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Spherical harmonics: a theoretical and graphical study

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Abstract

The topic of harmonic polynomials is briefly discussed to show that every polynomial on $\mathbb{R}^n$ can be decomposed into harmonic polynomials. Using this property it is proved that every function that is square integrable on the hypersphere can be represented by a series of spherical harmonics (harmonic polynomials restricted to the hypersphere), and that the series is converging with respect to the norm in this space. Explicit formulas for these functions and series are calculated for three dimensional euclidean space and used for graphical illustrations. By applying stereographic projection a way of graphically illustrating spherical harmonics in the plane and how a given function is approximated by a sum of spherical harmonics is presented.

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1 Introduction

From Fourier analysis it is known that an infinite set of orthogonal sine and cosine functions span the space of square integrable functions on the interval $[-\pi, \pi]$. By considering functions in n-dimensional space that solve Laplace's equation, a subclass of functions called spherical harmonics can be defined. These functions can be shown to be an analogue to the sine and cosine functions in the sense that spherical harmonics of different degrees form an orthogonal basis that spans the space of functions that are square integrable on the sphere.

This study is meant to present some of the available information on spherical harmonics in a way that appeals to a reader at the undergraduate level. The main aim is to establish a clear connection between the special cases of Fourier analysis in $\mathbb{R}^2$ and spherical harmonics in $\mathbb{R}^3$, both by using theory and by graphically illustrating the spherical harmonics in a number of ways.

Each section is structured around one or a few central results. These will be introduced at the beginning of each section, in the form of a discussion or as a stated theorem. After this is done, the tools needed to prove the relevant theorems will be introduced. The purpose of this layout is to give the reader an appreciation of the importance and consequences of the central theorems.

The first part of this study, presented in section 2 and 3, is concerned with the general topic of harmonic polynomials and how these can be restricted to the sphere to define spherical harmonics. Since this theoretical part is not greatly facilitated by only considering three dimensional euclidean space, it will include complex valued functions $f(x): \mathbb{R}^n \mapsto \mathbb{C}$. No explicit formulas for spherical harmonics are derived in this section, which can make it seem somewhat abstract. It is recommended that the reader looks through the graphical illustrations at the end of this study to get an intuitive understanding of spherical harmonics while reading the general theory. Later sections will take a more formal approach to these illustrations. The main result of section 2 is the orthogonal decomposition of functions that are square integrable on the sphere into spherical harmonics, which is presented in Theorem 2.6. In section 3 we focus on a way to calculate the unique spherical harmonics that decompose a given function, which is given by the formula in Theorem 3.2. However, as can be seen in later sections that include explicit calculations, this formula is of theoretical rather than practical value.

The second part (section 4) is theory applied to three dimensional euclidean space. The emphasis in this section is on finding the solution to Laplace's equation in spherical coordinates. The answer results in an explicit expression for spherical harmonics in three dimensions. In Theorem 4.4 it is summarized how to find the expansion of a given function into spherical harmonics. We will only consider the case of real-valued functions, thus finding formulas that can be applied directly in the final section containing the illustrations.

The last part (section 5) graphically illustrates the theoretical concepts from part two. Examples will be given of both traditional ways of doing this, as well as less common ways (not found in literature during the research of this study). To accomplish this, some theory about stereographic projection is presented.

The general disposition of section 2 and 3 are inspired heavily by [1, Chapter 5]. Some theorems have been added or chosen to be proved in a different way. If so, their source will be referred to in the text. The idea to section 4 is from [3, Chapter 10], [4] and [8].

2 Harmonic polynomials

2.1 Definitions and notations

In this study n will always denote a positive integer. A function f that is square integrable on $\mathbb{R}^n$ is written as $f \in L^2(\mathbb{R}^n)$. A function $f(x)$ defined on an open subset of $\mathbb{R}^n$ that is at least twice continuously differentiable and fulfills Laplace's equation (1)

$$\frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} \equiv 0, \tag{1}$$

is called *harmonic*. Defining *the Laplacian operator* Δ as the sum of all the second partial derivatives the above condition can be written as

$$\Delta f \equiv 0. \tag{2}$$

Note that this definition applies to complex valued functions, since it would only mean that the real and imaginary parts of f are both harmonic.

As is customary $\mathbb{R}$ denotes the real numbers and $\mathbb{C}$ the complex numbers. If a function f is continuous on a given set $\mathcal{K}$, this will be denoted $f \in C(\mathcal{K})$. The unit sphere is defined as the boundary of the unit ball, and denoted as S. It is understood that if dealing with a subset of $\mathbb{R}^n$, the surface that is the unit sphere has dimension $n-1$.

2.2 The orthogonal decomposition of polynomials

A polynomial $p(x)$ on $\mathbb{R}^n$ is called *homogeneous of degree* k if for a constant t it fulfills $p(tx) = t^k p(x)$. The space of polynomials that are homogeneous of degree k will be denoted $\mathcal{P}_k(\mathbb{R}^n)$ and the subspace of $\mathcal{P}_k(\mathbb{R}^n)$ containing those polynomials that are harmonic will be denoted $\mathcal{H}_k(\mathbb{R}^n)$. Note that every polynomial P of degree k on $\mathbb{R}^n$ can be written as $P = \sum_{j=0}^{k} p_j$, where each $p_j \in \mathcal{P}_j(\mathbb{R}^n)$. Since $\Delta P = \sum_{j=0}^{k} \Delta p_j$, we have that P is harmonic if and only if each $p_j \in \mathcal{H}_j(\mathbb{R}^n$. Given this fact, this section will focus on the polynomials $p_k \in \mathcal{H}_k(\mathbb{R}^n)$.

The main result of this section is about the decomposition of homogeneous polynomials. This is presented in the theorem below.

Theorem 2.1. *If $k \geq 2$, then*

$$\mathcal{P}_k(\mathbb{R}^n) = \mathcal{H}_k(\mathbb{R}^n) \oplus |x|^2 \mathcal{P}_{k-2}(\mathbb{R}^n).$$

However, before proving the statement above, let us consider some important consequences. Theorem 2.1 states that *every* homogeneous polynomial $p \in \mathcal{P}_k(\mathbb{R}^n)$ can be decomposed in this way. Naturally this argument can be transferred to a homogeneous polynomial $q \in \mathcal{P}_{k-2}(\mathbb{R}^n)$. Extending this to polynomials of lesser degree we get:

$$\begin{aligned}
&p = p_k + |x|^2 q, \text{ for some } p_k \in \mathcal{H}_k(\mathbb{R}^n),\ q \in \mathcal{P}_{k-2}(\mathbb{R}^n),\\
&q = p_{k-2} + |x|^2 s, \text{ for some } p_{k-2} \in \mathcal{H}_{k-2}(\mathbb{R}^n),\ s \in \mathcal{P}_{k-4}(\mathbb{R}^n),\\
&\vdots
\end{aligned}$$

This relation holds $\frac{k}{2}$ times (or the largest integer less than this), leaving the last term to contain either a polynomial of degree 1 or a constant. Substituting the above relations stepwise leads us to a corollary to Theorem 2.1.

Corollary 2.2. *Every $p \in \mathcal{P}_k(\mathbb{R}^n)$ can be uniquely written in the form*

$$p = p_k + |x|^2 p_{k-2} + \cdots + |x|^{2m} p_{k-2m},$$

where m denotes the largest integer less than or equal to $\frac{k}{2}$ (that is $k-2m$ equals 0 if k is even, 1 if k is odd) and $p_j \in \mathcal{H}_j(\mathbb{R}^n)$.

Proof. The proof has already been outlined in the argument above. Noting that $\mathcal{P}_k(\mathbb{R}^n) = \mathcal{H}_k(\mathbb{R}^n)$ for $k = 0, 1$, we see that the statement is true for these values of k. For $k \geq 2$, the proof is by induction assuming that the equality holds when k is replaced by $k - 2$. This holds because of Theorem 2.1, giving the above result.

For the uniqueness of the decomposition, assume that

$$p_k + |x|^2 q_{k-2} = \tilde{p}_k + |x|^2 \tilde{q}_{k-2},$$

where $p_k, \tilde{p}_k \in \mathcal{H}_k(\mathbb{R}^n)$ and $q_{k-2}, \tilde{q}_{k-2} \in \mathcal{P}_{k-2}(\mathbb{R}^n)$. This is equivalent to

$$p_k - \tilde{p}_k = |x|^2 \tilde{q}_{k-2} - |x|^2 q_{k-2}.$$

Since the left hand side of the equation above is a harmonic polynomial, this must also be true for the right hand side. But according to Theorem 2.1 the right hand side does not belong to $\mathcal{H}_k(\mathbb{R}^n)$. Hence the only way the equality can hold is if $q_{k-2} - \tilde{q}_{k-2} = 0$. $\square$

The importance of this corollary becomes apparent when considering polynomials that are restricted to the sphere. In this special case Corollary 2.2 becomes the following statement.

Corollary 2.3. *Any homogeneous polynomial $p \in \mathcal{P}_k(\mathbb{R}^n)$ restricted to the unit sphere can be uniquely written on the form*

$$p = p_k + p_{k-2} + \cdots + p_{k-2m},$$

where m denotes the largest integer less than or equal to $\frac{k}{2}$ (that is $k-2m$ equals 0 if k is even, 1 if k is odd) and $p_j \in \mathcal{H}_j(\mathbb{R}^n)$.

Proof. Just applying the fact that any power of the factor $|x|^2 = 1$ on S to Corollary 2.2, gives us the decomposition of p. We will only comment on the uniqueness by observing that the decomposition is harmonic and is the (unique) solution to the Dirichlet problem for the ball (see [1, p. 12]) when the boundary data is the restriction of p to the sphere. $\square$

From our initial discussion about homogeneous polynomials we know that Corollary 2.3 indirectly states that any polynomial on the sphere can be written as a sum of unique harmonic homogeneous polynomials (on the sphere). Here we have already hinted at the importance of Theorem 2.1 and that this leads to a special reason to study harmonic polynomials on the sphere. Since this topic will be more thoroughly discussed in section 2.4, we leave this special case for now.

So far we have only stated Theorem 2.1. For the proof it will be necessary to rely on some facts from linear algebra about the decomposition of dual spaces. In particular the following about adjoint mappings is used ([5, p. 204]).

Adjoint mappings. *Let E and F be inner product spaces. Then the linear map $\varphi : E \mapsto F$ induces a map $\tilde{\varphi} : F \mapsto E$ satisfying*

$$\langle \varphi x, y \rangle = \langle x, \tilde{\varphi} y \rangle \,, \tag{3}$$

where φ and $\tilde{\varphi}$ are said to be adjoint. By this relation F can be orthogonally decomposed as

$$F = \operatorname{Im} \varphi \oplus \ker \tilde{\varphi}. \tag{4}$$

For a thorough discussion on the definitions and the linear algebra used, see for instance [5, Chapter II]. We now prove Theorem 2.1 ([7, Theorem 4.1.1]).

Proof of Theorem 2.1. The goal of this proof is to find adjoint maps from $\mathcal{P}_k(\mathbb{R}^n) \to \mathcal{P}_{k-2}(\mathbb{R}^n)$ and $\mathcal{P}_{k-2}(\mathbb{R}^n) \to \mathcal{P}_k(\mathbb{R}^n)$, such that equation (4) can be used to determine the orthogonal decomposition of these spaces. To accomplish this an inner product is defined to suit this specific purpose. To facilitate this process we introduce multi-index notation at this point.

