact. □

Example 22. Prove compactness of $T : l^2 \longrightarrow l^2$ defined by $y = (\eta_j) = Tx$, where $\eta_j = \xi_j / j$ for $j = 1, 2, \cdots$.

Proof. Let $T_n : l^2 \longrightarrow l^2$ be defined by

$$T_n x = (\xi_1, \xi_2/2, \xi_3/3, \cdots, \xi_n/n, 0, 0, \cdots).$$

T_n is bounded, and it is compact by Theorem 9 because dim $R(T_n) = n < \infty$. Furthermore,

$$\begin{aligned}\|(T - T_n)x\|^2 &= \sum_{j=n+1}^{\infty} |\eta_j|^2 = \sum_{j=n+1}^{\infty} \frac{1}{j^2}|\xi_j|^2 \\ &\le \frac{1}{(n+1)^2} \sum_{j=n+1}^{\infty} |\xi_j|^2 \le \frac{\|x\|^2}{(n+1)^2},\end{aligned}$$

hence

$$\sup \frac{\|(T - T_n)x\|}{\|x\|} \le \frac{1}{n+1}$$

and

$$\|T - T_n\| \le \frac{1}{n+1}.$$

Hence $T_n \longrightarrow T$, and T is compact by Theorem 10. □

Example 23. Let Y be a Banach space and let $T_n : X \longrightarrow Y,\ n = 1, 2, \cdots$, be bounded operators of finite rank. If (T_n) is uniformly operator convergent, then the limit operator is compact.

Proof. Since dim $T_n(X) =$ rank $T_n < \infty$, it follows from Theorem 9 that T_n is compact for all n. Hence by Theorem 10 T is compact. □

2.4 Spectrum of compact operators

Definition 12. Let X be a normed space and $T : X \longrightarrow X$ a bounded linear operator. Denote

$$T_\lambda = T - \lambda I, \tag{24}$$

where λ is a complex number and I is the identity operator on X. If T_λ has no bounded inverse on X, then λ is called a spectral value, denoted $\lambda \in \sigma(T)$. If $T_\lambda x = 0$ for some $x \neq 0$, then λ is said to be an eigenvalue and x an eigenvector.

If λ is an eigenvalue, then $\lambda \in \sigma(T)$, but the converse need not be true.

Theorem 11. (Eigenvalues). The set of eigenvalues of a compact linear operator $T : X \longrightarrow X$ on a normed space X is countable (perhaps finite or even empty), and the only possible point of accumulation is $\lambda = 0$.

Proof. We suppose the opposite, and we get a contradiction. Let $k_0 > 0$, and suppose there is an infinite sequence (λ_n) of distinct eigenvalues such that $|\lambda_n| > k_0$ for all n.
For each n, let $T_{\lambda_n} x_n = 0,\ x_n \neq 0$. The set of all the x_n's is linearly independent (see Thm. 7.4-3 in [1] or Proposition 2 in [3]). Let M_n =span$\{x_1, \cdots, x_n\}$. Then each $x \in M_n$ has a unique representation

$$x = \alpha_1 x_1 + \cdots + \alpha_n x_n.$$

We apply $T - \lambda_n I$ and use $Tx_j = \lambda_j x_j$:

$$(T - \lambda_n I)x = \alpha_1(\lambda_1 - \lambda_n)x_1 + \cdots + \alpha_{n-1}(\lambda_{n-1} - \lambda_n)x_{n-1},$$

hence

$$(T - \lambda_n I)x \in M_{n-1} \text{ for all } x \in M_n.$$

The M_n's are finite dimensional and therefore closed (see Thm. 2.4-3 in [1]). By Riesz's Lemma there is a sequence (y_n) such that $y_n \in M_n$ and $\|y_n\| = 1,\ \ \|y_n - x\| \geq \frac{1}{2}$ for all $x \in M_{n-1}$.We show that

