SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

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av

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2009 - No 6

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Explicit growth functions of the Coxeter groups of Lannér and quasi-Lannér type

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Självständigt arbete i matematik 30 högskolepoäng, avancerad nivå

Handledare: Dimitry Leites

2009

Abstract

In 1950, in *Medd. Lunds Univ. Mat. Sem.*, Lannér classified Coxeter groups whose Coxeter diagram without one vertex is a disjoint union of Coxeter diagrams of spherical type. In 1980s, Vinberg and Shwartsman classified Coxeter groups of *quasi-Lannér type* whose Coxeter diagram without one vertex is a disjoint union of Coxeter diagrams of spherical or Euclidean type.

Solomon proved (1966) that the growth function of any Coxeter group is a rational function. The growth functions of Coxeter groups of spherical or Euclidean type are known. Here we give the explicit expressions of the growth functions of (quasi-)Lannér groups. For the Lannér groups with 4 and 5 generators, these series are known thanks to Worthington (1988) but, for 3 of 5 cases of the Lannér groups with 5 generators, his results are wrong. For quasi-Lannér groups, our results are new as well as corrected answers for Lannér groups.

We offer the virgin form of the growth function as a reliable tool for verification of results; it also helps to get an explicit expression of the zeros of the growth function.

The non-real poles of the growth functions of quasi-Lannér groups lie in a narrow annulus, as in the famous Eneström theorem, although the coefficients of the denominators of the growth functions do not satisfy conditions of Eneström's theorem.

Contents

Preface

> The values of the polynomial $n^2 + n + 41$ are primes for integers $n = 0$ through 39. This fact is, however, insufficient to make a tempting claim that the values of this polynomial are prime numbers at all integers. It suffices to set $n = 41$, which is obvious, or $n = 40$ if one is not observant but patient and perseverant.
>
> A well-known Euler's example ([FGI]).

D. Leites told me the above story in May 2008 as a warning, when I came to him and asked him to be my advisor of an *examentsarbete*. He gave me the translation into Russian of a book by Bourbaki [Bou] (published in 1968) where a certain statement is formulated as an exercise true for ANY Coxeter group but in an implicit form, and asked to learn the definitions and rewrite the recurrent expression explicitly, having added "Although it is not required from you, it is more interesting to get — on any level — new results than just verify old ones: life is too short".

Then, together with him and his friends, we started to skim through the literature and ask experts for advice. Vinberg told us that **in Lund, in 1950, Lannér classified the compact "hyperbolic" Coxeter groups**; in 1970s, Vinberg classified the non-compact (quasi-Lannér) ones.

In what follows, we present the results of our preliminary investigations (a more detailed text, a joint work with Leites and Stekolshchik, is in preparation for publication) — what one can add in Stockholm to Lund's discoveries 59 years after. Our result resembles **a discovery of yet another Swedish mathematician, Eneström**, 1893. What are the conditions on the coefficients of the real polynomial in order for its roots to behave as the poles of the growth functions of quasi-Lannér groups? This is a topic for future studies.

The mistakes I made at first, and the mistakes of different authors found in the literature, made the verification of the results our main concern. The results were double-checked by two independent codes.

Acknowledgements. I am thankful to D. Leites, who raised the problem, for encouragement and help, to R. Stekolshchik for useful references, questions, verification and other help, to A. Chapovalov and D. Chapovalov for their help with the computer program [DCh], and to V. Molotkov who verified the poles by means of another code, I am also thankful to R. Grigorchuk, B. Okun, O. Shwartsman, and É. Vinberg for helpful comments in answer to Leites's questions.

1 Introduction

In this work I will not go into geometry of which I know nothing and only consider algebraic aspects of the problem; geometric images and terms appear only for smoothness of our presentation and to give some sources to the reader interested in geometry.

The Coxeter groups are discrete groups generated by reflection acting on, respectively, the sphere, Euclidean space, and Lobachevsky (or hyperbolic) space. In the hyperbolic case, if the group divides the space into simplexes of finite volume, it is said to be of *Lannér* type if it acts cocompactly, and *quasi-Lannér* type otherwise. It was Vinberg who suggested the term *Lannér* in honor of Lannér [La] who was the first, it seems, to list all connected Lannér diagrams (see also [CW]); Shwartsman and Vinberg [VSh] listed all quasi-Lannér diagrams.

The growth functions of the Coxeter groups of spherical and Euclidean types are known. In this work, I explicitly compute the growth functions of certain particular Coxeter groups of hyperbolic types.

Except for the spherical Coxeter groups $I_2^{(m)}$ (for $m \neq 3, 4, 6$), H_3, and H_4, each spherical or Euclidean Coxeter group serves as the Weyl group $W_{\mathfrak{g}(A)}$ of, respectively, simple finite dimensional or affine Kac-Moody Lie algebra. The hyperbolic groups of (quasi-)Lannér type serve as the Weyl groups of what Leites suggested to call *almost affine* Lie algebra[1] $\mathfrak{g}(A)$, where A is a Cartan matrix; for definitions and even the list of almost affine Lie algebras, see the `arXiv` version of [CCLL]. We assume that all Cartan and Coxeter matrices are indecomposable, unless otherwise stated.

1.1 The two known facts and related problems.

Fact 1. *The growth function P_W is a rational function for ANY infinite Coxeter group (W, S) with finite set of generators S. The zeros of P_W lie on on the unit circle C centered at the origin but* **if W is not of spherical or Euclidean type, their precise values were unknown**. *The growth of the Coxeter groups of hyperbolic type is exponential, so there is a pole outside C and* **this is all that is known about poles in general**. (1)

In [So, Ste, Bou], an **implicit** recurrence expression (10) for P_W is given. From [So, Ste, Bou] nothing is clear about the poles of P_W. For the Coxeter groups of other than spherical and Euclidean types, the eigenvalues of the Coxeter transformations do not lie on C, are of the form difficult to describe (see, e.g., [St]), and, obviously, have nothing to do with the zeros of P_W.

We will show that the zeros of the growth functions are, nevertheless, easy to describe (without any computer, almost orally) if these functions are represented in a special **virgin** form.

The initial goal of this note was to give an explicit expression not only for the zeros of these rational functions (and try to compare them with the eigenvalues of the Coxeter

[1]These Lie algebras are currently known under other names: "hyperbolic" (also applied to Lorentzian Lie algebras which constitute a different set) as well as *overextended* (although it is the Dynkin diagrams that are extended twice, not the Lie algebras).

transformations) but also for their **poles** (not spoken about in [So, Ste, Bou] at all) for all *(quasi-)Lannér groups*. These groups are particular among all Coxeter groups, being most close, in a sense, to the Coxeter groups of spherical and Euclidean type: A given Coxeter group is *(quasi-)Lannér* if its Coxeter diagram is connected, neither spherical nor Euclidean, but any its connected proper subdiagram is spherical (resp. spherical or Euclidean).

Knowing a recurrence formula, the problem does not seem to be difficult ideologically but how to be sure that the result is correct? Our own mistakes we made at first, and those we found in the literature make this question more serious than we thought at first.

For the case of Coxeter diagrams with 3 vertices, see the paper by Wagreich [Wa].

For the Lannér diagrams on 4 and 5 vertices, the answers are known [Wo], but I obtained different answer in the three cases of five possible Lannér diagrams on 5 vertices.

This was where my job stopped until reliable means for verifications were found.

Using these means described below (subsec. 4.1), I've found that 3 out of 5 Worthington's answers are wrong.

To check our results, we need the correct results of Worthington [Wo], and so we reproduce them.

It seemED (it does not seem so anymore even for general hyperbolic Coxeter groups) that the denominators of the growth functions of Lannér groups do not admit a nice description except the following ([CW]):

> **Fact 2**. *"With the exception of a single real pair of poles, the poles of the growth function of any compact hyperbolic (Lannér) group with* 4 *generators lie on the unit circle* C*. This is not so for any of the* 5*-generator Lannér groups"*. (2)

The following problems arise:

1) Give reliable criteria for verification of the description of the growth functions.

2) Explicitly describe the poles of the growth function of the 5-generator Lannér groups.

3) Explicitly describe the poles of the growth function of quasi-Lannér groups. In particular, how does the number of the poles not lying on C grow with the number of generators?

1.1.1 On applications. Wagreich's paper also discusses several applications (e.g., due to J. Milnor and M. Gromov) giving motivation for this type of activity. For applications of growth functions of the Coxeter groups of spherical and Euclidean type in the theory of simple finite groups, see [So, St]. There are other types of applications of the growth functions of the hyperbolic groups, see, e.g., [BC, GNa, DDJO].

1.1.2 Main results. I give an explicit form of the growth functions (a.k.a. Hilbert-Poincaré series) of the Lannér groups with 5 generators and quasi-Lannér groups.

I offer reliable means for verifications of the correctness of the growth functions found.

For the Lannér diagrams and the corresponding growth functions, see Tables 3 – 5.

For the quasi-Lannér diagrams and the corresponding growth functions, see Tables 6 – 18.

We observe that (R. Stekolshchik gave an *a priori* explanation of the phenomena)

- If the number of vertices of a given quasi-Lannér diagram is even, than the Euler characteristic $\chi = P_W(1)^{-1}$ of the group W vanishes.

- The difference of degrees of the numerator and denominator of the growth functions is always ≤ 1 in the quasi-Lannér cases.

We have found out that the poles of the growth functions of the quasi-Lannér groups lie behave rather nicely:

1.2 Towards a generalization of the Eneström-Kakeya theorem.

1.2.1 Gal's formulation. For recent studies of the poles of the growth functions of Coxeter groups, see Gal's preprint [Gal]. Gal considered Coxeter diagrams for which the nerve N_W (whatever this is) of the corresponding Coxeter group W is a homology sphere[2]. Gal wondered how many real poles can the growth function of such a group have (he notes that the degree of the denominator of the growth function of certain of such Coxeter group may be however great). I do not know at the moment how to describe the nerve of the group or check if it is a homology sphere and can only cite Gal: "If W is an affine Coxeter group, then there is a unique real pole of order n at 1 [Bou]. If $\dim N_W = n \leq 3$, then there are exactly n positive real roots of the denominator of P_W [Par]. Moreover, in these two cases, all the non-real poles lie on the unit circle."

Gal writes that usually (but does not explain how often does this "usually" occur and what are the exceptions), if $\dim N_W \geq 3$, the non-real poles of the growth function fail to lie on the unit circle. Looking at the examples known to him Gal made the following observation (he writes that he "tested a number of groups whose nerve is a simplex or a product of simplexes" but, regrettably, did not specify the number and gave only two illustrations which, actually, are $L5_5$ and $QL10_2$):

several poles lie "near"the real positive half-line and the rest of the poles tend to lie "near"the unit circle. (3)

1.2.2 Quasi-Lannér cases. Having found the precise expressions of the growth functions and their poles we saw that the distribution of poles, which could have been random, does stick to the pattern (3) almost correctly described by Gal [Gal]. Let us forget for a moment the poles lying "near the real positive half-line"; the remaining poles do lie in a thin annulus concentric with and lying "near"the unit circle.

Our results and Gal's hints lead us to a result of G. Eneström [E]. His theorem (rediscovered by Kakeya [Kak]) says

1.2.3. Theorem. *Let $p(t) = a_0 + a_1 t + \dots + a_n t^n$ be a polynomial with positive coefficients, $m := \min\limits_{0\leq i<n} \frac{a_i}{a_{i+1}}$, and $M := \max\limits_{0\leq i<n} \frac{a_i}{a_{i+1}}$. Then all the roots of $p(t)$ lie in an annulus with bounding circles of radius m and M concentric with and containing the unit circle C centered at the origin.*

The coefficients of the denominators of the growth functions of the (quasi-)Lannér polynomials do not satisfy the conditions of the Eneström-Kakeya theorem but the zeros of these polynomials behave as if they do, or almost: all non-real roots lie in an annulus with the center at the origin (except that we do not know how to define m and M from the coefficients). It is natural, therefore, to try to find the conditions these coefficients satisfy in

[2] A *homology sphere* is an n-dimensional manifold having the same homology groups as S^n does.

order to derive a generalization of the Eneström-Kakeya theorem for polynomials whose real coefficients can be of any sign or vanish.

Leites asked V. Molotkov to check my results. Molotkov started with the study of the $L4_7$ case (as one of the simplest cases) and saw that the poles lying on C are hardly roots of unity (unlike the zeros of the numerator of the growth function of any Coxeter group). Molotkov also observed that, in contradistinction with what is depicted in Gal's illustration for $QL10_2$,

> when the number of vertices of the Coxeter diagram of the quasi-Lannér group becomes >4, NONE of the non-real roots lies on C itself; real poles (if any) lie near 1 or -1. (4)

Molotkov's results, more precise than Gal's, inspired us to verify and sharpen Gal's conjecture (3) as formulated in (4). To list all the poles in all (quasi-)Lannér cases is not very time-consuming but occupies many pages; besides, the poles we found numerically do not look as if they are simple-looking (for humans) algebraic numbers. Therefore we have summarized the answer by listing only the real roots and the extremal values of the absolute values of the complex roots, see Tables 19–26.

