SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Strong homotopy algebras

Introducing and explicitly constructing A_∞ and L_∞ algebras

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2009 - No 1

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Introducing and explicitly constructing A_∞ and L_∞ algebras

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Självständigt arbete i matematik 30 högskolepoäng, avancerad nivå

Handledare: Sergei Merkulov

2009

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Abstract

This essay is intended as an introduction to A_∞-algebra and L_∞-algebra for those not familiar, or partially familiar with concepts such as graded algebras, differentials and general homological algebra.

The first chapter defines some basic structures in homological algebra, and proves some of the basic theorems. This is covered to provide a basis for how to think about differentials and gradings, as well as a setting for many of the results related to strong homotopy algebras that are outside of the scope of this paper.

The second chapter defines some graded algebras with differentials. These structures may be regarded as special cases of the algebras in the last three chapters.

The third chapter introduces the tensor product, defines algebras using the product and proves results related to the structures.

The forth chapter introduces two subalgebras of the tensor algebra/coalgebra, called the symmetric and the Grassmannian algebras/coalgebras.

The fifth chapter defines the strong associative homotopy algebra, the A_∞-algebra. The algebra and related concepts are also studied and explicitly constructed in the chapter.

In the sixth chapter the strong Lie homotopic algebra, the L_∞-algebra, is defined. A few related concepts are introduced, and a geometric interpretation of the algebra is outlined in detail. This chapter and the essay ends with three explicit constructions of L_∞-algebras from three different polynomial rings.

Chapter 1

Chain complexes

1.1 Modules

A *module* is a generalisation of a vector space, where the field of scalars is replaced by scalars from a ring structure.

This means that just as in the case of a vector space, a module is an additive Abelian group, and in direct analogy with the vector space's product, it is equipped with a product between scalars from the ring and elements in the group.

Please note that many of the well-known results about vector spaces, such as the existence of a basis or duality, aren't necessarily true for a module over an arbitrary ring Λ.

Definition 1.1.1. A *left* Λ*-module* over a ring Λ is an Abelian group $(M, +)$ together with an operation $\Lambda \times M \to M$, called *scalar multiplication*, such that for all $a, b \in \Lambda$ and $x, y \in M$,

$$\begin{array}{lrcl} 1. & a(x+y) & = & ax + ay \\ 2. & (a+b)x & = & ax + bx \\ 3. & (ab)x & = & a(bx) \\ 4. & 1x & = & 1. \end{array}$$

A *right* Λ*-module* is defined as above, but with the scalar multiplication defined by $M \times \Lambda \to M$, and with the corresponding set of rules.

Conversely, consider a left Λ-module where Λ a field, then the module may of course also be regarded as a vector space.

1.2 Chain complexes & Homology modules

This section will discuss different ring homomorphisms. In general a homomorphism is a mapping that preserve relevant structure, the homomorphisms in question in this context will be ring and module homomorphisms. In order to simplify the language all homomorphism will often be referred to simply as homomorphism.

Definition 1.2.1. A *ring homomorphism* is a mapping $f : R \to S$, where R and S are rings and where the following preserving properties are respected,

$$\begin{aligned} f(a+b) &= f(a) + f(b), \\ f(ab) &= f(a)f(b), \\ f(1) &= 1, \end{aligned}$$

for all $a, b \in R$.

Definition 1.2.2. A *module homomorphism* is a mapping $f : M \to N$, where M and N are modules over the ring $\mathcal{K}$ and where the following preserving properties are respected,

$$\begin{aligned} f(a+b) &= f(a) + f(b), \\ f(ra) &= rf(a), \end{aligned}$$

for all $a, b \in M$ and $r, s \in \mathcal{K}$.

The homomorphisms has a corresponding set of classifications to the set-theoretic mappings. An "injective homomorphism" is called a monomorphism (or mono), a "surjective homomorphism" is called an epimorphism (or epi) and a "bijective homomorphism" is called an isomorphism (or iso).

To further remind ourselves, consider $f : M \to N$. We know that if and only if f is a monomorphism, $\ker f = 0$, and that if and only if f is an epimorphism, $\operatorname{im} f = N$, and finally that f is an isomorphism if and only if there exist an inverse function f^{-1}, such that $f \circ f^{-1} = f^{-1} \circ f = Id$.

In order to get a feel for the main topic of this essay we will now introduce and define a few simple, yet very central, concepts in homological algebra.

Definition 1.2.3. Let $f : L \to M$ and $g : M \to N$ be Λ-homomorphisms. Then

$$L \xrightarrow{f} M \xrightarrow{g} N$$

is a *short exact sequence* if

$$\text{im } f = \ker g.$$

The definition of exactness is generalised to longer sequences in the natural manner, that is, consider the sequence

$$E : 0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0.$$

E is *exact* if and only if the subsequences,

$$\begin{aligned} 0 &\longrightarrow L \xrightarrow{f} M, \\ L &\xrightarrow{f} M \xrightarrow{g} N, \\ M &\xrightarrow{g} N \longrightarrow 0, \end{aligned}$$

are short exact sequences.

Definition 1.2.4. A *chain-complex* $(C_\bullet, d_\bullet)$ is a sequence of Λ-modules $\ldots, C_{-2}, C_{-1}, C_0, C_1, C_2, \ldots$ with Λ-module homomorphisms $d_n : C_n \to C_{n-1}$, such that $d_{n-1} \circ d_n = 0$ for all n. That is

$$\cdots \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \longrightarrow \cdots \longrightarrow C_0 \xrightarrow{d_0} C_{-1} \longrightarrow \cdots.$$

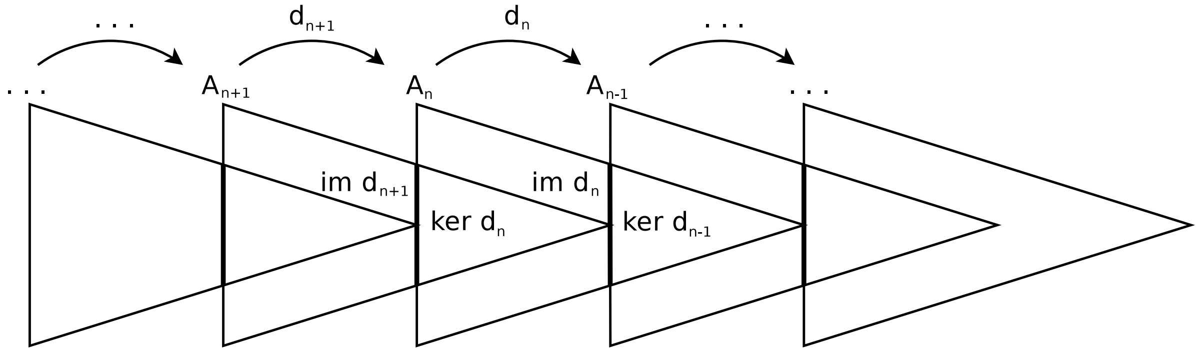
We may consider d_i for any $i \in \mathbb{Z}$ as a restriction of a general $d_\bullet$ to C_i, hence we may safely drop the subscripts when convenient. The function $d_\bullet$ is called *the differential of* $C_\bullet$.

The kernel of d_n is usually denoted $Z_n = Z_n(C_\bullet)$ and its members are called *cycles*. In the same manner the image of d_{n+1} usually is denoted $B_n = B_n(C_\bullet)$ and its members are called *boundaries*. Now, since $d_\bullet \circ d_\bullet = 0$, we know that

$$0 \subseteq B_n \subseteq Z_n \subseteq C_n$$

for all n.

A nice way to remember and think about chain-complex graphically is depicted in the diagram below. The thicker lines represents the im $d_i = \ker d_{i-1} \subset A_i$ and where the sharp-angle edge of each triangle indicates the zero-element.



Definition 1.2.5. A *chain-complex map* is a morphism $u : C_\bullet \to D_\bullet$, between two chain-complexes, such that it commutes with the differential d, in the sense that the diagram

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d_\bullet} & C_{n+1} & \xrightarrow{d_\bullet} & C_n & \xrightarrow{d_\bullet} & C_{n-1} & \xrightarrow{d_\bullet} & \cdots \\ & & \downarrow u & & \downarrow u & & \downarrow u & & \\ \cdots & \xrightarrow{d_\bullet} & D_{n+1} & \xrightarrow{d_\bullet} & D_n & \xrightarrow{d_\bullet} & D_{n-1} & \xrightarrow{d_\bullet} & \cdots \end{array}$$

commutes.

We can now construct another central concept in homological algebra, the homology module.

Definition 1.2.6. The n^{th} *homology module* of $C_\bullet$ is defined as $H_n(C_\bullet) = Z_n/B_n$ of C_n.

The purpose of the homology module is to measure "how far $C_\bullet$ is from being exact."

Definition 1.2.7. A chain-complex C is called a *subcomplex* of D if each C_n is a submodule of D_n.

In direct analogy to the concept of chain-complexes, cycles, boundaries and homology-modules, there are corresponding "co-constructions". The "co-constructions" are essentially the same as the "non-co", but with the arrows reversed and denoted with superscript.

To be more precise:

Definition 1.2.8. A *cochain-complex* is a family $\{C^n\}$ of Λ-modules, together with maps $d^n : C^n \to C^{n+1}$, such that $d^\bullet \circ d^\bullet = 0$.

$$\cdots \xleftarrow{d^{n+1}} C^{n+1} \xleftarrow{d^n} C^n \xleftarrow{d^{n-1}} C^{n-1} \xleftarrow{d^{n-2}} \cdots$$

We refer to the members of $Z^n(C^\bullet) = \ker d^n$ as *cocycles* and the members of $B^n(C^\bullet) = \text{im } d^{n-1}$ as *coboundaries* in direct analogy to the "non-co" case.

Definition 1.2.9. The n^{th} *cohomology module of* $C^\bullet$ is denoted $H^n(C^\bullet)$ and defined by

$$H^n(C^\bullet) = Z^n(C^\bullet)/B^n(C^\bullet).$$

1.3 Categories & Functors

Definition 1.3.1. *Categories* are constructed to describe mathematical entities and their relationships. A category $\mathcal{C}$ consists of a collection of objects, denoted $\text{Ob}(\mathcal{C})$ and a collection of morphisms denoted by $\text{Hom}(\mathcal{C})$. That is, every $f \in \text{Hom}(\mathcal{C})$ is a mapping $f : A \to B$ where $A, B \in \text{Ob}(\mathcal{C})$. $\text{Hom}(\mathcal{C})$ always includes the trivial identity mapping.

Categories are usually denoted by an abbreviation in a bold font. A few examples of categories are:

Set Which consists of all sets together with functions between the sets.

Ab The category of all Abelian groups and their group homomorphisms.

Ring Consisting of all rings and their ring homomorphisms.

We will later see that chain-complexes are also a category, and more specifically an Abelian category; it is denoted by **Ch**.

Definition 1.3.2. A *preadditive category* (or **Ab**-category) $\mathcal{A}$ is a category where for $A, B \in Ob(\mathcal{A})$ every $\text{Hom}_{\mathcal{A}}(A, B) \subset \text{Hom}(\mathcal{A})$ is given the structure of an Abelian group, where composition distributes over addition. That is for A, B, C and D in $\text{Ob}(\mathcal{A})$, $f, h \in \text{Hom}(\mathcal{A})$ and g, g' in $\text{Hom}_{\mathcal{A}}(A, B)$,

$$C \xrightarrow{f} A \underset{g'}{\overset{g}{\rightrightarrows}} B \xrightarrow{h} D,$$

says that $h(g + g')f = hgf + hgf'$.

Definition 1.3.3. An *additive category* $\mathcal{A}$ is a preadditive category that has a zero element and a biproduct $A \times B$. The notion of a biproduct is the natural generalisation of the direct sum to preadditive categories.

The *kernel* of a morphism in an additive category $f : A \to B$ is defined to be the map $i : I \to A$, such that $fi = 0$ and the *cokernel* of f is a map $e : B \to E$, such that $ef = 0$. Hence in $\mathcal{A}$ a map, $g : B \to A$, is a *monomorphism* if $ig = 0$ implies that $g = 0$ and an *epimorphism* if $he = 0$ implies that $h = 0$.

Definition 1.3.4. Every category $\mathcal{C}$ has an *opposite category* $\mathcal{C}^{op}$. The objects of $\mathcal{C}$ and $\mathcal{C}^{op}$ are the same. The morphisms however are reversed in the sense that for every morphism $f : C \to D$ in $\text{Hom}(\mathcal{C})$ there exist a corresponding morphism $f^{op} : D \to C$ in $\text{Hom}(\mathcal{C}^{op})$.

Further, if f is a monomorphism, then f^{op} will be an epimorphism and vice versa. Similarly the kernel and cokernel are each others corespondents in the same manner.

The $\mathcal{C}^{op}$ is also referred to as the *dual category* of $\mathcal{C}$.

Definition 1.3.5. A *functor*, $F : \mathcal{C} \to \mathcal{D}$, is a mapping between categories. That is a mapping such that for every $a \in \text{Ob}(\mathcal{C})$, $F(a) \in \text{Ob}(\mathcal{D})$, and for every $f : A \to B \in \text{Hom}(\mathcal{C})$, $F(f) : F(A) \to F(B) \in \text{Hom}(\mathcal{D})$. Also $F(\text{Id}_{\mathcal{C}}) = \text{Id}_{F(\mathcal{C})}$ and $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f : L \to K$ and $g : K \to M$, where $L, K, M \in \text{Ob}(\mathcal{C})$.

A *contravariant functor* is a functor where the depicted mappings are reversed. That is if the exact sequence

$$0 \longrightarrow L \xrightarrow{f} K \xrightarrow{g} M \longrightarrow 0$$

is depicted with a contravariant functor F we will get

$$F(L) \xleftarrow{F(f)} F(K) \xleftarrow{F(g)} F(M).$$

If a functor preserves exactness we say that the functor is exact and if the functor only preserves exactness from one direction we say that the functor is left or right exact respectively.

Definition 1.3.6. An *Abelian category* $\mathcal{A}$, is an additive category such that every map in $\text{Hom}(\mathcal{A})$ has a kernel and cokernel and every monomorphism in $\text{Hom}(\mathcal{A})$ is the kernel of its cokernel and in correspondence every epimorphism in $\text{Hom}(\mathcal{A})$ is the cokernel of its kernel.

Theorem 1.3.7. *The category of chain-complexes,* **Ch** *or* $\mathbf{Ch}(\mathcal{A})$*, is an Abelian category.*

Proof. Let $f : C_\bullet \mapsto D_\bullet$ be a complex chain map, and hence $f \in \text{Hom}(\mathbf{Ch})$. Now we will construct maps for all modules in the complex: the injective maps $i : \ker(f) \mapsto C_n$ and the projective map $e : D_n \mapsto \text{coker}\,(f)$. Hence the maps i and e will be the kernel and cokernel to f. □

1.4 Chain complex operations

Definition 1.4.1. Let C be a chain-complex and $n \in \mathbb{Z}$. Now let $\tau_{\geq n}C$ denote the subcomplex of C defined by

$$(\tau_{\geq n}C)_i = \begin{cases} 0, & i < n \\ Z_n, & i = n \\ C_i, & i > n. \end{cases}$$

The subcomplex defined above is called the (good) *truncation* of C below n and the quotient complex $\tau_{<n}C = C/(\tau_{\geq n}C)$ is called the (good) *truncation* of C above n, hence $H_i(\tau_{<n}C)$ is equal to $H_i(C)$ as $i < n$ and vanishes as $i \geq n$.

There is a variant of truncation called *brutal truncation* denoted with $\sigma_{\geq n}C$ and $\sigma_{<n}C$. The difference between the good and brutal truncation is that for $i = n$, $(\tau_{\geq n}C)_i = C_i$, therefore it is simpler to define, however the homology modules are distorted as $i = n$.

Definition 1.4.2. Let m be an integer and C be a chain-complex or cochain-complex. The *translation* operation is then denoted $C[m]$ and defined as $C[m]_n = C_{n+m}$ and $C[m]^n = C^{n-m}$ respectively, with differential $(-1)^m d$. The chain-complex $C[m]$ is referred to as the m^{th} translate of $C_\bullet$.

Definition 1.4.3. A chain-complex C is called *split* if there exist maps $s_n : C_n \to C_{n+1}$, such that $d_\bullet|_n = d_n s_n d_{n+1}$. The maps $\{s_n\}$ are called the *splitting maps.*

1.5 More on long exact sequences

The snake lemma is something as unusual as a pop-fiction lemma. Featured in at least two films, *The Graduate* from 1967 and *It's My Turn* from 1980[13].

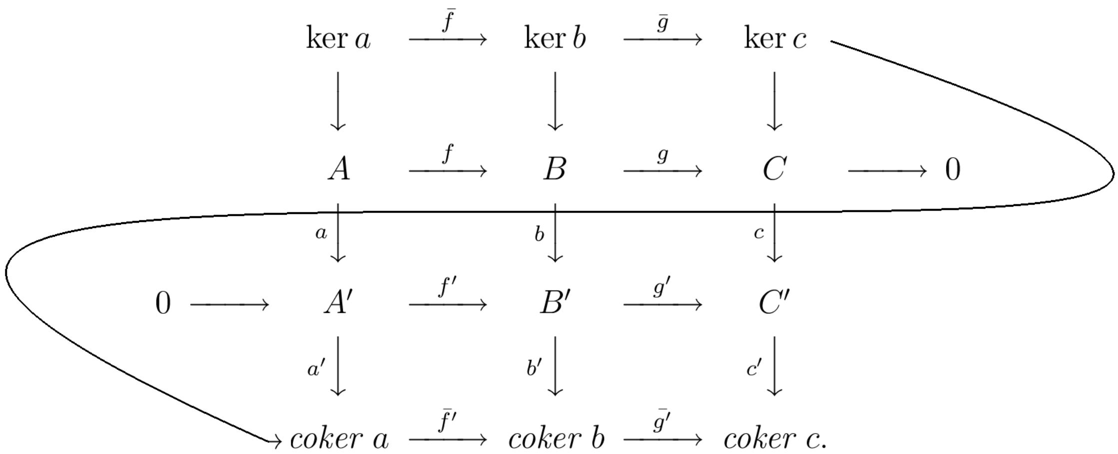
Lemma 1.5.1. The snake lemma.