$$\|Ty_n - Ty_m\| \geq \frac{1}{2}k_0 \text{ for all } n > m,$$

so that (Ty_n) has no convergent subsequence because $k_0 > 0$. Since T is compact and (y_n) is bounded, we get a contradiction.
Now let $m < n$, then we have $\lambda_m y_m \in M_{n-1}$ and $(T - \lambda_n I)y_n \in M_{n-1}$. Thus

$$x = \frac{1}{\lambda_n}(Ty_m - Ty_n) + y_n \in M_{n-1}.$$

Hence

$$\begin{aligned}\|Ty_n - Ty_m\| &= \|\lambda_n y_n - \lambda_n x\| \\ &= |\lambda_n| \|y_n - x\| \\ &\geq \frac{1}{2}|\lambda_n| \geq \frac{1}{2}k_0.\end{aligned}$$

□

Theorem 12. (Null space). Let $T : X \longrightarrow X$ be a compact linear operator on a normed space X. Then for every $\lambda \neq 0$ the null space $N(T_\lambda)$ of $T_\lambda = T - \lambda I$ is finite dimensional.

Proof. We shall apply Theorem 2 from which it follows that if the closed unit ball M in $N(T_\lambda)$ is compact, then $\dim N(T_\lambda) < \infty$. Now we show that M is compact.

Let (x_n) be in M, then

$$\|x_n\| \leq 1.$$

T is compact and (x_n) is bounded, hence (Tx_n) has a convergent subsequence (Tx_{n_k}) by Theorem 8.
Since $x_n \in M \subset N(T_\lambda)$,

$$T_\lambda x_n = Tx_n - \lambda x_n = 0,$$

so that

$$x_n = \lambda^{-1} T x_n.$$

Thus

$$x_{n_k} = \lambda^{-1} T x_{n_k}$$

is a convergent subsequence of (x_n), and the limit is in M because M is closed. This proves that M is compact. □

Example 24. Let $T : X \longrightarrow X$ be a compact linear operator on a normed space. If $\dim X = \infty$, then $0 \in \sigma(T)$.

Proof. We assume $0 \notin \sigma(T)$. Similarly as in the last theorem, we have that if the closed unit ball M in X is compact, then $\dim X < \infty$. Let (x_n) be in M. Since T is compact, there is a convergent subsequence (Tx_{n_k}) of (Tx_n), $Tx_{n_k} \longrightarrow y$. T^{-1} exists because $0 \notin \sigma(T)$, hence $x_{n_k} \longrightarrow T^{-1}y$. From Theorem 2 it follows that $\dim X < \infty$ which is a contradiction. Hence $0 \in \sigma(T)$. □

Lemma 4. Let $T : X \longrightarrow X$ be a compact linear operator on a normed space X. If $\lambda \neq 0$ is not eigenvalue, then $R(T_\lambda)$ is a closed subset of X.

Proof. We have to prove that for any convergent sequence $(T_\lambda x_n)$ the limit of it is in $R(T_\lambda)$.

Suppose

$$T_\lambda x_n = Tx_n - \lambda x_n \longrightarrow y.$$

First we will show that (x_n) must be bounded. Suppose (x_n) is unbounded, then we may assume passing to a subsequence that $\|x_n\| \longrightarrow \infty$.
Let $x'_n = \dfrac{x_n}{\|x_n\|}$. Then

$$T_\lambda x'_n = Tx'_n - \lambda x'_n \longrightarrow 0,$$

and since T is compact, there is y' such that

$$Tx'_n \longrightarrow y'$$

after passing to a subsequence once more. It follows that

$$\lambda x'_n \longrightarrow y'$$

and

$$|\lambda| = \|\lambda x'_n\| \longrightarrow \|y'\|.$$

So $y' \neq 0$. On the other hand,

$$\begin{aligned} T_\lambda y' &= T_\lambda(\lim \lambda x'_n) \\ &= \lambda \lim T_\lambda x'_n \\ &= \lambda \lim (Tx'_n - \lambda x'_n) \\ &= \lambda(y' - y') = 0. \end{aligned}$$