2 Precise setting of the problems

2.1 Generating functions. Generating functions of graded objects were introduced and studied by Hilbert and Poincaré at more or less the same time. Leaving touchy priority questions aside, Wikipedia informs us:

> "A *Hilbert-Poincaré series*, named after David Hilbert and Henri Poincaré, is an adaptation of the notion of dimension to the context of graded algebraic structures (where the dimension of the entire structure is often infinite). It is a formal power series in one indeterminate, say t, where the coefficient of t^n gives the dimension (or rank) of the sub-structure of elements homogeneous of degree n." (5)

2.1.1. Remark. Observe that in the above definition certain restrictions are taken for granted: the dimension of each homogeneous component must be finite, and usually only non-negative components are non-zero; "graded" is only assumed to be by means of $\mathbb{Z}$. For $\mathbb{Z}^k$-graded objects (under similar restrictions: The support of the degrees with non-zero components lies in the cone with non-negative coordinates and each component is finite-dimensional), we get series in several indeterminates, as in [McD, DDJO].

In the particular case of Coxeter groups stratified by the length of their elements, the term "Hilbert-Poincaré series" is usually replaced lately by the *growth function*. These functions in the particular case of Coxeter groups of (quasi-)Lannér type is the object of our study.

2.2 Coxeter groups. A *Coxeter group* is a pair (W, S) consisting of a group W and a set of generators $S \subset W$ subject to relations

$$(st)^{m_{s,t}} = 1, \text{ where } m_{s,s} = 1, \quad \text{and} \quad m_{s,t} = m_{t,s} \geq 2 \text{ for } s \neq t \text{ in } S. \tag{6}$$

If no relation occurs for a pair s, t, then it is assumed that $m_{s,t} = \infty$. The symmetric matrix $M = (m_{s,t})_{s,t \in S}$ is called a *Coxeter matrix*.

The presentation of every finitely generated Coxeter group can be illustrated by an undirected labeled graph, called *Coxeter diagram*, whose vertices correspond to the generators S of W and edges are as follows. If $m_{s,t} = 2$ then no edge joins s and t. If $m_{s,t} = 3$, then an edge joins s and t. The edge between the vertices corresponding to $s, t \in S$ is endowed with label $m_{s,t}$ if $m_{s,t} > 3$.

The growth function $P_{W,S}(t)$ of a group W relative to a finite generating set S is briefly denoted $P_W(t)$ and defined as follows. For any $g \in W$, define the *length* $l(g)$ to be the minimum length of all words in S representing $g \neq 1$ and $l(1) = 0$. Then

$$P_W(t) := \sum_{g \in W} t^{l(g)}. \tag{7}$$

2.2.1 Remarks 1) The Coxeter diagrams, so graphic in the spherical and Euclidean cases, are utterly useless if the Coxeter matrix is not sparse, as is the case of Lorentzian Lie algebras considered by Borcherds, and Gritsenko and Nikulin, see [GN]. In this note, we deal with the cases where graphs are helpful, but the reader should realize that *actually* we deal with Coxeter matrices.

2) Other notations used (less convenient, we think, if there are many cases of multiple edges): The edge between nodes s and t is often depicted as a multiple one of multiplicity $m_{s,t} - 2$, unless $m_{s,t} = \infty$; for $m_{s,t} = \infty$, the edge is usually depicted **thick**.

2.3 Exponents. Let W be a finite group generated by reflections r_i, where $i = 1, \dots, n$, in the Euclidean space or on the sphere. (For example, the Weyl group $W = W_{\mathfrak{g}}$ of a simple Lie algebra $\mathfrak{g}$ naturally acts in the root space of $\mathfrak{g}$.) Let $\mathbf{C} := \prod r_i$, called *the Coxeter transformation*, be the product of all generators (in any order; all these products are conjugate, see [St]). For the Weyl groups of simple finite dimensional and affine Kac-Moody Lie algebras, the eigenvalues of $\mathbf{C}$ are of the form ω^{m_i}, where $\omega = e^{2\pi i/h}$ and where $h = 1 + \max m_i$, called the *Coxeter number*, is the order of $\mathbf{C}$ ([CM], [OV], [Ste]). The numbers m_i are called the *exponents* of the Coxeter group W, see [Cox, Table 2].

We do not reproduce the list of spherical and Euclidean Coxeter diagrams (see [Vi]): They are easily obtained from the well-known Dynkin graphs and their Cartan matrices, see [Bou].

2.4 The growth functions (a.k.a. Hilbert-Poincaré series) of the Coxeter groups. Following Solomon, Bourbaki [Bou] gives an explicit expression of the growth function $P_{W_{\mathfrak{g}}}$ of the Weyl groups of simple finite dimensional Lie algebras in terms of exponents:

$$P_{W_{\mathfrak{g}}} = \prod \frac{1 - t^{m_i+1}}{1 - t}. \tag{8}$$

This formula is applicable not only to the Weyl groups of the simple finite dimensional Lie algebras, but to other Coxeter groups of spherical type, see Table 2.

The generalization of (8) to affine Weyl groups is due to Bott [Bo]; see also [Ste]. Bott writes about the loop groups and loop algebras (i.e., algebras of the form $\tilde{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[u^{-1}, u]$, where $\mathfrak{g}$ is any simple finite dimensional Lie algebra) but in reality he only considered the Weyl groups of the Lie algebras of these loop groups; since the exponents are defined up to dualization of the root system, the growth function of the Weyl groups of the "twisted" affine Kac-Moody algebras are covered by Bott's result. The answer is given by the formula

$$P_{W_{\tilde{\mathfrak{g}}}} = \prod \frac{1 - t^{m_i+1}}{(1 - t)(1 - t^{m_i})} = P_{W_{\mathfrak{g}}} \prod \frac{1}{1 - t^{m_i}}. \tag{9}$$

Let us now try to perform the next step — consider the Weyl groups of almost affine Lie algebras.

2.5 Digression: (Quasi-)Lannér groups are the Weyl groups of almost affine Lie algebras. There are several (intersecting but distinct) sets of Lie algebras whose elements are often called "hyperbolic" Lie algebras. We would like to carefully distinguish between these sets so need an appropriate name for each. We say that a submatrix of a square matrix is *principal* if it is obtained by striking out a row and column that intersect on the main diagonal. We say that Lie algebra with Cartan matrix whose entries belong to the ground field is *almost affine* if it is not finite dimensional or affine, and its subalgebra corresponding to any principal submatrix of the Cartan matrix is the sum of finite dimensional or affine Lie algebras.

Z. Kobayashi and J. Morita classified the almost affine Lie algebras with indecomposable symmetrizable Cartan matrix of size > 2 [KoMo]. Later, Li Wang Lai [Li] obtained a complete answer (for Cartan matrices of size > 2): there are 238 almost affine Lie algebras; 142 of these algebras have a symmetrizable Cartan matrix. Later Saçlioğlu [S] rediscovered the result of Kobayashi and Morita (with few omissions, see [BS]); his paper is devoted to physical applications and is very interesting.

Since nobody bothered to make the complete list of Cartan matrices or Dynkin diagrams of the almost affine Lie algebras accessible, and since it is sometimes needed (for example, for the multiparameter version of this work), we have reproduced it in [CCLL].

In this paper we derive explicit formulas for the growth functions of the groups most close in a sense to the Weyl groups of simple finite dimensional Lie algebras.

2.5.1. Remark. In the literature, in similar studies, the authors write sometimes that they are studying the Lie algebras or even the Lie groups having these Lie algebras, whereas they are only studying the Weyl groups of these Lie algebras. This subtlety is sometimes important: in particular, to list all the groups we are dealing with (Lannér and quasi-Lannér) is much easier than to list the Lie algebras whose Weyl groups they are. These are almost affine (a.k.a hyperbolic) Lie algebras; their complete list was unknown when the description of the growth functions of their Weyl groups has begun (and the classification of these Lie algebras is not needed in this particular study of their Weyl groups). There are several stages of generalization of simple finite dimensional Lie algebras (which all possess very particular Cartan matrices) to the Lie algebras with more-or-less arbitrary Cartan matrix. We intend to generalize the results on the growth functions known for the Weyl groups of simple finite dimensional and affine Kac-Moody Lie algebras to the case of Weyl groups of almost affine Lie algebras. These Lie algebras became of acute interest lately in connection with "cosmic billiards"; for details and further references, see [H], [BS]. The growth functions of the Weyl groups of almost affine Lie algebras are invariants of these Lie algebras that can be used further, see [Wa] and references therein. The set of almost affine Lie algebras has a non-empty intersection with the (different) set of Lorentzian Lie algebras, sometimes also called "hyperbolic". For applications of Lorentzian Lie algebras, see [RU], [GN]. For one of these applications Borcherds was awarded with Fields medal.

3 The growth functions (known facts)

3.1 The Solomon-Steinberg recursion: Eq. (10)**.** For a finite set X, let $\varepsilon(X) = (-1)^{\operatorname{card}(X)}$. Let $P_X(t)$ be the growth function (a polynomial or series) of the Coxeter group W_X whose Coxeter graph is X. If $\operatorname{card} W_D < \infty$, let $\mathcal{M}$ be the maximal length of the elements of W_D (there is only one element of maximal length).

Ex. 26 to §1 of Ch.4 [Bou] claims that f o r a n y C o x e t e r g r a p h D, w e h a v e (this formula is obviously due to Solomon [So]; Steinberg [Ste], Theorem 1.25 gave a simpler proof; see also an exposition of Steinberg's proof in [McD], where there are considered multi-parameter series t a k i n g i n t o a c c o u n t d i f f e r e n c e i n l e n g t h o f r o o t s[3]); here X is any **complete**[4] subgraph of D:

$$\sum_{X \subset D} \frac{\varepsilon(X)}{P_X(t)} = \begin{cases} \dfrac{t^{\mathcal{M}}}{P_D(t)} & \text{if } \operatorname{card} W_D < \infty, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

In this expression, the summand corresponding to the empty subgraph is equal to 1.

Recall that the *rational* (non-polynomial) function $P(t)$ is said to be *reciprocal* if $P(t^{-1}) = P(t)$; if $P(t^{-1}) = -P(t)$ the rational function $P(t)$ is often said to be *anti-reciprocal*.

The *polynomial* function $P(t)$ is said to be *reciprocal* (resp. *anti-reciprocal*) if

$$P(t) = t^{\mathcal{M}} P(t^{-1}), \quad (\text{resp. } P(t) = -t^{\mathcal{M}} P(t^{-1})), \text{ where } \mathcal{M} = \deg P.$$

The (anti-)reciprocal function is said to be $\pm$-*reciprocal*.

The recurrence (10) and $\pm$-reciprocity of $P_X(t)$ if $|W_X| < \infty$ imply the following sharpening of (10) due to Steinberg [Ste]: If $\operatorname{card} W_D = \infty$, then

$$\frac{1}{P_D(t^{-1})} = \sum_{X \subsetneq D \;|\; \operatorname{card} W_X < \infty} \frac{\varepsilon(X)}{P_X(t)}. \tag{11}$$

[3]Therefore, for this task, we need not just Coxeter graphs but the Dynkin diagrams, and hence the classification of almost affine (a.k.a. hyperbolic) Lie algebras due to [Li, S]; for the list of such diagrams, see also [CCLL].

[4]Recall that a *complete* subgraph is a subgraph such that each node is connected to every other node in the subgraph.

To begin the induction, recall the following facts:

0) If the Coxeter graph X is the disjoint union of connected components X_i, then $P_X(t) = \prod P_{X_i}(t)$. Hereafter it is advisable to simplify notations: For any $n \in \mathbb{N} \cup \{\infty\}$, set

$$[n] := \begin{cases} 1 + t + \dots + t^{n-1} & \text{for } n < \infty, \\ 1 + t + \dots = \dfrac{1}{1-t} & \text{for } n = \infty. \end{cases} \tag{12}$$

1) $P_\emptyset(t) = 1$ and $P_*(t) = 1 + t = [2]$ (that is, for the graph consisting of 1 vertex and 0 edges).