If $x \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ we define

$$\begin{aligned} x^\alpha &= x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}, \\ \alpha! &= \alpha_1! \alpha_2! \ldots \alpha_n!, \\ |\alpha| &= \alpha_1 + \alpha_2 + \cdots + \alpha_n \quad \text{and} \\ \frac{\partial^\alpha}{\partial x^\alpha} &= \frac{\partial}{\partial x_1^{\alpha_1}} \frac{\partial}{\partial x_2^{\alpha_2}} \cdots \frac{\partial}{\partial x_n^{\alpha_n}}. \end{aligned}$$

Any polynomial $p(x) \in \mathcal{P}_k(\mathbb{R}^n)$ can be written on the form

$$p(x) = \sum_\alpha c_\alpha x^\alpha, \quad \text{where } |\alpha| = k, \text{ and } c_\alpha \in \mathbb{C}.$$

Using the operator $p(D) = \sum_\alpha c_\alpha \frac{\partial^\alpha}{\partial x^\alpha}$, an inner product on $\mathcal{P}_k(\mathbb{R}^n)$ can be defined as follows.

$$\begin{aligned} \langle p, q \rangle &= p(D)[\overline{q}] \\ &= \sum_{|\alpha|=k} c_\alpha \frac{\partial^\alpha}{\partial x^\alpha} \left(\sum_{|\beta|=k} \overline{d_\beta} x^\beta \right) \\ &= \sum_{|\alpha|,|\beta|=k} c_\alpha \overline{d_\beta} \delta_{\alpha\beta} \alpha!, \end{aligned}$$

where $p, q \in \mathcal{P}_k(\mathbb{R}^n)$, c_α and d_β are (complex) constants and $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$ and $\delta_{\alpha\beta} = 0$ if $\alpha \neq \beta$.

Now assume that $p \in \mathcal{P}_k(\mathbb{R}^n)$ is orthogonal to $|x|^2 \mathcal{P}_{k-2}(\mathbb{R}^n)$, so that $\langle |x|^2 q, p \rangle =$

0 for all $q \in \mathcal{P}_{k-2}(\mathbb{R}^n)$. Then by the definition of the inner product we get

$$\begin{aligned}\langle |x|^2 q, p\rangle &= (|x|^2 q)(D)[\overline{p}] \\ &= \left(\sum_{j=1}^{n} \frac{\partial^2}{\partial^2 x_j^2} \sum c_\alpha \frac{\partial^\alpha}{\partial x^\alpha}\right)(\overline{p}) \\ &= \Delta q(D)[\overline{p}] = q(D)[\Delta\overline{p}] \\ &= q(D)[\overline{\Delta p}] = \langle q, \Delta p\rangle = 0.\end{aligned}$$

Since $\Delta p \in \mathcal{P}_{k-2}(\mathbb{R}^n)$, the calculation above indicates that Δp is orthogonal to *every* $q \in \mathcal{P}_{k-2}(\mathbb{R}^n)$. This can only be true if $\Delta p \equiv 0$, and therefore $p \in \mathcal{H}_k(\mathbb{R}^n)$.

Now consider the map

$$\varphi : \mathcal{P}_k(\mathbb{R}^n) \to \mathcal{P}_{k-2}(\mathbb{R}^n) \quad \text{such that} \quad p \mapsto \Delta p.$$

According to the above argument and equation (3) this has the adjoint map

$$\tilde{\varphi} : \mathcal{P}_{k-2}(\mathbb{R}^n) \to \mathcal{P}_k(\mathbb{R}^n) \quad \text{such that} \quad q \mapsto |x|^2 q.$$

With $\operatorname{Im}\tilde{\varphi} = \{|x|^2 q;\ q \in \mathcal{P}_{k-2}(\mathbb{R}^n)\}$ and $\ker\varphi = \{p;\ p \in \mathcal{H}_k(\mathbb{R}^n)\}$, equation (4) shows that

$$\mathcal{P}_k(\mathbb{R}^n) = \mathcal{H}_k(\mathbb{R}^n) \oplus \mathcal{P}_{k-2}(\mathbb{R}^n).$$

□

Finally, note that the orthogonality in Theorem 2.1 implies that no polynomial times the factor $|x|^2$ is harmonic.

2.3 The dimension of homogeneous harmonic polynomials

This section is dedicated to finding $\dim\mathcal{H}_k(\mathbb{R}^n)$. We start by considering the case $n = 2$. From complex analysis it is known that any polynomial $p(z) = a_0 + a_1 z + a_2 z^2 ...$ (where $a_0, a_1, a_2 \ldots$ are complex constants) can be written on the complex form $p(x, y) = u(x, y) + iv(x, y)$. Since every polynomial $p(z)$ is analytic it follows that $u(x, y)$ and $v(x, y)$ are both harmonic functions. For a homogeneous polynomial of degree k we have that $u = \frac{a_k z^k + \overline{a_k}\overline{z}^k}{2}$ and $v = \frac{a_k z^k - \overline{a_k}\overline{z}^k}{2i}$. This indicates that both z^k and $\overline{z}^k$ are homogeneous harmonic polynomials. Hence every homogeneous harmonic polynomial p_k can be written as a complex linear combination of $\{z^k, \overline{z}^k\}$. From this we can see that $\dim\mathcal{H}_k(\mathbb{R}^2) = 2$ for all values of $k \geq 1$. When $k = 0$ we have that p_0 is a constant function and only has dimension equal to one.

For $n > 2$, we note that Theorem 2.1 gives that the dimension of $\mathcal{H}_k(\mathbb{R}^n)$ is equal to $\dim\mathcal{P}_k(\mathbb{R}^n)$ minus $\dim\mathcal{P}_{k-2}(\mathbb{R}^n)$. Hence all that is needed is to find $\dim\mathcal{P}_k(\mathbb{R}^n)$. This can be accomplished through combinatorics, as is shown in the proof of the proposition below. For the values $k = 0, 1$ every homogeneous polynomial is harmonic, so we restrict our attention to $k \geq 2$.

Proposition 2.4. *If $k \geq 2$, then*

$$\dim\mathcal{H}_k(\mathbb{R}^n) = \binom{n+k-1}{n-1} - \binom{n+k-3}{n-1}$$

Proof. If we use the multi-index notation introduced in the proof of Theorem 2.1, we are looking for all the unique monomials

$$x^\alpha \quad \text{such that } |\alpha| = k.$$

This set of combinations can be seen as a basis for the space of homogeneous polynomials of degree k, since every $p \in \mathcal{P}_k(\mathbb{R}^n)$ is a linear combination of these elements.

In other words, we are asking the question "*What is the number of unordered selections, with repetition, of k objects from a set of n objects that can be made?*" ([2, Theorem 11.2]). The answer is found in combinatorics and is

$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}.$$

Hence the expression above equals $\dim\mathcal{P}_k(\mathbb{R}^n)$. Similarly $\dim\mathcal{P}_{k-2}(\mathbb{R}^n) = \binom{n+k-3}{n-1}$. Since $\dim\mathcal{H}_k(\mathbb{R}^n) = \dim\mathcal{P}_k(\mathbb{R}^n) - \dim\mathcal{P}_{k-2}(\mathbb{R}^n)$, this finishes the proof. $\square$

We can easily calculate $\dim\mathcal{H}_k(\mathbb{R}^n)$ for $n = 2$, by using the formula from Theorem 2.2.

$$\dim\mathcal{H}_k(\mathbb{R}^2) = \binom{k+1}{1} - \binom{k-1}{1} = (k+1) - (k-1) = 2$$

for values of $k \geq 2$. This confirms our previous argument that any $p \in \mathcal{H}_k(\mathbb{R}^2)$ is in the complex linear span of $\{z^k, \overline{z}^k\}$. For $n = 3$ a similar calculation show that $\dim\mathcal{H}_k(\mathbb{R}^3) = 2k + 1$, so the dimension of homogeneous harmonic polynomials increase linearly with the degree. Further calculations can be made and will reveal that if $n = 4$ the dimension will increase in a quadratic manner, if $n = 5$ in a cubic manner etc.

Now that we have discussed the basic properties of homogeneous harmonic polynomials, we are ready to study what the results will be if these are restricted to the sphere. This will be the main purpose of the next section, which contains the general theory of spherical harmonics.

2.4 Spherical Harmonics

In section 2.2 we concluded that the restriction of harmonic polynomials to the sphere resulted in important consequences, which motivates the following definition.

Definition 2.5. *A homogeneous harmonic polynomials of degree k on $\mathbb{R}^n$ restricted to the unit sphere is called a spherical harmonic of degree k. The set of spherical harmonics of degree k is denoted $\mathcal{H}_k(S^{n-1})$. If the situation permits, the dimension of the sphere will be omitted and the set will be denoted $\mathcal{H}_k(S)$.*

The aim of this section is to find an (infinite) orthogonal set of functions that span the space $L^2(S)$. That the set of spherical harmonics should be such a set can be motivated by the following argument considering Fourier analysis. Usually one thinks of a Fourier series as an expansion of a given function $f(x) \in L^2[-\pi, \pi]$ into trigonometric functions on the interval $[-\pi, \pi]$. This is usually written as

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

where a_0, a_k and b_k are constants. However, from trigonometry we know that the sine and cosine functions are functions defined on the unit circle, and that these can be written on exponential form as $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$ (where θ is the angle from the positive x-axis). This gives us the above Fourier series in exponential form

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ik\theta}, \tag{5}$$

where c_k are complex constants that are directly related to a_k and b_k from the ordinary Fourier series. From Parseval's equation and Fourier analysis we conclude that the (complex) linear span of $\{e^{ik\theta}, e^{-ik\theta}\}_{k=0}^{\infty}$ is dense in the space of $L^2[-\pi, \pi]$([3], p. 191). But when $\theta \in [-\pi, \pi]$ the set $\{e^{ik\theta}, e^{-ik\theta}\}_{k=0}^{\infty}$ is just the restriction of $\{z^k, \overline{z}^k\}_{k=0}^{\infty}$ to the one dimensional subspace that is the unit circle.

Hence the Fourier series in equation (5) can be seen as an expansion into spherical harmonics. Thus, in the special case $n = 2$, a Fourier expansion into spherical harmonics and Parseval's equation shows us that $L^2(S) = \oplus_{k=0}^{\infty}\mathcal{H}_k(S)$. This leads us to state the main theorem of this section.

Theorem 2.6. *The infinite set $\{\mathcal{H}_k(S)\}_{k=0}^{\infty}$ is an orthogonal decomposition of the space $L^2(S)$ so that*

$$L^2(S) = \oplus_{k=0}^{\infty}\mathcal{H}_k(S).$$

As has already been shown in Corollary 2.3, any polynomial restricted to the sphere can be written as a sum of spherical harmonics. However, we want to expand the concept of spherical harmonics being an orthogonal basis in the space of polynomials restricted to the sphere to spherical harmonics being an orthogonal basis in the space $L^2(S)$, as is presented in Theorem 2.6. A theorem of great importance in accomplishing this task is the Stone-Weierstrass theorem (S-W). To get a first idea how this is done we will again think about the case of Fourier analysis. The S-W for one variable is as follows ([11, Theorem 7.26]).

If f is a continuous complex function on $[a, b]$, there exists a sequence of polynomials P_n such that

$$\lim_{n\to\infty} P_n(x) = f(x)$$

uniformly on $[a, b]$.

Together with Corollary 2.3 and Parseval's equation, the S-W results in $L^2[-\pi, \pi] = \oplus_{k=0}^{\infty}\mathcal{H}_k(S)$. An analogous result of the S-W holds in higher dimension ([11, Theorem 7.33]).