Hence y' is an eigenvector, but this is a contradiction because λ is not an eigenvalue. This shows that (x_n) cannot be unbounded.
Now (x_n) is bounded, T is compact, hence there is $z \in X$ such that

$$Tx_n \longrightarrow z$$

after passing to a subsequence, so

$$\lambda x_n \longrightarrow z - y.$$

Thus we have

$$\begin{aligned} T_\lambda(\frac{1}{\lambda}(z-y)) &= T_\lambda(\frac{1}{\lambda}(\lim_{n\to\infty} \lambda x_n)) \\ &= \lim_{n\to\infty} T_\lambda x_n \\ &= \lim_{n\to\infty} (Tx_n - \lambda x_n) \\ &= z - (z - y) \\ &= y. \end{aligned}$$

So $y \in R(T_\lambda)$. $\square$

Since $T_\lambda(X) \subset X$, it follows that $T_\lambda^2(X) \subset T_\lambda(X)$. Continuing, we see that

$$X = T_\lambda^0(X) \supset T_\lambda(X) \supset T_\lambda^2(X) \supset \cdots ,$$

where $T_\lambda^0 = I$.

Lemma 5. Let $T : X \longrightarrow X$ be a compact linear operator on a normed space X and let $\lambda \neq 0$. Then there exists a smallest $q \geq 0$ such that

$$T_\lambda^q(X) = T_\lambda^{q+1}(X).$$

Proof. We assume the opposite, i. e.,

$$T_\lambda^n(X) \supset T_\lambda^{n+1}(X)$$

for all n and all inclusions are proper. We shall find a bounded sequence (x_n) such that (Tx_n) has no convergent subsequence, and this contradicts the compactness of T. Hence $T_\lambda^q(X) = T_\lambda^{q+1}(X)$ for some $q \geq 0$.
By the preceding lemma, $T_\lambda^{n+1}(X)$ is a closed subspace of $T_\lambda^n(X)$, so it follows from Riesz's Lemma that there is a sequence (x_n), $x_n \in T_\lambda^n(X)$, such that

$$\|x_n\| = 1, \ \|x_n - x\| \geq \frac{1}{2} \ \text{ for all } x \in T_\lambda^{n+1}(X). \tag{25}$$

We will show

$$\|Tx_n - Tx_m\| \geq \frac{1}{2}|\lambda| \ \text{ for all } m > n.$$

We have

$$Tx_n - Tx_m = \lambda x_n + T_\lambda x_n - T_\lambda x_m - \lambda x_m.$$

Since

$$x_m \in T_\lambda^m(X) \subset T_\lambda^{n+1}(X),$$

$$T_\lambda x_m \in T_\lambda^{m+1}(X) \subset T_\lambda^{n+1}(X)$$

and

$$T_\lambda x_n \in T_\lambda^{n+1}(X),$$

it follows that

$$x = \frac{1}{\lambda}(T_\lambda x_m - T_\lambda x_n - \lambda x_m) \in T_\lambda^{n+1}(X).$$

Thus (25) gives

$$\begin{aligned}\|Tx_n - Tx_m\| &= \|\lambda x_n - \lambda x\| \\ &= |\lambda|\|x_n - x\| \\ &\geq \frac{1}{2}|\lambda| > 0.\end{aligned}$$

□

Theorem 13. (Eigenvalues). Let $T : X \longrightarrow X$ be a compact linear operator on a Banach space X. Then every spectral value $\lambda \neq 0$ of T (if it exists) is an eigenvalue of T.

Proof. Suppose $\lambda \neq 0$ is not an eigenvalue. Then $T_\lambda x = 0$ if and only if $x = 0$, hence

$$T_\lambda^{-1} \ : \ T_\lambda(X) \longrightarrow X$$

exists. Now we show

$$T_\lambda(X) = X,$$

that is, $q = 0$ in Lemma 5. Suppose $q \geq 1$, then the inclusion $T_\lambda^q(X) \subset T_\lambda^{q-1}(X)$ is proper, and hence there exists

$$y \in T_\lambda^{q-1}(X) - T_\lambda^q(X). \tag{26}$$

We have

$$y = T_\lambda^{q-1}x \ \text{ for some } x.$$

Since

$$T_\lambda y = T_\lambda^q x \in T_\lambda^q(X) = T_\lambda^{q+1}(X),$$
$$T_\lambda y = T_\lambda^{q+1} z \ \text{ for some } z. \tag{27}$$