2) If X has two vertices joined by $m - 2$ edges, then

$$P_X(t) = \begin{cases} \dfrac{(1+t)(1-t^{m+2})}{1-t} = [2][m+1] & \text{if } 3 \leq m < \infty \text{ (for } I_2^{(m)}), \\ \dfrac{1+t}{1-t} = [2][\infty] & \text{if } m = \infty \text{ (for } I_2^{(\infty)}). \end{cases} \tag{13}$$

3) The growth functions of the 3-generator Coxeter group $G_{p,q,r}$ with diagram $L3$ or $QL3$ (if $|G_{p,q,r}| < \infty$, then $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$):

$$P_{G_{p,q,r}}(t) = \frac{[2][p][q][r]}{[2][p][q][r] - 3[p][q][r] + [p][q] + [p][r] + [q][r]} \times \begin{cases} (t^{\mathcal{M}} + 1) & \text{if } |G_{p,q,r}| < \infty, \\ 1 & \text{otherwise}, \end{cases} \tag{14}$$

where, as before, $\mathcal{M}$ is the length of the element of maximal length in $G_{p,q,r}$.

We summarize the results needed to explicitly compute (11) in Table 2.

$L3$: Each diagrams on 3 vertices is a triangle with edges labeled by p, q, r such that $2 \leq p, q, r < \infty$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. One (only one) of the labels p, q, r may be equal to 2, and then the graph is not, actually, a triangle.

$QL3$: The graphs look as those for $L3$ but any of the labels p, q, r may be (and at least one is) equal to ∞.

3.2 Lannér and quasi-Lannér diagrams on > 3 vertices. In the literature we saw, these diagrams are seldom identified (the only exception known to us is an interesting paper [JKRT] with too complicated[5] names for them), so we simply number them for convenience. The first to list these diagrams was, it seems, Lannér [La], see also [CW50].

3.2.1 Worthington's results. For the Lannér diagrams with 4 vertices, Worthington computed the growth functions, and we confirm them in Tables 3–4. For the Lannér diagrams with 5 vertices, Worthington computed the growth functions, but in 3 of 5 cases his answers are wrong.

[5]In addition to overcomplicated proper names, called Witt symbols, there are given in [JKRT] also Coxeter symbols that encode the Coxeter graphs, but can not be used as short names, either, and are not clearly defined for an arbitrary diagram in either [CM] or [JKRT] (try to reconstruct the rules for, e.g., $\overline{DP}_3$, $\overline{M}_3$ or $\overline{N}_4$).

4 The growth functions of the Lannér and quasi-Lannér groups (new results)

Having computed something different from Worthington's results, we realized that means of verification are badly needed. Besides, our goal was not to refute (or verify) somebody's results but to say something new. At first, we could only say something negative ("there is no reciprocity", "not all poles lie on C, the unit circle centered at the origin", etc.), which was not appealing. Let me concentrate on my own results.

4.1 The virgin form of the numerator. The numerator of $P_D(t)$ is equal to the denominator of the sum $\sum\limits_{X\subsetneq D}\frac{\varepsilon(X)}{P_X(t)}$. By (8), for the finite Coxeter group W_X with exponents

$$m_1, m_2, \ldots, m_k$$

the growth function P_X is a polynomial of the form

$$[m_1+1][m_2+1]\ldots[m_k+1]. \tag{15}$$

The least common multiple

$$\mathrm{Virg}(D) := \underset{X\subsetneq D \text{ such that } |W_X|<\infty}{\mathrm{LCM}} P_X(t) \tag{16}$$

is said to be the *virgin form* of (the numerator of) $P_D(t)$.

The expression of $P_D(t)$ in as an irreducible fraction is said to be a *reduced form*.

4.1.1. Lemma. *The growth function $P_D(t)$ can be expressed as a rational fraction whose numerator is* $\mathrm{Virg}(D)$.

We say that a subgroup W_J of the Coxeter group (W, S) is *special* if it is generated by a subset $J \subset S$. [6]

Proof. The statement is obvious if all special subgroups W_X are finite: then the numerator of $P_D(t)$ is equal to the denominator of the sum $\sum\limits_{X\subsetneq D}\frac{\varepsilon(X)}{P_X(t)}$ and all denominators of its summands are polynomials of the form (15). The general case is done by induction on $|X|$. □

4.1.1a. Corollary. *Let $\frac{\varepsilon(X)}{P_X(t)}$ be expressed as an irreducible fraction. Then the LCM of all denominators in the sum $\sum\limits_{X\subsetneq D}\frac{\varepsilon(X)}{P_X(t)}$ is equal to* $\mathrm{Virg}(D)$.

Proof. Indeed, if $|W_X| = \infty$, then the denominator of the irreducible fraction $\frac{\varepsilon(X)}{P_X(t)}$ divides $\mathrm{Virg}(X)$ and $\mathrm{Virg}(X)$ divides $\mathrm{Virg}(D)$. If $|W_X| < \infty$, then $P_X(t)$ divides $\mathrm{Virg}(D)$ by definition. Hence, the LCM of denominators divides $\mathrm{Virg}(D)$.

Implication in the opposite direction: divisibility of the LCM of denominators by $\mathrm{Virg}(D)$ ia obvious. □

[6] In some works such a group is called *parabolic*, but in other works the *parabolic group* means wW_Jw^{-1} for some $w \in W$, where W_J is the subgroup generated by $J \subset S$. Besides, the term *parabolic group* is already occupied in the Lie group theory. On top of this, some say that there are Coxeter groups of elliptic, hyperbolic and *parabolic* type, so the term is overused.

If $|W_X| < \infty$, then $P_X(t)$ is of the form (15). We would like to represent $\mathrm{Virg}(D)$ in the same form, but this is not always possible: if m and n are not relatively prime, then $[m]$ and $[n]$ are not relatively prime. On the other hand, each such polynomial can be represented as the product of irreducible over $\mathbb{Q}$ polynomials $\Phi_n(t)$, where $n = 2, 3, \dots$, namely

$$[n] = \prod_{i|n,\ i>1} \Phi_i(t). \tag{17}$$

Therefore, it is natural to compute $\mathrm{Virg}(D)$ in the form of the product of the $\Phi_i(t)$. It is convenient to introduce one more notation:

$$[n'] := 1 + t^n; \quad \text{observe that } [n][n'] = [2n]. \tag{18}$$

4.1.1b. Remark. At first, we thought that the virgin form is only useful to control the computations. But we got more: The answer shows that, for (quasi-)Lanner groups, the virgin form coincides with the reduced form except for $QL8_1$, $QL8_2$, $QL8_4$.

5 The code and means of control

We used the *Mathematica*-based code `subg` due to D. Chapovalov [DCh] and R. Stekolshchik double-checked the result with his own code. The codes compute the growth functions w i t h n u m e r a t o r s i n t h e v i r g i n f o r m of the Coxeter groups given by Coxeter graphs.

5.1 Code `subg`. We rewrite the expression (11) in the following form

$$P_D(t) = \frac{-\varepsilon(D)}{\sum\limits_{X \subsetneq D} \frac{\varepsilon(X)}{P_X(t)}}. \tag{19}$$

Given a graph D, the code `subg` generates a list of variables, one for each complete subgraph of D, the graph D itself including. To isomorphic graphs one variable corresponds (so the code compares graphs).

In the file `input.txt`, to the variables that correspond to finite Coxeter groups of spherical type listed in Table 1 the values equal to the growth functions of the corresponding groups are already assigned; to the other variables the value is generated:

1. if the graph is disconnected, the value is the product of the values corresponding to the connected components;

2. if the graph is connected, the value of the variable is computed in terms of the values of the variables corresponding to its complete subgraphs according to (19).

5.1.1 Encoding Coxeter diagrams. First of all, we need an economic way of encoding/decoding the Coxeter graph to eliminate the chances for a "human error" to creep in. There are several ways to document the presentation of a given Coxeter group; the Coxeter matrix, Coxeter diagram as well as the Coxeter and Witt symbols (see [CM, JKRT]) are the most used ones but the first symbol is too long, while the latter two, especially the Witt symbol, are unclear for humans and computers alike. To encode the Cartan matrix $M = (m_{ij})_{i,j=1}^n$ it suffices to punch in not n^2 elements but just $\frac{1}{2}n(n-1)$ (say, above the diagonal), thanks to symmetry.

Instead of the Coxeter matrix we use the incidence matrix of the Coxeter graph $K = (k_{ij})_{i,j=1}^n$, where $k_{ij} = m_{ij} - 2$,, more precisely, its part above the diagonal and to encode M we write:

$$mk_{12} \dots k_{1n}k_{23} \dots k_{2n} \dots\dots\dots k_{n-1,n}. \tag{20}$$

For example, the Coxeter matrix, the incidence matrix and the variable for the graph B_3 are

$$\begin{pmatrix} * & 3 & 2 \\ * & * & 4 \\ * & * & * \end{pmatrix}, \qquad \begin{pmatrix} * & 1 & 0 \\ * & * & 2 \\ * & * & * \end{pmatrix}, \qquad m102.$$

This way is universal but not practical since it is difficult for a human to recognize a given graph if $n > 4$ (given a Coxeter matrix). Therefore, we introduced several auxiliary variables helping to overview formulas and recognize graphs. Since we intend to deal only with graphs on ≤ 10 vertices, the following suffices:

1. The variables $a1$, $a2$, ..., $a9$, $b2$, $b3$, ..., $b9$, $d4$, $d5$, ..., $d9$, $h3$, $h4$, $i25$, $i26$, $f4$, $e6$, $e7$, $e8$ correspond to their namesakes-groups. These variables are short and graphic and are used instead of rather long m-variables wherever possible.

2. Any chain-graph is denoted by y followed by an ordered list of multiplicities. For example, $b4$ can be expressed as $y112$ or $y211$.

3. The cycle-graph can be encoded as z followed by an ordered list of multiplicities. For example, $L4_7$ can be expressed as $z2121$ or $z1212$.

4. Let a given graph have vertices of degree ≥ 3 (i.e., with ≥ 3 edges emitted from the vertex) but no two such vertices belong to one cycle. Such a graph can be obtained by gluing several graphs at this vertex. The variable of this graph can be encoded by juxtaposition of the variables of the subgraphs glued. For example, $d4$ is the result of gluing a chain with two edges and two chains of one edge each, so it can be expressed as $y11y1y1$, or $y1y1y11$, or $y1y11y1$. The graph $QL4_{20}$ can be encoded as $y4z111$, the graph $QL7_2$ can be encoded as $y1y11m100110$. By means of such short notation one can encode any of quasi-Lannér graphs, except for $QL4_{21}$, $QL4_{22}$ and $QL5_9$

5.1.2 The procedure. For example, let us compute the growth function of $QL4_2$ to which the variable $y214$ corresponds. We copy `input.txt` into the file `ql42.txt` (we have to create), in the same catalog where the code `subg` lies. In the command line we write

`subg y214 -fql42`.

This means that we do NOT have to generate variables already computed in the file `ql42.txt`. The result will be written in the same file `ql42.txt`:

$$
\begin{aligned}
& m1 = a2 \\
& m2 = b2 \\
& m3 = i25 \\
& m4 = i26 \\
& m002 = (b2 * a1) \\
& m004 = (a1 * i26) \\
& m011 = a3 \\
& m012 = b3 \\
& m013 = h3 \\
& m014 = 1/(1 + 1/i26 + 1/a2 - 3/a1 + 1/m0) \\
& m001011 = d4 \\
& m010011 = a4 \\
& m010012 = f4 \\
& m100012 = b4 \\
& m100013 = h4 \\
& m200014 = -1/(1 + 1/i26 - 1/b3 + 1/b2 + 1/a2 - 4/a1 - 1/m014- \\
& \qquad\qquad\qquad 1/m004 - 1/m002 + 3/m0) \\
& \ldots \quad (\text{"}\ldots\text{" is what we do not need for the moment }) \\
& y214 = m200014
\end{aligned} \tag{21}
$$