Stone-Weierstrass Theorem 2.7. *Suppose $\mathcal{A}$ is a self-adjoint algebra of complex continuous functions on a compact set $\mathcal{K}$. If*

- *to every distinct pair of points $x_1, x_2 \in \mathcal{K}$, there corresponds a function $f \in \mathcal{A}$ such that $f(x_1) \neq f(x_2)$ ($\mathcal{A}$ separates points on $\mathcal{K}$) and*

- *to each $x \in \mathcal{K}$ there corresponds a function $g \in \mathcal{A}$ such that $g(x) \neq 0$ ($\mathcal{A}$ vanishes at no point of $\mathcal{K}$),*

then $\mathcal{A}$ is dense in $C(\mathcal{K})$.

If we can confirm that the space of polynomials ($\mathcal{A}$ in S-W) restricted to the sphere ($\mathcal{K}$ in S-W) fulfills the conditions in Theorem 2.7, we can say that these are dense in the space of continuous functions restricted to the sphere.

- If $\mathcal{P}_k(S)$ is the complex vector space of homogeneous polynomials of degree k restricted to the sphere, then any polynomial $p_k \in \mathcal{P}_k(S)$ can be written as its real and imaginary part $p_k = q_k + is_k$ where $q_k, s_k \in \mathcal{P}_k(S)$. Hence $\overline{p}_k = q_k - is_k$ also belongs to $\mathcal{P}_k(S)$ and the space of homogeneous polynomials is self-adjoint. Since every polynomial can be decomposed into a sum of homogeneous polynomials, this also applies to $\mathcal{P}(S)$.

- To show that $\mathcal{P}(S)$ separates any distinct points $x, y \in S,\ x \neq y$, consider the set of functions $\{p|p = x_k;\ k = 0, 1 \ldots n\}$ where $(x_1, x_2, \ldots, x_n)$ is the basis of $\mathbb{R}^n$. If x, y vary with x_1, then by continuity $p(x) \neq p(y)$. The same argument holds for $x_2, \ldots, x_n$, so the space of polynomials separates points.

- A function $p = c$ constant on the sphere , where $p \in \mathcal{P}(S)$ and $c \neq 0$, never vanishes.

Hence spherical harmonics are dense in $C(S)$. To show the orthogonal decomposition of the space $L^2(S)$, we need to introduce an inner product. The (standard) inner product of this space is defined by

$$\langle f, g \rangle = \int_S f\overline{g}\ d\sigma, \tag{6}$$

where σ denotes the normalized surface-area measure on S. To connect to the case $n = 2$, consider the set of spherical harmonics $\{e^{ik\theta}, e^{-ik\theta}\}_{k=0}^{\infty}$ that are used in equation (5) for the Fourier series. Applying the inner product to these gives

$$\left\langle e^{ik\theta}, e^{im\theta} \right\rangle = \int_{-\pi}^{\pi} e^{i(k-m)\theta}\ \frac{d\theta}{2\pi} = \begin{cases} 1 & \text{if } k = m \\ 0 & \text{if } k \neq m. \end{cases}$$

By this we can see that not only are the functions of the Fourier series dense in $L^2(S)$, but they are an orthogonal (even orthonormal) decomposition of that space. To show that this also is true if $n \geq 2$, we use a special case of Green's identity ([1, p. 79]) for the ball.

$$\int_B (u\Delta v - v\Delta u)\ dV = \int_S (uD_{\mathbf{n}}v - vD_{\mathbf{n}}u)\ ds,$$

where B is the unit ball, dV is the volume measure and ds is the surface measure. $D_{\mathbf{n}}$ refers to differentiation with respect to the outwards unit normal, $\mathbf{n}$. Note that the left hand side of Green's identity equals 0 if the two functions u, v are harmonic. If Green's identity is applied to $p \in \mathcal{H}_k(S)$ and $q \in \mathcal{H}_m(S)$, $k \neq m$ then we have

$$\int_S qD_{\mathbf{n}}p - pD_{\mathbf{n}}q\ d\sigma = 0.$$

When $x \in S$ the vector $\mathbf{n}$ is only rx, where $r = 1$ on the unit sphere. So differentiating with respect to r is the same as differentiating with respect to $\mathbf{n}$. Hence, because of the homogeneity of p,

$$(D_{\mathbf{n}}p)(x) = \frac{d}{dr}p(rx) = \frac{d}{dr}(r^k p(x)) = kr^{k-1}p(x) = kp(x).$$

The same applies to q, and Green's identity then gives us

$$(k-m)\int_S pq\, d\sigma = 0. \tag{7}$$

Since $k \neq m$ we conclude that the integral in the equation above must equal zero.

With the properties of $\mathcal{H}_k(S)$ that follow from the Stone-Weierstrass theorem and Green's identity we have enough to prove the main theorem of this section. The conditions for the direct sum used in the proof of Theorem 2.6, come from Hilbert space theory (which deals with euclidean spaces of infinite dimension, as we are studying here). For the most part these conditions are quite intuitive, and the theory behind them will not be discussed in detail.

Proof of Theorem 2.6. The conditions that must hold for Theorem 2.6 to be true are:

1. $\mathcal{H}_k(S)$ is a closed subspace of $L^2(S)$ for every k,
2. $\mathcal{H}_k(S)$ is orthogonal to every $\mathcal{H}_m(S)$, if $k \neq m$ and
3. $\bigcup_{k=0}^{\infty} \mathcal{H}_k(S)$ is dense in $L^2(S)$.

Because $\mathcal{H}_k(S)$ is finite dimensional for every k, this space is closed. It also is a subspace of $L^2(S)$, therefore condition 1 holds. Condition 2 was showed to be true in equation (7) by Green's identity. Similarly condition 3 will hold because the space of continuous functions $C(S)$ is dense in $L^2(S)$. That $\bigcup_{k=0}^{\infty} \mathcal{H}_k$ is dense in $C(S)$ was a result of applying Theorem 2.7 to spherical harmonics. □

Hence any function $f \in L^2(S)$ can be expressed as an infinite sum of spherical harmonics of different degrees. What remains to be shown is which *specific* spherical harmonics are included in this sum when the function f is given.

3 Zonal Harmonics

3.1 Zonal Harmonics in the series expansion of a given function

We will now try to find an analogue in $\mathbb{R}^n$ to what the Fourier coefficients are in $\mathbb{R}^2$. That there exists a unique series of spherical harmonics for any given function $f \in L^2(S)$ is clear from Theorem 2.6. Hence

$$f = \sum_{k=0}^{\infty} p_k \quad \text{where } p_k \in \mathcal{H}_k(S). \tag{8}$$

In a Fourier series, we have a big advantage compared to when $n > 2$, namely we know that an explicit orthogonal basis for $\mathcal{H}_k(S)$ is $\{e^{ik\theta}, e^{-ik\theta}\}$, and that this is valid for all k. Thus only two operations are needed in order to determine the unique Fourier coefficients A_k, B_k that specify p_k as a linear combination in these bases. If we would try to extend this concept to $n > 2$, two immediate problems emerge. First, nothing in our study so far indicates that there is an obvious basis for $\mathcal{H}_k(\mathbb{R}^n)$ that we could use to determine the coefficients. Secondly, just considering the case of $n = 3$ where $\dim \mathcal{H}_k(S) = 2k + 1$, the amount of calculations necessarily increases with k (in section 4.5 we will see that this leads to a double summation).

From the above discussion we ideally need a function that determines p_k, but so that the function itself should be *independent* of the choice of basis of $\mathcal{H}_k(S)$. To find such a function, consider a fixed point $x \in S$ and the linear map $\varphi : \mathcal{H}_k(S) \to \mathbb{C}$ defined by $\varphi(p_k) = p_k(x)$. This linear map has the property we are looking for. Using the inner product defined in equation (6), we define a spherical harmonic with the property that fulfills the map φ (we wait by showing its existence until later in the section).

Definition 3.1. *For a fixed point $x \in S$, the zonal harmonic of degree k with pole x, $\mathcal{Z}_k(\cdot, x)$, is defined to be the unique spherical harmonic that fulfills the reproducing property*

$$p_k(x) = \langle p_k, \mathcal{Z}_k(\cdot, x)\rangle = \int_S p_k(y)\overline{\mathcal{Z}_k(y, x)}\, d\sigma(y). \tag{9}$$

With the zonal harmonic, we can state the main result of this section, which presents a way to calculate the spherical harmonics of the series expansion in equation (8).

Theorem 3.2. *If $f \in L^2(S)$ and*

$$f(x) = \sum_{k=0}^{\infty} p_k(x) \quad \textit{where } p_k \in \mathcal{H}_k(S),$$

then each p_k is calculated by

$$p_k(x) = \langle f, \mathcal{Z}_k(\cdot, x)\rangle .$$

Since the proof of the above theorem follows almost directly form the definition of zonal harmonics, we state it here.

Proof. If $f \in L^2(S)$, then according to Theorem 2.6 f can be written as

$$f(y) = \sum_{i=0}^{\infty} p_i(y) \quad \text{where } p_i \in \mathcal{H}_k(S).$$

We use the reproducing property of zonal harmonics and the orthogonality of spherical harmonics to get

$$\begin{aligned}\langle f(y), \mathcal{Z}_k(y,x)\rangle &= \left\langle \sum_{i=0}^{\infty} p_i(y), \mathcal{Z}_k(y,x) \right\rangle \\ &= \sum_{i=0}^{\infty} \langle p_i(y), \mathcal{Z}_k(y,x)\rangle = p_k(x).\end{aligned}$$

To show that the spherical harmonic $p_k(x)$ is uniquely determined by f, consider an expansion of $f = \sum_{j=0}^{\infty} q_j$ for some $q_j \in \mathcal{H}_j(S)$. Then we have that

$$p_k(x) = \langle f, \mathcal{Z}_k(\cdot, x)\rangle = \left\langle \sum_{j=0}^{\infty} q_j, \mathcal{Z}_k(\cdot, x) \right\rangle = q_k(x).$$

Hence the expansion stated in Theorem (3.2) is true and unique. □

What remains to be proved is the existence of the zonal harmonic used in the proof of Theorem 3.2. This is done with the aid of a theorem from Hilbert space theory that is stated below. The theorem is modified from [10, Theorem 4.4] for the purpose here, but the original theorem applies to general Hilbert spaces.

Riesz Representation Theorem 3.3. *Let H be a finite Hilbert space and consider the linear map $\varphi : H \to \mathbb{C}$. Then there exists a unique $z \in H$ such that $\varphi(p) = \langle p, z\rangle$ for all $p \in H$.*

If we set the finite Hilbert space H to be $\mathcal{H}_k(S)$, p to be $p_k \in \mathcal{H}_k(S)$ and the unique element z to be $\mathcal{Z}_k(\cdot, x) \in \mathcal{H}_k(S)$, applying Theorem 3.3 with the inner product defined in equation (6) gives the relation in Definition 3.1.