By (26) and since $T_\lambda^q z \in T_\lambda^q(X)$,

$$y - T_\lambda^q z \neq 0.$$

But by (27)

$$T_\lambda(y - T_\lambda^q z) = 0$$

which is impossipole because λ is not an eigenvalue. So $q = 0$ and $T_\lambda(X) = X$. Hence T_λ^{-1} is defined on the whole space X and T_λ^{-1} must be bounded by the bounded inverse theorem 4.12-2 in [1]. It follows that $\lambda \notin \sigma(T)$. □

Definition 13. (Equicontinuous). Let (x_n) be a sequence in $C[a,b]$. We say it is equicontinuous if for any $\epsilon > 0$ there is a $\delta > 0$ depending on ϵ, such that for all $s_1, s_2 \in [a,b]$, $|s_1 - s_2| < \delta$, we have

$$|x_n(s_1) - x_n(s_2)| < \epsilon \quad \text{for all } n.$$

Ascoli's Theorem (Equicontinuous sequence). A bounded equicontinuous sequence (x_n) in $C[a,b]$ has a convergent subsequence.

Theorem 14. (Compact integral operator). Let $J = [a,b]$ be any compact interval and suppose that k is continuous on $J \times J$. Then the operator $T : X \longrightarrow X$ defined by

$$(Tx)(s) = \int_a^b k(s,t)x(t)dt, \tag{28}$$

where $X = C[a,b]$, is a compact operator.

Proof. We shall prove T is bounded, hence (Tx_n) is bounded when (x_n) is bounded. Then we show that (Tx_n) is equicontinuous, so Ascoli's theorem will imply that (Tx_n) has a convergent subsequence, and so T is compact. Boundedness of T follows from

$$\|Tx\| = \max_{s\in J} |\int_a^b k(s,t)x(t)dt| \leq \|x\| \max_{s\in J} \int_a^b |k(s,t)|dt,$$

which is of the form $\|Tx\| \leq M\|x\|$. Now let (x_n) be any bounded sequence in $X, \|x_n\| \leq c$, and let $y_n = Tx_n$. We show that (y_n) is equicontinuous. Since k is continuous, for all $\epsilon > 0$ there is $\delta > 0$ such that for all $t \in J$ and all $s_1, s_2 \in J$ satisfying $|s_1 - s_2| < \delta$ we have

$$|k(s_1,t) - k(s_2,t)| < \frac{\epsilon}{(b-a)c}.$$

Consequently, for $|s_1 - s_2| < \delta$ and every n we obtain

$$\begin{aligned} |y_n(s_1) - y_n(s_2)| &= |\int_a^b [k(s_1,t) - k(s_2,t)]x_n(t)dt| \\ &< (b-a)\frac{\epsilon}{(b-a)c}c = \epsilon. \end{aligned}$$

This proves equicontinuity of (y_n). $\square$

Finally we consider the boundary value probem

$$x''(t) + \mu q(t)x(t) = f(t), \quad x(0) = x(1) = 0, \tag{29}$$

where $\mu \in C$, q is a given real-valued function, $q(t) > 0$ for all $t \in [0, 1]$ and $f \in C[0, 1]$. We shall show that for most values of μ, (29) has a unique solution for each f. Before showing that, let us consider the following lemma.