Let us now switch *Mathematica* on and copy in it from the file `Virgin-mdk.txt` the following formulas needed to compute $a1 - e8$ in terms of $v1, v2, \ldots, vn, \ldots$ (corresponding to the respective polynomials $\Phi_n(t)$):

$$
\begin{array}{l}
p2 = v2\\
p3 = v3\\
p4 = v2 * v4\\
p5 = v5\\
p6 = v2 * v3 * v6\\
p7 = v7\\
p8 = v2 * v4 * v8\\
p9 = v3 * v9\\
p10 = v2 * v5 * v10\\
p12 = v2 * v3 * v4 * v6 * v12\\
p14 = v2 * v7 * v14\\
p16 = v2 * v4 * v8 * v16\\
p18 = v2 * v3 * v6 * v9 * v18\\
p20 = v2 * v4 * v5 * v10 * v20\\
p24 = v2 * v3 * v4 * v6 * v8 * v12 * v24\\
p30 = v2 * v3 * v5 * v6 * v10 * v15 * v30\\
\\
a1 = p[[2]]\\
a2 = p[[2]] * p[[3]]\\
a3 = (a2 * p[[4]])\\
a4 = (a3 * p[[5]])\\
a5 = (a4 * p[[6]])\\
a6 = (a5 * p[[7]])\\
a7 = (a6 * p[[8]])\\
a8 = (a7 * p[[9]])\\
a9 = (a8 * p[[10]])\\
b2 = p[[2]] * p[[4]]\\
b3 = b2 * p[[6]]\\
b4 = b3 * p[[8]]\\
b5 = b4 * p[[10]]\\
b6 = b5 * p[[12]]\\
b7 = b6 * p[[14]]\\
b8 = b7 * p[[16]]\\
b9 = b8 * p[[18]]\\
d4 = b3 * p[[4]]\\
d5 = b4 * p[[5]]\\
d6 = b5 * p[[6]]\\
d7 = b6 * p[[7]]\\
d8 = b7 * p[[8]]\\
d9 = b8 * p[[9]]\\
h3 = p[[2]] * p[[6]] * p[[10]]\\
h4 = p[[2]] * p[[12]] * p[[20]] * p[[30]]\\
i25 = p[[2]] * p[[5]]\\
i26 = p[[2]] * p[[6]]\\
f4 = p[[2]] * p[[6]] * p[[8]] * p[[12]]\\
e6 = f4 * p[[5]] * p[[9]]\\
e7 = f4 * p[[10]] * p[[14]] * p[[18]]\\
e8 = p[[2]] * p[[8]] * p[[12]] * p[[14]] * p[[18]] * p[[20]] * p[[24]] * p[[30]]
\end{array}
\tag{22}
$$

and order "compute". Now we copy formulas from `ql42`, type `Simplify[y214]`, and order "compute". The preliminary result is ready:

$$
\begin{array}{l}
(v2^3 * v3 * v4 * v6)/(1 + v4 + v3 * v6 - v2 * v3 * v6 - 2v2 * v3 * v4 * v6 + v2^2 * v3 * v4 * v6)\\
Virg(QL42) = v2^3 * v3 * v4 * v6
\end{array}
\tag{23}
$$

To compute the corresponding denominator, we copy from the file `Virgin-mdk.txt` to

Mathematica the formulas expressing the vi's in terms of t:

$$\begin{array}{l}
p = Table[Cancel[(1-t^n)/(1-t)], n, 1, 30] \\
v2 = p[[2]] \\
v3 = p[[3]] \\
v4 = Cancel[p[[4]]/v2] \\
v5 = p[[5]] \\
v6 = Cancel[p[[6]]/(v2 * v3)] \\
v7 = p[[7]] \\
v8 = Cancel[p[[8]]/(v2 * v4)] \\
v9 = Cancel[p[[9]]/v3] \\
v10 = Cancel[p[[10]]/(v2 * v5)] \\
v12 = Cancel[p[[12]]/(v2 * v3 * v4 * v6)] \\
v14 = Cancel[p[[14]]/(v2 * v7)]v15 = Cancel[p[[15]]/(v3 * v5)] \\
v16 = Cancel[p[[16]]/(v2 * v4 * v8)] \\
v18 = Cancel[p[[18]]/(v2 * v3 * v6 * v9)] \\
v20 = Cancel[p[[20]]/(v2 * v4 * v5 * v10)] \\
v24 = Cancel[p[[24]]/(v2 * v3 * v4 * v6 * v8 * v12)] \\
v30 = Cancel[p[[30]]/(v2 * v3 * v5 * v6 * v10 * v15)]
\end{array} \tag{24}$$

and perform the calculations.

Now we compute the denominator:

$$Expand[Simplify[(1 + v4 + v3v6 - v2v3v6 - 2v2v3v4v6 + v22v3v4v6)]] \tag{25}$$

The answer

$$1 - t - t^3 - t^5 + t^6 + t^8$$

is ready!

6 Tables

Table 1: The exponents, Coxeter number, and the maximal length of the elements in the spherical Coxeter groups

Coxeter group	Lie algebra	exponents m_i	maximal length $l(w_0) = \sum m_i$	Coxeter number h
A_n	$\mathfrak{sl}(n+1)$	$1, 2, 3, \ldots, n$	$\frac{n(n+1)}{2}$	$n+1$
B_n	$\mathfrak{o}(2n)$ for $n \geq 2$	$1, 3, \ldots, 2n-1$	n^2	$2n$
C_n	$\mathfrak{sp}(2n)$ for $n \geq 2$	$1, 3, \ldots, 2n-1$	n^2	$2n$
D_n	$\mathfrak{o}(2n+1)$	$1, 3, \ldots, 2n-3;\ n-1$	$n(n-1)$	$2(n-1)$
G_2	$\mathfrak{g}_2$	$1, 5$	6	6
F_4	$\mathfrak{f}_4$	$1, 5, 7, 11$	24	12
E_6	$\mathfrak{e}_6$	$1, 4, 5, 7, 8, 11$	36	12
E_7	$\mathfrak{e}_7$	$1, 5, 7, 9, 11, 13, 17$	63	18
E_8	$\mathfrak{e}_8$	$1, 7, 11, 13, 17, 19, 23, 29$	120	30
$I_2^{(m)}$ for $m > 6$ or $m = 5$	—	$1, m-1$	m	m
H_3	—	$1, 5, 9$	15	10
H_4	—	$1, 11, 19, 29$	60	30

Note. The groups $I_2^{(m)}$ are the non-crystallographic dihedral groups for $m = 5$ and $m > 6$. For $m = 3, 4,$ and 6, respectively, we have the crystallographic dihedral group as follows:

$$A_2 = I_2^{(3)}, \qquad B_2 = C_2 = I_2^{(4)}, \qquad G_2 = I_2^{(6)}.$$

Table 2: The growth functions of the spherical Coxeter groups with connected Coxeter diagram

Coxeter group	its growth function
A_n	$[2]\dots[n+1]$
B_n	$[2][4]\dots[2n]$
D_n	$[2][4]\dots[2n-2][n]$
G_2	$[2][6]$
F_4	$[2][6][8][12]$
E_6	$[2][5][6][8][9][12]$
E_7	$[2][6][8][10][12][14][18]$
E_8	$[2][8][12][14][18][20][24][30]$
$I_2^{(m)}$ for $5 \leq m \leq \infty$	$[2][m]$
H_3	$[2][6][10]$
H_4	$[2][12][20][30]$

Table 3: The Lannér diagrams on 4 vertices and growth functions

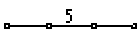
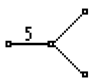
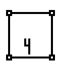
Label	Diagram Degrees	Growth function $\chi = 0$ in all cases
$L4_1$	 $(11, 11)$	$\dfrac{[2][6][5']}{t^{11} - 2t^{10} + t^9 - t^7 + 2t^6 - 2t^5 + t^4 - t^2 + 2t^1 - 1}$
$L4_2$	 $(15, 15)$	The numerator is $[4][10][3']$ The denominator is $t^{15} - 2t^{14} + 2t^{13} - 2t^{12} + t^{11} - t^{10}$ $+t^9 - t^8 + t^7 - t^6 + t^5 - t^4 + 2t^3 - 2t^2 + 2t - 1$
$L4_3$	 $(13, 13)$	$\dfrac{[2][10][3']}{t^{13} - 2t^{12} + t^{11} - t^9 + t^8 - t^7 + t^6 - t^5 + t^4 - t^2 + 2t - 1}$
$L4_4$	 $(11, 11)$	$\dfrac{[4][3'][5']}{t^{11} - 3t^{10} + 4t^9 - 4t^8 + 3t^7 - 2t^6 + 2t^5 - 3t^4 + 4t^3 - 4t^2 + 3t - 1}$
$L4_5$	 $(7, 7)$	$\dfrac{[2][4][3']}{t^7 - 2t^6 + 2t^4 - 2t^3 + 2t - 1}$

Table 4: The Lannér diagrams on 4 vertices and growth functions (cont.)

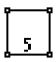
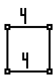
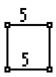
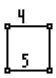
Label	Diagram Degrees	Growth function $\chi = 0$ in all cases
$L4_6$	 $(11, 11)$	$\dfrac{[4][3'][5']}{t^{11} - 3t^{10} + 3t^9 - t^8 - 2t^7 + 4t^6 - 4t^5 + 2t^4 + t^3 - 3t^2 + 3t - 1}$
$L4_7$	 $(7, 7)$	$\dfrac{[2][4][3']}{t^7 - 2t^6 + t^4 - t^3 + 2t - 1}$
$L4_8$	 $(9, 9)$	$\dfrac{[2][3'][5']}{t^9 - 3t^8 + 2t^7 + t^6 - 3t^5 + 3t^4 - t^3 - 2t^2 + 3t - 1}$
$L4_9$	 $(11, 11)$	$\dfrac{[4][3'][5']}{t^{11} - 3t^{10} + 3t^9 - 2t^8 + 2t^6 - 2t^5 + 2t^3 - 3t^2 + 3t - 1}$

Table 5: The Lannér diagrams on 5 vertices and growth functions

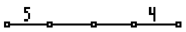
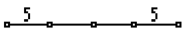
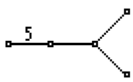
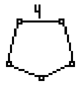
$L5_1$	 $\chi = 1/14400$ $\deg = (60,60)$	The numerator is [2][12][20][30] The denominator is $t^{60} - t^{59} - t^{53} + t^{52} - t^{51} + t^{50} - t^{49} + t^{46} - t^{45}$ $+t^{44} - 2t^{43} + 2t^{42} - t^{41} + t^{40} - t^{39} + t^{38} - t^{37} + 2t^{36}$ $-2t^{35} + 2t^{34} - 2t^{33} + 2t^{32} - t^{31} + t^{30} - t^{29} + 2t^{28} - 2t^{27}$ $+2t^{26} - 2t^{25} + 2t^{24} - t^{23} + t^{22} - t^{21} + t^{20} - t^{19} + 2t^{18}$ $-2t^{17} + t^{16} - t^{15} + t^{14} - t^{11} + t^{10} - t^{9} + t^{8} - t^{7} - t + 1$
$L5_2$	 $\chi = 17/28880$ $\deg = (64,64)$	The numerator is [2][12][20][30][4′] The denominator is $t^{64} - t^{63} - t^{61} + 2t^{60} - 2t^{59} + t^{58} - 3t^{57} + 3t^{56} - 3t^{55}$ $+3t^{54} - 5t^{53} + 5t^{52} - 5t^{51} + 6t^{50} - 7t^{49} + 8t^{48} - 8t^{47} + 9t^{46}$ $-9t^{45} + 11t^{44} - 11t^{43} + 12t^{42} - 11t^{41} + 14t^{40} - 13t^{39} + 14t^{38}$ $-13t^{37} + 16t^{36} - 14t^{35} + 15t^{34} - 14t^{33} + 17t^{32} - 14t^{31} + 15t^{30}$ $-14t^{29} + 16t^{28} - 13t^{27} + 14t^{26} - 13t^{25} + 14t^{24} - 11t^{23} + 12t^{22}$ $-11t^{21} + 11t^{20} - 9t^{19} + 9t^{18} - 8t^{17} + 8t^{16} - 7t^{15} + 6t^{14} - 5t^{13}$ $+5t^{12} - 5t^{11} + 3t^{10} - 3t^{9} + 3t^{8} - 3t^{7} + t^{6} - 2t^{5} + 2t^{4} - t^{3} - t + 1$
$L5_3$	 $\chi = 13/7200$ $\deg = (60,60)$	The numerator is [2][12][20][30] The denominator is $t^{60} - t^{59} - t^{57} - t^{53} - t^{51} + 2t^{50} - 2t^{49} + 2t^{48} - 2t^{47}$ $+2t^{46} + 2t^{44} - 2t^{43} + 2t^{42} - 2t^{41} + 6t^{40} - 3t^{39} + 4t^{38} - 3t^{37}$ $+4t^{36} + 4t^{34} - 3t^{33} + 4t^{32} - 3t^{31} + 8t^{30} - 3t^{29} + 4t^{28} - 3t^{27}$ $+4t^{26} + 4t^{24} - 3t^{23} + 4t^{22} - 3t^{21} + 6t^{20} - 2t^{19} + 2t^{18} - 2t^{17}$ $+2t^{16} + 2t^{14} - 2t^{13} + 2t^{12} - 2t^{11} + 2t^{10} - t^{9} - t^{7} - t^{3} - t + 1$
$L5_4$	 $\chi = 17/14400$ $\deg = (60,60)$	The numerator is [2][12][20][30] The denominator is $t^{60} - t^{59} - t^{57} + t^{56} - t^{55} - t^{53} + t^{52} - t^{51} - t^{49} + 2t^{48}$ $-t^{47} + t^{46} - t^{45} + 2t^{44} - t^{43} + t^{42} - t^{41} + 3t^{40} - t^{39}$ $+2t^{38} - t^{37} + 3t^{36} - t^{35} + 2t^{34} - t^{33} + 3t^{32} - t^{31} + 3t^{30}$ $-t^{29} + 3t^{28} - t^{27} + 2t^{26} - t^{25} + 3t^{24} - t^{23} + 2t^{22} - t^{21}$ $+3t^{20} - t^{19} + t^{18} - t^{17} + 2t^{16} - t^{15} + t^{14} - t^{13} + 2t^{12}$ $+t^{11} - t^{9} + t^{8} - t^{7} - t^{5} + t^{4} - t^{3} - t^{1} + 1$
$L5_5$	 $\chi = 11/5760$ $\deg = (28,28)$	The numerator is [2][5][6][8][12] The denominator is $t^{28} - t^{26} - t^{25} - t^{24} - 2t^{23} - 2t^{22} - t^{21} + t^{20} + t^{19} + 2t^{18}$ $+2t^{17} + 3t^{16} + 2t^{15} + 3t^{14} + 2t^{13} + 3t^{12} + 2t^{11} + 2t^{10} + t^{9} + t^{8}$ $-t^{7} - 2t^{6} - 2t^{5} - t^{4} - t^{3} - t^{2} + 1$

Table 6: The quasi-Lannér diagrams on 4 vertices and growth functions, none of them reciprocal

In what follows, the column **Inf.gr.** contains # of infinite special groups.