Let us try to reconnect to Fourier analysis by calculating what the zonal harmonic is when $n = 2$. From our argument following equation (5), we know that any spherical harmonic of degree k is a linear combination of $\{e^{ik\theta}, e^{-ik\theta}\}$. So for a fixed point $e^{i\varphi} \in S$ we can write $\mathcal{Z}_k(e^{i\theta}, e^{i\varphi}) = ae^{ik\theta} + be^{-ik\theta}$ where a, b are constants. The relation in Definition (3.1) gives

$$\begin{aligned}ce^{ik\varphi} + de^{-ik\varphi} &= \int_0^{2\pi} (ce^{ik\theta} + de^{-ik\theta})(\overline{a}e^{-ik\theta} + \overline{b}e^{ik\theta}) \frac{d\theta}{2\pi} \\ &= \overline{a}c + \overline{b}d\end{aligned}$$

for all c, d. Hence it must be that

$$\begin{aligned}a &= e^{-ik\varphi} \\ b &= e^{ik\varphi}\end{aligned}$$

which gives

$$\mathcal{Z}_k(e^{i\theta}, e^{i\varphi}) = e^{ik(\theta-\varphi)} + e^{ik(\varphi-\theta)} = 2\cos k(\theta-\varphi).$$

When $k = 0$ this reduces to $\mathcal{Z}_0 = 1$. By assuming that the function f is represented by the series in equation (8), we can check by calculation that

$$\begin{aligned}\langle f, 2\cos k(\theta-\varphi)\rangle &= 2A_k \int_0^{2\pi} \cos k\theta \cos k(\theta-\varphi)\frac{d\theta}{2\pi} \\ &\quad + 2B_k \int_0^{2\pi} \sin k\theta \cos k(\theta-\varphi)\frac{d\theta}{2\pi} \\ &= A_k \cos k\varphi + B_k \sin k\varphi,\end{aligned}$$

where the factor $\frac{1}{2\pi}$ comes from the standard measure on the circle. This shows that when $n = 2$, Theorem 3.2 is the ordinary Fourier series. Hence, using the zonal harmonic does not specifically calculate the coefficients A_k and B_k, but provides us with a way to determine p_k as a whole. So we have reduced the number of necessary operations from two to one. Similarly, using zonal harmonics when $n > 2$ reduces theses operations from $\dim\mathcal{H}_k(S)$ for each k to one.

Later in our study (section 4.2) we will calculate the explicit formula for a fixed $x \in S$ for the zonal harmonic when $n = 3$. This will be an essential step in finding a formula for spherical harmonics in $\mathbb{R}^3$.

3.2 Properties of zonal harmonics

Zonal harmonics possess some special properties. Only a few of these will be necessary for our calculations in the upcoming sections, so we limit our focus to the relevant characteristics. To prove Proposition 3.5 stated below, we will need a property of harmonic functions, namely the rotational invariance of harmonic functions. We denote the set of orthogonal (orthonormal) transformations as $O(n)$ and state the following lemma.

Lemma 3.4. *Let $T \in O(n)$. Then f is a harmonic if and only if $(f \circ T)$ is harmonic.*

Proof. We outline this proof by using the mean-value property of harmonic functions (see [1, Theorem 1.4]). This states that if f is harmonic on the closed ball $\overline{B}(a,r)$ (closed ball centered at the point a with radius r), then $f(a)$ equals the average of f over the closure of $B(a,r)$. Since the closure of $B(a,r)$ is a sphere of radius r and the mean over spheres does not change with rotation, $(f \circ T)$ is harmonic. Since $T^{-1} \in O(n)$, the converse is also true. □

Proposition 3.5. *Suppose $x, y \in S$, $T \in O(n)$ and $k \geq 0$, then*

1. *$\mathcal{Z}_k$ is real valued,*
2. *$\mathcal{Z}_k(y, T(x)) = \mathcal{Z}_k(T^{-1}(y), x)$,*
3. *$\mathcal{Z}_k(x,x) = dim\mathcal{H}_k(S)$ and*
4. *$|\mathcal{Z}_k(y,x)| \leq dim\mathcal{H}_k(S)$.*

Proof. To prove property 1, we assume that $p_k \in \mathcal{H}_k(S)$ and is real valued. This gives that

$$\begin{aligned} 0 &= \mathrm{Im}\ p_k(x) \\ &= \mathrm{Im} \int_S p_k(y)\overline{\mathcal{Z}_k(y,x)}\ d\sigma(y). \end{aligned}$$

Now if we define $p_k(y) = \mathrm{Im}\mathcal{Z}_k(y,x)$, then the above statement implies that

$$\int_S (\mathrm{Im}\mathcal{Z}_k(y,x))^2\ d\sigma(y) = 0,$$

which, because of the inner product defined, gives that $\mathrm{Im}\mathcal{Z}_k(y,x) = 0$.

To prove property 2, note that it applies for all $p_k \in \mathcal{H}_k$ that

$$\begin{aligned} p_k(T(x)) &= (p_k \circ T)(x) \\ &= \int_S p_k(T(y))\mathcal{Z}_k(y,x)\ d\sigma(y) \quad = \int_S p_k(y)\mathcal{Z}_k(T^{-1}(y),x)\ d\sigma(y). \end{aligned}$$

The last equality is due to the rotational invariance property of both the spherical harmonics and the standard surface measure σ. On the other hand we can also write

$$p_k(T(x)) = \int_S p_k(y)\mathcal{Z}_k(y,T(x))\ d\sigma(y).$$

Thus

$$\int_S p_k(y)\mathcal{Z}_k(y,T(x))\ d\sigma(y) = \int_S p_k(y)\mathcal{Z}_k(T^{-1}(y),x)\ d\sigma(y).$$

Due to the uniqueness of zonal harmonics asserted by Theorem 3.3, we conclude that property 2 is true.

For property 3, let $e_1, \dots, e_{h_k}$ be an orthonormal basis of $\mathcal{H}_k(S)$. Then the linear combination of $\mathcal{Z}_k(\cdot,x)$ in this basis is

$$\mathcal{Z}_k(\cdot,x) = \sum_{j=1}^{j=h_k} \langle \mathcal{Z}_k(\cdot,x), e_j\rangle\, e_j = \sum_{j=1}^{j=h_k} \overline{e_j(x)}e_j,$$

where the last equality is due to the reproducing property of zonal harmonics. Hence we have that

$$\mathcal{Z}_k(x,x) = \sum_{j=1}^{j=h_k} \overline{e_j(x)}e_j(x) = \sum_{j=1}^{j=h_k} |e_j(x)|^2.$$

By property 2 we have that $\mathcal{Z}_k(T(x),T(x)) = \mathcal{Z}_k(x,x)$, so that the function $x \mapsto \mathcal{Z}_k(x,x)$ is constant on S. Integrating the equation above over the sphere and using the orthonormal properties of the basis gives that

$$\mathcal{Z}_k(x,x) = \int_S \mathcal{Z}_k(x,x) = \int_S \left(\sum_{j=1}^{j=h_k} |e_j(x)|^2\right) d\sigma(x) = h_k = \mathrm{dim}\mathcal{H}_k.$$

To prove property 4, note that property 3 and the reproducing property of zonal harmonics give that

$$||\mathcal{Z}_k(\cdot, x)||_2^2 = \langle \mathcal{Z}_k(\cdot, x), \mathcal{Z}_k(\cdot, x)\rangle = \mathcal{Z}_k(x, x) = \dim \mathcal{H}_k,$$

where $|| \; ||_2$ denotes the norm in $L^2(S)$. Using the Cauchy-Schwarz inequality we get that

$$|\mathcal{Z}_k(y, x)| = |\langle \mathcal{Z}_k(\cdot, x), \mathcal{Z}_k(\cdot, y)\rangle| \leq ||\mathcal{Z}_k(\cdot, x)||_2 ||\mathcal{Z}_k(\cdot, y)||_2 = \dim \mathcal{H}_k.$$

□

Note that a direct consequence of Proposition 3.5 is that zonal harmonics are constant on the intersection of S and hyperplanes perpendicular to the pole vector. That is, the value of a zonal harmonic in a given point $x \in S$ depends *only* on the distance of x to the pole. This is the explanation to the name they have been given. To see that this is true, let $T \in O(n)$ be $T(x) = x$. Hence a function f is dependent only on the distance from x if and only if it satisfies the relation $f \circ T^{-1} = f$ (and $f \circ T = f$). From Proposition 3.5 we can see that for these types of T applied to zonal harmonics, it gives that

$$\mathcal{Z}_k(y, x) = \mathcal{Z}_k(y, T(x)) = \mathcal{Z}_k(T^{-1}(y), x).$$

The last property, which we here will only comment on, is that any spherical harmonic possessing the properties of a zonal harmonic, must be a zonal harmonic times a scalar constant. For a detailed discussion of this topic see [1, p. 101 -103].

We have now with Theorem 3.2 accomplished what we set out to do. We have shown that the concept of Fourier analysis can be extended to n-dimensional space with spherical harmonics playing the role of the infinite set of orthogonal functions and zonal harmonics as the tool to determine what specific spherical harmonics are to be used in a series expansion. The remainder of this study will focus on developing explicit formulas for spherical and zonal harmonics and applying these in a series in the special case when $n = 3$. In section 5 we illustrate these functions graphically in different ways to develop an intuitive understanding of the theoretical concepts from the previous sections.

4 Spherical Harmonics in Spherical Coordinates

4.1 Eigenfunctions to Laplace's equation

In this section we will be performing direct calculations, so it is only natural to find a suitable coordinate system. Applying the theory from the previous sections to *real-valued* functions will lead to expressions of spherical and zonal harmonics in the spherical coordinate system.

Consider a harmonic polynomial $p(x, y, z) \in \mathcal{H}_k(\mathbb{R}^3)$. A change of variables

$$x = r \sin\theta \cos\varphi, \qquad y = r \sin\theta \sin\varphi, \qquad z = r\cos\theta$$

where

r is the radius of the sphere,

φ is the angle from the positive x-axis to the projection in the xy-plane,

θ is the angle from the positive z-axis,

gives the polynomial $p(r, \varphi, \theta)$ in spherical coordinates. Because $p(x, y, z)$ is homogeneous of degree k, it is possible to factor out r^k so that $p(x, y, z) \to p(r, \varphi, \theta) = r^k f(\varphi, \theta)$. By applying the laplacian for spherical coordinates (for a derivation, see Appendix A.1),

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right), \tag{10}$$

to $p(r, \varphi, \theta)$ we obtain the relation (11).

$$\Delta p = \frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\Delta_s p = 0, \tag{11}$$

where Δ_s is the spherical Laplace operator defined as

$$\Delta_s = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

The partial derivatives of $p(r, \varphi, \theta)$ with respect to r can be calculated to be

$$\frac{\partial p}{\partial r} = k r^{k-1} f(\varphi, \theta) = \frac{k}{r} p \text{ and}$$
$$\frac{\partial^2 p}{\partial r^2} = k(k-1) r^{k-2} f(\varphi, \theta) = \frac{k(k-1)}{r^2} p.$$

Inserting the expressions of the partial derivatives into equation (11) gives

$$k(k+1)p + \Delta_s p = 0. \tag{12}$$

Since the spherical Laplace operator Δ_s does not act in the variable r, we can divide equation (12) by r^k resulting in the expression for spherical harmonics

$$k(k+1) f(\varphi, \theta) + \Delta_s f(\varphi, \theta) = 0. \tag{13}$$

Equation (13) is a partial differential equation that can be solved by separation of variables. Let $f(\varphi, \theta) = \Phi(\varphi)\Theta(\theta)$. If equation (13) is divided by $f = \Phi\Theta$ we get

$$k(k+1) + \frac{\Delta_s(\Phi\Theta)}{\Phi\Theta} = 0.$$

From the above relation we can see that spherical harmonics are the eigenfunctions to the spherical Laplace operator Δ_s. Inserting the full expression for Δ_s and simplifying we get that

$$k(k+1) + \frac{1}{\sin\theta\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2} = 0. \tag{14}$$

Equation (14) suggests that its last term is dependent only on the variable θ. Therefore it holds that

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2} = -n^2, \tag{15}$$

where n is a positive constant. By standard tools for solving ordinary differential equations, the solution to equation (15) is

$$\Phi = A\cos n\varphi + B\sin n\varphi \qquad \text{where } A \text{ and } B \text{ are real constants.}$$

Note that if Φ would be complex valued n would not need to be positive for Φ to be periodic, since with $A, B \in \mathbb{C}$ the solution to $\Phi(\varphi)$ could be written as a complex linear combination of $e^{in\varphi}$ and $e^{-in\varphi}$. Inserting the expression in equation (15) into equation (14) and multiplying by Θ results in

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \left(k(k+1) - \frac{n^2}{\sin^2\theta}\right)\Theta = 0. \tag{16}$$

If the function Θ is assumed to solve equation (16), then the solution to equation (13) are functions of the kind

$$f(\varphi,\theta) = (A\cos n\varphi + B\sin n\varphi)\,\Theta, \tag{17}$$

which are spherical harmonics of degree k. Combining this with the facts that $p(r,\varphi,\theta) = r^k f(\varphi,\theta)$ and $f = \Phi\Theta$ gives us the harmonic polynomials

$$p(r,\varphi,\theta) = r^k\,(A\cos n\varphi + B\sin n\varphi)\,\Theta.$$

For further calculations it is convenient to make a change of a variable. If $z = \cos\theta$ and $y(z) = \Theta(\theta)$, we can write equation (16)

$$(1-z^2)y''(z) - 2zy'(z) + \left(k(k+1) - \frac{n^2}{1-z^2}\right)y(z) = 0. \tag{18}$$

Equation (18) is called *Legendre's associated equation* and the solutions to this will be the topic of the next section.