Given a diagram of Λ*-modules and* Λ*-module homomorphisms,*

$$\begin{array}{ccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow b & & \downarrow c & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & & \end{array}$$

where the two rows are exact sequences, we can create the following exact sequence,

$$\ker a \xrightarrow{\bar{f}} \ker b \xrightarrow{\bar{g}} \ker c \xrightarrow{\delta} coker\ a \xrightarrow{\bar{f}'} coker\ b \xrightarrow{\bar{g}'} coker\ c,$$



Proof. Since $\bar{f}$ and $\bar{g}$ are defined as restricted versions of the original f and g and $\ker a \subseteq A, \ker b \subseteq B$, $\ker c \subseteq C$, it is clear that $\bar{g}\bar{f} = 0$, which implies that $\text{im } \bar{f} \subseteq \ker \bar{g}$.

We now want to show that the upper sequence is exact, which is clear if $\bar{f}$ is a monomorphism. Let $\beta \in \ker \bar{g}$. Then $\bar{g}(\beta) = g(\beta) = 0$, hence since $\text{im } f = \ker g$, there is a $\alpha \in A$ such that $\beta = f(\alpha)$.

Thereby we can conclude that,

$$(f'a)(\alpha) = (bf)(\alpha) = b(\beta) = 0,$$

since $\beta \in \ker \bar{g} \subseteq \ker b$. But we know that f' is a monomorphism, hence $a(\alpha) = 0$ and $\alpha \in \ker a$. Thereby we have established that $\bar{f}$ is a monomorphism and $\beta = \bar{f}(\alpha)$.

By the same type of *diagram chasing* argument one may show that the bottom sequence is exact too.

It remains to show that

$$\ker b \xrightarrow{\bar{g}} \ker c \xrightarrow{\delta} \text{coker } a \xrightarrow{\bar{f}'} \text{coker } b,$$

is exact.

We start by showing that $\text{im } \bar{g} \subseteq \ker \delta$. Suppose $v \in \text{im } \bar{g}$, that is for all $\beta \in \ker b$, $v = g(\beta)$. Now, since f' is a monomorphism $\beta' = b(\beta) = 0$ and hence $\delta(v) = 0$.

Next we will show that $\text{im } \bar{g} \supseteq \ker \delta$. Suppose $\delta(v) = 0$. As we know that $\text{im } f = \ker g$ and the fact that g is an epimorphism, we may choose β such that $v = g(\beta)$. It is clear that,

$$b(\beta) = (f'a)(\alpha) = (bf)(\alpha),$$

by casual diagram chasing. Now consider $\beta - f(\alpha)$. We have that $b(\beta - f(\alpha)) = 0$. Then $\beta - f(\alpha) \in \ker b$ and $g(\beta - f(\alpha)) = g(\beta) - (gf)(\alpha) = g(\beta) = v$, such that $v \in \text{im } \bar{g}$.

Now we will show im $\delta \subseteq \ker \bar{f}'$. Suppose $\bar{\alpha'} = \delta(v)$. Choose β and α' as in the previous construction of δ. Then $\bar{f}'(\bar{\alpha'}) = (a'b)(\beta) = 0$, since $a'b = 0$.

Finally we will show that im $\delta \supseteq \ker \bar{f}'$ Now suppose that $\bar{f}'(\bar{\alpha'}) = 0$. Choose α', such that $a'(\alpha') = \bar{\alpha'}$. Then $(b'f')(\alpha') = (\bar{f}'a')(\alpha') = 0$ which implies that $f'(\alpha') \in \ker b' = \text{im } b$. Therefore $f'(\alpha') = m(\beta)$. Let $v = g(\beta)$. It is clear that $\delta(v) = \bar{\alpha'}$ and we are done. □

Utilizing the snake lemma we will now be able to prove the following powerful theorem.

Proposition 1.5.2. *Let*

$$E_\bullet : 0 \longrightarrow A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \longrightarrow 0$$

be an exact sequence of chain-complexes, then there exist canonical homomorphisms called the connecting homomorphisms,

$$\delta_n(E_\bullet) : H_n(C_\bullet) \to H_{n-1}(A_\bullet),$$

such that the sequence,

$$\begin{array}{l} \ldots \longrightarrow H_{n+1}(C_\bullet) \xrightarrow{\delta_{n+1}} H_n(A_\bullet) \xrightarrow{\bar{f}_\bullet} \\ \xrightarrow{\bar{f}_\bullet} H_n(B_\bullet) \xrightarrow{\bar{g}_\bullet} H_n(C_\bullet) \xrightarrow{\delta n} H_{n-1}(A_\bullet) \longrightarrow \ldots, \end{array}$$

is exact.

Proof. If we expand the exact sequence $E_\bullet$ it is evident that the diagram,

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle d^A_{n+2}} & & \downarrow{\scriptstyle d^B_{n+2}} & & \downarrow{\scriptstyle d^C_{n+2}} & & \\
0 & \longrightarrow & A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d^A_{n+1}} & & \downarrow{\scriptstyle d^B_{n+1}} & & \downarrow{\scriptstyle d^C_{n+1}} & & \\
0 & \longrightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \longrightarrow & 0, \\
 & & \downarrow{\scriptstyle d^A_n} & & \downarrow{\scriptstyle d^B_n} & & \downarrow{\scriptstyle d^C_n} & & \\
0 & \longrightarrow & A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} & \xrightarrow{g_{n-1}} & C_{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d^A_{n-1}} & & \downarrow{\scriptstyle d^B_{n-1}} & & \downarrow{\scriptstyle d^C_{n-1}} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

commutes and have exact rows.

Consider the sequence

$$0 \longrightarrow Z_n(A_\bullet) \xrightarrow{\bar{f}} Z_n(B_\bullet) \xrightarrow{\bar{g}} Z_n(C_\bullet),$$

where $\bar{f}$ and $\bar{g}$ are given by f and g as in the snake lemma (Lemma 1.5.1), which also establishes this sequence's exactness. In the same lemma we induce the mappings $\bar{f}'$ and $\bar{g}'$. Consider these mappings and let $X_\bullet$ be either of $A_\bullet, B_\bullet$ or $C_\bullet$. Now let,

$$Z'_n(X_\bullet) = \text{coker } d^X_{n+1} = \frac{X_n}{\text{im } d^X_{n+1}} = \frac{X_n}{B_n}(X_\bullet),$$

then we may consider the following sequence,

$$Z'_n(A_\bullet) \xrightarrow{\bar{f}'} Z'_n(B_\bullet) \xrightarrow{\bar{g}'} Z'_n(C_\bullet),$$

which we know commutes and is exact, again by virtue of the snake lemma (Lemma 1.5.1).

We can connect these two sequences using a chain-complex map induced by the differentials. Consider $X_n \xrightarrow{d^X_n} X_{n-1}$ and recall that im $d^X_{n+1} = \ker d^X_n$, this induces a map $Z'_n(X_\bullet) \xrightarrow{\tilde{d^X_n}} Z_{n-1}(X_\bullet)$ for any chain-complex $X_\bullet$.

Using these maps, $\tilde{d^X_n}$, we get the following commutative diagram,

$$\begin{array}{ccccccc} & & Z'_n(A) & \xrightarrow{\bar{f}'} & Z'_n(B) & \xrightarrow{\bar{g}'} & Z'_n(C) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d^A_n} & & \downarrow{\scriptstyle d^B_n} & & \downarrow{\scriptstyle d^C_n} & & \\ 0 & \longrightarrow & Z_n(A) & \xrightarrow{\bar{f}} & Z_n(B) & \xrightarrow{\bar{g}} & Z_n(C). & & \end{array}$$

But

$$\ker \tilde{d^X_n} = \ker(X_n/\text{im } d^X_{n+1} \to X_{n-1}) = \tag{1.1}$$
$$= \ker(X_n \to X_{n-1})/\text{im } d^X_{n+1} = \tag{1.2}$$
$$= Z_n/B_n(X_\bullet) = H_n(X_\bullet) \tag{1.3}$$

and

$$\text{coker } \tilde{d^X_n} = Z_{n-1}(X_\bullet)/\text{im } (X_n/\text{im } d^X_{n+1} \to X_{n-1}) = \tag{1.4}$$
$$= Z_{n-1}(X_\bullet)/\text{im } (X_n \to X_{n-1}) = \tag{1.5}$$
$$= Z_{n-1}/B_{n-1}(X_\bullet) = H_{n-1}(X_\bullet), \tag{1.6}$$

so we get the sought-after long exact sequence,

$$\begin{aligned} \ldots \longrightarrow H_{n+1}(C_\bullet) &\xrightarrow{\delta_{n+1}} H_n(A_\bullet) \xrightarrow{\bar{f}_\bullet} H_n(B_\bullet) \xrightarrow{\bar{g}_\bullet} \\ \xrightarrow{\bar{g}_\bullet} H_n(C_\bullet) &\xrightarrow{\delta n} H_{n-1}(A_\bullet) \longrightarrow \ldots, \end{aligned}$$

once again by the snake lemma (Lemma 1.5.1), and we are done. □

1.6 Homotopy

In topology two functions are homotopic if they can be continuously transformed in to each other. To be more precise:

Definition 1.6.1. Let $f, g : A \to B$, where A, B is topological spaces. f and g is *homotopic* if we can define a continuous function $H : A \times [0,1] \to B$, such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in A$.

We can construct splitting maps from any chain-complex of vector spaces over a field. Let

$$C_n = Z_n \oplus B'_n, \qquad B'_n \simeq C_n/Z_n = d_\bullet(C_n) = B_{n-1},$$
$$Z_n = B_n \oplus H'_n, \qquad H'_n \simeq Z_n/B_n = H_n(C).$$

Thus

$$C_n \to Z_n \to B_n \simeq B'_{n+1} \subseteq C_{n+1}$$

and we got the splitting maps $s_n : C_n \to C_{n+1}$, where $d_\bullet = d_\bullet s d_\bullet$.

Definition 1.6.2. A chain map $f : C \to D$ is called *null homotopic* if there are maps $s_n : C_n \to D_{n+1}$, such that $f = d_\bullet s + s d_\bullet$. The $\{s_n\}$ maps of a null homotopic f are called a chain contraction of f.

To get a better understanding of null homotopic chain-complexes, we will study the following diagram, where $f = d_{n+1}s_n + s_{n-1}d_n$:

$$\begin{array}{ccccc} C_{n+1} & \xrightarrow{d_\bullet} & C_n & \xrightarrow{d_\bullet} & C_{n-1} \\ & \swarrow & \downarrow f & \swarrow & \\ D_{n+1} & \xrightarrow{d_\bullet} & D_n & \xrightarrow{d_\bullet} & D_{n-1} \end{array}$$

Now consider the composition df (once again dropping the subscripts), we get that

$$d_\bullet f = d_\bullet(d_\bullet s + s d_\bullet) = d_\bullet s d_\bullet = (d_\bullet s + s d_\bullet) d_\bullet = f d_\bullet.$$

Thus $f = d_\bullet s + sd_\bullet$ is a chain map from $C_\bullet \to D_\bullet$.

If $g : C_\bullet \to D_\bullet$ is any chain map, then so is the map $g + (d_\bullet s + sd_\bullet)$ for any choice of splitting maps. Note that g and $g + (d_\bullet s + sd_\bullet)$ are closely related.

Definition 1.6.3. Let $f, g : C_\bullet \to D_\bullet$ be two chain maps. We say that f and g are *chain homotopic* if their difference $f - g$ is null homotopic, i.e. that there exist splitting maps $\{s_n\}$ such that

$$f - g = d_\bullet s + sd_\bullet,$$

and the set of splitting maps $\{s_n\}$ is in this case called a *chain homotopy* from f to g.

Finally we construct an equivalence relation called *chain homotopy equivalence*. We say that $f : C_\bullet \to D_\bullet$ and $g : D_\bullet \to C_\bullet$ are equivalent if gf and fg are both chain homotopic to their respective identity maps.

Lemma 1.6.4. *If $f : C_\bullet \to D_\bullet$ is a null homotopic chain-complex mapping then every map $f' : H_n(C) \to H_n(D)$ is zero.*

Proof. Let $f = d_\bullet s + sd_\bullet$ and $x \in H_n(C)$, then x is also in $Z_n(C)$ and hence a n-cycle. That means that $f(x) = d_\bullet s(x) + sd_\bullet(x) = d_\bullet s(x)$, hence $f(x) \in B_n(D)$. Thus $f(x)$ is represented by 0 in $H_n(D)$. □

Corollary 1.6.5. *If $f, g : C \to D$ are chain homotopic, then they induce the same maps $H_n(C) \to H_n(D)$.*

Proof. The corollary follows directly from the previous lemma (Lemma 1.6.4). □

Chapter 2

Differential Graded Algebras

2.1 $\mathbb{Z}$-graded vector spaces

Definition 2.1.1. A $\mathbb{Z}$-graded vector space V is a vector space V together with a direct sum decomposition

$$V = \bigoplus_{i \in \mathbb{Z}} V^i,$$

elements $\varrho \in V^i$ are called homogeneous of degree i, which we denote by $|\varrho| := i$.

An example of an $\mathbb{Z}$-graded vector space is the set of all polynomials where every linear combination of monomials of degree n is a homogeneous elements of degree n.

Definition 2.1.2. Let $V = \bigoplus_{i \in \mathbb{Z}} V^i$ and $W = \bigoplus_{i \in \mathbb{Z}} W^i$ be $\mathbb{Z}$-graded vector spaces. A linear map $f : V \to W$ is called homogeneous of degree k if $\forall \varrho \in V^i, f(\varrho) \in W^{i+k}$.

The set of all homogeneous linear maps of degree k from V to W is itself a vector space, we denote it by $\mathrm{Hom}_k(V, W)$.

Definition 2.1.3. If $V = \bigoplus_{i \in \mathbb{Z}} V^i$ is a $\mathbb{Z}$-graded vector space, $\forall p \in \mathbb{N}$, then

$$V[p] := \bigoplus_{i \in \mathbb{Z}} V[p]^i,$$

is also a $\mathbb{Z}$-graded vector space with $V[p]^i := V^{i+p}$.

2.2 Lie algebras

Definition 2.2.1. A *Lie algebra* is a vector space V over a field $\mathcal{K}$ together with a binary operation

$$[\cdot,\cdot] : V \times V \to V$$

referred to as the *Lie bracket* and satisfying the following axioms:

- Bilinearity

$$[ax + by, z] = a[x, z] + b[y, z], [x, ay + bz] = a[x, y] + b[x, z],$$

  for all scalars $a, b \in \mathcal{K}$ and all elements $x, y, z \in V$.

- Anti-commutativity

$$[x, y] = -[y, x],$$

  for all elements $x, y \in V$.

- Jacobi-identity

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0,$$

  for all elements $x, y, z \in V$.

We can construct a Lie algebra from any associate algebra, A, by defining the Lie bracket as their commutator,

$$[a, b] = a \cdot b - b \cdot a,$$

and the resulting Lie algebra is then denoted $L(A)$. The associate algebra A is now referred to as the *enveloping algebra* of $L(A)$.

2.3 Differential Graded Algebra

Definition 2.3.1. A *differential graded algebra*, or briefly a *DG-algebra*, over a field $\mathcal{K}$ constitutes of a $\mathbb{Z}$-graded vector space, $A = \bigoplus_{i\in\mathbb{Z}} A^i$ together with a morphism,

$$\mu : A \otimes A \to A,$$

such that,

$$\mu \otimes (\mathrm{Id}_A \otimes \mu) = \mu \otimes (\mu \otimes \mathrm{Id}_A).$$

Definition 2.3.2. A *differential graded algebra morphism* from (A, μ_A) to (B, μ_B) is a morphism $\phi : A \to B$ such that,

$$\mu_B(\phi \otimes \phi) = \phi\mu_A.$$

An example of a DG-algebra is the *tensor algebra*, that we will study in details in the next chapter.

2.4 Differential Graded Coalgebra

Definition 2.4.1. A *differential graded coalgebra*, or briefly *DG-coalgebra*, over a field $\mathcal{K}$ constitutes of a $\mathbb{Z}$-graded vector space, $A = \bigoplus_{i\in\mathbb{Z}} A^i$, together with a morphism,

$$\Delta : A \to A \otimes A,$$

such that,

$$(\Delta \otimes \mathrm{Id}_A) \otimes \Delta = (\mathrm{Id}_A \otimes \Delta) \otimes \Delta.$$

Definition 2.4.2. A *differential graded coalgebra morphism* from (A, Δ_A) to (B, Δ_B) is a morphism $\phi : A \to B$ such that,

$$(\phi \otimes \phi)\Delta_B = \Delta_A \phi.$$

2.5 Differential Graded Lie Algebra

Definition 2.5.1. A *differential graded Lie algebra*, or briefly *DGL-algebra*, over a field $\mathcal{K}$, consists of a $\mathbb{Z}$-graded vector space, $A = \bigoplus_{i\in\mathbb{Z}} A^i$, together with a morphism,

$$[\cdot,\cdot] : A \otimes A \to A,$$

such that it fulfills the bilinearity, anti-commutativity and Jacobi-identity.