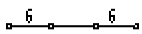
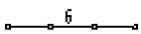
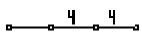
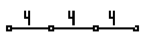
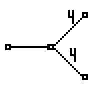
Label	Diagram Degrees	Growth function $\chi = 0$ in all cases	Inf.gr
$QL4_1$	 $(9,8)$	$\dfrac{[2][4][6]}{t^8 - t^4 - t + 1}$	1
$QL4_2$	 $(9,8)$	$\dfrac{[2][4][6]}{t^8 + t^6 - t^5 - t^3 - t + 1}$	1
$QL4_3$	 $(15,14)$	$\dfrac{[2][6][10]}{t^{14} + t^{12} - t^5 - t^3 - t + 1}$	1
$QL4_4$	 $(7,7)$	$\dfrac{[2]^2[6]}{t^7 + t^6 - t^5 - t^2 - t + 1}$	2
$QL4_5$	 $(6,6)$	$\dfrac{[2][6]}{t^6 - t^4 + t^2 - 2t + 1}$	2
$QL4_6$	 $(9,8)$	$\dfrac{[2][4][6]}{t^8 - t^3 - t + 1}$	1
$QL4_7$	 $(5,5)$	$\dfrac{[2]^2[4]}{t^5 + t^4 - t^3 - t^2 - t + 1}$	2
$QL4_8$	 $(7,6)$	$\dfrac{[2][4][3']}{t^6 - t^5 + t^4 - t^3 + t^2 - 2t + 1}$	1

Table 7: The quasi-Lannér diagrams on 4 vertices and growth functions, none of them reciprocal (cont.)

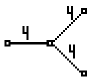
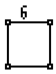
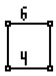
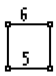
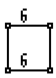
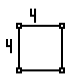
Label	Diagram	Growth function $\chi = 0$ in all cases	Inf.gr.
$QL4_9$	 $(5,5)$	$\dfrac{[2]^2[4]}{2t^5 + t^4 - 2t^3 - t^2 - t + 1}$	3
$QL4_{10}$	 $(8,8)$	$\dfrac{[4][6]}{t^8 + t^6 - t^5 - t^3 + t^2 - 2t + 1}$	2
$QL4_{11}$	 $(9,9)$	$\dfrac{[2][4][6]}{t^9 + t^8 + t^7 + t^6 - t^5 - t^4 - t^3 - t^2 - t + 1}$	2
$QL4_{12}$	 $(11,11)$	$\dfrac{[2][6][5']}{t^{11} - 2t + 1}$	2
$QL4_{13}$	 $(6,6)$	$\dfrac{[2][6]}{3t^6 - 2t^5 - 2t + 1}$	4
$QL4_{14}$	 $(7,6)$	$\dfrac{[2][4][3']}{t^6 - t^4 + t^3 - 2t^1 + 1}$	1

Table 8: The quasi-Lannér diagrams on 4 vertices and growth functions, none of them reciprocal (cont.)

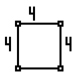
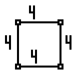
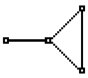
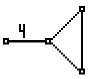
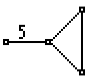
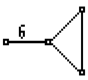
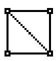
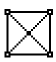
Label	Diagram	Growth function $\chi = 0$ in all cases	Inf.gr.
$QL4_{15}$	 $(7,7)$	$\frac{[2][4][3']}{t^7-2t+1}$	2
$QL4_{16}$	 $(4,4)$	$\frac{[2][4]}{3t^4-2t^3-2t+1}$	4
$QL4_{17}$	 $(5,4)$	$\frac{[3][4]}{t^4-t^3+t^2-2t+1}$	1
$QL4_{18}$	 $(9,8)$	$-\frac{[2][4][6]}{t^8+t^6-t^4-t^2-t+1}$	1
$QL4_{19}$	 $(11,10)$	$\frac{[2][6][5']}{t^{10}-t^9+t^8-t^6+t^5-t^4+t^3-2t+1}$	1
$QL4_{20}$	 $(6,6)$	$\frac{[2][6]}{2t^6-t^5-t^4+t^3-2t+1}$	3
$QL4_{21}$	 $(5,5)$	$\frac{[3][4]}{t^5-2t+1}$	2
$QL4_{22}$	 $(3,3)$	$\frac{[2][3]}{3t^3-2t^2-2t^1+1}$	4

Table 9: The quasi-Lannér diagrams on 5 vertices and growth functions, none of them reciprocal

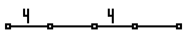
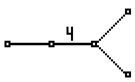
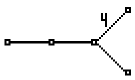
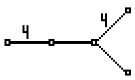
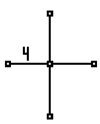
Label	Diagram χ, Degrees	Growth function D:=denominator	Inf.gr.
$QL5_1$	 $(24, 23)$ $\chi = -1/1152$	The numerator is [2][6][8][12] $D := t^{23} + t^{19} - t^{18} - t^{16} + t^{15} - 2t^{14}$ $+t^{13} - 2t^{12} + t^{11} - 2t^{10} + 2t^9 - 2t^8$ $+t^7 - t^6 + 2t^5 - t^4 + t^3 + t - 1$	1
$QL5_2$	 $(24, 23)$ $\chi = -1/576$	The numerator is [2][6][8][12] $D := t^{23} + t^{20} - t^{17} - t^{15} - t^{13} - t^{12}$ $-t^{11} - t^8 + t^5 + t^3 + t^1 - 1$	1
$QL5_3$	 $(20, 19)$ $\chi = -1/1920$	The numerator is [2][4][5][6][8] $D := t^{19} + t^{18} + t^{17} + t^{16} - t^{14} - t^{13} - 2t^{12} - 2t^{11}$ $-2t^{10} - t^9 - t^8 + t^7 + t^6 + 2t^5 + t^4 + t^3 - 1$	1
$QL5_4$	 $(16, 16)$ $\chi = -1/384$	The numerator is [2][4][6][8] $D := t^{16} + t^{15} - t^{12} - 2t^{10} + t^9 - 3t^8$ $+t^7 - 2t^6 + 2t^5 - t^4 + 2t^3 + t - 1$	2
$QL5_5$	 $\chi = -1/192$ $(12, 12)$	The numerator is $[2][4]^2[6]$ The denominator is $2t^{12} + t^{11} - 4t^8 - t^7 - 3t^6 + 2t^5 - t^4 + 3t^3 + t^1 - 1$	3

Table 10: The quasi-Lannér diagrams on 5 vertices and growth functions, none of them reciprocal (cont.)

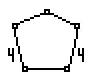
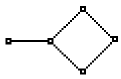
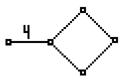
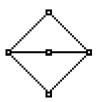
Label	Diagram χ, Degrees	Growth function D:=denominator	Inf.gr.
$QL5_6$	 $(24, 23)$ $\chi = -1/144$	The numerator is [2][6][8][12] $D := t^{23} + t^{22} + t^{20} - t^{17} - 2t^{16} - t^{15} - t^{14} - t^{13}$ $-3t^{12} - t^{11} - t^{10} - 3t^{8} + t^{5} + t^{3} + t^{2} + t - 1$	1
$QL5_7$	 $(16, 15)$ $\chi = -1/960$	The numerator is $[2][4]^2[5][6]$ $D := t^{15} + t^{14} + t^{13} + t^{12} - t^{11} - 2t^{10}$ $-2t^{9} - 3t^{8} - t^{7} + t^{5} + 2t^{4} + t^{3} + t^{2} - 1$	1
$QL5_8$	 $(16, 16)$ $\chi = -1/192$	The numerator is [2][4][6][8] $D := t^{16} + t^{15} - t^{11} - t^{10} - t^{9}$ $-2t^{8} - t^{7} + t^{3} + t^{2} + t - 1$	2
$QL5_9$	 $(12, 12)$ $\chi = -1/192$	The numerator is $[2][4]^2[6]$ $D := 2t^{12} + t^{11} - t^{10} + t^{9} - 3t^{8} - 2t^{7}$ $-t^{6} - 2t^{5} + t^{3} + 2t^{2} + t^{1} - 1$	3

Table 11: The quasi-Lannér diagrams on 6 vertices and growth functions

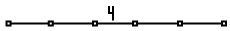
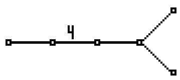
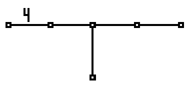
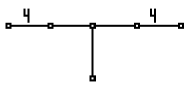
Label	Diagram Degrees	Growth function $\chi = 0$ in all cases D:=denominator	Inf.gr.
$QL6_1$	 $(25, 25)$	The numerator is $[2]^2[6][8][12]$ $D := t^{25} + t^{24} - t^{23} - t^{22} - t^{19} - t^{18} + 2t^{16}$ $-t^{15} - t^{14} + t^{13} + t^{12} - t^{11} + t^{10} + t^7$ $+t^6 - 2t^5 + t^3 - t^2 - t + 1$	2
$QL6_2$	 $(33, 32)$	The numerator is $[2][6][8][10][12]$ $D := t^{32} - t^{26} - t^{23} - t^{22} - t^{19} + t^{18}$ $+t^{16} + t^{14} + t^{12} - t^5 - t + 1$	1
$QL6_3$	 $(25, 25)$	The numerator is $[2]^2[6][8][12]$ $D := t^{25} + t^{24} - t^{23} - t^{22} - t^{19} - t^{18} + 2t^{16} - t^{15}$ $-t^{14} + 2t^{12} - t^{11} + 2t^8 - t^5 + t^4 - t^2 - t + 1$	2
$QL6_4$	 $(25, 25)$	The numerator is $[2]^2[6][8][12]$ $D := 2t^{25} + t^{24} - 2t^{23} - t^{22} - 2t^{19} - t^{18} + 3t^{16}$ $-2t^{15} - t^{14} + 3t^{12} - 2t^{11} + t^{10} + 2t^8 + t^6$ $-2t^5 + t^4 - t^2 - t + 1$	3
$QL6_5$	 $(23, 22)$	The numerator is $[2][4][8][10][3']$ The denominator is $D := t^{22} - t^{21} + t^{19} - 2t^{18} + t^{17} - t^{15} + t^{14}$ $-t^{13} + t^{11} - t^9 + 2t^8 - t^7 + t^5 - t^4 + t^2 - 2t^1 + 1$	1
$QL6_6$	 $(17, 17)$	The numerator is $[2]^2[4][6][8]$ $D := 2t^{17} + t^{16} - 2t^{15} - 2t^{14} - t^{13}$ $-t^{11} + 3t^8 + t^6 + t^4 - t^3 - t^2 - t^1 + 1$	3

Table 12: The quasi-Lannér diagrams on 6 vertices and growth functions (continue)