4.2 The Legendre Polynomial

We start our search of solutions to equation (18) with the simplified special case of $n = 0$, which then becomes

$$(1-z^2)y''(z) - 2zy'(z) + k(k+1)y(z) = 0. \tag{19}$$

When $n = 0$ in equation (17), the spherical harmonic reduces to $f(\varphi,\theta) = A\Theta$. From the discussion in section 3 we know that this solution can only be a zonal harmonic times some real constant. Hence we derive a formula for $\mathcal{Z}_k(S)$ in

$\mathbb{R}^3$. Equation (19) is well known in the field of ordinary differential equations and special functions and is known as *Legendre's equation*. The solution to Legendre's equation is here only outlined, but a more extensive discussion can be found in for instance [3, Ch. 10].

ODE theory tells us that $z = 0$ is an ordinary point of equation (19). This means that we should seek a series solution to Legendre's equation in the form

$$y = \sum_{j=0}^{\infty} a_j z^j.$$

Differentiating this series term by term, and plugging the series expressions of y, y' and y'' into equation (19) gives a recurrence relation between a_j and a_{j+2}

$$a_{j+2} = \frac{j(j+1) - k(k+1)}{(j+2)(j+1)} a_j. \tag{20}$$

The important thing to note about the recurrence relation in (20) is that when $j = k$, the coefficients $a_{k+2} = a_{k+4} = a_{k+6} = \cdots = 0$. Hence one of the independent solutions will always be a polynomial of degree k (with only even or odd powers of z, depending on the value of k), while the other is an infinite series. The polynomial solution is called the *Legendre polynomial of degree k* and will be denoted $P_k(z)$. The series solution is named the *Legendre function of the second kind*, and denoted $Q_k(z)$. By applying the Cauchy ratio test to the recurrence relation in (20) it can be shown that this series converges for $-1 < z < 1$, however it diverges for $|z| \geq 1$. This can be seen by observing that the series converges to a function for a specific k. For instance $Q_0 = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right)$, which diverges at the points $z = \pm 1$ ([6, p. 710]). Since spherical harmonics are defined on the whole sphere, the solution that we seek must be continuous on the closed interval $\cos\theta = z \in [-1, 1]$. By the continuity of polynomials, the Legendre polynomial is continuous for $z \in [-1, 1]$. Hence we focus on $P_k(z)$ to be the solution. We now try to find a formula for $P_k(z)$.

The recurrence relation

$$a_j = -\frac{(j+2)(j+1)}{(k-j)(k+j+1)} a_{j+2}$$

follows directly from (20). If we start with the value $j + 2 = k$, and use the above formula i times (where i is a positive integer), we can find the following expression

$$a_{k-2i} = \frac{(-1)^i}{2^i i!} \cdot \frac{k(k-1)\cdots(k-2i+1)}{(2k-1)(2k-3)\cdots(2k-2i+1)} a_k. \tag{21}$$

To complete the polynomial expression we still have to decide the value of a_0 (or a_1). A normalization value of a_0 is chosen so that the k:th coefficient in $P_k(z)$ has the value $a_k = \frac{(2k)!}{2^k (k!)^2}$. This is specially chosen so that $P_k(1) = 1$ for all k. If this is inserted into equation (21) we get that

$$a_{k-2i} = \frac{1}{2^k} \cdot \frac{(-1)^i}{i!} \cdot \frac{(2k-2i)!}{(k-2i)!(k-i)!}.$$

It follows that if k is even the Legendre polynomial has $\frac{k}{2}$ terms, or $\frac{(k-1)}{2}$ if k is odd. This gives the sum

$$P_k(z) = \frac{1}{2^k}\sum_{i=0}^{m}\frac{(-1)^i}{i!}\cdot\frac{(2k-2i)!}{(k-2i)!(k-i)!}z^{k-2i}, \quad k = 0,1,2\ldots, \tag{22}$$

where

$$m = \begin{cases} k/2 & \text{if } k \text{ is even,} \\ (k-1)/2 & \text{if } k \text{ is odd.} \end{cases}$$

We know from section 3 that $P_k(z)$ times some constant c is a zonal harmonic and that this set of polynomials is orthogonal. From the variable change $z = \cos\theta$ we can see that this particular zonal harmonic has its pole in the point $(x, y, z) = (0, 0, 1)$. Using the notation for the unit vector in the z-axis direction $\hat{z}$ we can write $\mathcal{Z}_k(\theta, \hat{z}) = cP_k(\cos\theta)$. The following proposition shows the relation between the zonal harmonic and its corresponding Legendre polynomial.

Proposition 4.1. *Let $P_k(\cos\theta)$ be a Legendre polynomial of degree k with pole $\hat{z}$. Then*

1. $\mathcal{Z}_k(\theta, \hat{z}) = (2k+1)P_k(\cos\theta)$,
2. $\int_{-1}^{1}[P_k(z)]^2\ dz = \frac{2}{2k+1}$.

Proof. For 1, note that the coefficient a_0 in the Legendre polynomial is chosen so that $P_k(1) = 1$. Since $z = \cos\theta = 1$ at the pole $\hat{z}$, we can see from Proposition 3.5 that

$$\mathcal{Z}_k(\hat{z}, \hat{z}) = \dim\mathcal{H}_k(S) = cP_k(1).$$

Together with Proposition 2.4 this gives that

$$c = \dim\mathcal{H}_k(S) = 2k+1,$$

which proves 1.

To prove 2, we again use Proposition 3.5 together with the result above.

$$\begin{aligned} ||\mathcal{Z}_k(\cdot, \hat{z})||_2^2 &= \langle\mathcal{Z}_k(\cdot, \hat{z}), \mathcal{Z}_k(\cdot, \hat{z})\rangle = \langle(2k+1)P_k(z), (2k+1)P_k(z)\rangle \\ &= \int_0^{2\pi}\int_{-1}^{1}[(2k+1)P_k(z)]^2\ d\sigma(z)d\sigma(\varphi) = 2k+1, \end{aligned}$$

where the last equality holds since $\dim\mathcal{H}_k(S) = 2k+1$. The last line in the equation above indicates that

$$\int_{-1}^{1}[(2k+1)P_k(z)]^2\ \frac{d(z)}{2} = 2k+1,$$

which gives that

$$\int_{-1}^{1}[P_k(z)]^2\ dz = \frac{2}{2k+1}.$$

This completes the proof. □

Calculating the first few Legendre polynomials shows that

$$\begin{array}{ll} P_0(z) = 1, & P_1(z) = z \\ P_2(z) = \frac{1}{2}(3z^2 - 1), & P_3(z) = \frac{1}{2}(5z^3 - 3z), \\ P_4(z) = \frac{1}{8}(35z^4 - 30z^2 + 3), & P_5(z) = \frac{1}{8}(63z^5 - 70z^3 + 15z). \end{array}$$

From Proposition 4.1 we know that the Legendre polynomials are zonal harmonics times a constant. Hence we have that $P_k(z) \in \mathcal{H}_k(S)$. However, if we consider a Legendre polynomial, for instance $P_5(z)$,

$$P_5(x, y, z) = \frac{1}{8}(63z^5 - 70z^3 + 15z),$$

this polynomials does not look to be either homogeneous of degree 5 or harmonic. By observing that if a function is restricted to the unit sphere any power of $(x^2 + y^2 + z^2)$ is equal to one, we can expand $P_5(z)$ as follows

$$P_5(x, y, z) = \frac{1}{8}(63z^5 - 70z^3(x^2 + y^2 + z^2) + 15z(x^2 + y^2 + z^2)^2).$$

This indeed is a homogeneous harmonic polynomial of degree 5, so $P_5(x, y, z) \in \mathcal{H}_5(S)$ which was to be expected from the previous theoretical discussion.

4.3 Oscillations of the Zonal Harmonic

By the properties of polynomials we expect $P_k(z)$ to have k roots. Before we present a proposition about the oscillatory behavior of Legendre polynomials (from [9, Theorem 2.1.2]), we note that the set $\{P_k(z)\}_{k=0}^n$ is a basis for the space of polynomials up to degree n. To see this, assume that the previous statement is not true. Then there exists a linear combination such that for some $0 \leq k \leq n$, $P_k(z) = a_0 P_0(z) + a_1 P_1(z) + \cdots + a_{k-1} P_{k-1}(z) + a_{k+1} P_{k+1}(z) \cdots + a_n P_n(z)$, where $a_j \in \mathbb{R}$ for $0 \leq j \leq n,\ j \neq k$. The orthogonality of Legendre polynomials gives that

$$\int_{-1}^{1} [P_k(z)]^2 \, dz = \int_{-1}^{1} \sum_{j=0}^{k-1} P_k(z) P_j(z) \, dz + \int_{-1}^{1} \sum_{j=k+1}^{n} P_k(z) P_j(z) \, dz = 0,$$

which can not be true according to Proposition 4.1. An immediate consequence is that any polynomial $p \in \mathbb{P}_{k-1}(\mathbb{R})$, where $\mathbb{P}_{k-1}(\mathbb{R})$ denotes the space of real-valued polynomials of degree less than or equal to $k - 1$, is orthogonal to the Legendre polynomial $P_k(z)$ on the interval $[-1, 1]$.

Proposition 4.2. *Let $P_k(z)$ be as in equation (22). Then the zeros of $P_k(z)$ are real, distinct and occur k times on the interval $(-1, 1)$. Furthermore, the zeros are symmetric around $z = 0$, and if k is an odd integer, $z = 0$ is a zero itself.*

Proof. By the orthogonality of Legendre polynomials we can see that

$$\int_{-1}^{1} P_0(z) P_k(z) \, dz = \int_{-1}^{1} P_k(z) \, dz = 0,$$

which implies that $P_k(z)$ has at least one zero in the interval $(-1, 1)$. To see that the zeros in this interval are distinct, assume that $z = z_1$ is a multiple zero.