Definition 2.5.2. A *differential graded Lie algebra morphism* from $(A, [\cdot,\cdot]_A)$ to $(B, [\cdot,\cdot]_B)$ is a morphism $\phi : A \to B$ such that,

$$[\cdot,\cdot]_B(\phi \otimes \phi) = \phi[\cdot,\cdot]_A.$$

Chapter 3

Tensor Algebras and Coalgebras

3.1 Tensor Product

Definition 3.1.1. Let $L_1, \ldots, L_p$ be $\mathbb{Z}$-graded vector spaces over the field $\mathcal{K}$. A vector space $\mathcal{M}$ is defined as the set of all functions $f : L_1 \times \cdots \times L_p \to \mathcal{K}$ such that $f(l) = 0$ for all but finitely many $l \in L_1 \times \cdots \times L_p$. The $\mathcal{M}$-space is clearly a linear space over $\mathcal{K}$ under the ordinary operations of point-wise addition and the multiplication of a scalar.

A basis for the space $\mathcal{M}$ consists of the so-called Kronecker delta-functions, δ_l, defined as $\delta_l(k) = 1$ if $k = l$ and $\delta_l(k) = 0$ if $k \neq l$. There is an obvious one-to-one correspondence between the delta-functions and the elements in $L_1 \times \cdots \times L_p$ and we will therefore omit the symbol δ to simplify notations. Now all elements of $\mathcal{M}$ can be written as finite linear combinations of $(l_1, \ldots, l_p) \in L_1 \times \cdots \times L_p$ and the space as the set,

$$\mathcal{M} = \left\{ \sum a_{l_1 \cdots l_p}(l_1, \ldots, l_p) | a_{l_1 \cdots l_p} \in \mathcal{K} \right\}.$$

Definition 3.1.2. Let $\mathcal{M}_0$ be a subspace of $\mathcal{M}$ generated by the vectors on the form,

$$(l_1, \ldots, l'_j + l''_j, \ldots, l_p) - (l_1, \ldots, l'_j, \ldots, l_p) - (l_1, \ldots, l''_j, \ldots, l_p)$$

and

$$(l_1, \ldots, al_j, \ldots, l_p) - a(l_1, \ldots, l_j, \ldots, l_p), \text{ where } a \in \mathcal{K} \text{ and } 1 \leq j \leq p.$$

Definition 3.1.3. We define

$$L_1 \otimes \cdots \otimes L_p = \mathcal{M}/\mathcal{M}_0.$$

Now since the right hand side is a quotient space there exist a natural projection

$$\pi : \mathcal{M} \to \mathcal{M}/\mathcal{M}_0 = L_1 \otimes \cdots \otimes L_p$$

and we may therefore denote

$$\pi(l_1, \ldots, l_p) = l_1 \otimes \cdots \otimes l_p.$$

It is clear from the fact that $(l_1, \ldots, l_p)$ is a basis of $\mathcal{M}$ that the tensors $l_1 \otimes \cdots \otimes l_p$ form a basis of $L_1 \otimes \cdots \otimes L_p$.

We will now study some of the principal properties of the tensor product mapping.

Theorem 3.1.4. *Let*

$$t : L_1 \times \cdots \times L_p \to L_1 \otimes \cdots \otimes L_p$$

a) The mapping t is multilinear, that is linear in each variable, i.e. for any constant c,

$$t(l_1, \ldots, c(l'_k) + l''_k, \ldots, l_n) = ct(l_1, \ldots, l'_k, \ldots, l_n) + t(l_1, \ldots, l''_k, \ldots, l_n).$$

b) The mapping t is universal.
By universal we mean that for any linear space M over the field $\mathcal{K}$ and any multilinear mapping $s : L_1 \times \cdots \times L_p \to M$ it exist an unique linear mapping $f : L_1 \otimes \cdots \otimes L_p \to M$ such that $s = f \circ t$.

Proof. The proof of part a) is fairly trivial as the criteria for multilinearity,

$$l_1 \otimes \cdots \otimes (l_k + l''_k) \otimes \cdots \otimes l_n = \\ = l_1 \otimes \cdots \otimes l'_k \otimes \cdots \otimes l_n + l_1 \otimes \cdots \otimes l''_k \otimes \cdots \otimes l_n,$$

and

$$l_1 \otimes \cdots \otimes al_k \otimes \cdots \otimes l_n = a(l_1 \otimes \cdots \otimes l_n),$$

correspond exactly to the definition of the denominator of the quotient space $\mathcal{M}/\mathcal{M}_0$.

To show part b), the universality, we consider a linear mapping,

$$g : \mathcal{M} \to M,$$

determined by the following mapping for the base elements,

$$g(l_1, \ldots, l_p) = s(l_1, \ldots, l_p),$$

that is in general,

$$g(\sum a_{l_1 \cdots l_p}(l_1, \ldots, l_p)) = \sum a_{l_1 \cdots l_p} s(l_1, \ldots, l_p).$$

Now consider what the multilinear nature of s implies for g. That is

$$\begin{aligned} g((l_1, \ldots, l'_k + l''_k, \ldots, l_p) - (l_1, \ldots, l'_j, \ldots, l_p) - (l_1, \ldots, l''_j, \ldots, l_p)) &= \\ s(l_1, \ldots, l'_k + l''_k, \ldots, l_p) - s(l_1, \ldots, l'_j, \ldots, l_p) - s(l_1, \ldots, l''_j, \ldots, l_p) &= 0, \end{aligned}$$

and

$$g((l_1, \ldots, al_k, \ldots, l_p) - a(l_1, \ldots, l_p)) = s(l_1, \ldots, al_k, \ldots, l_p) - as(l_1, \ldots, l_p) = 0,$$

which makes clear that $\mathcal{M}_0 \subset \ker g$, now this clearly induces a function

$$f : \mathcal{M}/\mathcal{M}_0 = L_1 \otimes \cdots \otimes L_p \to M,$$

such as

$$f(l_1 \otimes \cdots \otimes l_p) = s(l_1, \ldots, l_p),$$

which settles the proof if f is unique. However we know that t generates $L_1 \otimes \cdots \otimes L_p$, that implies that f is uniquely determined as $s = f \circ t$. □

Theorem 3.1.5. *The tensor product over a field $\mathcal{K}$ is associative.*

Proof. Let L_i be a $\mathbb{Z}$-graded vector space over $\mathcal{K}$ and $l_i \in L_i$. To prove the tensor product's associativity we construct mappings using the universal property of the tensor product,

$$\Psi_i : L_1 \otimes \cdots \otimes L_p \to (L_1 \otimes L_2) \cdots \otimes L_p),$$

where the domain has an arbitrary distribution of parenthesis. We can conclude using an inductive argument that the mappings Ψ_i are isomorphisms, since it is clear that,

$$l_1 \otimes l_2 \otimes l_3 \mapsto (l_1 \otimes l_2) \otimes l_3,$$

is a simple transformation. □

If the tensor product is a new acquaintance for you and you need further assistance to grasp the concept you might find Timothy Gowers comforting guide[1] helpful.

Corollary 3.1.6. *There exist a canonical isomorphism between $Hom(L_1 \times \cdots \times L_k, M)$ and $Hom(L_1 \otimes \cdots \otimes L_k, M)$.*

Proof. In the proof of Theorem 3.1.4, multilinear maps $f : L_1 \times \cdots \times L_k \to M$ was split into $s = f \circ t$. The mapping is surjective since for any linear mapping f the mapping s is multilinear, it is injective since $s \neq 0$ implies that $f \circ t \neq 0$ and thus $f \neq 0$, and hence it is therefore an isomorphisms. □

Definition 3.1.7. The tensor product for two chain-complexes $C = (C_\bullet, d_C)$ and $D = (D_\bullet, d_D)$ is given by

$$(C \otimes D)_n = \bigotimes_{i+j=n} C_i \otimes D_j,$$

with the differential

$$d_{C \otimes D}(x \otimes y) = d_C(x) \otimes y + (-1)^i x \otimes d_D(y),$$

for all $x \in C_i$ and $y \in D_j$.

Theorem 3.1.8. *Let L_i be finite dimensional linear spaces, then there is a canonical isomorphism between $L_1^* \otimes \cdots \otimes L_k^*$ and $(L_1 \otimes \cdots \otimes L_k)^*$.*

Proof. Every element $(f_1, \ldots, f_k) \in L_1^* \times \cdots \times L_k^*$ can be used to construct a multilinear function from $L_1 \times \cdots \times L_k$ to the base field $\mathcal{K}$, by multiplying $f_1(l_1) \cdots f_k(l_k) : L_1^* \times \cdots \times L_k^*$. By Theorem 3.1.4 we know that it may be constructed through a function from $L_1^* \otimes \cdots \times L_k^*$.

Now, let this function be called F, then

$$F : L_1^* \times \cdots \times L_k^* \to \mathrm{Hom}(L_1 \times \cdots \times L_k, \mathcal{K}),$$

and by Theorem 3.1.4 there is maps g, t such that $F = g \circ t$, where

$$t : L_1^* \otimes \cdots \otimes L_k^* \to \mathrm{Hom}(L_1 \times \cdots \times L_k, \mathcal{K}).$$

By Theorem 3.1.6 we know that $\mathrm{Hom}(L_1 \times \cdots \times L_k, \mathcal{K})$ is isomorphic to $\mathrm{Hom}(L_1 \otimes \cdots \otimes L_k, \mathcal{K}) = (L_1 \otimes \cdots \otimes L_k)^*$.

We have now constructed a mapping between $L_1^* \otimes \cdots \otimes L_k^*$ and $(L_1 \otimes \cdots \otimes L_k)^*$, it remains to show that it is an isomorphism.

It is clear that $dim(L_1^* \otimes \cdots \otimes L_k^*) = dim((L_1 \otimes \cdots \otimes L_k)^*)$, and hence it is enough to show that the mapping is surjective.

The surjectiveness follows as the when F runs through a basis of $f_i \in L_i^*$, the image thanks to the multilinear nature of the function is also going to run through the basis of $\mathrm{Hom}(L_1 \times \cdots \times L_k, \mathcal{K})$, which again is isomorphic to $(L_1 \otimes \cdots \otimes L_k)^*$, by Theorem 3.1.6. □

3.2 Tensor Algebra

Definition 3.2.1. An *associative algebraic structure*, or *associative algebra*, on a $\mathbb{Z}$-graded vector space $A = \bigoplus_{i\in\mathbb{Z}} A^i$ is formed by associating a map μ of degree 0,

$$\mu : A \otimes A \to A,$$

such that the following diagram

$$\begin{array}{ccc} A\otimes A\otimes A & \xrightarrow{\mathrm{Id}\otimes\mu} & A\otimes A \\ {\scriptstyle \mu\otimes\mathrm{Id}}\downarrow & & \downarrow{\scriptstyle \mu} \\ A\otimes A & \xrightarrow{\mu} & A \end{array}$$

commutes. This means that for all homogeneous $a, b, c \in A$

$$|\mu(a, b)| = |a| + |b|$$

and

$$\mu(a, (\mu(b, c))) = \mu(\mu(a, b), c).$$

Definition 3.2.2. Let $A = \bigoplus_{i\in\mathbb{Z}} A^i$ be a $\mathbb{Z}$-graded associative algebra. A *derivation* of A of degree k is a linear map

$$D : A \to A$$

of degree k such that for all homogeneous $a, b \in A$,

$$D(a \otimes b) = D(a) \otimes b + (-1)^{k|a|} a \otimes D(b).$$

Definition 3.2.3. A differential D in a $\mathbb{Z}$-graded associative algebra A is a derivation of degree 1 such that

$$D^2 = 0$$

To simplify the equations we introduce the following notation,

$$L^{\otimes q} := \underbrace{L \otimes \cdots \otimes L}_{q \text{ times}}.$$

Definition 3.2.4. Let $L = \bigoplus_{i\in\mathbb{Z}} L^i$ be a $\mathbb{Z}$-graded vector space and q an integer. We call the space

$$A := TL = \bigoplus_{q\geq 0} L^{\otimes q}$$

together with the function

$$\mu : A \otimes A \to A$$

given by

$$\mu(a_1 \otimes \cdots \otimes a_p, b_1 \otimes \cdots \otimes b_q) = a_1 \otimes \cdots \otimes a_p \otimes b_1 \otimes \cdots \otimes b_q$$

the *tensor algebra* of L. Note that μ is the ordinary tensor multiplication.

Lemma 3.2.5. *The tensor algebra of a vector space L, (TL, μ), is an associative algebra.*

Proof. It is clear from Theorem 3.1.4 that μ is multilinear or more specifically bilinear.

Let a, b, c be elements of A. It is easy to see that μ fulfills the commutative diagram in Definition 3.2.1,

$$\begin{aligned}
\mu(a, \mu(b, c)) &= \mu(a_1 \otimes \cdots \otimes a_l, \mu(b_1 \otimes \cdots \otimes b_m, c_1 \otimes \cdots \otimes c_n)) = \\
&= a_1 \otimes \cdots \otimes a_l \otimes b_1 \otimes \cdots \otimes b_m \otimes c_1 \otimes \cdots \otimes c_n = \\
&= \mu(\mu(a_1 \otimes \cdots \otimes a_l, b_1 \otimes \cdots \otimes b_m), c_1 \otimes \cdots \otimes c_n) \\
&= \mu(\mu(a, b), c).
\end{aligned}$$

We conclude that A together with μ is an associative algebra. □

Theorem 3.2.6. *There is a one-to-one correspondence between degree k derivations of TL and degree k linear maps $L \to TL \in Hom_k(L, TL)$*

Proof. Let $a_i = l_1 \otimes \cdots \otimes l_p \in TL$, and let $D : TL \to TL$ be a derivation of degree k. It is clear from the definition of the derivation D that

$$D(a_1 \otimes a_2) = D(a_1) \otimes a_2 + (-1)^{k|a_1|} a_1 \otimes D(a_2).$$

Using this in an induction argument, we can prove that for all $a \in TL$, $D(a)$ is determined by a linear combination of $D(a_i)$ and a_j, where $D(a_i), a_j \in TL$.

Construct a function $f : L \to TL$ by $f := D|_L$. The restriction of a homogeneous function of degree k is again a homogeneous function of degree k, hence f is a homogeneous function of degree k and therefore an element of $\mathrm{Hom}_k(L, TL)$. Further all elements in $\mathrm{Hom}_k(L, TL)$ can be described as $D|_L$ since the restricted function is an arbitrary linear map from which the non-restricted D may be unfolded. □

Definition 3.2.7. Let
$$\bar{T}L = \bigoplus_{k\geq 1} L^{\otimes k}$$
together with μ from Definition 3.2.4 be an algebra. This algebra is called the *bar tensor algebra* of L. The algebra is an associative by the same arguments as in Lemma 3.2.5.

3.3 Tensor Coalgebra

Definition 3.3.1. A *coassociative coalgebraic structure*, or *coassociative coalgebra*, is a $\mathbb{Z}$-graded vector space, $A = \bigoplus_{i\in\mathbb{Z}} A_i$, together with a linear map of degree 0,
$$\Delta : A \to A \otimes A$$
such that the diagram
$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A\otimes A \\ \Delta\big\downarrow & & \big\downarrow \mathrm{Id}\otimes\Delta \\ A\otimes A & \xrightarrow{\Delta\otimes\mathrm{Id}} & A\otimes A\otimes A \end{array}$$
commutes. This means that for any $a \in A$, $\Delta a = \sum_i a'_i \otimes a''_i \in A \otimes A$ and $a'_i, a''_i \in A$ and
$$\sum_i \Delta(a'_i) \otimes a''_i = \sum_i a'_i \otimes \Delta(a''_i).$$

Definition 3.3.2. Let $A = \bigoplus_{i\in\mathbb{Z}} A^i$ be a $\mathbb{Z}$-graded coalgebra. A coderivation on A of degree k is a linear map
$$D : A \to A$$
of degree k such that for all $a \in A$
$$\Delta \circ D(a) = (\mathrm{Id} \otimes D + D \otimes \mathrm{Id})\Delta a$$
i.e. if $\Delta a = \sum_i a'_i \otimes a''_i$ then
$$\Delta D(a) = \sum_i D(a'_i) \otimes a''_i + (-1)^{k|a'_i|} \sum_i a'_i \otimes D(a''_i).$$
We denote the set of coderivations on A with $CoDer(A)$.