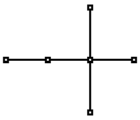
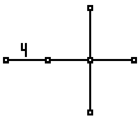
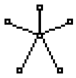
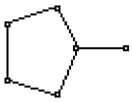
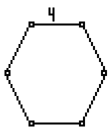
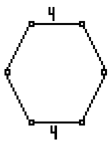
Label	Diagram Degrees	Growth function $\chi = 0$ in all cases D:=denominator	Inf.gr.
$QL6_7$	 $(17,16)$	The numerator is $[2]^2[4][6][8]$ $D := t^{16} - t^{14} - t^{12} + t^8 - t^7$ $+2t^5 + t^4 - t^3 - t^2 - t + 1$	1
$QL6_8$	 $(17,17)$	The numerator is $[2]^2[4][6][8]$ $D := 3t^{17} + t^{16} - 3t^{15} - 3t^{14} - t^{13} + t^{12} - 2t^{11}$ $+4t^8 - t^7 + 2t^6 + 2t^4 - 2t^3 - t^2 - t + 1$	4
$QL6_9$	 $(11,11)$	The numerator is $[2]^2[4]^2[3']$ $D := 4t^{11} - 3t^{10} - 5t^9 + 2t^8 + t^6$ $-t^5 + 5t^4 - 2t^3 - 2t + 1$	5
$QL6_{10}$	 $(17,16)$	The numerator is $[2][5][6][8]$ $D := t^{16} - t^{15} - t^{11} + t^9 - t^8 + t^7 + t^3 - 2t + 1$	1
$QL6_{11}$	 $(29,29)$	The numerator is $[2][6][8][12][5']$ $D := t^{29} - t^{26} + t^{23} - 2t^{22} - 2t^{19} + t^{18} - t^{15}$ $+t^{14} + 2t^{12} - t^{11} + 2t^8 - t^6 + t^3 - 2t + 1$	2
$QL6_{12}$	 $(24,24)$	The numerator is $[2][6][8][12]$ $D := 5t^{24} - 4t^{23} - 2t^{21} + 3t^{20} - 6t^{19} + 3t^{18}$ $-6t^{17} + 7t^{16} - 4t^{15} + 2t^{14} - 6t^{13} + 8t^{12} - 4t^{11}$ $+2t^{10} - 2t^9 + 5t^8 - 2t^7 + 3t^6 - 2t^5 + t^4 - 2t^1 + 1$	6

Table 13: The quasi-Lannér diagrams on 7 vertices and growth functions

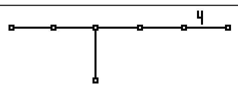
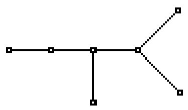
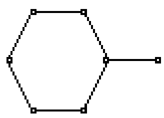
Label	Diagram χ, Degrees	Growth function D:=denominator	Inf.gr.
$QL7_1$	 1/414720, $(44, 43)$	The numerator is [2][4][6][8][9][10][12] $D = t^{43} + t^{42} + t^{41} + t^{40} - t^{37} - t^{36} - 2t^{35} - 3t^{34}$ $-3t^{33} - 3t^{32} - 3t^{31} - 2t^{30} - t^{29} + t^{28} + 2t^{27} + 3t^{26}$ $+4t^{25} + 4t^{24} + 5t^{23} + 5t^{22} + 4t^{21} + 2t^{20} + t^{19} - t^{18}$ $-t^{17} - 3t^{16} - 3t^{15} - 4t^{14} - 3t^{13} - 3t^{12} - t^{11} - t^{10}$ $+t^9 + t^7 + t^6 + t^5 + t^4 + t^3 - 1$	1
$QL7_2$	 1/207360, $(42, 41)$	The numerator is $[2]^2$[6][8][9][10][12] $D = t^{41} + t^{40} - t^{35} - t^{34} - t^{33} - 2t^{32} - 2t^{31} - 2t^{30}$ $-t^{29} - t^{28} + 2t^{26} + 2t^{25} + t^{24} + 2t^{23} + 3t^{22} + 4t^{21}$ $+3t^{20} + t^{19} - t^{15} - 2t^{14} - 2t^{13} - 2t^{12} - 2t^{11} - t^{10}$ $+t^4 + t^3 + t^2 - 1$	1
$QL7_3$	 13/725760, $(48, 47)$	The numerator is $[2]^2$[6][7][8][9][10][12] $D = t^{47} + 2t^{46} + 2t^{45} + 2t^{44} + 2t^{43} + 2t^{42} + t^{41}$ $-2t^{40} - 4t^{39} - 6t^{38} - 9t^{37} - 11t^{36} - 12t^{35} - 13t^{34}$ $-11t^{33} - 7t^{32} - 3t^{31} + t^{30} + 6t^{29} + 12t^{28} + 18t^{27}$ $+21t^{26} + 22t^{25} + 21t^{24} + 19t^{23} + 17t^{22} + 13t^{21} + 6t^{20}$ $-5t^{18} - 8t^{17} - 10t^{16} - 12t^{15} - 12t^{14} - 10t^{13} - 9t^{12}$ $-6t^{11} - 2t^{10} + t^8 + 2t^7 + 2t^6 + 2t^5 + 2t^4$ $+2t^3 + t^2 - t^1 - 1$	1

Table 14: The quasi-Lannér diagrams on 8 vertices and growth functions

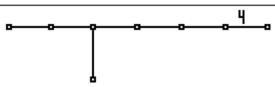
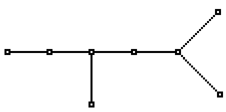
Label	Diagram Degrees	Growth function $\chi = 0$ in all cases D:=denominator
$QL8_1$	 $(55, 54)$	The numerator is $[4][6][8][12][18][5'][7']$ $t^{54} - 2t^{53} + 2t^{52} - 2t^{51} + t^{50} + t^{49} - 3t^{48} + 5t^{47} -$ $7t^{46} + 7t^{45} - 6t^{44} + 5t^{43} - 3t^{42} + 2t^{40} - 2t^{39} + 2t^{38} -$ $t^{37} - 2t^{36} + 5t^{35} - 6t^{34} + 7t^{33} - 6t^{32} + 3t^{31} - 3t^{29}$ $+6t^{28} - 7t^{27} + 6t^{26} - 5t^{25} + 3t^{24} - t^{22} - t^{21} +$ $2t^{20} - 4t^{19} + 8t^{18} - 11t^{17} + 12t^{16} - 12t^{15} + 10t^{14} -$ $7t^{13} + 4t^{12} - t^{11} - t^{10} + 2t^{9} - t^{8} - t^{7}$ $+3t^{6} - 5t^{5} + 6t^{4} - 5t^{3} + 4t^{2} - 3t + 1$
$QL8_2$	 $(51, 50)$	The numerator is $[4][6][8][10][12][7'][9']$ $D = t^{50} - 2t^{49} + 2t^{48} - 2t^{47} + t^{46} - t^{44} + 3t^{43} -$ $5t^{42} + 7t^{41} - 9t^{40} + 10t^{39} - 11t^{38} + 11t^{37} - 10t^{36} +$ $10t^{35} - 9t^{34} + 8t^{33} - 7t^{32} + 6t^{31} - 6t^{30} + 6t^{29} -$ $5t^{28} + 4t^{27} - 2t^{26} + 2t^{24} - 6t^{23} + 10t^{22} - 13t^{21} +$ $15t^{20} - 16t^{19} + 17t^{18} - 18t^{17} + 17t^{16} - 16t^{15} + 14t^{14} -$ $12t^{13} + 12t^{12} - 11t^{11} + 10t^{10} - 10t^{9} + 9t^{8} - 9t^{7} +$ $9t^{6} - 7t^{5} + 6t^{4} - 5t^{3} + 4t^{2} - 3t + 1$

Table 15: The quasi-Lannér diagrams on 8 vertices and growth functions

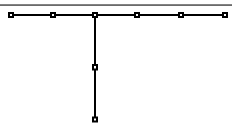
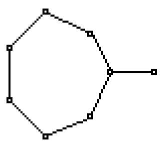
Label	Diagram Degrees	Growth function $\chi = 0$ in all cases D:=denominator
$QL8_3$	 $(63, 62)$	The numerator is [2][6][8][10][12][14][18] $D = t^{62} - t^{56} - t^{54} - t^{52} - t^{50} - t^{49} + t^{48}$ $-t^{47} + t^{46} + t^{44} + 2t^{42} + t^{40} + 2t^{39} + t^{37}$ $+2t^{36} - t^{34} + 2t^{33} - 2t^{32} - t^{29} - 2t^{28} + t^{27}$ $-2t^{26} - 2t^{25} + t^{24} - 2t^{23} - t^{22} - 2t^{19} + 3t^{18}$ $-t^{17} + t^{15} + t^{14} + t^{12} + t^{11} - t^{10}$ $+2t^9 - t^8 - t^4 - t + 1$
$QL8_4$	 $(57, 56)$	The numerator is [4][6][8][10][12][14][9′] $D = t^{56} - t^{55} + t^{54} - t^{53} - t^{50} - t^{48} + 2t^{47} - 3t^{46}$ $+3t^{45} - 3t^{44} + 2t^{43} - t^{41} + 3t^{40} - 3t^{39} + 5t^{38} - 4t^{37}$ $+6t^{36} - 4t^{35} + 4t^{34} - 2t^{33} + t^{32} + 2t^{31} - 3t^{30} + 5t^{29}$ $-6t^{28} + 6t^{27} - 7t^{26} + 5t^{25} - 5t^{24} + 2t^{23} - 3t^{21} + 3t^{20}$ $-6t^{19} + 5t^{18} - 7t^{17} + 6t^{16} - 4t^{15} + 2t^{14} + t^{13} - 2t^{12}$ $+2t^{11} - 3t^{10} + 3t^9 - 2t^8 + 3t^7 - t^6 + t^5 - t^3 + t^2 - 2t^1 + 1$

Table 16: The quasi-Lannér diagrams on 9 vertices and growth functions

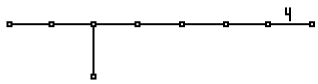
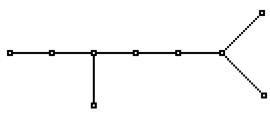
Label	Diagram χ, Degrees	Growth function D:=denominator
$QL9_1$	 17/1393459200, $(128, 127)$ The number of infinite special subgroups: 1	The numerator is [2][12][14][16][18][20][24][30] $D = t^{127} - t^{117} + t^{116} - 2t^{115} + t^{114} - 3t^{113}$ $+2t^{112} - 4t^{111} - 12t^{103} + 3t^{110} - 6t^{109} + 6t^{108}$ $-8t^{107} + 7t^{106} - 10t^{105} + 11t^{104} + 13t^{102} - 15t^{101}$ $+18t^{100} - 17t^{99} + 21t^{98} - 21t^{97} + 26t^{96} - 23t^{95}$ $+29t^{94} - 26t^{93} + 33t^{92} - 28t^{91} + 35t^{90} - 31t^{89}$ $+38t^{88} - 32t^{87} + 38t^{86} - 33t^{85} + 38t^{84} - 33t^{83}$ $+36t^{82} - 32t^{81} + 32t^{80} - 30t^{79} + 28t^{78} - 26t^{77}$ $+22t^{76} - 22t^{75} + 14t^{74} - 16t^{73} + 6t^{72} - 9t^{71}$ $-3t^{70} - t^{69} - 13t^{68} + 7t^{67} - 22t^{66} + 17t^{65}$ $-32t^{64} + 26t^{63} - 40t^{62} + 35t^{61} - 49t^{60} + 44t^{59}$ $-55t^{58} + 52t^{57} - 62t^{56} + 59t^{55} - 65t^{54} + 65t^{53}$ $-69t^{52} + 69t^{51} - 69t^{50} + 72t^{49} - 72t^{48} + 73t^{47}$ $-68t^{46} + 73t^{45} - 68t^{44} + 71t^{43} - 63t^{42} + 68t^{41}$ $-61t^{40} + 63t^{39} - 55t^{38} + 59t^{37} - 52t^{36} + 53t^{35}$ $-45t^{34} + 47t^{33} - 42t^{32} + 40t^{31} - 35t^{30} + 35t^{29}$ $-32t^{28} + 29t^{27} - 26t^{26} + 24t^{25} - 23t^{24} + 19t^{23}$ $-18t^{22} + 15t^{21} - 15t^{20} + 12t^{19} - 11t^{18} + 9t^{17}$ $-10t^{16} + 7t^{15} - 6t^{14} + 5t^{13} - 5t^{12} + 4t^{11} - 3t^{10}$ $+3t^{9} - 3t^{8} + 2t^{7} - t^{6} + t^{5} - t^{4} + t^{3} + t - 1$
$QL9_2$	 17/696729600 $(120, 119)$ The number of infinite special subgroups: 1	The numerator is [2][8][12][14][18][20][24][30] $D = t^{119} - t^{111} - t^{109} + t^{108} - 2t^{107} - 2t^{105} + t^{104} - 2t^{103}$ $+t^{102} - 3t^{101} + 2t^{100} - 2t^{99} + 2t^{98} - 2t^{97} + 3t^{96} - 2t^{95}$ $+2t^{94} + 4t^{92} + 2t^{90} + t^{89} + 3t^{88} + 3t^{87} + 2t^{86} + 3t^{85}$ $+t^{84} + 5t^{83} - t^{82} + 6t^{81} - 2t^{80} + 6t^{79} - 5t^{78} + 8t^{77}$ $-5t^{76} + 7t^{75} - 8t^{74} + 7t^{73} - 10t^{72} + 8t^{71} - 10t^{70} + 6t^{69}$ $-12t^{68} + 6t^{67} - 12t^{66} + 5t^{65} - 12t^{64} + 3t^{63} - 12t^{62} + 3t^{61}$ $-11t^{60} + t^{59} - 9t^{58} - 8t^{56} - 4t^{54} - 2t^{53} - 4t^{52} - 2t^{51}$ $-2t^{49} - 3t^{47} + 3t^{46} - 2t^{45} + 4t^{44} - 2t^{43} + 6t^{42} - 2t^{41}$ $+4t^{40} - t^{39} + 7t^{38} - t^{37} + 4t^{36} + 6t^{34} + 3t^{32} + 3t^{30}$ $+t^{29} + t^{28} + t^{27} + t^{26} + t^{25} - 2t^{24} + t^{23} + t^{21}$ $-3t^{20} + t^{19} - t^{18} + t^{17} - 3t^{16} + t^{15} - 2t^{14} + t^{13}$ $-2t^{12} + t^{11} - t^{10} + t^{9} - 2t^{8} + t^{7} + t^{5} - t^{4} + t^{3} + t - 1$

Table 17: The quasi-Lannér diagrams on 9 vertices and growth functions (cont.)