In this case it follows that $\frac{P_k(z)}{(z-z_1)^2} \in \mathbb{P}_{k-2}(\mathbb{R})$, which is orthogonal to $P_k(z)$. This gives that

$$0 = \int_{-1}^{1} \frac{P_k(z)}{(z-z_1)^2} P_k(z)\ dz = \int_{-1}^{1} \left(\frac{P_k(z)}{(z-z_1)} \right)^2\ dz.$$

Since the last the expression in the last integral is greater or equal to zero on $[-1, 1]$, the above equation is a contradiction. If we assume that $P_k(z)$ has $j \geq 1$ distinct zeros in $(-1, 1)$ and that $j < k$ we have that

$$P_k(z) = (z-z_1)(z-z_2)\dots(z-z_j)p(z) = q(z)p(z),$$

where $p(z)$ is a polynomial of constant sign on $(-1, 1)$ and $q(z) \in \mathbb{P}_j(\mathbb{R})$. Using the orthogonality of $q(z)$ and $P_k(z)$ we get that

$$0 = \int_{-1}^{1} q(z)P_k(z)\ dz = \int_{-1}^{1} [q(z)]^2 p(z)\ dz.$$

This can not be true, since the expression $[q(z)]^2p(z)$ has constant sign on $(-1, 1)$. This gives that $j = k$ and $p(x) = 1$. $\square$

The relation between the Legendre polynomials and zonal harmonics and the above theorem tells us that $\mathcal{Z}_k(\cdot, \hat{z})$ vanishes on k circles perpendicular to the z-axis.

4.4 Solutions to Legendre's associated equation

We have seen that the special case of $n = 0$ gives a formula for zonal harmonics. Now we attempt to find a solution to equation (18) for $n \geq 0$. If this can be accomplished, we have found the formula for spherical harmonics.

The approach of finding a solution is somewhat experimental in nature. The general idea is that a relation between Legendre's associated equation (18) and Legendre's equation (19) can be found by differentiating the latter n times. We start by trying to find a closed formula for the n:th derivative of Legendre's equation. By the usual notation $\frac{dy}{dz}$ is y', $\frac{d^2y}{dz^2}$ is y'' and so on.

$$(1-z^2)\frac{d^2y'}{dz^2} - 2\cdot 2z\frac{dy'}{dz} + (k(k+1) - 2)\, y' = 0 \quad \text{first derivative,}$$

$$(1-z^2)\frac{d^2y''}{dz^2} - 2\cdot 3z\frac{dy''}{dz} + (k(k+1) - 2(1+2))\, y'' = 0 \quad \text{second derivative,}$$

$$(1-z^2)\frac{d^2y'''}{dz^2} - 2\cdot 4z\frac{dy'''}{dz} + (k(k+1) - 2(1+2+3))\, y''' = 0 \quad \text{third derivative.}$$

Continuing this process gives the general formula for differentiating n times.

$$(1-z^2)\frac{d^2y^{(n)}}{dz^2} - 2(n+1)z\frac{dy^{(n)}}{dz} + (k(k+1) - n(n+1))\, y^{(n)} = 0 \quad n\text{:th derivative.} \tag{23}$$

Since we know that $P_k(z)$ solves Legendre's equation, a solution to equation (23) is the n:th derivative of this function, $\frac{d^nP_k(z)}{dz^n}$.

If we make the substitution $y = (1-z^2)^{\frac{n}{2}} u$ in Legendre's associated equation (18) we get

$$(1-z^2)\frac{d^2u}{dz^2} - 2\cdot 2z\frac{du}{dz} + (k-n)(k+n+1)u = 0.$$

This equation is the same as the the relation arrived at in equation (23) with u instead of $y^{(n)}$. By the substitution made and relation just mentioned we can conclude that a solution to Legendre's associated equation (18) is

$$P_k^n(z) = (1-z^2)^{\frac{n}{2}}\frac{d^nP_k(z)}{dz^n}. \tag{24}$$

In a similar way it can be shown that $(1-z^2)^{\frac{n}{2}}\frac{d^nQ_k(z)}{dz^n}$ is the other solution to equation (18), independent of $P_k^n(z)$. However, the series solution is not of any particular interest when deriving an expression for spherical harmonics. $P_k^n(z)$ is called the *associated Legendre function of degree k and order n*, and as will be shown in the coming section, this function will be important when expanding a function $f \in L^2(S)$ into a series of spherical harmonics.

4.5 Series of Spherical Harmonics

From equation (17), we see that we can write a general expression for a spherical harmonic as a function of z and φ, and that this solves equation (13).

Definition 4.3. *The general solution to Laplace's equation that takes the form*

$$\tilde{Y}_k^n(\varphi, z) = (A\cos n\varphi + B\sin n\varphi)\,(1-z^2)^{\frac{n}{2}}\frac{d^nP_k(z)}{dz^n}, \tag{25}$$

where A and B are real constants, $z = \cos\theta$ and $0 \leq n \leq k$, is called a spherical harmonic of degree k and order n.

We can see that there are $2k+1$ types of spherical harmonics of degree k, since there are two types $(\sin n\varphi(1-z^2)^{\frac{n}{2}}\frac{d^nP_k(z)}{dz^n}$ and $\cos n\varphi(1-z^2)^{\frac{n}{2}}\frac{d^nP_k(z)}{dz^n})$ when $1 \leq n \leq k$, and only one when $n = 0$ (the Legendre polynomial). This is consistent with our result for $\dim\mathcal{H}_k(S)$ from section 2.3.

From the theory covered in previous sections we know that $\tilde{Y}_k^n$ is a homogeneous harmonic polynomial of degree k in the variables x, y and z. However, it is not obvious that the expression given in Definition 4.3 is a polynomial. We know that $\frac{d^nP_k(z)}{dz^n}$ is a polynomial in z, since this is just the n:th derivative of the Legendre polynomial of degree k. Furthermore, since $z = \cos\theta$, we see that the remaining part of the expression takes the form $(A\cos n\varphi + B\sin n\varphi)\sin^n\theta$. Focusing on the cosine part of this last expression (remembering that our initial calculations restricted the function $r^k\tilde{Y}_k^n$ to the sphere), we multiply this by r^n to get

$$\begin{aligned} r^n\cos n\varphi\sin^n\theta &= \frac{1}{2}\left[\left(r\sin\theta e^{i\varphi}\right)^n + \left(r\sin\theta e^{-i\varphi}\right)^n\right] \\ &= \frac{1}{2}\left[(x+iy)^n + (x-iy)^n\right]. \end{aligned}$$

By the Binomial Theorem the last term in the above calculation is a (homogeneous) polynomial in x and y. A similar argument can be made for $\sin n\varphi\sin^n\theta$. This shows that the restriction of $r^k\tilde{Y}_k^n$ to the unit sphere is a polynomial.

From section 2.4 we know that any function $f(z,\varphi) \in L^2(S)$ can be developed in an orthogonal series of spherical harmonics. For it to be possible to express *any* spherical harmonic of degree k in the series, we need all the $2k+1$ spherical harmonics of degree k to be available. This leads to the double summation in the series expansion of f, presented in the final theorem of this study.

Theorem 4.4. *Let $f(\theta,\varphi) \in L^2(S)$ be a real-valued function defined on the unit sphere. Then f can be expressed as a series of spherical harmonics, such that*

$$f(\theta,\varphi) = \sum_{k=0}^{\infty}\left[A_{0,k}P_k(\cos\theta) + \sum_{n=1}^{n=k} A_{n,k}\cos n\varphi P_k^n(\cos\theta) + B_{n,k}\sin n\varphi P_k^n(\cos\theta)\right],$$

where

$$\begin{aligned}
A_{0,k} &= \frac{2k+1}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(\theta,\varphi)\cos n\varphi P_k(\cos\theta)\sin\theta\ d\theta\ d\varphi,\\
A_{n,k} &= \frac{2k+1}{2\pi}\frac{(k-n)!}{(k+n)!}\int_0^{2\pi}\int_0^{\pi} f(\theta,\varphi)\cos n\varphi P_k^n(\cos\theta)\sin\theta\ d\theta\ d\varphi \quad \text{and}\\
B_{n,k} &= \frac{2k+1}{2\pi}\frac{(k-n)!}{(k+n)!}\int_0^{2\pi}\int_0^{\pi} f(\theta,\varphi)\sin n\varphi P_k^n(\cos\theta)\sin\theta\ d\theta\ d\varphi.
\end{aligned}$$

Note that in the theorem above the substitution $z = \cos\theta$ has been made, resulting in

$$\int_{-1}^{1} f(z)\ dz = \int_0^{\pi} f(\theta)\sin\theta\ d\theta.$$

Before proving this theorem, we compare it to the main result of section 3 presented in Theorem 3.2. Applied to three dimensional space, this theorem states that the spherical harmonic of degree k presented in Theorem 4.4 can be calculated by the inner product of $f(\varphi,\theta)$ and the Legendre polynomial $P_k(\cos\theta)$ (the constant relating the Legendre polynomial to the zonal harmonic is here omitted for readability). Hence the constants $A_{0,k}$, $A_{n,k}$ and $B_{n,k}$ would not have to be calculated. However, this would only calculate the series expansion in a fixed point of the sphere, namely the pole of the Legendre polynomial. This means that we would have to perform *infinitely* many operations if f is to be expressed as a function with the pole (of $P_k(\cos\theta)$) as its variable. Comparing this to Theorem 4.4, where $2k+1$ operations are necessary for each spherical harmonic of degree k, the practical use of zonal harmonics to calculate the series expansion of f is limited.

Before we can calculate the constants of the series in Theorem 4.4, we need the orthogonality property of associated Legendre functions of different degrees. This is presented in the proposition below.

Proposition 4.5. *Let $P_k^n(z)$ and $P_l^n(z)$ be the associated Legendre functions of order n and degrees k and l respectively. Then*

$$\int_{-1}^{1} P_k^n(z)P_l^n(z)\ dz = \begin{cases} 0 & \text{if } k \neq l,\\ \frac{2}{2k+1}\frac{(k+n)!}{(k-n)!} & \text{if } k = l.\end{cases}$$

Proof. By equation (24) and integration by parts we get

$$\int_{-1}^{1}(1-z^2)^n\frac{d^nP_l(z)}{dz^n}\frac{d^nP_k(z)}{dz^n}dz = \tag{26}$$
$$-\int_{-1}^{1}\frac{d^{n-1}P_l(z)}{dz^{n-1}}\cdot\frac{d}{dz}\left[(1-z^2)^n\frac{d^nP_k(z)}{dz^n}\right]\,dz.$$

Inspired by the differentiation formula for associated Legendre functions, we substitute n by $n-1$ in equation (23) (remember that $\frac{d^{n-1}P_k(z)}{dz^{n-1}}$ is a solution to this equation) to get

$$(1-z^2)\frac{d^{n+1}P_k(z)}{dz^{n+1}}-2nz\frac{d^nP_k(z)}{dz^n}+(k(k+1)-n(n-1))\frac{d^{n-1}P_k(z)}{dz^{n-1}}=0.$$

Further, multiplying the above formula by $(1-z^2)^{n-1}$ leads to

$$\frac{d}{dz}\left[(1-z^2)^n\frac{d^nP_k(z)}{dz^n}\right]=-(k+n)(k-n+1)(1-z^2)^{n-1}\frac{d^{n-1}P_k(z)}{dz^{n-1}}.$$