Definition 3.3.3. A codifferential D in a $\mathbb{Z}$-graded associative algebra A is a derivation of degree 1 such that

$$D^2 = 0$$

Definition 3.3.4. Let L be a $\mathbb{Z}$-graded vector space over a field $\mathcal{K}$, then

$$A := T^c L = \mathcal{K} \oplus L \oplus L^{\otimes 2} \oplus L^{\otimes 3} \oplus \cdots$$

is a coalgebra together with the linear map of degree 0

$$\Delta : A \to A \otimes A$$

defined by

$$\begin{aligned} \Delta(a_1 \otimes \cdots \otimes a_p) &:= 1 \otimes (a_1 \otimes \cdots \otimes a_p) + (a_1 \otimes \cdots \otimes a_p) \otimes 1 \\ &+ \sum_{i=1}^{p-1} (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_p), \end{aligned}$$

where $p \geq 1$ and $\Delta(1) = 1 \otimes 1$. This coalgebra is referred to as the *tensor coalgebra.*

Lemma 3.3.5. *The tensor coalgebra of a vector space L, $(T^c L, \Delta)$, is a coassociative coalgebra.*

Proof. Let $a = (a_1 \otimes \cdots \otimes a_p) \in A$. It is clear from Theorem 3.3.4 and Theorem 3.1.4 that Δ is multilinear. Thus it only remains to show associativity, which in turn is clear by,

$$\begin{aligned} (\Delta \otimes \mathrm{Id}) \circ \Delta a &= \sum_{i=1}^{p-1} \Big(\sum_{j=1}^{j=i} (a_1 \otimes \cdots \otimes a_{j-1}) \otimes (a_j \otimes \cdots \otimes a_i)\Big) \otimes (a_{i+1} \otimes \cdots \otimes a_p) \\ &= \underbrace{a_1 \otimes \cdots \otimes a_p + \ldots + a_1 \otimes \cdots \otimes a_p}_{p \text{ terms}} \\ &= \sum_{i=1}^{p-1} (a_{i+1} \otimes \cdots \otimes a_p) \otimes \Big(\sum_{j=1}^{j=i} (a_1 \otimes \cdots \otimes a_{j-1}) \otimes (a_j \otimes \cdots \otimes a_i)\Big), \\ &= (\mathrm{Id} \otimes \Delta) \circ \Delta a, \end{aligned}$$

and Theorem 3.1.5, that the tensor coalgebra is a coassociative coalgebra. □

Theorem 3.3.6. *If $A = T^c L$, then as vector spaces $CoDer(A) = Hom(T^c L, L)$.*

Proof. Let $n \geq 0$. Given a homogeneous map $\varrho_n|_{L^{\otimes n}} : L^{\otimes n} \to L$, which can be considered as a map $\varrho_n : T^cL \to L \in \mathrm{Hom}(T^cL, L)$, by letting the function vanish at all points not in $L^{\otimes n}$. Now let

$$\tilde{\varrho}_n(l_1 \otimes \cdots \otimes l_k) := 0,$$

for $k < n$, and

$$\begin{aligned} \tilde{\varrho}_n(l_1 \otimes \cdots \otimes l_k) \;\; := \;\; & \sum_{i=0}^{k-n} (-1)^{|\varrho_n|(|l_1|+\ldots+|l_i|)} (l_1 \otimes \cdots \otimes l_i \otimes \varrho_n(l_{i+1} \otimes \cdots \otimes l_{i+n}) \\ & \otimes l_{i+n+1} \otimes \cdots \otimes l_k), \end{aligned}$$

for $k \geq n$. Let $\tilde{\varrho}_n^j : T^cL \to L^{\otimes j}$ be the j:th output-component of $\tilde{\varrho}_n$. To show that $\tilde{\varrho}_n$ is a uniquely determined coderivation we study

$$\begin{aligned} \Delta(\tilde{\varrho}_n(l_1 \otimes \cdots \otimes l_k)) &= (\tilde{\varrho}_n \otimes \mathrm{Id} + \mathrm{Id} \otimes \tilde{\varrho}_n)(\Delta(l_1 \otimes \cdots \otimes l_k)) = \\ &= (\tilde{\varrho}_n \otimes \mathrm{Id} + \mathrm{Id} \otimes \tilde{\varrho}_n)(\sum_{i=0}^{k} (l_1 \otimes \cdots \otimes l_i) \otimes \\ &\qquad \otimes (l_{i+1} \otimes \cdots \otimes l_k)) = \\ &= \sum_{i=0}^{k} \tilde{\varrho}_n(l_1 \otimes \cdots \otimes l_i) \otimes (l_{i+1} \otimes \cdots \otimes l_k) + \\ &\quad + (-1)^{|\tilde{\varrho}_n|(|l_1|+\ldots+|l_i|)} (l_1 \otimes \cdots \otimes l_i) \otimes \tilde{\varrho}_n(l_{i+1} \otimes \cdots \otimes l_k), \end{aligned}$$

and project both sides to $\bigoplus_{i+j=m} L^{\otimes i} \otimes L^{\otimes j} \subset T^cL \otimes T^cL$ which yields

$$\begin{aligned} \Delta(\tilde{\varrho}_n^m(l_1 \otimes \cdots \otimes l_k)) \;\; = \;\; & \sum_{i=0}^{k} \tilde{\varrho}_n^{m+i-k}(l_1 \otimes \cdots \otimes l_i) \otimes (l_{i+1} \otimes \cdots \otimes l_k) \\ & + (-1)^{|\tilde{\varrho}_n|(|l_1|+\ldots+|l_i|)} (l_1 \otimes \cdots \otimes l_i) \otimes \varrho_n^{m-i}(l_{i+1} \otimes \cdots \otimes l_k). \end{aligned}$$

This shows that the the right hand side depends solely on $\tilde{\varrho}_n^j$ if $j < m$, and falls back on the trivial case if $j = m$. An induction argument now shows that $\tilde{\varrho}_n^m$ vanishes for elements in $V^{\otimes k}$ where $k \neq m + n - 1$. For elements in $V^{\otimes m+n-1}$,

$$\begin{aligned} \tilde{\varrho}_n^m(l_1 \otimes \cdots \otimes l_k) = \;\; & \sum_{i=0}^{m-1} (-1)^{|\varrho_n|(|l_1|+\ldots+|l_i|)} (\;\; l_1 \otimes \cdots \otimes l_i \otimes \varrho_n(l_{i+1} \otimes \cdots \otimes l_{i+n}) \\ & \qquad \otimes l_{i+n+1} \otimes \cdots \otimes l_{m+n-1}), \end{aligned}$$

hence $\tilde{\varrho}_n|_{L^{\otimes k}} : L^{\otimes k} \to L^{\otimes k-n+1}$.

Noting that coderivation is a linear property it is clear that the sum of coderivations is again a coderivation. Thus the map

$$\alpha : \{\{\varrho_i : L^{\otimes i} \to L\}_{i\geq 0}\} \to CoDer(T^c L), \quad \{\varrho_i : L^{\otimes i} \to L\}_{i\geq 0} \mapsto \sum_{i\geq 0} \tilde{\varrho}_i$$

is well defined. Let the mapping β be given by

$$\sum_{i\geq 0} \tilde{\varrho}_i \mapsto \{\{pr_L \circ D|_{L^{\otimes i}}\}_{i\geq 0}\} = \{\varrho_i : L^{\otimes i} \to L\}_{i\geq 0},$$

then $\beta \circ \alpha = \mathrm{Id}$. The uniqueness of the mapping between ϱ and $\tilde{\varrho}$ gives that $\alpha \circ \beta = \mathrm{Id}$ and hence implies the isomorphism. □

Definition 3.3.7. Let

$$\bar{T}^c L = \bigoplus_{k\geq 1} L^{\otimes k}$$

together with the linear map of degree 0

$$\Delta : \bar{T}^c L \to \bar{T}^c L$$

given by

$$\Delta(a_1 \otimes \cdots \otimes a_n) = \begin{cases} \sum_{i=1}^{p-1} (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_p) & , n \geq 2 \\ 0 & , n = 1 \end{cases}$$

be a coalgebra. This algebra is usually referred to as the *bar symmetric coalgebra*.

Lemma 3.3.8. *$\bar{T}^c L$ is a coassociative algebra.*

Proof. Let $a \in L \subset \bar{T}^c L$. It is easy to confirm coassociativity for such a since,

$$((\Delta \otimes \mathrm{Id}) \circ \Delta)a = (\Delta \otimes \mathrm{Id})0 = 0 \otimes 0 = (\mathrm{Id} \otimes \Delta)0 = ((\mathrm{Id} \otimes \Delta) \circ \Delta)a.$$

For $a \in \bar{T}^c L \setminus L$, the commutativity and hence the theorem follows from the same argument as in Theorem 3.3.5. □

Theorem 3.3.9. *If $A = \bar{T}^c L$ then as vector spaces $CoDer(A) = Hom(\bar{T}^c L, L)$.*

Proof. This proof is almost identical to Theorem 3.3.6, just modified to compensate for the difference between $T^c L$ and $\bar{T}^c L$. Let $n \geq 1$. Given a homogeneous map $\varrho_n|_{L^{\otimes n}} : L^{\otimes n} \to L$, we may construct a map, $\varrho_n : \bar{T}^c L \to L \in$

$\mathrm{Hom}(\bar{T}^c L, L)$, by letting the function vanish for all points not in $L^{\otimes n}$. Now let

$$\tilde{\varrho}_n(l_1 \otimes \cdots \otimes l_k) := 0,$$

for $k < n$, and

$$\begin{aligned}\tilde{\varrho}_n(l_1 \otimes \cdots \otimes l_k) \; &:= \; \sum_{i=0}^{k-n} (-1)^{|\varrho_n|(|l_1|+\ldots+|l_i|)} (l_1 \otimes \cdots \otimes l_i \otimes \varrho_n(l_{i+1} \otimes \cdots \otimes l_{i+n}) \\ &\qquad \otimes l_{i+n+1} \otimes \cdots \otimes l_k),\end{aligned}$$

for $k \geq n$. Let $\tilde{\varrho}_n^j : \bar{T}^c L \to L^{\otimes j}$ be the j:th output-component of $\tilde{\varrho}_n$. To show that $\tilde{\varrho}_n$ is a uniquely determined coderivation we study,

$$\begin{aligned}\Delta(\tilde{\varrho}_n(l_1 \otimes \cdots \otimes l_k)) &= (\tilde{\varrho}_n \otimes \mathrm{Id} + \mathrm{Id} \otimes \tilde{\varrho}_n)(\Delta(l_1 \otimes \cdots \otimes l_k)) = \\ &= (\tilde{\varrho}_n \otimes \mathrm{Id} + \mathrm{Id} \otimes \tilde{\varrho}_n)(\sum_{i=1}^{k-1}(l_1 \otimes \cdots \otimes l_i) \otimes \\ &\qquad \otimes (l_{i+1} \otimes \cdots \otimes l_k)) = \\ &= \sum_{i=1}^{k-1} \tilde{\varrho}_n(l_1 \otimes \cdots \otimes l_i) \otimes (l_{i+1} \otimes \cdots \otimes l_k) + \\ &\quad + (-1)^{|\tilde{\varrho}_n|(|l_1|+\ldots+|l_i|)}(l_1 \otimes \cdots \otimes l_i) \otimes \tilde{\varrho}_n(l_{i+1} \otimes \cdots \otimes l_k),\end{aligned}$$

and project both sides to $\bigoplus_{i+j=m} L^{\otimes i} \otimes L^{\otimes j} \subset \bar{T}^c L \otimes \bar{T}^c L$ which yields

$$\begin{aligned}\Delta(\tilde{\varrho}_n^m(l_1 \otimes \cdots \otimes l_k)) &= \sum_{i=1}^{k-1} \tilde{\varrho}_n^{m+i-k}(l_1 \otimes \cdots \otimes l_i) \otimes (l_{i+1} \otimes \cdots \otimes l_k) \\ &\quad + (-1)^{|\tilde{\varrho}_n|(|l_1|+\ldots+|l_i|)}(l_1 \otimes \cdots \otimes l_i) \otimes \varrho_n^{m-i}(l_{i+1} \otimes \cdots \otimes l_k)\end{aligned}$$

this shows that the right hand side depends solely on $\tilde{\varrho}_n^j$ if $j < m$, and if $j = m$ it is trivial. An induction argument now shows us that $\tilde{\varrho}_n^m$ vanishes for elements in $V^{\otimes k}$ where $k \neq m + n - 1$. For elements in $V^{\otimes m+n-1}$,

$$\begin{aligned}\tilde{\varrho}_n^m(l_1 \otimes \cdots \otimes l_k) &= \sum_{i=0}^{m-1} (-1)^{|\varrho_n|(|l_1|+\ldots+|l_i|)}(l_1 \otimes \cdots \otimes l_i \\ &\qquad \otimes \varrho_n(l_{i+1} \otimes \cdots \otimes l_{i+n}) \otimes l_{i+n+1} \otimes \cdots \otimes l_{m+n-1}),\end{aligned}$$

hence $\tilde{\varrho}_n|_{L^{\otimes k}} : L^{\otimes k} \to L^{\otimes k-n+1}$.

And once again by noting that coderivation is a linear property it is clear that the sum of coderivations is again a coderivation. Thus the map

$$\alpha : \{\{\varrho_i : L^{\otimes i} \to L\}_{i \geq 1}\} \to CoDer(\bar{T}^c L), \quad \{\varrho_i : L^{\otimes i} \to L\}_{i \geq 1} \mapsto \sum_{i \geq 1} \tilde{\varrho}_i$$

is well defined. The mapping β given by

$$\sum_{i\geq 1} \tilde{\varrho}_i \mapsto \{\{pr_L \circ D|_{L^{\otimes i}}\}_{i\geq 1}\} = \{\varrho_i : L^{\otimes i} \to L\}_{i\geq 1}$$

fulfills $\beta \circ \alpha = \mathrm{Id}$. The uniqueness of the mapping between ϱ and $\tilde{\varrho}$ gives that $\alpha \circ \beta = \mathrm{Id}$, hence the mapping is an isomorphism

$\square$

Chapter 4

Symmetric Tensor Algebras & Grassmann Algebras

4.1 Symmetric Tensor Algebra

Definition 4.1.1. Let S_n be the set of permutation mappings of the numbers 1 to n, and let $Sh(p, q)$ denote the set of (p, q)-shuffles, that is, those permutations $\sigma \in S_{p+q}$ that fulfills $\sigma(1) < \sigma(2) < \ldots < \sigma(p)$ and $\sigma(p+1) < \sigma(p+2) < \ldots < \sigma(p+q)$.

Definition 4.1.2. Let $l = (l_1, \ldots, l_q)$ be an element in the tensor algebra of the $\mathbb{Z}$-graded vector space L and let

$$f_\sigma(l) = f_\sigma(l_1 \otimes \cdots \otimes l_q) = l_{\sigma(1)} \otimes \cdots \otimes l_{\sigma(q)}.$$

We will call a tensor $l \in L^{\otimes q}$ *symmetric* if $f_\sigma(l) = l$ for any permutation $\sigma \in S_q$. For convenience we will also include all scalars in the category of symmetric tensors.

We will denote the subspace of symmetric tensors in $L^{\otimes q}$ with $L^{\odot q}$.

Proposition 4.1.3. *Let $\sigma \in S_q$, and let L be a linear space, then the mapping*

$$S : TL \to TL$$

given by

$$S(L^{\otimes q}) = \frac{1}{q!} \sum_{\sigma \in S_q} f_\sigma : L^{\otimes q} \to L^{\otimes q}.$$

has the following properties $S^2 = S$ and im $S = L^{\odot q}$. S is referred to as the symmetrisation mapping*.*

Proof. It is obvious that im $S \subset L^{\odot q}$. It is also clear that the symmetrisation works as an identity map for all symmetric elements, $s \in L^{\odot q}$, that is $s = S(s)$. and hence im $S = L^{\odot q}$ and $S^2 = S$. $\square$

Remark 4.1.4. Note that the symmetrisation using S of the basis $e_1, \ldots, e_n$ of $L^{\otimes q}$ generates $L^{\odot q}$.

We introduce the following notation

$$S(e_{i_1} \otimes \cdots \otimes e_{i_q}) = e_{i_1} \odot \cdots \odot e_{i_q}.$$

As the product does not change under any permutation of the indices, $\{i_j\}$ we may in general simplify the notation further by

$$l_1^{a_1} \odot \cdots \odot l_n^{a_n},$$

where $a_i \geq 0, a_1 + \cdots + a_n = q$ denotes how many times the vector l_i is a factor in $l_{i_1} \otimes \cdots \otimes l_{i_q}$.

Definition 4.1.5. Let L be a $\mathbb{Z}$-graded vector space, then

$$SL = \bigoplus_{q \geq 0} L^{\odot q}$$

together with

$$\mu : SL \times SL \rightarrow SL$$

given by

$$\mu(A_1, A_2) = S(A_1 \otimes A_2),$$

where S is the symmetrisation mapping, is called the *symmetric algebra* of the $\mathbb{Z}$-graded vector space L. We will hereinafter denote the operator μ with $\odot$.

Lemma 4.1.6. *The symmetric algebra is an associative algebra.*

Proof. Since the tensor product is multilinear our operator μ clearly is bilinear. It remains to show associativity, let $a, b, c \in SL$, then

$$\begin{aligned}
\mu(\mu(a,b),c) &= \mu(S(a \otimes b), c) = S(S(a \otimes b) \otimes c) = S(a \otimes b \otimes c) = \\
&= S(a \otimes S(b \otimes c)) = S(a \otimes \mu(b,c)) = \mu(a, \mu(b,c)).
\end{aligned}$$

The third and forth equality follows from the fact that the outer application of the S does the symmetrisation fully, with or without the inner S. $\square$

Remark 4.1.7. The definition of derivations and differentials of an $\mathbb{Z}$-graded algebra can be found in Definition 3.2.2 and 3.2.3.

Theorem 4.1.8. *There is a one-to-one correspondence between degree k derivations of $A = SL$ and degree k linear maps $L \to SL \in Hom_k(L, SL)$*

Proof. This follows by the same arguments as in Theorem 3.2.6. □

Definition 4.1.9. Let

$$\bar{S}L = \bigoplus_{k \geq 1} L^{\odot k}$$

together with μ from Definition 4.1.5 be an algebra. The algebra is an associative algebra by the same arguments as in Lemma 3.2.5, and is called the *bar symmetric algebra* of L.