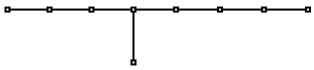
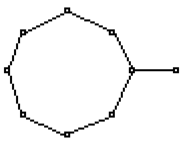
Label	Diagram χ, Degrees	Growth function D:=denominator
$QL9_3$	 $1/696729600$ $(120, 119)$ The number of infinite special subgroups: 1	The numerator is [2][8][12][14][18][20][24][30] $D = t^{119} - t^{111} - t^{107} - t^{105} - t^{103} + t^{102} - 2t^{101}$ $+t^{100} - t^{99} + t^{98} - 2t^{97} + 3t^{96} - 3t^{95} + 2t^{94}$ $-t^{93} + 3t^{92} - 3t^{91} + 5t^{90} - 3t^{89} + 3t^{88} - t^{87}$ $+5t^{86} - 5t^{85} + 6t^{84} - 2t^{83} + 3t^{82} - 2t^{81} + 6t^{80}$ $-6t^{79} + 5t^{78} - t^{77} + 2t^{76} - 4t^{75} + 6t^{74} - 5t^{73}$ $+t^{72} + t^{70} - 5t^{69} + 3t^{68} - 2t^{67} - 3t^{66} + t^{65}$ $-4t^{63} - t^{62} + 2t^{61} - 6t^{60} + t^{59} - t^{58} - t^{57}$ $-5t^{56} + 7t^{55} - 7t^{54} + 2t^{53} - 2t^{52} + 4t^{51} - 8t^{50}$ $+9t^{49} - 6t^{48} + 3t^{47} - 3t^{46} + 8t^{45} - 9t^{44} + 9t^{43}$ $-4t^{42} + 4t^{41} - 5t^{40} + 10t^{39} - 8t^{38} + 7t^{37} - 3t^{36}$ $+5t^{35} - 6t^{34} + 9t^{33} - 6t^{32} + 4t^{31} - 3t^{30} + 5t^{29}$ $-6t^{28} + 6t^{27} - 4t^{26} + 2t^{25} - 3t^{24} + 4t^{23} - 5t^{22}$ $+3t^{21} - 2t^{20} + t^{19} - 3t^{18} + 3t^{17} - 3t^{16} + t^{15}$ $-t^{14} + t^{13} - 2t^{12} + 2t^{11} - t^{10} + t^{7} - t^{6} + t^{5} + t^{1} - 1$
$QL9_4$	 $(120, 120)$ $17/43545600$ The number of infinite special subgroups: 2	The numerator is [2][8][12][14][18][20][24][30] $D = t^{120} + t^{119} - t^{115} + t^{114} - t^{113} - t^{112} - 2t^{111}$ $-2t^{109} - 3t^{107} - t^{106} - 2t^{105} + t^{104} - 3t^{103} + 2t^{102}$ $-2t^{101} + 2t^{100} + 6t^{98} - t^{97} + 6t^{96} + t^{95} + 9t^{94}$ $+3t^{93} + 11t^{92} + 3t^{91} + 12t^{90} + 5t^{89} + 12t^{88} + 6t^{87}$ $+14t^{86} + 4t^{85} + 11t^{84} + 6t^{83} + 11t^{82} + 3t^{81} + 6t^{80}$ $+5t^{78} + t^{77} - 2t^{76} - 5t^{75} - 5t^{74} - 7t^{73} - 13t^{72}$ $-8t^{71} - 16t^{70} - 14t^{69} - 22t^{68} - 14t^{67} - 26t^{66} - 15t^{65}$ $-31t^{64} - 19t^{63} - 31t^{62} - 16t^{61} - 36t^{60} - 17t^{59} - 33t^{58}$ $-16t^{57} - 33t^{56} - 9t^{55} - 31t^{54} - 10t^{53} - 27t^{52} - 5t^{51}$ $-23t^{50} + t^{49} - 22t^{48} + t^{47} - 12t^{46} + 7t^{45} - 12t^{44}$ $+11t^{43} - 6t^{42} + 10t^{41} - 3t^{40} + 15t^{39} - t^{38} + 15t^{37}$ $+2t^{36} + 14t^{35} + 4t^{34} + 16t^{33} + t^{32} + 12t^{31} + 4t^{30}$ $+11t^{29} + 3t^{28} + 10t^{27} + t^{26} + 6t^{25} + t^{24} + 5t^{23}$ $-t^{22} + 4t^{21} - t^{20} + t^{19} - t^{18} + t^{17} - 3t^{16} - 2t^{14}$ $-t^{13} - t^{12} - t^{10} - t^{9} - t^{8} + t^{2} + t^{1} - 1$

Table 18: The quasi-Lannér diagrams on 10 vertices and growth functions

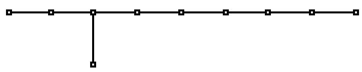
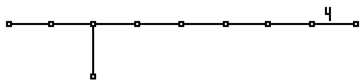
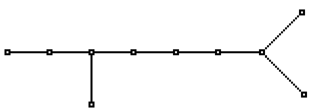
Label	Degrees Diagram	Growth function in all cases $\chi = 0$; D:=denominator
$QL10_1$	$(129, 128)$ 	The numerator is $[2]^2[12][14][16][18][20][24][30]$ $D = -1 + t + t^2 - t^3 + t^7 - t^8 - t^9 + t^{10} + t^{11} - t^{13} - t^{18} + t^{19} - t^{21} + t^{22} - t^{24} - t^{26} + t^{27} + t^{28} - t^{30} + t^{32} - t^{36} + 2t^{37} + t^{38} - t^{39} + t^{43} - t^{45} + t^{46} + t^{47} - 2t^{48} + t^{50} - t^{51} - t^{54} + t^{55} - 2t^{57} + t^{59} - 2t^{60} + t^{62} - t^{63} - t^{66} + t^{67} - t^{69} + t^{71} + t^{74} - t^{75} + 2t^{77} - t^{81} + 2t^{82} + t^{83} - t^{84} + t^{86} - t^{93} + t^{94} + t^{95} - 2t^{96} - t^{97} + t^{98} - t^{101} - t^{102} + t^{103} - t^{105} + t^{116} + t^{126} - t^{128}$
$QL10_2$	$(129, 129)$ 	The numerator is $[2]^2[12][14][16][18][20][24][30]$ $D = -1 + t + t^2 - t^4 + t^7 - 2t^8 + t^9 + t^{11} - 2t^{12} + t^{13} - t^{14} + 2t^{15} - 4t^{16} + 2t^{17} - 2t^{18} + 4t^{19} - 4t^{20} + 3t^{21} - 3t^{22} + 5t^{23} - 7t^{24} + 6t^{25} - 4t^{26} + 8t^{27} - 8t^{28} + 7t^{29} - 6t^{30} + 11t^{31} - 11t^{32} + 9t^{33} - 7t^{34} + 13t^{35} - 14t^{36} + 12t^{37} - 9t^{38} + 14t^{39} - 17t^{40} + 12t^{41} - 11t^{42} + 17t^{43} - 18t^{44} + 11t^{45} - 12t^{46} + 17t^{47} - 22t^{48} + 12t^{49} - 12t^{50} + 15t^{51} - 18t^{52} + 10t^{53} - 13t^{54} + 15t^{55} - 18t^{56} + 7t^{57} - 9t^{58} + 13t^{59} - 15t^{60} + 5t^{61} - 4t^{62} + 9t^{63} - 11t^{64} + 2t^{65} - t^{66} + 8t^{67} - 5t^{68} - 3t^{69} + 5t^{70} + 5t^{71} - 2t^{72} - 5t^{73} + 9t^{74} + 4t^{76} - 6t^{77} + 11t^{78} - 2t^{79} + 4t^{80} - 11t^{81} + 15t^{82} - 3t^{83} + 6t^{84} - 11t^{85} + 13t^{86} - 6t^{87} + 7t^{88} - 11t^{89} + 10t^{90} - 6t^{91} + 7t^{92} - 12t^{93} + 11t^{94} - 5t^{95} + 2t^{96} - 10t^{97} + 8t^{98} - 5t^{99} + 4t^{100} - 8t^{101} + 5t^{102} - 3t^{103} + 2t^{104} - 7t^{105} + 4t^{106} - t^{107} + t^{108} - 4t^{109} + 4t^{110} - t^{111} - 2t^{113} + 2t^{114} + t^{116} - t^{117} + 2t^{118} + t^{119} - t^{120} - t^{121} + t^{122} + t^{126} + t^{127} - t^{128} - t^{129}$
$QL10_3$	$(129, 129)$ 	The numerator is $[2]^2[12][14][16][18][20][24][30]$ $D = -1 + t + t^2 - t^4 + 2t^7 - 3t^8 + 3t^{11} - 3t^{12} + t^{13} - 2t^{14} + 4t^{15} - 6t^{16} + 3t^{17} - 4t^{18} + 8t^{19} - 8t^{20} + 5t^{21} - 5t^{22} + 11t^{23} - 13t^{24} + 9t^{25} - 9t^{26} + 17t^{27} - 16t^{28} + 14t^{29} - 13t^{30} + 22t^{31} - 22t^{32} + 17t^{33} - 16t^{34} + 28t^{35} - 29t^{36} + 24t^{37} - 21t^{38} + 31t^{39} - 34t^{40} + 24t^{41} - 25t^{42} + 37t^{43} - 38t^{44} + 24t^{45} - 26t^{46} + 38t^{47} - 44t^{48} + 24t^{49} - 26t^{50} + 34t^{51} - 37t^{52} + 21t^{53} - 26t^{54} + 33t^{55} - 36t^{56} + 13t^{57} - 16t^{58} + 27t^{59} - 30t^{60} + 9t^{61} - 6t^{62} + 17t^{63} - 19t^{64} + t^{65} + 13t^{67} - 7t^{68} - 9t^{69} + 13t^{70} + 5t^{71} - t^{72} - 14t^{73} + 22t^{74} - 5t^{75} + 10t^{76} - 16t^{77} + 24t^{78} - 8t^{79} + 13t^{80} - 26t^{81} + 31t^{82} - 10t^{83} + 13t^{84} - 23t^{85} + 29t^{86} - 14t^{87} + 14t^{88} - 24t^{89} + 22t^{90} - 12t^{91} + 14t^{92} - 23t^{93} + 21t^{94} - 10t^{95} + 5t^{96} - 18t^{97} + 16t^{98} - 10t^{99} + 6t^{100} - 13t^{101} + 10t^{102} - 5t^{103} + 3t^{104} - 12t^{105} + 7t^{106} - 2t^{107} + t^{108} - 5t^{109} + 6t^{110} - 2t^{111} - 3t^{113} + 3t^{114} + t^{116} - t^{117} + 3t^{118} + 2t^{119} - 2t^{120} - 2t^{121} + 2t^{122} + t^{126} + 2t^{127} - t^{128} - 2t^{129}$

Table 19: The real poles and the extremal absolute values of the non-real poles of the growth functions. Lannér cases

$L5_1$	$L5_2$	$L5_3$	$L5_4$	$L5_5$
0.833415	0.720106	0.659358	yes, correct:	0.61621
0.94166	0.898971	0.875566	no	0.85384
1.06195	1.11238	1.14212	real	1.17118
1.19988	1.38868	1.51663	roots	1.62282
$m = 0.97149$	$m = 0.96401$	$m = 0.93176$	$m = 0.94718$	$m = 0.89454$
$M = 1.02935$	$M = 1.03734$	$M = 1.07324$	$M = 1.05577$	$M = 1.11788$

Table 20: The real poles and the extremal absolute values of the non-real poles of the growth functions. Quasi-Lannér cases (The trivial pole 1 is not indicated)

$QL4_1$	$QL4_2$	$QL4_3$	$QL4_4$	$QL4_5$
0.771327		0.639025	-1.61803	0.667961
$m = 0.950357$	$m = 1$	$m = 0.960217$	0.618034	$m = 0.910638$
$M = 1.103357$	$M = 1.210606$	$M = 1.142917$	$m = M = 1$	$M = 1.343628$