Inserting the above relation into equation (26) gives the reducing formula

$$\int_{-1}^{1}(1-z^2)^n\frac{d^nP_l(z)}{dz^n}\frac{d^nP_k(z)}{dz^n}dz =$$
$$-(k+n)(k-n+1)\int_{-1}^{1}(1-z^2)^{n-1}\frac{d^{n-1}P_l(z)}{dz^{n-1}}\frac{d^{n-1}P_k(z)}{dz^{n-1}}\,dz.$$

Applying this formula n times results in

$$\int_{-1}^{1}P_l^n(z)P_k^n(z)\,dz=\frac{(k+n)!}{(k-n)!}\int_{-1}^{1}P_l(z)P_k(z)\,dz.$$

By the orthogonality of Legendre polynomials (Proposition 4.1) this equals zero if $k\neq l$ and $\frac{2}{2k+1}\frac{(k+n)!}{(k-n)!}$ if $k=l$, ending the proof. □

To facilitate the calculations of the actual constants, we use the series

$$f(z,\varphi)=\sum_{m=0}^{\infty}\left[A_{0,m}P_m(z)+\sum_{l=1}^{l=m}A_{l,m}\cos l\varphi P_m^l(z)+B_{l,m}\sin l\varphi P_m^l(z)\right], \tag{27}$$

which is the series in Theorem 4.4 after the substitution $\cos\theta=z$ and a change of indices. Suppose we want to find $A_{n,k}$ in equation (27). It is natural to multiply the series by $\cos n\varphi P_k^n(z)$ and integrate over the sphere. Since $\cos n\varphi P_k^n(z)$ is a spherical harmonic of degree k, the orthogonality of spherical harmonics immediately gives that all terms in the outer sum where $m\neq k$ equal zero. Hence the only possible non-zero inner products are the components of the k:th term of the outer sum in equation (27). These are of the form

$\int_0^{2\pi}\int_{-1}^{1}\cos n\varphi P_k^n(z)\cos l\varphi P_k^l(z)\,dz\,d\varphi$, $\int_0^{2\pi}\int_{-1}^{1}\cos n\varphi P_k^n(z)\sin l\varphi P_k^l(z)\,dz\,d\varphi$,
$\int_0^{2\pi}\int_{-1}^{1}\cos n\varphi\sin n\varphi[P_k^n(z)]^2\,dz\,d\varphi$, $\int_0^{2\pi}\int_{-1}^{1}\cos^2 n\varphi[P_k^n(z)]^2\,dz\,d\varphi$,

where $n,l\leq k$ and $n\neq l$. Using the orthogonality of the $\sin\varphi$ and $\cos\varphi$ functions over the interval $[0,2\pi]$ (used in Fourier analysis) and the Proposition

4.5, we can see that the only non-zero term above is the last. By Proposition 4.5 and that $\int_0^{2\pi} \cos^2 n\varphi \; d\varphi = \pi$, the last integral equals $\frac{2\pi}{2k+1}\frac{(k+n)!}{(k-n)!}$. Thus we get that all the terms except the one involving $A_{n,k}$ equals zero and

$$A_{n,k} = \frac{2k+1}{2\pi}\frac{(k-n)!}{(k+n)!}\int_0^{2\pi}\int_{-1}^{1} f(z,\varphi)\cos n\varphi P_k^n(z) \; dz \; d\varphi$$

We finally resubstitute $\cos\theta = z$ so that

$$A_{n,k} = \frac{2k+1}{2\pi}\frac{(k-n)!}{(k+n)!}\int_0^{2\pi}\int_0^{\pi} f(\theta,\varphi)\cos n\varphi P_k^n(\cos\theta)\sin\theta \; d\theta \; d\varphi.$$

We can calculate the constants $A_{0,k}$ and $B_{n,k}$ in a similar way. This proves Theorem 4.4.

5 Graphical illustrations

In this section we use the theory from section 4 to create graphical illustrations of the spherical harmonics. Hopefully the following figures will help to increase the readers intuition about these.

Several ways of representing the spherical harmonics have been chosen. For one kind of illustration we emphasize the link to Fourier analysis by letting the value of the spherical harmonic be represented as distance from a reference sphere. This corresponds to the representation of the sine and cosine functions on the unit circle. Secondly a more common representation is used where the values of the spherical harmonic takes on different colors on the unit sphere. A third way of illustration is chosen for purposes of literature reference, namely the squared absolute magnitude of the harmonic. In this illustration the value at a given point is represented by the distance from the origin.

Finally a less common way to represent spherical harmonics that has not been found in literature will be attempted. By using stereographic projection, we can view harmonics as is commonly done when considering a function in the plane. In Fourier analysis a projection of the unit circle onto the line is easily accomplished by using the 2π-periodicity of a function defined on the circle. However, for a function defined on the unit sphere, there is no easy way of representing the periodicity in the plane. The stereographic projection of a function corresponds to projection a function on the circle to the real line in Fourier analysis. It is this way that is chosen to illustrate the main result of the previous section: the approximation of a given function on the sphere by a sum of spherical harmonics.

5.1 Illustrations

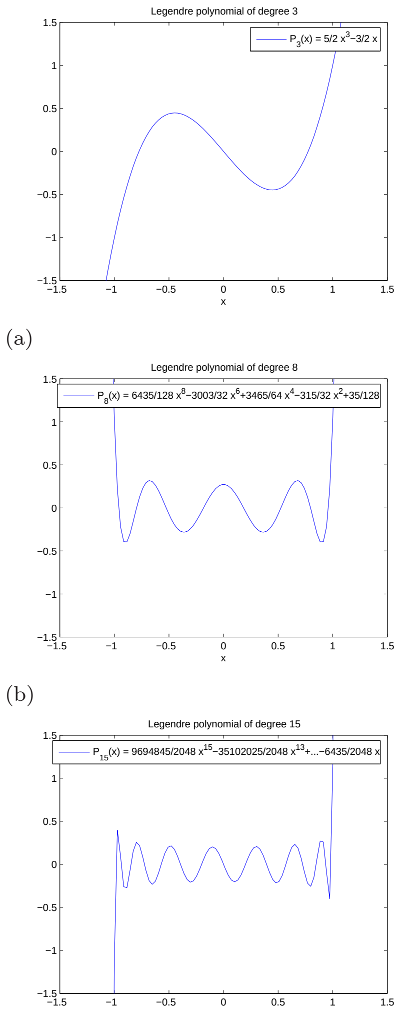
5.1.1 Legendre polynomials and Zonal Harmonics

We start this section by illustrating a few of the Legendre polynomials. Note that these are well defined on the whole real axis. Here it becomes visually clear that the polynomials oscillate only in the interval $[-1, 1]$ and that they are symmetric around $z = 0$. By construction all the polynomials also pass through the point (1,1). From the figures one can see that on the interval $[-1, 1]$, the behavior of Legendre polynomials is similar to both sine and cosine functions. The amplitude is smaller than the standard sine/cosine function (this can be remedied by a constant) but the oscillatory behavior is related. The polynomials plotted in Figure 1 are

$$\begin{aligned} P_3(z) &= \frac{5}{2}z^3 - \frac{3}{2}z, \quad P_8(z) = \frac{6435}{128}z^8 - \frac{3003}{32}z^6 + \frac{3465}{64}z^4 - \frac{315}{32} \\ P_{15}(z) &= \frac{9694845}{2048}z^{15} - \frac{35102025}{2048}z^{13} + \frac{50702925}{2048}z^{11} - \frac{37182145}{2048}z^9 \\ &\quad + \frac{14549535}{2048}z^7 - \frac{2909907}{2048}z^5 + \frac{255255}{2048}z^3 - \frac{6435}{2048}z \end{aligned}$$

From Proposition 4.1 it follows that the Legendre polynomials restricted to the interval $[-1, 1]$ times a normalizing constant are really zonal harmonics. In Figure 2, to the right of Figure 1, the zonal harmonics $\mathcal{Z}_3$, $\mathcal{Z}_8$ and $\mathcal{Z}_{15}$ corresponding to $P_3(z)$, $P_8(z)$ and $P_{15}(z)$ are plotted. By corresponding, we mean that the normalization constant from Proposition 4.1 has been used in such a way that the zonal harmonics plotted in this study attain the same values as the Legendre polynomials. In Figure 2a the squared absolute magnitude of $\mathcal{Z}_3$ is plotted. Because the value of the plotted zonal harmonics never surpass $|\mathcal{Z}_k| \leq 1$ (since the normalization constant is used) we can use the unit sphere as a reference surface (just like the unit circle is a reference for sine and cosine) as a way of plotting $\mathcal{Z}_8$. This is shown in Figure 2b. The last zonal harmonic is illustrated in Figure 2c as values on the unit sphere.

Note that when comparing the Legendre polynomials in Figure 1 to the zonal harmonics in Figure 2, that because of the substitution $\cos\theta = z$, the right hand side of the graphs is the north pole in the three dimensional figures. From Figures 1 and 2 one can see that there is no clear advantage of the three dimensional way of illustrating a zonal harmonic versus the polynomial representation in the Legendre polynomials. The connection between two and three dimensions is even easier to imagine when seeing the oscillating pattern of Figure 1. Specially Figure 2a does not aid us in any way to visualize the zonal harmonic, and Figures 2b and 2c do not give us any additional information that is not contained in the corresponding graphs.

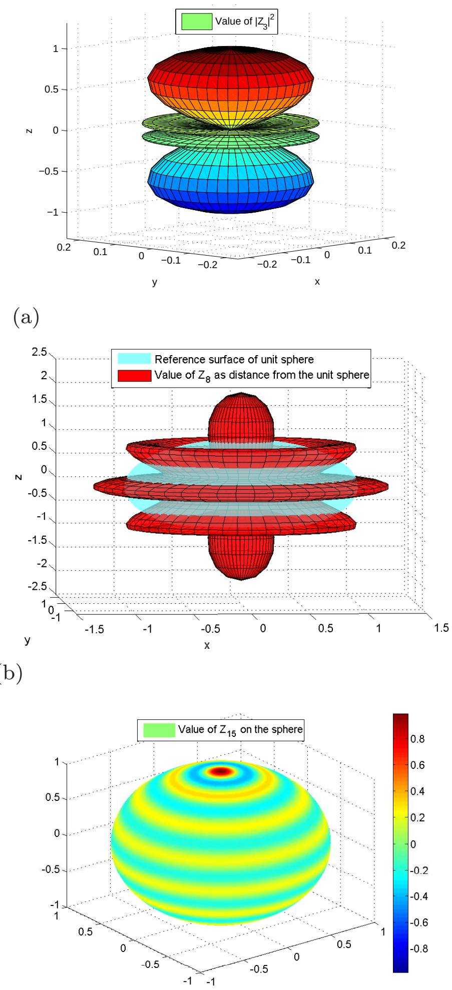


(a)

(b)

(c)

Figure 1: Legendre polynomials.
a) $P_3(z)$, b) $P_8(z)$ and c) $P_{15}(z)$.



(a)

(b)

(c)

Figure 2: Zonal harmonics.
a) $\mathcal{Z}_3$, b) $\mathcal{Z}_8$ and c) $\mathcal{Z}_{15}$.