4.2 Symmetric Tensor Coalgebra

Definition 4.2.1. Let L be a $\mathbb{Z}$-graded vector space over a field $\mathcal{K}$ and let $\epsilon(\sigma, l_1, \ldots, l_k)$ be defined such that $l_{\sigma(1)} \odot \cdots \odot l_{\sigma(k)} = \epsilon(\sigma) l_1 \odot \cdots \odot l_k$ and let $\bar{S}L$ be the same set as in the algebra defined in Definition 4.1.9. Now

$$\bar{S}^c L := \bar{S}L$$

is a coalgebra with

$$\Delta : \bar{S}^c L \to \bar{S}^c L \otimes \bar{S}^c L$$

given by

$$\Delta(l_1 \odot \cdots \odot l_k) = \sum_{i=1}^{k-1} \sum_{\sigma \in S_k} \epsilon(\sigma, l_1, \ldots, l_k) l_{\sigma(1)} \odot \cdots \odot l_{\sigma(i)} \otimes l_{\sigma(i+1)} \odot \cdots \odot l_{\sigma(k)}.$$

Lemma 4.2.2. *The symmetric coalgebra is a coassociative algebra.*

Proof. It is clear from Theorem 4.2.1 that Δ is bilinear. Now, as

$$\begin{aligned}
&(\mathrm{Id} \otimes \Delta) \circ \Delta(l_1 \odot \cdots \odot l_k) = \\
&= (\mathrm{Id} \otimes \Delta)(\sum_{i=1}^{n-1} \sum_{\sigma \in S_k} \epsilon(\sigma, l_1, \ldots, l_k) l_{\sigma(1)} \odot \cdots \odot l_{\sigma(i)} \otimes l_{\sigma(i+1)} \odot \cdots \odot l_{\sigma(k)}) = \\
&= \sum_{i=1}^{n-1} \sum_{\sigma \in S_k} \epsilon(\sigma, l_1, \ldots, l_k) l_{\sigma(1)} \odot \cdots \odot l_{\sigma(i)} \otimes \\
&\quad \otimes(\sum_{j=\sigma(1)}^{\sigma(n-i-1)} \sum_{\tau \in S_{n-i}} l_{i+\tau(1)} \odot \cdots \odot l_{i+\tau(j)} \otimes l_{i+\tau(j+1)} \odot \cdots \odot l_{i+\tau(n-i)}) = \\
&= (\Delta \otimes \mathrm{Id})(\sum_{i=1}^{n-1} \sum_{\sigma \in S_k} \epsilon(\sigma, l_1, \ldots, l_k) l_{\sigma(1)} \odot \cdots \odot l_{\sigma(i)} \otimes l_{\sigma(i+1)} \odot \cdots \odot l_{\sigma(k)}) = \\
&= (\Delta \otimes \mathrm{Id}) \circ \Delta(l_1 \odot \cdots \odot l_k),
\end{aligned}$$

it is clear that the symmetric tensor coalgebra is an coassociative coalgebra □

Theorem 4.2.3. *Let $\bar{S}^c L$ be a symmetric tensor coalgebra, then $CoDer(\bar{S}^c L) = Hom(\bar{S}^c L, L)$ as vector spaces.*

Proof. Let $n \geq 1$. Given a homogeneous map $\varrho_n|_{L^{\odot n}} : L^{\odot n} \to L$, which can be considered as a map $\varrho_n : \bar{S}^c L \to L \in \mathrm{Hom}_k(\bar{S}^c L, L)$ letting the function vanish at all points not in $L^{\odot n}$. Now let

$$\tilde{\varrho}_n(l_1 \odot \cdots \odot l_k) := 0,$$

for $k < n$, and

$$\begin{aligned}
\tilde{\varrho}_n(l_1 \odot \cdots \odot l_k) &:= \sum_{i=1}^{n} \sum_{\sigma \in Sh(i,k)} \epsilon(\sigma) \varrho_n(l_{\sigma(1)} \odot \cdots \odot l_{\sigma(i)}) \odot \\
&\quad \odot l_{\sigma(i+1)} \odot \cdots \odot l_{\sigma(k)}),
\end{aligned}$$

for $k \geq n$. Let $\tilde{\varrho}_n^j : \bar{S}^c L \to L^{\odot j}$ be the j:th output-component of $\tilde{\varrho}_n$. To show that $\tilde{\varrho}_n$ is a uniquely determined coderivation we study

$$\begin{aligned}\Delta(\tilde{\varrho}_n(l_1 \odot \cdots \odot l_k)) &= (\tilde{\varrho}_n \otimes \mathrm{Id} + \mathrm{Id} \otimes \tilde{\varrho}_n)(\Delta(l_1 \odot \cdots \odot l_k)) = \\ &= (\tilde{\varrho}_n \otimes \mathrm{Id} + \mathrm{Id} \otimes \tilde{\varrho}_n)(\sum_{i=1}^{k-1} \sum_{\sigma \in Sh(i,k)} \epsilon(\sigma)(l_{\sigma(1)} \odot \cdots \odot l_{\sigma(i)}) \otimes \\ &\qquad \otimes (l_{\sigma(i+1)} \odot \cdots \odot l_{\sigma(k)})) = \\ &= \sum_{i=1}^{k-1} \sum_{\sigma \in Sh(i,k)} \epsilon(\sigma)\tilde{\varrho}_n(l_{\sigma(1)} \odot \cdots \odot l_{\sigma(i)}) \otimes \\ &\qquad \otimes (l_{\sigma(i+1)} \odot \cdots \odot l_{\sigma(k)}) + \\ &\quad + \sum_{\sigma \in Sh(i,k)} \epsilon(\sigma)(-1)^{|\tilde{\varrho}_n|(|l_1|+\ldots+|l_i|)}(l_{\sigma(1)} \odot \cdots \odot l_{\sigma(i)}) \otimes \\ &\qquad \otimes \tilde{\varrho}_n(l_{\sigma(i+1)} \odot \cdots \odot l_{\sigma(k)}),\end{aligned}$$

and project both sides to $\bigoplus_{i+j=m} L^{\odot i} \otimes L^{\odot j} \subset \bar{S}^c L \otimes \bar{S}^c L$ which yields

$$\begin{aligned}\Delta(\tilde{\varrho}_n^m(l_1 \odot \cdots \odot l_k)) &= \sum_{i=1}^{k-1} \sum_{\sigma \in Sh(i,k)} \epsilon(\sigma)\tilde{\varrho}_n^{m+i-k}(l_{\sigma(1)} \odot \cdots \odot l_{\sigma(i)}) \otimes \\ (l_{\sigma(i+1)} \odot \cdots \odot l_{\sigma(k)}) + \\ &\quad + \sum_{\sigma \in Sh(i,k)} \epsilon(\sigma)(-1)^{|\tilde{\varrho}_n|(|l_1|+\ldots+|l_i|)}(l_{\sigma(1)} \odot \cdots \odot l_{\sigma(i)}) \otimes \\ &\qquad \otimes \varrho_n^{m-i}(l_{\sigma(i+1)} \odot \cdots \odot l_{\sigma(k)}),\end{aligned}$$

this shows that the right hand side depends solely on $\tilde{\varrho}_n^j$ if $j < m$ or the trivial case where $j = m$. An induction argument now shows us that $\tilde{\varrho}_n^m$ vanishes for elements in $V^{\odot k}$ where $k \neq m+n-1$. For elements in $V^{\odot m+n-1}$,

$$\begin{aligned}\tilde{\varrho}_n^m(l_1 \odot \cdots \odot l_k) &= \sum_{i=0}^{m-1} \sum_{\sigma \in Sh(i,m+n-1)} \epsilon(\sigma)(-1)^{|\varrho_n|(|l_1|+\ldots+|l_i|)}(l_{\sigma(1)} \odot \cdots \odot l_{\sigma(i)} \otimes \\ &\qquad \otimes \varrho_n(l_{\sigma(i+1)} \odot \cdots \odot l_{\sigma(i+n)}) \otimes l_{\sigma(i+n+1)} \odot \cdots \odot l_{\sigma(m+n-1)}),\end{aligned}$$

and hence $\tilde{\varrho}_n|_{L^{\odot k}} : L^{\odot k} \to L^{\odot m} = L^{\odot k-n+1}$.

Noting that being a coderivation is a linear property it is clear that the sum of coderivations is again a coderivation. Thus the map

$$\alpha : \{\{\varrho_i : L^{\odot i} \to L\}_{i \geq 0}\} \to CoDer(\bar{S}^c L), \quad \{\varrho_i : L^{\odot i} \to L\}_{i \geq 0} \mapsto \sum_{i \geq 0} \tilde{\varrho}_i$$

is well defined. The mapping β given by

$$\sum_{i\geq 0} \tilde{\varrho}_i \mapsto \{\{pr_L \circ D|_{L^{\odot i}}\}_{i\geq 0}\} = \{\varrho_i : L^{\odot i} \to L\}_{i\geq 0}$$

fulfills $\beta \circ \alpha = \text{Id}$. The uniqueness of the mapping between ϱ and $\tilde{\varrho}$ gives that $\alpha \circ \beta = \text{Id}$, hence the mapping is an isomorphism. □

4.3 Grassmann Algebra

Definition 4.3.1. Let l be an element of the tensor algebra of a $\mathbb{Z}$-graded vector space L. If $f_\sigma(l) = \epsilon(\sigma)l$ for all ϵ and all permutations σ, we call l *skew-symmetric* (or *anti-symmetric*).

The Grassmann algebra, also referred to as the exterior algebra, is closely related to the symmetric tensor algebra, It may be considered as the skew-symmetric tensor algebras.

In direct analogy with the symmetric tensor algebra we define a projection

Proposition 4.3.2. *Let q be an integer and let σ be a permutations of the integers 1 to q. Now let L be a linear space, then the mapping*

$$A : TL \to TL$$

given by

$$A = \frac{1}{q!}\sum_{\sigma} \epsilon(\sigma) f_\sigma : TL^q \to TL^q.$$

has the following properties $A^2 = A$ and im $A = L^{\wedge q}$. A is referred to as the skew-symmetrisation (or anti-symmetrisation) mapping.

Proof. It is obvious that im $A \subset L^{\wedge q}$. Conversely, if a tensor $s' \in L^{\wedge q}$, then $s' = A(s')$, hence im $A = L^{\wedge q}$. It remains to show that $A^2 = A$, which follows from the equation below, as $\epsilon(\sigma)^2 = 1$,

$$\begin{aligned} f_\sigma(A(s')) &= f_\sigma\left(\frac{1}{q!}\sum_{\tau\in S_q}\epsilon(\tau)f_\tau(T)\right) = \frac{1}{q!}\sum_{\tau\in S_q}\epsilon(\tau)f_{\sigma\tau}(T) = \\ &= \epsilon(\sigma)\frac{1}{q!}\sum_{\tau\in S_q}\epsilon(\sigma\tau)f_{\sigma\tau}(T) = \epsilon(\sigma)A(s') \end{aligned}$$

And thereby we can rewrite the square, A^2, back on the form of A,

$$A^2 = \frac{1}{(q!)^2}\sum_{\sigma,\tau\in S_q}\epsilon(\sigma\tau)f_{\sigma\tau} = \frac{1}{q!}\sum_{\rho\in S_q}\epsilon(\rho)f_\rho = A.$$

□

Definition 4.3.3. Let L be a $\mathbb{Z}$-graded vector space, then

$$\bigwedge L = \bigoplus_{q \geq 0} L^{\wedge q}$$

together with the linear map of degree 0

$$\mu : \bigwedge L \otimes \bigwedge L \to \bigwedge L$$

given by

$$\mu(B_1, B_2) = A(B_1 \otimes B_2),$$

where B_1, B_2 is elements in $\bigwedge L$, are called the *Grassmann algebra* of the $\mathbb{Z}$-graded vector space L. We will hereinafter denote the operator μ with $\wedge$.

Lemma 4.3.4. *The Grassmann algebra is an associative algebra.*

Proof. Since the tensor product is multilinear, μ is clearly bilinear. Let $a, b, c \in \bigwedge L$, then

$$\begin{aligned} \mu(\mu(a,b),c) &= \mu(A(a \otimes b), c) = A(A(a \otimes b) \otimes c) = A(a \otimes b \otimes c) = \\ &= A(a \otimes A(b \otimes c)) = A(a \otimes \mu(b,c)) = \mu(a, \mu(b,c)). \end{aligned}$$

The third and forth equality follows from the fact that the outermost S does the symmetrisation fully, with or without the inner S. □

4.4 Grassmann Coalgebra

Definition 4.4.1. Let L be a $\mathbb{Z}$-graded vector space, then

$$\bigwedge L = \bigoplus_{q \geq 0} L^{\wedge q}$$

together with the linear map of degree 0

$$\Delta : \bigwedge L \to \bigwedge L$$

given by

$$\Delta(a_1 \wedge \cdots \wedge a_k) = \sum_{i=1}^{k-1} \sum_{\sigma \in Sh(i,k)} (-1)^{\sigma} \epsilon(\sigma) a_{\sigma(1)} \wedge \cdots \wedge a_{\sigma(i)} \otimes a_{\sigma(i+1)} \wedge \cdots \wedge a_{\sigma(k)}$$

is called the *Grassmann coalgebra* of L.

Lemma 4.4.2. *The Grassmann coalgebra is a coassociative coalgebra.*

Proof. It is clear from Theorem 4.2.1 that Δ is bilinear, and as

$$\begin{aligned}
&(\mathrm{Id} \otimes \Delta) \circ \Delta(l_1 \wedge \cdots \wedge l_k) = \\
&= (\mathrm{Id} \otimes \Delta)(\sum_{i=1}^{n-1} \sum_{\sigma \in Sh(i,k)} \epsilon(\sigma, l_1, \ldots, l_k) l_{\sigma(1)} \wedge \cdots \wedge l_{\sigma(i)} \otimes l_{\sigma(i+1)} \wedge \cdots \wedge l_{\sigma(k)}) = \\
&= \sum_{i=1}^{n-1} \sum_{\sigma \in Sh(i,k)} \epsilon(\sigma, l_1, \ldots, l_k) l_{\sigma(1)} \wedge \cdots \wedge l_{\sigma(i)} \otimes \\
&\quad \otimes(\sum_{j=\sigma(1)}^{\sigma(n-i-1)} \sum_{\tau \in Sh(j,n-i)} l_{i+\tau(1)} \wedge \cdots \wedge l_{i+\tau(j)} \otimes l_{i+\tau(j+1)} \wedge \cdots \wedge l_{i+\tau(n-i)}) = \\
&= (\Delta \otimes \mathrm{Id})(\sum_{i=1}^{n-1} \sum_{\sigma \in Sh(i,k)} \epsilon(\sigma, l_1, \ldots, l_k) l_{\sigma(1)} \wedge \cdots \wedge l_{\sigma(i)} \otimes l_{\sigma(i+1)} \wedge \cdots \wedge l_{\sigma(k)}) = \\
&= (\Delta \otimes \mathrm{Id}) \circ \Delta(l_1 \wedge \cdots \wedge l_k),
\end{aligned}$$

it is clear that the symmetric tensor coalgebra is an coassociative coalgebra

□

Theorem 4.4.3. *If $A := \bigwedge L$, then as vector spaces $CoDer(A) = Hom(\bigwedge L, L)$*

Proof. Let $n \geq 0$. Given a homogeneous map $\varrho_n|_{L^{\wedge n}} : L^{\wedge n} \to L$, which can be considered as a map $\varrho_n : \bigwedge^c L \to L \in \mathrm{Hom}(\bigwedge^c L, L)$ letting the function vanish at all points not in $L^{\wedge n}$. Now let

$$\tilde{\varrho}_n(l_1 \wedge \cdots \wedge l_k) := 0,$$

for $k < n$ and

$$\begin{aligned}
\tilde{\varrho}_n(l_1 \wedge \cdots \wedge l_k) \ &:= \ \sum_{i=0}^{k-n} (-1)^{|\varrho_n|(|l_1|+\ldots+|l_i|)} (l_{\sigma(1)} \wedge \cdots \wedge l_{\sigma(i)} \otimes \varrho_n(l_{\sigma(i+1)} \wedge \cdots \wedge l_{\sigma(i+n)}) \\
&\qquad \otimes l_{\sigma(i+n+1)} \wedge \cdots \wedge l_{\sigma(k)}),
\end{aligned}$$

for $k \geq n$. Let $\tilde{\varrho}_n^j : \bigwedge^c L \to L^{\wedge j}$ be the j:th output-component of $\tilde{\varrho}_n$. To show that $\tilde{\varrho}_n$ is a uniquely determined coderivation we study

$$\begin{aligned}\Delta(\tilde{\varrho}_n(l_1 \wedge \cdots \wedge l_k)) &= (\tilde{\varrho}_n \otimes \mathrm{Id} + \mathrm{Id} \otimes \tilde{\varrho}_n)(\Delta(l_1 \wedge \cdots \wedge l_k)) = \\ &= (\tilde{\varrho}_n \otimes \mathrm{Id} + \mathrm{Id} \otimes \tilde{\varrho}_n)(\sum_{i=1}^{k-1}(l_{\sigma(1)} \wedge \cdots \wedge l_{\sigma(i)}) \otimes \\ &\qquad \otimes(l_{\sigma(i+1)} \wedge \cdots \wedge l_{\sigma(k)})) = \\ &= \sum_{i=1}^{k-1} \tilde{\varrho}_n(l_{\sigma(1)} \wedge \cdots \wedge l_{\sigma(i)}) \otimes (l_{\sigma(i+1)} \wedge \cdots \wedge l_{\sigma(k)}) + \\ &\quad +(-1)^{|\tilde{\varrho}_n|(|l_1|+\ldots+|l_i|)}(l_{\sigma(1)} \wedge \cdots \wedge l_{\sigma(i)}) \otimes \\ &\qquad \otimes\tilde{\varrho}_n(l_{\sigma(i+1)} \wedge \cdots \wedge l_{\sigma(k)}).\end{aligned}$$

and project both sides to $\bigoplus_{i+j=m} L^{\wedge i} \otimes L^{\wedge j} \subset \bigwedge^c L \otimes \bigwedge^c L$ which yields

$$\begin{aligned}\Delta(\tilde{\varrho}_n^m(l_1 \wedge \cdots \wedge l_k)) &= \sum_{i=1}^{k-1} \tilde{\varrho}_n^{m+i-k}(l_{\sigma(1)} \wedge \cdots \wedge l_{\sigma(i)}) \otimes (l_{\sigma(i+1)} \wedge \cdots \wedge l_{\sigma(k)}) \\ &\quad +(-1)^{|\tilde{\varrho}_n|(|l_1|+\ldots+|l_i|)}(l_{\sigma(1)} \wedge \cdots \wedge l_{\sigma(i)}) \otimes \\ &\qquad \otimes\varrho_n^{m-i}(l_{\sigma(i+1)} \wedge \cdots \wedge l_{\sigma(k)})\end{aligned}$$

this shows that the right hand side depends solely on $\tilde{\varrho}_n^j$ if $j < m$ or the trivial case where $j = m$. An induction argument now shows us that $\tilde{\varrho}^m$ vanishes for elements in $V^{\wedge k}$ where $k \neq m+n-1$. For elements in $V^{\wedge m+n-1}$,

$$\begin{aligned}\tilde{\varrho}_n^m(l_1 \wedge \cdots \wedge l_k) &= \sum_{i=0}^{m-1}(-1)^{|\varrho_n|(|l_1|+\ldots+|l_i|)}(l_{\sigma(1)} \wedge \cdots \wedge l_{\sigma(i)} \otimes \\ &\quad \varrho_n(l_{\sigma(i+1)} \wedge \cdots \wedge l_{\sigma(i+n)}) \otimes l_{\sigma(i+n+1)} \wedge \cdots \wedge l_{\sigma(m+n-1)}),\end{aligned}$$

hence $\tilde{\varrho}_n|_{L^{\wedge k}} : L^{\wedge k} \to L^{\wedge k-n+1}$.