Lemma 6. The boundary value problem

$$x''(t) = g(t), \quad x(0) = x(1) = 0, \tag{30}$$

where $g \in C[0, 1]$, has a unique solution

$$x(t) = \int_0^1 G(s, t)g(s)ds, \tag{31}$$

where G is defined by

$$G(s, t) = \begin{cases} s(t-1), & 0 \le s \le t \le 1 \\ t(s-1), & 0 \le t \le s \le 1. \end{cases}$$

Proof. Since G is continuous for $s, t \in [0, 1]$, (31) can be written as

$$\begin{aligned} x(t) &= \int_0^1 G(s, t)g(s)ds \\ &= \int_0^t s(t-1)g(s)ds + \int_t^1 t(s-1)g(s)ds \\ &= (t-1)\int_0^t sg(s)ds + t\int_t^1 (s-1)g(s)ds. \end{aligned}$$

If x satisfies (30), then $x(0) = 0, x(1) = 0$,

$$\begin{aligned} x'(t) &= \int_0^t sg(s)ds + (t-1)tg(t) + \int_t^1 (s-1)g(s)ds - t(t-1)g(t) \\ &= \int_0^t sg(s)ds + \int_t^1 (s-1)g(s)ds \end{aligned}$$

and

$$\begin{aligned} x''(t) &= tg(t) - (t-1)g(t) \\ &= g(t). \end{aligned}$$

Hence x a is solution of (30). If x satisfies (30), then integration gives

$$x'(\tau) = \int_0^\tau g(s)ds + c_1,$$

$$x(t) = \int_0^t \int_0^\tau g(s)dsd\tau + c_1 t + c_2.$$

$x(0) = 0$ gives $c_2 = 0$ and $x(1) = 0$ gives

$$c_1 = -\int_0^1 \int_0^\tau g(s)dsd\tau.$$

Hence

$$x(t) = \int_0^t \int_0^\tau g(s)dsd\tau - t\int_0^1 \int_0^\tau g(s)dsd\tau.$$

Changing the order of integration,

$$\begin{aligned} x(t) &= \int_0^t \int_s^t g(s)d\tau ds - t\int_0^1 \int_s^1 g(s)d\tau ds \\ &= \int_0^t (t-s)g(s)ds - \int_0^1 t(1-s)g(s)ds \\ &= \int_0^t (t-s)g(s)ds - \int_0^t t(1-s)g(s)ds - \int_t^1 t(1-s)g(s)ds \\ &= \int_0^t s(t-1)g(s)ds + \int_t^1 t(s-1)g(s)ds \\ &= \int_0^1 G(s,t)g(s)ds. \end{aligned}$$

Hence x is a solution of (30). □

Theorem 15. The boundary value problem (29) has a unique solution for each $f \in C[0,1]$ and each $\mu \in C\backslash A$, where $A = \{\mu_1, \mu_2, \cdots\}$ is a countable set such that $|\mu_n| \longrightarrow \infty$ as $n \longrightarrow \infty$.

Proof. The boundary value problem

$$x''(t) + \mu q(t)x(t) = 0, \quad x(0) = x(1) = 0 \tag{32}$$

is equivalent to

$$x(t) = -\mu \int_0^1 G(s,t)q(s)x(s)ds$$

(set $g(t) = -\mu q(t)x(t)$ in Lemma 6). Since $k(s,t) = -G(s,t)q(s)$ is continuous, it follows from Theorem 14 that the operator $T : C[0,1] \longrightarrow C[0,1]$ given by $(Tx)(t) = \int_0^1 k(s,t)x(s)ds$ is compact. Let $\lambda = 1/\mu$. Then x is a solution of (32) if and only if

$$T_\lambda x = Tx - \lambda x = 0.$$

By Theorems 11 and 13, the spectrum of T is a set (λ_n) which is either finite or $\lambda_n \longrightarrow 0$. But here $|\mu_n| = 1/|\lambda_n| \longrightarrow \infty$. Using Lemma 6 once more, we can re-write (29) as

$$x = \mu Tx + h,$$

where $h(t) = \int_0^1 G(s,t)f(s)ds$. Let $\lambda = 1/\mu$. Then (29) becomes

$$T_\lambda x = Tx - \lambda x = -\lambda h.$$

Let $A = \{\mu_1, \mu_2, \cdots\}$, where $\mu_n = 1/\lambda_n$. If $\mu \notin A$, then T_λ is invertible and $x = -T_\lambda^{-1}(\lambda h)$ is the unique solution of (29). $\square$

Let us finally remark that it is known from the theory of ordinary differential equations that if $q > 0$, then the set A is infinite and all μ_n are real and positive.

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