$QL4_6$	$QL4_7$	$QL4_8$	$QL4_9$	$QL4_{10}$
0.708134	-1.61803	0.636883	-1.33552	0.561856
$m = 0.957686$	0.618034	$m = 1.099895$	0.552965	$m = 0.909844$
$M = 1.146305$	$m = M = 1$	$M = 1.139254$	$m = M = 0.822833$	$M = 1.287859$

$QL4_{11}$	$QL4_{12}$	$QL4_{13}$	$QL4_{14}$	$QL4_{15}$
-1.29065	-1.11231			-1.19004
0.51879	0.500245	0.492432	0.551753	0.504138
$m = 1$	$m = 1.032895$	$m = 0.902209$	$m = 1.076010$	$m = 1.103491$
$M = 1.222085$	$M = 1.107983$	$M = 0.911924$	$M = 1.251157$	$M = 1.169974$

$QL4_{16}$	$QL4_{17}$	$QL4_{18}$	$QL4_{19}$
		0.588985	0.552531
0.469396	0.682328	$m = 0.962999$	$m = 0.986410$
$m = M = 0.842693$	$m = M = 1.210606$	$M = 1.231827$	$M = 1.243136$

$QL4_{20}$	$QL4_{21}$	$QL4_{22}$
		-0.767592
0.537613	-1.29065	0.434259
$m = 0.942397$	0.51879	$m = 0.94718$
$M = 1.023332$	$m = M = 1.222085$	$M = 1.05577$

Table 21: The real poles and the extremal absolute values of the non-real poles of the growth functions. Quasi-Lannér cases

$QL5_1$	$QL5_2$	$QL5_3$	$QL5_4$	$QL5_5$
0.698956 0.891273 1.09813 $m = 0.957885$ $M = 1.154830$	-1.236 -1.05414 0.654741 0.872516 1.12047 $m = 0.914538$ $M = 1.161598$	0.72899 0.903396 1.08431 $m = 0.948289$ $M = 1.150045$	-1.62934 0.627864 0.862852 1.12033 $m = 0.915917$ $M = 1.131353$	-1.46751 0.579431 0.842435 1.1368 $m = 0.849730$ $M = 1.074701$

$QL5_6$	$QL5_7$	$QL5_8$	$QL5_9$
0.537456 0.820844 1.18562 $m = 0.903486$ $M = 1.344774$	0.662566 0.876238 1.11333 $m = 0.947039$ $M = 1.119572$	$-1.162980.55887$ 0.831791 1.15607 $m = 0.920577$ $M = 1.106633$	-1.38639 0.491695 0.800368 1.18595 $m = 0.864826$ $M = 1.082570$

Table 22: The real poles and the extremal absolute values of the non-real poles of the growth functions. Quasi-Lannér cases. (The trivial pole 1 is not indicated)

$QL6_1$	$QL6_2$	$QL6_3$	$QL6_4$	$QL6_5$
-1.41222	0.801198		-1.35548	0.744209
0.741226	0.896819		0.667522	0.8651
0.864041	1.07859		0.822211	1.10943
			1.13349	
$m = 0.947515$	$m = 0.966866$	$m = 0.9459183$	$m = 0.940570$	$m = 0.961618$
$M = 1.137513$	$M = 1.130358$	$M = 1.136787$	$M = 1.068224$	$M = 1.147472$

$QL6_6$	$QL6_7$	$QL6_8$	$QL6_9$	$QL6_{10}$
-1.30069	0.702245	-1.28976	-1.25535	0.657119
0.634641	0.840655	0.59287	0.542596	0.814442
0.802625	1.13578	0.777193	0.744994	1.1665
1.15397		1.17396	1.20423	
$m = 0.915468$	$m = 0.881099$	$m = 0.897402$	$m = 0.747308$	$m = 0.993132$
$M = 1.051409$	$M = 1.177095$	$M = 0.981604$	$M = 0.932831$	$M = 1.106255$

$QL6_{11}$	$QL6_{12}$
-1.06958	0.533802
0.604368	0.742603
0.784304	1.19555
1.17725	
$m = 0.907820$	$m = 0.889811$
$M = 1.237682$	$M = 1.010751$

Table 23: The real poles and the extremal absolute values of the non-real poles of the growth functions. Quasi-Lannér cases

$QL7_1$	$QL7_2$	$QL7_3$
0.755431	0.726032	0.656509
0.847161	0.827211	0.780576
0.94682	0.939344	0.92162
1.0582	1.06657	1.08703
1.11047	1.13199	1.18705
$m = 0.943848$	$m = 0.937174$	$m = 0.892428$
$M = 1.145506$	$M = 1.141785$	$M = 1.160250$

Table 24: The real poles and the extremal absolute values of the non-real poles of the growth functions. Quasi-Lannér cases. (The trivial pole 1 is not indicated)

$QL8_1$	$QL8_2$	$QL8_3$	$QL8_4$
−1.25799	−1.29534	0.77866	0.657583
−1.04243	−1.0366	0.84753	0.760101
0.763804	0.741982	0.92114	0.873128
0.837519	0.821394		
0.915858	0.907002		
$m = 0.939234$	$m = 0.921189$	$m = 0.932965$	$m = 0.926349$
$M = 1.100237$	$M = 1.114182$	$M = 1.134982$	$M = 1.171854$

Table 25: The real poles and the extremal absolute values of the non-real poles of the growth functions. Quasi-Lannér cases

$QL9_1$	$QL9_2$	$QL9_3$	$QL9_4$
-1.28534	-1.23784	-1.19828	-1.03174
-1.02229	-1.0607	-1.01676	0.659124
0.770075	0.753304	0.826149	0.746793
0.831719	0.818656	0.873169	0.841212
0.896113	0.88767	0.922226	0.944445
0.964269	0.96124	0.973477	1.05875
1.03702	1.04029	1.02722	
$m = 0.935688$	0.913000	$m = 0.958697$	$m = 0.878261$
$M = 1.108580$	1.121820	$M = 1.093674$	$M = 1.347490$

Table 26: The real poles and the extremal absolute values of the non-real poles of the growth functions. Quasi-Lannér cases. (The trivial pole 1 is not indicated)

$QL10_1$	$QL10_2$	$QL10_3$
-1.14077	-1.01583	-1.014947
-1.008	0.774744	0.7615867
0.878674	0.827863	0.8172635
0.907888	0.882622	0.8751652
0.93783	0.939841	0.9359320
0.968518	1.06238	1.0662355
1.03182		
$m = 0.968106$	$m = 0.932746$	$m = 0.933795$
$M = 1.068399$	$M = 1.342964$	$M = 1.219335$

References

[BC] Bartholdi L., Ceccherini-Silberstein T. G., Salem numbers and growth series of some hyperbolic graphs. Geom. Dedicata **90** (2002), 107–114; `arXiv:math.GR/9910067 v2`

[Bo] Bott R., An application of the Morse theory to the topology of Lie-groups. Bull. Soc. Math. France **84** (1956), 251–281

[Bou] Bourbaki, N. *Lie groups and Lie algebras. Chapters 4–6.* Translated from the 1968 French original by A. Pressley. Elements of Mathematics (Berlin). Springer, Berlin, 2002. xii+300 pp.

[BS] de Buyl S., Schomblond Ch., Hyperbolic Kac Moody algebras and Einstein billiards. J. Math. Phys. **45** (2004) 4464–4492. `arXiv: hep-th/0403285`

[DCh] `http://sasja.shap.homedns.org/subg/subg.rar` The package for computing growth series of Coxeter groups, with instruction.

[CCLL] Chapovalov D., Chapovalov M., Lebedev A., Leites D., The classification of almost affine (hyperbolic) Lie superalgebras. `arXiv: ??`

[CW] Cannon J. W., Wagreich Ph., Growth functions of surface groups. Math. Ann., **293**(2), 1992, 239–257

[CM] Coxeter H.S.M, Mozer W.O.J., *Generators and relations for the discrete groups.* 4th ed. Springer, NY, 1984

[Cox] Coxeter H. S. M., The product of the generators of a finite group generated by reflections. Duke Math. J. **18** (1951), 765–782.

[CW50] Coxeter H. S. M., Whitrow G. J., World-structure and non-Euclidean honeycombs, Proc. Royal Soc. London **A201** (1950), 417–437.

[DDJO] Davis M., Dymara J., Januszkiewicz T., Okun B. Weighted L^2-cohomology of Coxeter groups, Geometry & Topology **11** (2007), 47–138

[E] Eneström G. Härledning av en allmän formel för antalet pensionärer, som vid en godtycklig tidpunkt förefinnas inom en sluten pensionskassa [Derivation of a general formula for the number of pensioners in a closed pension fund at a given time], Öfversigt af Kongl. Vetenskaps-Akademiens Förhandlingar (Stockholm) **50** (1893) 405–415. (in Swedish)

id., Remarque sur une théorème relatif aux racines de l'equation $a_n x^n + \cdots + a_0 = 0$ où tous les coefficients sont réeles et positifs. Tohôku Math. J. **18** (1920), 34–36 (kind of translation of the above paper)

[FGI] Fomin D., Genkin S., Itenberg I. *Mathematical circles (Russian experience).* AMS, 1996, 272pp.

[Gal] Gal Ś. On the poles of growth series of Coxeter groups. `http://www.math.uni.wroc.pl/ sgal` June, 2004

[GNa] Grigorchuk, R., Nagnibeda T., Complete growth functions of hyperbolic groups. Invent. Math. **130** (1997) no. 1, 159–188

[GN] Gritsenko V., Nikulin V., On classification of Lorentzian Kac-Moody algebras. Russian Math. Surveys **57** (2002), no. 5, 921–979. `arXiv:math.QA/0201162`

[JKRT] Johnson N. W., Kellerhals R., Ratcliffe J. G., and Tschantz S. T. The size of a hyperbolic Coxeter simplex. Transform. Groups **4** (1999), 329–353

[H] Henneaux M., Kac-Moody algebras and the structure of cosmological singularities: A New light on the Belinskii-Khalatnikov-Lifshitz analysis. `arXiv:hep-th/0806.4670`

[Kak] Kakeya S., On the limits of roots of an algebraic equation with positive coefficients, Tohôku Math. J. **2** (1912) 140–142.

[KoMo] Kobayashi Z., Morita J., Automorphisms of certain root lattices. Tsukuba J. Math., **7** (1983) 323–336

[La] Lannér F., On complexes with transitive groups of automorphisms, Medd. Lunds Univ. Mat. Sem. [Comm. Sem. Math. Univ. Lund], **11** (1950), 1–71

[Li] Li Wang Lai, Classification of generalized Cartan matrices of hyperbolic type. Chin. Ann. Math. **9B** (1988) 68–77

[McD] Macdonald, I.G, The Poincaré Series of a Coxeter Group. Math. Ann., **199** (1972) 161–174

[OV] Onishchik A., Vinberg E., *Lie groups and algebraic groups.* Translated from the Russian and with a preface by D. Leites. Springer Series in Soviet Mathematics. Springer, Berlin, 1990. xx+328 pp.

[Par] Parry W., Growth series of Coxeter groups and Salem numbers, J. of Algebra **154** (1993), 406–415.

[RU] Ray, U., *Automorphic Forms and Lie Superalgebras*, Springer, 2006, IX, 285 pp.

[S] Saçlioğlu C., Dynkin diagrams for hyperbolic Kac–Moody Algebras, J. Phys. A: Math. Gen. **22** (1989) 3753–3769.

[So] Solomon, L., The orders of the finite Chevalley groups. J. Algebra **3** (1966) 376–393

[Ste] Steinberg R. Endomorphisms of linear algebraic groups. Memoirs Amer. Math. Soc. **80**, (1968), 1–108; also available in *Collected papers*. Amer. Math. Soc. (1997)

[St] Stekolshchik R., *Notes on Coxeter transformations and the McKey correspondence.* Springer, Berlin et al, 2008, XIII+ 239 pp.

[Vi] Vinberg E. B., Discrete linear groups that are generated by reflections. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 1072–1112. English translation in: Math. USSR Izvestia **35** (1971), 1083–1119

[VSh] Vinberg E. B., Shvartsman O. V., Discrete groups of motions in spaces of constant curvature, Itogi Nauki i Tekhniki Sovr. Prob. Mat. **29** (1988), 147–259. (Russian) English translation: Geometry, II, Encyclopaedia Math. Sci., **29**, Springer, Berlin, 1993, 139–248

[Wa] Wagreich P., The growth function of a discrete group, Group actions and vector fields (Vancouver, B.C., 1981), Lecture Notes in Math., **956**, Springer, Berlin, 1982, 125–144.

[Wo] Worthington R. L., The growth series of compact hyperbolic Coxeter groups with 4 and 5 generators. Canad. Math. Bull. **41** (2), 1998, 231–239