Noting that coderivation is a linear property it is clear that the sum of coderivations is again a coderivation. Thus the map

$$\alpha : \{\{\varrho_i : L^{\wedge i} \to L\}_{i\geq 0}\} \to CoDer(\bigwedge^c L), \quad \{\varrho_i : L^{\wedge i} \to L\}_{i\geq 0} \mapsto \sum_{i\geq 0} \tilde{\varrho}_i$$

is well defined. It is clear that the mapping β given by

$$\sum_{i\geq 0} \tilde{\varrho}_i \mapsto \{\{pr_L \circ D|_{L^{\wedge i}}\}_{i\geq 0}\} = \{\varrho_i : L^{\wedge i} \to L\}_{i\geq 0}$$

fulfills $\beta \circ \alpha = \mathrm{Id}$. The uniqueness of the mapping between ϱ and $\tilde{\varrho}$ gives that $\alpha \circ \beta = \mathrm{Id}$, hence the mapping is an isomorphism. □

Chapter 5

A_∞-algebra

5.1 Definitions

Definition 5.1.1. An A_∞ *structure* on a $\mathbb{Z}$-graded vector space L is the coalgebra $\bar{T}^c(L[1])$ together with its codifferential D. The structure is called the A_∞-algebra of L.

A_∞-algebras are also referred to as strong homotopy associative algebras or sha-algebras.

Theorem 5.1.2. *An A_∞-algebra (L, D), can also be defined as the $\mathbb{Z}$-graded vector space L together with the mapping*

$$D : L[1] \oplus (L[1])^{\otimes 2} \oplus \cdots = \bar{T}^c L[1] \to \bar{T}^c L[1],$$

where $D^2 = 0$. Or equivalently as the vector space together with a set of mappings

$$D_k : L[1]^{\otimes k} \to L[1], \qquad |D_k| = 1, \forall k \geq 1$$

where $D_k^2 = 0$. Or again equivalently, as the vector space together with a set of mapping

$$m_k : L^{\otimes k} \to L. \qquad |m_k| = 2 - k, \forall k \geq 1$$

which satisfies the following system of equations

$$\begin{aligned}
m_1(m_1(a_1)) &= 0,\\
m_1(m_2(a_1,a_2)) - m_2(m_1(a_1),a_2) - (-1)^{|a_1|}m_2(a_1,m_1(a_2)) &= 0,\\
m_1(m_3(a_1,a_2,a_3)) - m_2(m_2(a_1,a_2),a_3) + m_2(a_1,m_2(a_2,a_3)) + &\\
+m_3(m_1(a_1),a_2,a_3) + (-1)^{|a_1|}m_3(a_1,m_1(a_2),a_3) + &\\
(-1)^{|a_1|+|a_2|}m_3(a_1,a_2,m_1(a_3)) &= 0,\\
&\vdots\\
\sum_{i=1}^{k}\sum_{j=0}^{k-i+1}(-1)^{\epsilon}m_{k-i+1}(a_1,\dots,m_i(a_j,\dots,a_{j+i-1}),\dots,a_k) &= 0,
\end{aligned}$$

where $\epsilon = i\sum_{l=1}^{j-1}|a_l| + (j-1)(i+1) + k - i$.

Proof. We know by Theorem 3.3.9 that there exist a mapping that takes all coderivations to a set of functions $\{D_k : L[1]^{\otimes k} \to L[1]\}_{k\geq 1}$. Hence we can decompose all coderivations D into a set of D_k and thereby the condition $D^2 = 0$ translates directly to $\forall k, D_k^2 = 0$.

Let $\uparrow$ be the canonical map $L \to L[1]$ of degree -1 and $\downarrow = \uparrow^- 1$, that is, a map $L[1] \to L$ of degree $+1$.

Let $D_k = \uparrow \circ m_k \circ \downarrow^k$, then

$$\begin{aligned}
D_k(\uparrow a_1 \otimes \cdots \otimes \uparrow a_k) &= \uparrow \circ m_k \circ \downarrow^k (\uparrow a_1 \otimes \cdots \otimes \uparrow a_k) =\\
&= (-1)^{\sum_{j=1}^{k-1}(|a_j|+1)} \uparrow \circ m_k \circ (\downarrow^{k-1} \otimes \mathrm{Id})\\
&\qquad (\uparrow a_1 \otimes \cdots \otimes \uparrow a_{k-1}, \downarrow\uparrow a_k) =\\
&= (-1)^{\sum_{j=1}^{k-2} 2(|a_j|+1)+|a_{k-1}|+1} \uparrow \circ m_k \circ (\downarrow^{k-2} \otimes (\mathrm{Id})^{\otimes 2})\\
&\qquad (\uparrow a_1 \otimes \cdots \otimes \uparrow a_{k-2}, \downarrow\uparrow a_{k-1} \downarrow\uparrow a_k) =\\
&\vdots\\
&= (-1)^{\sum_{j=1}^{k-j}(|a_j|+1)} \uparrow \circ m_k(a_1 \otimes \cdots \otimes a_k).
\end{aligned}$$

We can from the above system of equations conclude that D_k and m_k only differs by a linear transformation and they are therefore isomorphic. To study what $D_k^2 = 0$ translates to after this transformation we study

$$\begin{aligned} pr_{L[1]} \circ D^2(\uparrow a_1, \ldots, \uparrow a_k) &= \sum_{i=1}^{k} \sum_{j=0}^{k-i+1} (-1)^{\sum_{i=1}^{j-1}(|a_1|+1)} \\ &\quad D_{k-i+1}(\uparrow a_1, \ldots, D_i(\uparrow a_j, \ldots, \uparrow a_{j+i-1}), \ldots, \uparrow a_k) = \\ &= \sum_{i=1}^{k} \sum_{j=0}^{k-i+1} (-1)^{\sum_{l=1}^{j-1}(|a_l|+1)+\sum_{l=j}^{j+i-1}(j+i-l-1)(|a_l|+1)} \\ &\quad D_{k-i+1}(\uparrow a_1, \ldots, \uparrow \circ m_i(a_j, \ldots, a_{j+i-1}), \ldots, \uparrow a_k) = \\ &= \sum_{i=1}^{k} \sum_{j=0}^{k-i+1} (-1)^{\epsilon} m_{k-i+1}(a_1, \ldots, m_i(a_j, \ldots, a_{j+i-1}), \ldots, a_k), \end{aligned}$$

where

$$\epsilon = \sum_{l=1}^{k}(k-l)(|a_l|+1) + k + j - 1.$$

To verify the exact sign, i. e. ϵ, one may study cases of k. This is carried out in detail in [11]. □

We can easily construct examples of A_∞-algebras, since every differential graded algebra (DG-algebra), say (A, d, μ), can also be viewed as an A_∞-algebra structure by letting $m_1 := d, m_2 := \mu$ and $m_i := 0$ for all $k \geq 3$. The equations proposed by the A_∞-structure given these definitions implies precisely the defining conditions of a DG-algebra,

$$d^2(a) = 0,$$

$$d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b),$$

$$(a \cdot b) \cdot c = a \cdot (b \cdot c),$$

for all $a, b, c \in A$ and there are no higher equations since $m_i = 0$ when $i > 2$.

5.2 Morphisms of A_∞-algebras

Definition 5.2.1. An *A_∞-algebra morphism* between two A_∞-algebras (A, m) and (A', m') consists of a family of functions

$$f_n : A^{\otimes n} \to B, \qquad n \geq 1,$$

homogeneous of degree 0 such that, for all $n \geq 1$, we have

$$\sum_{i+j+k=n} f_{i+1+k} \circ (1^{\otimes i} \otimes b_j \otimes 1^{\otimes k}) = \sum_{i_1+\ldots+i_s=n} b_s \otimes (f_{i_1} \otimes \cdots \otimes f_{i_s}).$$

5.3 An explicit construction of A_∞-algebra

Definition 5.3.1. A *commutator of a graded module*, M, is an operation, $[\cdot,\cdot] : M \times M \to M$, such that for all $m, n \in M$,

$$[m,n] = mn - (-1)^{|m||n|} nm.$$

This explicit construction uses the same method as Merkulov[7], but with a few of the steps described in further detail. Let (V, d) be a differential graded associative algebra and let $(W, d) \subset (V, d)$ such that there exist an odd operator

$$Q : V \to V,$$

that for any $v \in V$ the element $(1 - [d, Q])v$ lies in the subspace W and $[\cdot,\cdot]$ is the commutator.

The construction will use the following recursively defined tensors

$$\lambda_n : V^{\otimes n} \to V, \qquad n \geq 2,$$

initialized by

$$\lambda_2 := v_1 \cdot v_2,$$

and with $\lambda_1 := -Q^{-1}$, the recursive formula for $n \geq 2$ is

$$\begin{aligned}\lambda_n(v_1, \ldots, v_n) \;=\; & -\sum_{\substack{k+l=n+1\\ k,l\geq 1}} (-1)^{k+(l-1)(|v_1|+\ldots+|v_k|)} [Q\lambda_k(v_1, \ldots, v_k)] \cdot \\ & \cdot [Q\lambda_l(v_{k+1}, \ldots, v_n)].\end{aligned}$$

Lemma 5.3.2. *The tensors λ_k, $k \geq 2$, satisfies the identities,*

$$\begin{aligned}\Phi_n(v_1, \ldots, v_n) \;=\; & \sum_{\substack{k+l=n+1\\ k,l\geq 2}} \sum_{j=0}^{k-1} (-1)^r \\ & \lambda_k(v_1, \ldots v_j, \lambda_l(v_{j+1}, \ldots, v_{j+l}), v_{j+l+1}, \ldots, v_n) \\ =\; & 0,\end{aligned}$$

where

$$r = l(|v_1| + \ldots + |v_j|) + j(l-1) + (k-1)l,$$

for any $n \geq 3$ and any $v_1, \ldots, v_n \in V$.

Proof. As a first step we split Φ_n into three sums,

$$\begin{aligned}\Phi_n(v_1,\dots,v_n) &= \sum_{\substack{k+l=n+1\\k,l\geq 2}} (-1)^{(k-1)l}\lambda_k(\lambda_l(v_1,\dots,v_l),v_{l+1},\dots,v_n) + \\ &\quad \sum_{\substack{k+l=n+1\\k,l\geq 2}} (-1)^{l(|v_1|+\dots+|v_{k-1}|)+k-1}\lambda_k(v_1,\dots,v_{k-1},\lambda_l(v_k,\dots,v_n)) + \\ &\quad \sum_{\substack{k+l=n+1\\k,l\geq 2}} \sum_{j=1}^{k-2} \lambda_k(v_1,\dots,v_j,\lambda_l(v_{j+1},\dots,v_{j+l}),v_{j+l+1},\dots,v_n) \\ &= 0.\end{aligned}$$

Given the recursive definition of λ_i and expanding the λ_l not directly coupled with Q it is clear that

$$\begin{aligned}&- \sum_{\substack{k+l=n+1\\k,l\geq 2}} (-1)^{l+k(|v_1|+\dots+|v_l|)}\lambda_l(v_1,\dots,v_l)\cdot Q\lambda_{k-1}(v_{l+1},\dots,v_n) \\ &+ \sum_{\substack{k+l=n+1\\k+l\geq 2}} (-1)^{l(|v_1|+\dots+|v_{k-1}|)}Q\lambda_{k-1}(v_1,\dots,v_{k-1})\lambda_l(v_k,\dots,v_n) \quad = \quad 0.\end{aligned}$$

Thus the first two sums in the split Φ_n reduces to the following expression

$$\begin{aligned}&- \sum_{\substack{k+l=n+1\\k,l\geq 2}} \sum_{\substack{s+t=k\\s\geq 2\\t\geq 1}} (-1)^p[Q\lambda_s(\lambda_l(v_1,\dots,v_l),v_{l+1},\dots,v_{s+l-1}))]\cdot[Q\lambda_t(v_{l+s},\dots,v_n)] \\ &\quad - \sum_{\substack{k+l=n+1\\k,l\geq 2}} \sum_{\substack{s+t=k\\s\geq 1\\t\geq 1}} (-1)^q Q\lambda_s(v_1,\dots,v_s)\cdot Q\lambda_t(v_{s+1},\dots,v_{k-1},\lambda_l(v_k,\dots,v_n)),\end{aligned}$$

where $p = k(l-1)+s+(t-1)(|v_1|+\ldots+|v_{s+l-1}|+l)$ and $q = l(|v_1|+\ldots+|v_{k-1}|)+k-1+s+(t-1)(|v_1|+\ldots+|v_s|)$.

The third and last sum splits into the following two sums

$$\begin{aligned}&\sum_{\substack{k+l=n+1\\k,l\geq 2}} \sum_{j=1}^{k-2} \sum_{\substack{s+t=k\\1\leq s\leq j\\t\geq 1}} (-1)^a Q\lambda_s(v_1,\dots,v_s)\cdot \\ &\qquad \cdot Q\lambda_t(v_{s+1},\dots,v_j,\lambda_l(v_{j+1},\dots,v_{j+l}),v_{j+l-1},\dots,v_n) \\ &- \sum_{\substack{k+l=n+1\\k,l\geq 2}} \sum_{j=1}^{k-2} \sum_{\substack{s+t=k\\s\geq j+1\\t\geq 1}} (-1)^b Q\lambda_s(v_1,\dots,v_j,\lambda_l(v_{j+1},\dots,v_{j+l}),v_{j+l+1},\dots,v_{s+l-1})\cdot \\ &\qquad \cdot Q\lambda_t(v_{s+l},\dots,v_n),\end{aligned}$$

where $a = r + s - 1 + (t-1)(|v_1| + \ldots + |v_s|)$ and $b = r + s + (t-1)(|v_1| + \ldots + |v_{s+l-1}| + l)$.

Substituting these expressions back into the original equation we get the following recursive formula

$$\begin{aligned}\Phi_n(v_1,\ldots,v_n) &= \sum_{\substack{k+l=n\\k\geq 3\\l\geq 1}} (-1)^{(l-1)(|v_1|+\ldots+|v_k|)+k} Q\Phi_k(v_1,\ldots,v_k)\cdot Q\lambda_l(v_{k+1},\ldots,v_n)\\ &\quad - \sum_{\substack{k+l=n\\k\geq 1\\l\geq 3}} (-1)^{l(|v_1|+\ldots+|v_k|)} Q\lambda_k(v_1,\ldots,v_k)\cdot Q\Phi_l(v_{k+1},\ldots,v_n),\end{aligned}$$

where $n \geq 4$. Finally to cover all cases we compute $\Phi_3(v_1, v_2, v_3) = (v_1 \cdot v_2) \cdot v_3 - v_1 \cdot (v_2 \cdot v_3) = 0$. Thus $\Phi_n = 0$ for all $n \geq 3$. □

Lemma 5.3.3. *The λ_k defined above satisfies the following identities,*

$$\begin{aligned}\Theta_n(v_1,\ldots,v_n) &= d\lambda_n(v_1,\ldots,v_n) + \sum_{j=0}^{n-1}(-1)^{n-1+|v_1|+\ldots+|v_j|}\\ &\qquad \lambda_n(v_1,\ldots,v_j,dv_{j+1},v_{j+2},\ldots,v_n)\\ &\quad - \sum_{\substack{k+l=n+1\\k,l\geq 2}}\sum_{j=0}^{k-1}(-1)^r\lambda_k(v_1,\ldots,v_j,[d,Q]\\ &\qquad \lambda_l(v_{j+1},\ldots,v_{j+l}),v_{j+l+1},\ldots,v_n)\\ &= 0,\end{aligned}$$

where

$$r = l(|v_1| + \ldots + |v_j|) + j(l-1) + (k-1)l,$$

for any $n \geq 2$ and any $v_1, \ldots, v_n \in V$.

Proof. Using the same path of execution as in the previous Lemma, this result follows. That is single out the terms where $j = 0$ and $j = k-1$ in the sums and expand the λ_k not coupled with Q as above, do cancelling where possible and then it is possible to derive the following recursion formula,

$$\begin{aligned}\Theta_n(v_1,\ldots,v_n) &= \sum_{\substack{k+l=n\\k\geq 2\\l\geq 1}} (-1)^{(l-1)(|v_1|+\ldots+|v_k|)+k} [Q\Theta_k(v_1,\ldots,v_k)]\cdot [Q\lambda_l(v_{k+1},\ldots,v_n)]\\ &\quad - \sum_{\substack{k+l=n\\k\geq 1\\l\geq 2}} (-1)^{l(|v_1|+\ldots+|v_k|)} [Q\lambda_k(v_1,\ldots,v_k)]\cdot [Q\Theta_l(v_{k+1},\ldots,v_n)],\end{aligned}$$

where $n \geq 3$. And as

$$\Theta_2(v_1, v_2) = d(v_1 \cdot v_2) - (dv_1) \cdot v_2 - (-1)^{|v_1|} v_1 \cdot (dv_2),$$

is the Leibniz identity for d it is clear that $\Theta_2 = 0$, and thus $\Theta_n = 0$ for all $n \geq 3$ too. □

Theorem 5.3.4. *The the linear maps*

$$\mu_k : \bigotimes_k W \to W, k \geq 1,$$

defined by

$$\begin{aligned} \mu_1 &:= d, \\ \mu_k &:= (1 - [d, Q])\lambda_k, k \geq 2, \end{aligned}$$

with

$$\lambda_1 := -Q^{-1}$$

and

$$\lambda_n(v_1, \ldots, v_k) = -\sum_{\substack{k+l=n+1 \\ k,l \geq 1}} (-1)^{k+(l-1)(\tilde{v_1}+\ldots+\tilde{v_k})} [Q\lambda_k(v_1 \ldots, v_k)][Q\lambda_l(v_{k+1}, \ldots, v_n)],$$

where $n \geq 2$ and λ_n satisfy the so-called higher order associativity condition,

$$\sum_{k+l=n+1} \sum_{j=0}^{k-1} (-1)^r \mu_k(v_1, \ldots, v_j, \mu_l(v_{j+1}, \ldots, v_{j+l}), v_{j+l+1}, \ldots, v_n) = 0,$$

where $r = l(|v_1| + \ldots + |v_j|) + j(l-1) + (k-1)l$.

Thus there exist an A_∞-algebra structure on W.

Proof. We denote the high order associative equation with Ψ_n, that is,

$$\Psi_n = \sum_{k+l=n+1} \sum_{j=0}^{k-1} (-1)^r \mu_k(v_1, \ldots, v_j, \mu_l(v_{j+1}, \ldots, v_{j+l}), v_{j+l+1}, \ldots, v_n),$$

where $r = l(|v_1| + \ldots + |v_j|) + j(l-1) + (k-1)l$.

Since (V, d) is a differential algebra and (W, d) is a subcomplex of (V, d) it is clear that Ψ_1 and Ψ_2 vanishes. For $n \geq 3$ we may consider the equivalent expression

$$\Psi_n = (1 - [d, Q])(\Phi_n + \Theta_n),$$

with Φ_n and Θ_n has been defined in Lemma 5.3.2 and 5.3.3. By the same Lemmas it is also clear that Φ_n and Θ_n vanishes for all $n \geq 3$, and this completes the proof. □

Chapter 6

L_∞-Algebras

6.1 Definitions

Definition 6.1.1. An L_∞-*structure* on a $\mathbb{Z}$-graded vector space L is a symmetric coalgebra $(\bar{S}(L[1]), \Delta)$ together with a codifferential Q, hereinafter referred to as the L_∞ algebra (L, Q).

L_∞-algebras are also referred to as strong homotopy Lie algebras or shl-algebras.

Theorem 6.1.2. *An L_∞-algebra, (L, Q), can also be regarded as a $\mathbb{Z}$-graded vector space L, together with the mapping*

$$Q : \bar{S}^c L[1] \to \bar{S}^c L[1],$$

or equivalently as the vector space together with a set of mappings

$$Q_n : L[1]^{\odot n} \to L[1], \qquad |Q_n| = 1, \forall n \geq 1,$$

where $Q_n^2 = 0$, that is

$$(Q^2)_n(a_1, \dots, a_n) = \sum_{k+l=n+1} \sum_{\sigma \in Sh(l,n)} \epsilon(\sigma) Q_l(Q_k(a_{\sigma(1)}, \dots a_{\sigma(l)}), a_{\sigma(l+1)}, \dots, a_{\sigma(n)} = 0.$$

Or again equivalently as the vector space together with a set of mappings

$$\mu_n : \bigwedge^n L \to L, \qquad |\mu_n| = 2 - n,$$

related to Q_n as,

$$Q_n = (-1)^{n(n-1)/2} \downarrow \circ \mu_n \circ \uparrow^n : L^{\cdot n} \to L$$

and satisfy the following condition,

$$\sum_{s+t=n+1} \sum_{\sigma \in Sh(s,n)} \epsilon(\sigma) Q_t(Q_s(l_{\sigma(1)}, \ldots, l_{\sigma(k)}), l_{\sigma(k+1)}, \ldots l_{\sigma(n)}) = 0,$$

for each $n \geq 0$ and $l_1, \ldots, l_n \in L$.

Proof. The one-to-one correspondence between a coderivation of degree k and a series of linear maps follows from Theorem 4.2.3, and hence we may construct a set of $Q_n : L[1]^{\odot n} \to (L[1])[1]$ such that

$$Q|_{L[1]^{\odot n}}(a_1, \ldots, a_n) = \sum_{l=0}^{n} \sum_{\sigma \in Sh(l,n)} \epsilon(\sigma) Q_l(a_{\sigma(1)}, \ldots a_{\sigma(l)}) \odot a_{\sigma(l+1)} \odot \ldots \odot a_{\sigma(n)},$$

with $l = 0$ interpreted as $Q_0(1) \odot a_1 \odot \ldots \odot a_n$.

Considering the next equivalence we need to consider how $Q^2 = 0$ translates to the last formulation.

$$\begin{aligned}(\mu^2)_n(a_1, \ldots, a_n) &= \sum_{k+l=n+1} \sum_{\sigma \in Sh(l,n)} (-1)^{k(l-1)} \epsilon(\sigma) \\ &\quad Q_l(Q_k(a_{\sigma(1)}, \ldots a_{\sigma(l)}), a_{\sigma(l+1)}, \ldots, a_{\sigma(n)} = 0.\end{aligned}$$

Now considering the Q_k as functions $L[1]^{\odot k} \to L[2]$, then there is a natural isomorphism to the mapping μ_k from the Grassmann algebra $L^{\wedge k} \to L[2-n]$. □

Just as in the case with A_∞-algebras there is a direct relation between differential graded counterparts and hence easy to construct examples. Let $(A, d, [\cdot, \cdot])$ be an arbitrary DG Lie-algebra, then it can also be seen as an L_∞-algebra by letting $\mu_1 = d$, $\mu_2 = [\cdot, \cdot]$ and for $i > 2, \mu_i = 0$. It is easy to see that the higher order Jacobi identities suggests the functionality of a differential d and the normal Jacobi identity $[\cdot, \cdot]$ for μ_1 and μ_2 respectively.

6.2 Morphisms of L_∞-algebras

Definition 6.2.1. An L_∞-morphism is a mapping between two L_∞-algebras, (L, Δ) and (L', Δ'), such that the there is a morphism between the differential graded coalgebras

$$((\bar{S}^c(V_1[1]), D_1), \Delta_{\bar{S}^c(V_1[1])}) \to ((\bar{S}^c(V_2[1], D_2), \Delta_{\bar{S}^c(V_2[1])}).$$

That is, that there exist a mapping

$$F : \bar{S}^c(V_1[1]) \to \bar{S}^c(V_2[1]),$$

such that the following two diagrams commute with Δ being the respective comultiplication,

$$\begin{array}{ccc}
\bar{S}^c(V_1[1]) & \xrightarrow{\Delta} & \bar{S}^c(V_1[1]) \otimes \bar{S}^c(V_1[1]) \\
{\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F\otimes F} \\
\bar{S}^c(V_2[1]) & \xrightarrow{\Delta} & \bar{S}^c(V_2[1]) \otimes \bar{S}^c(V_2[1])
\end{array}$$

and

$$\begin{array}{ccc}
\bar{S}^c(V_1[1]) & \xrightarrow{F} & \bar{S}^c(V_2[1]) \\
{\scriptstyle D_1}\downarrow & & \downarrow{\scriptstyle D_2} \\
\bar{S}^c(V_2[1]) & \xrightarrow{F} & \bar{S}^c(V_2[1]).
\end{array}$$

6.3 A geometrical interpretation of L_∞-algebras

Lemma 6.3.1. *Let V be a graded finite dimensional vector space, then there is a one-to-one correspondence between the symmetric coalgebra $\bar{S}^c(V[1])$ and its dual, the symmetric algebra $\bar{S}(V^*[-1])$, as vector spaces.*

Proof. By Theorem 3.1.8 we know that for vector spaces $L_1, \ldots, L_p$, there is a canonical isomorphism from $L_1^* \otimes \cdots \otimes L_p^*$ to $(L_1 \otimes \cdots \otimes L_p)^*$.

Since the symmetric algebra is a subalgebra of the tensor algebra this result obviously implies that $L_1^* \odot \cdots \odot L_p^*$ is isomorphic to $(L_1 \odot \cdots \odot L_p)^*$, hence the result follows given that the derivation is defined as the corresponding coderivation in the symmetric coalgebra. □

Theorem 6.3.2. *Let V be a graded finite dimensional vector space, then there is a one-to-one correspondence between coderivations of degree k over the symmetric coalgebra $\bar{S}^c(V[1])$ and derivations of degree k over the symmetric algebra $\bar{S}(V^*[-1])$.*

Proof. By Theorem 4.2.3, $CoDer(\bar{S}^c(V[1])) = \mathrm{Hom}(\bar{S}^c(V[1]), V[1])$ as vector spaces. Further, we know by Lemma 6.3.1 that $\bar{S}^c(V[1])$ is isomorphic to $\bar{S}(V^*[-1])$ and as it is obvious that $V[1] \simeq V^*[-1]$, we can conclude that $\mathrm{Hom}(\bar{S}^c(V[1]), V[1])$ must be isomorphic to $\mathrm{Hom}(V^*[-1], \bar{S}(V^*[-1]))$. Which in turn is isomorphic to $\bar{S}(V^*[-1])$ by Theorem 4.1.8.

Letting the index shift into negatives when we do the dualisation, it is clear that for $D \in \mathrm{Hom}(\bar{S}^c(V[1]), V[1])$ and $D^* \in \mathrm{Hom}(V^*[-1], \bar{S}(V^*[-1]))$, that $|D| = |D^*|$ as the associated dual bases form the Kronecker delta $\delta^i_j = d^*_{-i}(d_j)$, and therefore obviously are of degree 0.

It now remains to show that $\mathrm{Hom}(V^*[-1], \bar{S}(V^*[-1]))$ is isomorphic to $Der(\bar{S}(V^*[-1])$ to prove the claim.

A derivation $D \in Der(\bar{S}(V^*[-1])$ can always be restricted to a function $D|_{V^*[-1]} : V^*[-1] \to \bar{S}(V^*[-1]) \in \mathrm{Hom}(V^*[-1], \bar{S}(V^*[-1]))$. Conversely it is clear that the non-restricted derivation D is fully described by the restriction $D|_{V^*[-1]}$, as all derivations D by definition recursively fulfill

$$D(a \odot b) = D(a) \odot b \pm a \odot D(b).$$

$\square$

Lemma 6.3.3. *Let V be a graded finite dimensional vector space, and let $\{x^\alpha\}_{\alpha \in I}$ be a basis of its shifted dual $V^*[-1]$, then there is an isomorphism of algebras $\bar{S}(V^*[-1]) \simeq \mathbb{K}[x^\alpha]$.*

Proof. As $\{x^\alpha\}_{\alpha \in I}$ is a basis of $V^*[-1]$, every element $a \in \bar{S}(V^*[-1])$ can be expressed as

$$a = \sum_i \lambda_i x^{\alpha_1} \odot x^{\alpha_2} \odot x^{\alpha_3} \cdots ,$$

where λ_i are scalars in the base field $V^*[-1]$.

Further, the symmetric tensor product operates in the same manner as the normal commutative product in a formal polynomial ring, hence it is clear that if we just denote $x^\alpha \odot x^\beta$ by $x^\alpha x^\beta$ it is possible to identify every $a \in \bar{S}(V^*[-1])$ with an element in the formal polynomial ring $\mathbb{K}[x^\alpha]$. $\square$

Now we are ready to use the above results to show the geometrical interpretation.

Theorem 6.3.4. *An L_∞-algebra over a finite dimensional vector space V is isomorphic to the formal polynomial ring $\mathbb{K}[x^\alpha]$ together with a derivation $D = \sum_{\alpha \in I} V^\alpha(x) \frac{\partial}{\partial x^\alpha}$ that fulfills $\sum_{\alpha,\beta \in I} V^\alpha(x) \frac{\partial V^\beta(x)}{\partial x^\alpha} = 0$, $|V^\alpha(x)| = |x^\alpha| + 1$ and $|\frac{\partial}{\partial x^\alpha}| = -|x^\alpha|$.*

Proof. It is well-known from the field of differential geometry that all derivations D' of $\mathbb{K}[x^\alpha]$ must be on the form

$$D' = \sum_{\alpha \in I} (-1)^{|x^\alpha|} V^\alpha(x) \frac{\partial}{\partial x^\alpha},$$

where $V^\alpha(x)$ (hereinafter also referred to as V^α) is a polynomial over $\mathbb{K}$.

In a symmetric algebra A, a differential $D : A \to A$ needs to fulfill $D^2 = 0$, that is $D(Df') = 0$ for all $f' \in A$. Now this condition translates in the formal polynomial ring, and for an arbitrary f, to

$$D'^2 = D'(D'f) = D'(\sum_{\alpha\in I_1}(-1)^{|x^\alpha|}V^\alpha\frac{\partial f}{\partial x^\alpha}) =$$
$$\sum_{\beta\in I_2}(-1)^{|x^\beta|}V^\beta\frac{\partial}{\partial x^\beta}\left(\sum_{\alpha\in I_1}(-1)^{|x^\alpha|}V^\alpha\frac{\partial f}{\partial x^\alpha}\right) = 0,$$

where $D' : \mathbb{K}[x^\alpha] \to \mathbb{K}[x^\alpha]$ is the corresponding function to the coderivation D.

Applying product law of derivations,

$$\frac{\partial}{\partial x}(fg) = \frac{\partial f}{\partial x}g + (-1)^{|x||f|}f\frac{\partial g}{\partial x},$$

this leaves us with,

$$\sum_{\alpha\in I_1,\beta\in I_2}(-1)^{|x^\alpha|+|x^\beta|}V^\beta(x)\left(\frac{\partial V^\alpha(x)}{\partial x^\beta}\frac{\partial f}{\partial x^\alpha} + (-1)^{|V^\alpha(x)||\frac{\partial f}{\partial x^\alpha}|}V^\alpha(x)\frac{\partial^2 f}{\partial x^\beta\partial x^\alpha}\right) = 0.$$

Now, by the symmetry of the second derivation we may conduct the derivation by x^α or x^β in any order, given that we adjust the grading, that is if we study an isolate part of the sum over the second term,

$$\sum_{\alpha,\beta}V^\beta V^\alpha\frac{\partial^2 f}{\partial x^\beta\partial x^\alpha} = \frac{1}{2}\sum_{\alpha,\beta}(V^\beta V^\alpha\frac{\partial^2 f}{\partial x^\beta\partial x^\alpha} + (-1)^{|V^\beta||V^\alpha|+|\frac{\partial}{\partial x^\beta}||\frac{\partial}{\partial x^\alpha}|}(V^\alpha V^\beta\frac{\partial^2 f}{\partial x^\alpha\partial x^\beta})$$

Let r be the exponent of (-1), that is

$$\begin{aligned} r &= |V^\beta||V^\alpha| + |\frac{\partial}{\partial x^\beta}||\frac{\partial}{\partial x^\alpha}| = (|x^\alpha|+1)(|x^\beta|+1) + (-|x^\alpha|)(-|x^\beta|) \\ &= = 2|x^\alpha||x^\beta| + |x^\alpha| + |x^\beta| + 1. \end{aligned}$$

As we can choose the order of the indices α and β we may choose them such that $|x^\alpha| = |x^\beta|$ and hence $r \equiv 1 \mod 2$. If we now rewrite the expression into two separate sums,

$$\sum_{\alpha\in I_1,\beta\in I_2}(-1)^{|x^\alpha|+|x^\beta|}V^\beta\left(\frac{\partial V^\alpha}{\partial x^\beta}\frac{\partial f}{\partial x^\alpha}\right) +$$
$$+\sum_{\alpha\in I_1,\beta\in I_2}(-1)^{|x^\alpha|+|x^\beta|}\frac{1}{2}V^\beta\left(\sum_{\alpha,\beta}V^\beta V^\alpha\frac{\partial^2 f}{\partial x^\beta\partial x^\alpha} + (-1)^r(V^\beta V^\alpha\frac{\partial^2 f}{\partial x^\beta\partial x^\alpha})\right),$$

it is due to the oddness of r clear that the second term in the outermost summation will vanish.

It is therefore safe to ignore the last term in the initial expression, and hence the following condition is sufficient to guarantee that $D^2 = 0$,

$$\sum_{\alpha, \in I_1, \beta \in I_2} (-1)^{|x^\alpha| + |x^\beta|} V^\beta \frac{\partial V^\alpha}{\partial x^\beta} = 0,$$

and that the conditions on grading in turn implies that $|D| = 1$, as $|V^\alpha| + |\frac{\partial V^\beta(x)}{\partial x^\alpha}| = 1$.

Using the isomorphism in Lemma 6.3.3 it is now clear that $\mathbb{K}[x^\alpha]$ together with the subset of derivations D' circumscribed by the conditions above will be exactly the differentials D in the symmetric algebra $\bar{S}(V^*[-1])$.

Finally, Lemma 6.3.1 gives the isomorphism $\bar{S}^c(V[1]) \simeq \bar{S}(V^*[-1])$ that will take these differentials to the codifferential Q in the symmetric coalgebra structure $((\bar{S}^c(V[1]), \Delta), Q)$ which by Definition 6.1.1 is the L_∞-algebra over V. □

6.4 Explicit construction of L_∞-algebras

6.4.1 From the polynomial ring $\mathbb{S}^2$

Example 6.4.1. Consider the polynomial ring $\mathbb{K}[x, y]$, where $|x| = 2$ and $|y| = 3$, and consider a function $d : \mathbb{K}[x, y] \to \mathbb{K}[x, y]$, such that $d(x) = 0$, $d(y) = x^2$ and $|d| = 1$.

Using Lemma 6.3.3 we have an isomorphism from $\mathbb{K}[x, y]$ to $\bar{S}V^*[-1]$. Let the isomorphism map $x \mapsto v_1^*[-1]$ and $y \mapsto v_2^*[-1]$, and the gradings are preserved, that is $|v_1^*[-1]| = 2$ and $|v_2^*[-1]| = 3$, and $v_1^*[-1], v_2^*[-1]$ spans $V^*[-1]$.

Let the function d over the polynomial ring be transferred using the isomorphism in Lemma 6.3.3 into $\hat{d} : \bar{S}V^*[-1] \to \bar{S}V^*[-1]$, it is then clear that $\hat{d}$ is a derivation over $\bar{S}V^*[-1]$. As $\hat{d}^2 = 0$ and $|\hat{d}| = 1$ it is also clear that $\hat{d}$ is a differential of degree 1 over $\bar{S}V^*[-1]$.

Now, for the set of derivations over $\bar{S}V^*[-1]$, $Der(\bar{S}V^*[-1])$, we know from the definition of a derivation, such as $\hat{d}$ (Definition 3.2.2) that $\hat{d}(a \odot b) = \hat{d}(a) \odot b + (-1)^{|a|} a \odot \hat{d}(b)$, and hence by the obvious recursive argument all values of $\hat{d}$ are determined.

The mapping of d therefore explicitly transforms into,

$$\hat{d} := \begin{cases} v_1^*[-1] \mapsto 0 \\ v_2^*[-1] \mapsto v_1^*[-1] \odot v_1^*[-1], \end{cases} .$$

And, as $\hat{d}$ is fully determined by the restriction of $\hat{d}|_{V^*[-1]}$, there is a natural isomorphism from $Der(\bar{S}V^*[-1])$ to $Hom(V^*[-1], \bar{S}V^*[-1])$. This means that if we use the following list as the basis for $SV^*[-1]$

$$\begin{array}{lcl} e_1^* & = & v_1^*[-1] \\ e_2^* & = & v_2^*[-1] \\ e_3^* & = & v_1^*[-1] \odot v_1^*[-1] \\ e_4^* & = & v_1^*[-1] \odot v_2^*[-1] \\ e_5^* & = & v_2^*[-1] \odot v_2^*[-1] \\ e_6^* & = & v_1^*[-1] \odot v_1^*[-1] \odot v_1^*[-1] \\ e_7^* & = & v_1^*[-1] \odot v_1^*[-1] \odot v_2^*[-1] \\ e_8^* & = & v_1^*[-1] \odot v_2^*[-1] \odot v_2^*[-1] \\ e_9^* & = & v_2^*[-1] \odot v_2^*[-1] \odot v_2^*[-1] \\ e_{10}^* & = & v_1^{*\odot 4}[-1] \\ & \vdots & \end{array}$$

then we can represent $\hat{d}$ using the following matrix,

$$D^* = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \end{pmatrix}.$$

In order to continue the reasoning we need the following remark about one of the basic theorem about duals.

Remark 6.4.2. It is well known that given a linear mapping between linear spaces $f : L \to M$ there exist a dual mapping $f^* : M^* \to L^*$ that satisfies the following equation,

$$(f^*(m^*))(l) = m^*(f(l))$$

Let A be the matrix representing f, and let $\vec{m^*}$ and $\vec{l}$ denote the coordinate vector representatives of m^* and l. Now

$$m^*(f(l)) = \vec{m^*}^t(A\vec{l})$$

and

$$(f^*(m^*))(l) = (B\vec{m^*})^t\vec{l} = (\vec{m^*}^t B^t)\vec{l},$$

this implies by the uniqueness of f^* that $A = B^t$

We may now use the Remark 6.4.2 and Lemma 6.3.1 to explicitly calculate the dual of D^*, $D \in \mathrm{Hom}(\bar{S}^c V[1], V[1])$ as $D = D^{*t}$. That is using the corresponding system of bases for $S^c V[1]$ as we used for $SV^*[-1]$, we will get,

$$D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \end{pmatrix}.$$

As we dualise the base elements in $V^*[1]$ the associated dual basis will form the Kronecker delta $v_i^*[-1](v_j[1]) = \delta_j^i$. As the Kronecker delta has degree 0, this implies that $|v_1[1]| = -|v_1^*[-1]|$ and $|v_2[1]| = -|v_2^*[-1]|$. We let indices become negative in the dualised direct sum, i. e. if $D^* : V^i \to V^{i+1}$, it will be dualised to $D : V_{-i+1} \leftarrow V_{-i}$, and hence the degree of D will be equal to the degree of D^*.

To continue the quest to explicitly construct an L_∞-algebra out of the original d over the polynomial ring, one may study the details of the proof of Theorem 4.2.3, which gives an isomorphism between $\mathrm{Hom}(\bar{S}^c V[1], V[1])$ and $CoDer(\bar{S}^c V[1])$.

Looking closer at the proof of Theorem 4.2.3, it gives the explicit details on how to map a set of mappings $\varrho_n|_{L^{\odot n}} : L^{\odot n} \to L$ to a coderivator map $Q : \bar{S}^c L \to \bar{S}^c L$.

Let $Q \in \mathrm{Hom}(\bar{S}^c V[1], V[1])$ be the linear map determined by the matrix D, the function will have the same degree, hence $|Q| = 1$ Splitting Q into the restricted mappings $Q_i : L^{\odot i} \to L$. It is clear from the matrix that $Q_i = 0$ if $i \neq 2$, and that

$$Q_2 : V[1]^{\odot 2} \to V[1], \qquad Q_2 : \begin{cases} \lambda v_1[1] \odot v_1[1] & \to \quad \lambda v_2[1][1] = v_2[2] \\ \text{else} & \to \quad 0 \end{cases}$$

As Q_2 is of degree 1 one may construct a $\tilde{Q} : V[1]^{\odot 2} \to V[2]$, just by internalising the grading.

Now applying the natural isomorphism,

$$V[1]^{\odot n} \simeq V^{\wedge n}[n], \qquad v_1[1] \odot \cdots \odot v_n[1] \mapsto (-1)^{\sum_{i=0}^{n}(n-i)|v_i|+n}(v_1 \wedge \cdots \wedge v_n)[n],$$

between the exterior and symmetric algebra one gets the following function

$$\mu_2 : V^{\wedge 2} \to V[2-2] = V, \qquad \mu_2 : \begin{cases} \lambda v_1 \wedge v_1 & \to \quad \lambda v_2 \\ \text{else} & \to \quad 0 \end{cases}$$

As all Q_i where $i > 2$ vanishes, the L_∞ algebra constructed is a normal graded Lie algebra, and the Lie bracket function is the Q_2, that is, $[v_1, v_1] = v_2, [v_1, v_2] = 0$ and $[v_2, v_2] = 0$.

6.4.2 From the polynomial ring $\mathbb{CP}^2$

This explicit construction will follow the same path of execution as the previous one and will therefore not dwell on details in the same extent as the first.

The polynomial ring in this example is $\mathbb{K}[x, y]$, where $|x| = 2$ and $|y| = 5$. This implies that $y^2 = 0$, and the derivation d of degree 1 is therefore determined by the following characteristics,

$$d : \begin{cases} x \mapsto 0 \\ y \mapsto x^3 \end{cases} .$$

Using the same steps and basis as in the previous example one will need to construct a matrix describing the linear transformation $D^* : V[-1] \to SV^*[-1]$ that corresponds to our initial d given above. Which is, by the same kind of reasoning as in the previous example is,

$$D^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \end{pmatrix}$$

Now, once again using Remark 6.4.2 and Lemma 6.3.1, one may explicitly calculate the dual of D^*, $D \in \mathrm{Hom}(\bar{S}^c V[1], V[1])$ as $D = D^{*t}$. That is using the corresponding system of bases as for $SV^*[-1]$,

$$D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \end{pmatrix} .$$

As in the first example most Q_i will vanish for all $i \neq 3$, and

$$Q_3 : V[1]^{\odot 3} \to V[1], \qquad Q_3 : \begin{cases} v_1[1] \odot v_1[1] \odot v_1[1] & \to & v_2[1][1] = v_2[2] \\ \text{else} & \to & 0 \end{cases}$$

which translates via the natural isomorphism used in the previous example to

$$\mu_3 : V^{\wedge 3} \to V[2-3] = V[-1], \qquad \mu_3 : \begin{cases} v_1 \odot v_1 \odot v_1 & \to & v_2[-1] \\ \text{else} & \to & 0 \end{cases},$$

and the $\mu_i = 0$ where $i \neq 3$. These μ_i is thereby an L_∞-algebra over V.

6.4.3 From a slightly more complex polynomial ring

Once again this explicit construction is in most parts analog in execution to the first one, and the redundant details has been left out of this construction.

The polynomial ring for this construction example is $\mathbb{K}[c_1, c_2, u, v]$, where $|c_1| = 2, |c_2| = 4, |u| = 5$ and $|v| = 7$. This implies that $u^2 = v^2 = 0$. The derivation d of degree 1 over the ring is determined by,

$$d : \begin{cases} c_1 \mapsto 0 \\ c_2 \mapsto 0 \\ u \mapsto c_1^3 - 2c_1c_2 \\ v \mapsto c_2c_1^2 - c_2^2 \end{cases}.$$

In order to construct the matrix for D^* as in the examples above we need to specify a basis for our $V^*[-1]$ and $SV^*[-1]$. Let the basis for $V^*[-1]$ be given by $v_1^*[-1], v_2^*[-1], v_3^*[-1], v_4^*[-1]$ relating to the polynomial rings basis as $c_1 \mapsto v_1^*[-1], c_2 \mapsto v_2^*[-1], u \mapsto v_3^*[-1]$ and $v \mapsto v_4^*[-1]$, and the basis for $SV^*[-1]$ be,

$$\begin{array}{lllllll}
e_1^* & = & v_1^*[-1] & & e_{12}^* & = & v_3^*[-1] \odot v_4^*[-1] \\
e_2^* & = & v_2^*[-1] & & e_{13}^* & = & v_1^*[-1] \odot v_1^*[-1] \odot v_1^*[-1] \\
e_3^* & = & v_3^*[-1] & & e_{14}^* & = & v_1^*[-1] \odot v_1^*[-1] \odot v_2^*[-1] \\
e_4^* & = & v_4^*[-1] & & e_{15}^* & = & v_1^*[-1] \odot v_1^*[-1] \odot v_3^*[-1] \\
e_5^* & = & v_1^*[-1] \odot v_1^*[-1] & & e_{16}^* & = & v_1^*[-1] \odot v_1^*[-1] \odot v_4^*[-1] \\
e_6^* & = & v_1^*[-1] \odot v_2^*[-1] & & e_{17}^* & = & v_1^*[-1] \odot v_2^*[-1] \odot v_2^*[-1] \\
e_7^* & = & v_1^*[-1] \odot v_3^*[-1] & & e_{18}^* & = & v_1^*[-1] \odot v_2^*[-1] \odot v_3^*[-1] \\
e_8^* & = & v_1^*[-1] \odot v_4^*[-1] & & e_{19}^* & = & v_1^*[-1] \odot v_2^*[-1] \odot v_4^*[-1] \\
e_9^* & = & v_2^*[-1] \odot v_2^*[-1] & & e_{20}^* & = & v_1^*[-1] \odot v_3^*[-1] \odot v_4^*[-1] \\
e_{10}^* & = & v_2^*[-1] \odot v_3^*[-1] & & e_{21}^* & = & v_1^{*\odot 4}[-1] \\
e_{11}^* & = & v_2^*[-1] \odot v_4^*[-1] & & e_{22}^* & = & v_1^*[-1] \odot v_1^*[-1] \odot v_1^*[-1] \odot v_2^*[-1] \\
 & & & & & \vdots &
\end{array}.$$

Now the linear mapping D^* may be described using the following matrix,

$$D^* = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots
\end{pmatrix}$$

Using Remark 6.4.2 and Lemma 6.3.1, we can explicitly calculate the dual of D^*, $D \in \mathrm{Hom}(\bar{S}^c V[1], V[1])$ as $D = D^{*t}$.

This implies that we will have a function $Q : S^c V[1] \to S^c V[1]$ such that if we split it into Q_i.

Then $Q_i = 0$ for $i \neq 2$ and $i \neq 3$,

$$Q_2 : V[1]^{\odot 2} \to V[1], \qquad Q_2 : \begin{cases} v_1[1] \odot v_2[1] & \to & -2v_3[1][1] = -2v_3[2] \\ v_2[1] \odot v_2[1] & \to & -v_4[1][1] = -v_3[2] \\ \text{else} & \to & 0, \end{cases}$$

and

$$Q_3 : V[1]^{\odot 3} \to V[1], \qquad Q_3 : \begin{cases} v_1[1] \odot v_1[1] \odot v_1[1] & \to & v_3[1][1] = v_3[2] \\ v_1[1] \odot v_1[1] \odot v_2[1] & \to & v_4[1][1] = v_4[2] \\ \text{else} & \to & 0, \end{cases},$$

which in turn translates via the natural isomorphism to

$$\mu_2 : V^{\wedge 2} \to V[2-2] = V, \qquad \mu_2 : \begin{cases} v_1 \wedge v_2 & \to & -2v_3 \\ v_2 \wedge v_2 & \to & -v_4 \\ \text{else} & \to & 0, \end{cases}$$

and

$$\mu_3 : V^{\wedge 3} \to V[2-3] = V[-1], \qquad \mu_3 : \begin{cases} v_1 \wedge v_1 \wedge v_1 & \to & v_3[-1] \\ v_1 \wedge v_1 \wedge v_2 & \to & v_4[-1] \\ \text{else} & \to & 0, \end{cases}.$$

The set of μ_i form the L_∞-algebra over V, and as in the previous examples the μ_i not explicitly given above vanishes for all input, which is easy to see in the matrix D.

Bibliography

[1] Timothy Gowers. How to lose your fear of tensor products. 2007. [Online; accessed 2007-11-15].

[2] Ezra Getzler & John D. S. Jones. A_∞-algebras and the cyclic bar complex. 1990.

[3] Bernhard Keller. Introduction to A-infinity algebras and modules. math.RA/9910179v2, 2001.

[4] Bernhard Keller. A-infinity algebras, modules and functor categories. math.RT/0510508v3, 2006.

[5] Alexei I. Kostrikin & Yu I. Manin. *Linear Algebra and Geometry.* Taylor & Francis Ltd, 1989.

[6] Sergei A. Merkulov. L_∞-algebra of an unobstructed deformation functor. math.AG/9907031v2, 1999.

[7] Sergei A. Merkulov. Strong homotopy algebras of a Kähler manifold. math.AG/9809172v3, 2001.

[8] Michael Penkava. L_∞-algebras and their cohomology. math.QA/9512014v1, 1995.

[9] Frank Schuhmacher. Deformation of L_∞-algebras. math.QA/0405485v1, 2004.

[10] Tom Lada & Jim Stasheff. Introduction to sh lie algebras for physicists. UNC-MATH-92/2, 1992.

[11] Thomas Tradler. Infinity-inner-products on A-infinity-algebras. math.AT/0108027v2, 2002.

[12] Charles A. Weibel. *An introduction to homological algebra.* Cambridge University Press, 1994.

[13] Wikipedia. Snake lemma — wikipedia, the free encyclopedia, 2008. [Online; accessed 22-January-2009].