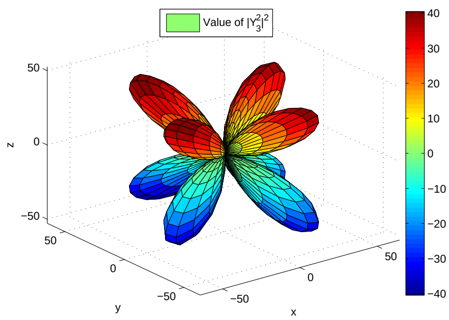
5.1.2 Spherical Harmonics

For the graphical representation of general spherical harmonics, we will use the function $\tilde{Y}_k^n$ from Definition 4.3, with constants $A, B = 1$. The value of the constants A and B does not significantly change the general appearance of the spherical harmonic, they only change its magnitude and rotate the harmonic in the variable φ. Since spherical harmonics can take on values greater than one, we can not use the same approach as with the zonal harmonics. In order to keep the same type of representation as in Figure 2b, we approach this problem by increasing the radius of the reference sphere to the maximum of $|\tilde{Y}_k^n|$. Hence the minimum value of the spherical harmonic will again be close to the origin. Note that when the order of the spherical harmonic equals the degree, that is $n = k$ in relation (25)

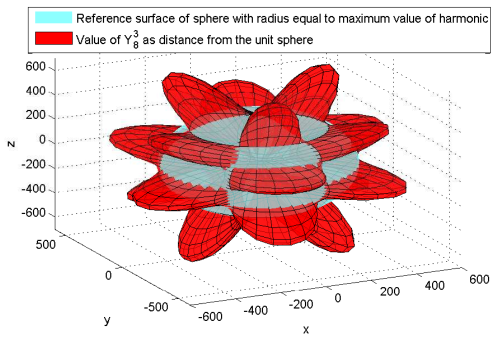
$$\tilde{Y}_k^k = (\cos k\varphi + \sin k\varphi)\,(\sin^k \theta) A_P,$$

where A_P is the constant resulting from the k:th derivative of the Legendre polynomial. This gives that the positions on the sphere where $\tilde{Y}_k^k$ equals zero is dependent only on φ, since the factor $\sin\theta$ is zero only at the poles. Hence we have a spherical harmonic that has a zonal structure that is perpendicular to zonal harmonics. This special case of a spherical harmonic is called a *sectorial harmonic* ([8, p. 132]), and is plotted in Figure 4b.

In Figure 3a the squared absolute magnitude of $\tilde{Y}_3^2$ is illustrated. This representation of a spherical harmonic is mostly used when $\tilde{Y}_k^n$ is complex-valued, since it is the only effective way of plotting the imaginary part of $\tilde{Y}_k^n$. Since we have restricted our graphical illustrations to real-valued spherical harmonics, this representation does not help us in any way to see the connection between the trigonometrical functions and spherical harmonics. The illustration in Figure 3b is an analogue to the trigonometrical functions on the circle, and it is an effective way to show the oscillatory behavior of a spherical harmonic. The disadvantage is that it is hard to tell what value the harmonic takes in a specific point. On the other hand, this is adequately represented in Figure 4a. The compromise one makes with this type of illustration, which is a more accurate way of presenting the values of the spherical harmonic, is that the general shape of the function can be harder to make out. From Figure 4b we can see that for the sectorial harmonic of degree 5, there are 10 sectors (a sector being a part of the sphere where the spherical harmonics is only positive or negative valued) and that these are $\frac{\pi}{5}$ apart (in the variable φ).

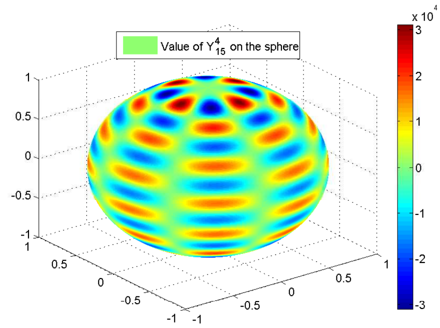


(a)

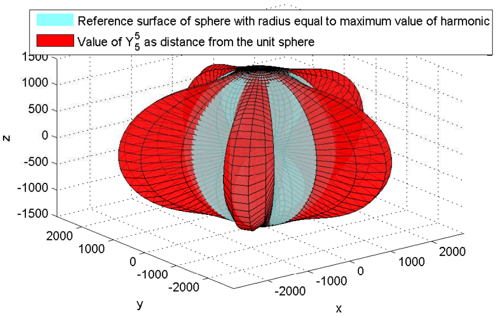


(b)

Figure 3: Spherical harmonics.
a) $|\tilde{Y}_3^2|^2$ and b) $\tilde{Y}_8^3$ on a reference sphere.



(a)



(b)

Figure 4: Spherical harmonics.
a) $\tilde{Y}_{15}^4$ on the unit sphere and b) $\tilde{Y}_5^5$, a sectorial harmonic.

5.1.3 Approximation of functions in stereographic coordinates

So far we have followed the natural choice of expressing spherical harmonics in spherical coordinates. However, the surface of the sphere can be mapped to a plane through the means of stereographic projection. This can be accomplished by fixing a point on the surface of the sphere, the projection point, and defining a plane onto which points on the sphere can be projected. The plane is chosen so that its normal goes through both the origin and the projection point.

The main reason for exploring spherical harmonics in a stereographic coordinate system is the reduction of discontinuities. In a spherical coordinate system there exist discontinuities in two points, one when the angle $\theta \pm n \cdot 2\pi = 0$ and the other at $\theta \pm n \cdot 2\pi = \pi$, where $n = 0, 1, \ldots$. At these points the angle φ can not be single-valued. In a stereographic projection a discontinuity exists in only one point, the projection point. If this point is to be projected onto the plane, its distance from the origin would be infinite and it would be defined in every direction.

For the projection of spherical harmonics the unit sphere with the projection point $(x, y, z) = (0, 0, 1)$ is chosen. The plane onto which the points of the sphere are projected is the xy-plane, that is the equatorial plane. From the reasoning above it is clear that stereographic projection is a mapping from $S^2 \to \mathbb{R}^2$ and that a point on the sphere $P = (x, y, z)$ is projected to a point $P' = (x', y', 0)$ or $P' = (x', y')$. By basic geometry and the fact that the radius of the sphere equals one, it is possible to derive the following relations in Cartesian coordinates.

$$\frac{x}{1-z} = \frac{x'}{1} \quad \Longleftrightarrow \quad x' = \frac{x}{1-z}.$$

Similarly

$$y' = \frac{y}{1-z}.$$

The purpose of this section is to investigate the change of variables in the spherical coordinate system that lead to stereographic coordinates. The relations for this transformation are derived in a similar way as for the Cartesian system described above. It is suitable to use the standard polar coordinate system to describe the position of a point in the plane, that is $P' = (\rho, \varphi)$. Since the azimuthal angle does not change when projected onto the plane, it remains to find ρ as a function of θ. The geometry of triangles can be used to see that

$$\frac{\sin\theta}{1-\cos\theta} = \frac{\rho}{1} \quad \Longleftrightarrow \quad \rho = \frac{\sin\theta}{1-\cos\theta}.$$

Furthermore, using the trigonometrical identities $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = 1 - 2\sin^2\theta$, the following useful expression for θ can be derived.

$$\frac{\sin 2\frac{\theta}{2}}{1-\cos 2\frac{\theta}{2}} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1-(1-2\sin^2\frac{\theta}{2})} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}.$$

According to the relation between spherical and stereographic projections above we have that

$$\cot\frac{\theta}{2} = \rho.$$

Hence, given the calculations above, the important relations between spherical and stereographic coordinates can be summarized as follows:

$$(\rho, \varphi) = \left(\frac{\sin\theta}{1-\cos\theta}, \varphi\right) = \left(\cot\frac{\theta}{2}, \varphi\right) \quad \text{and}$$
$$\theta = 2\tan^{-1}\frac{1}{\rho}. \tag{28}$$

For a derivation of the spherical Laplace operator in stereographic coordinates, see Appendix A.1.1

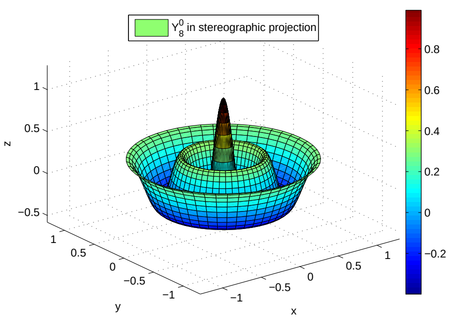
To avoid the discontinuity at the projection point, we choose to only project the bottom part of the sphere (and spherical harmonic). That is we restrict $\pi/2 \leq \theta \leq \pi$. This means that the harmonic will parametrized by the unit disc, and that the z-axis in the stereographic projection represents the value of the spherical harmonic. In Figures 5a and 5b the stereographic projections of $\mathcal{Z}_8$ from Figure 6a and $\tilde{Y}_3^2$ from 6b are shown. We can see that the zonal harmonic becomes a radial function. This is to be expected, since a given latitude on the sphere (θ is fixed) is projected onto a circle of radius $\tan\frac{\theta}{2}$, according to the relation above. Since zonal harmonics are constant on latitudes of the sphere (see Figures 1 and 2), the projected zonal harmonic will be constant on concentric circles.

When approximating a given function f we use only a finite number of terms from the spherical harmonic series expansion of f. In this section the approximation of a given function is a sum of spherical harmonics up to degree k, for a chosen fixed k. To see how these sums of spherical harmonics approximate the chosen functions, different numbers of terms from the series are included. The graphical representations presented below are plotted using the stereographic projection and are a direct application of the theory from section 4.5.

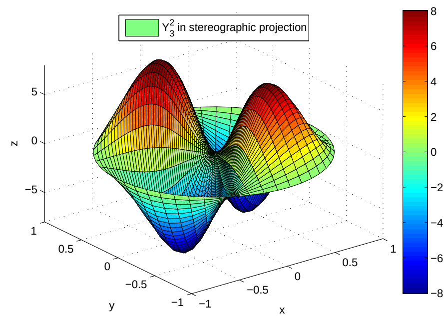
We start by approximating a smooth function defined on the sphere $f(\theta, \varphi) = \frac{\cos 4\theta}{2} + \sin\frac{\theta}{2}$, shown in Figure 7a. The choice of using a function dependent only on the variable θ is to avoid the problem of φ not being single-valued at the poles of the sphere, which can cause the function to be not smooth in the areas surrounding these. The two approximations of the chosen function (Figure 7b and 8a) are expansions of f up to degree 3 and 14. The difference plotted in Figure 8b is f minus the approximation of f. We will denote the approximation of f up to degree k as f_{Ak} and the difference $f - f_{Ak}$ as f_{Dk}. From Figure 7 we can see that the first approximation f_{A3} has not yet started to resemble the original function. This is because of the small number of terms included in the approximating sum. However, from the plot of f_{A14} we can visually confirm that after only 14 spherical harmonics the approximation is almost identical to f. This is also evident in Figure 8b, where the magnitude of f_{D14} is in the order of 10^{-4}, which is small compared to the values taken by f.

In Figure 9 we approximate a function that is not continuous on the sphere, namely $f(\theta, \varphi) = \frac{\pi}{2} - \varphi$. This is to graphically examine what happens near points where the function is discontinuous. Figure 9 shows the stereographic projection of f as well as f_{A3}, and in Figure 10 f_{A14} and f_{D14} are plotted. One can notice two things about the approximations. First, in the illustration of the approximating function in Figure 10a it looks like the section connecting the surfaces at the line of discontinuity only takes on two distinct color-values. This is because MATLAB can only assign one color to each partition of the plotted surface. Near the discontinuity, the connecting section is so steep that only two partitions fit the surface. This is not true, since spherical harmonics are continuous and take all values between 0 and 2π. Secondly, in Figure 10b, it becomes apparent that something similar to the Gibbs phenomenon in Fourier analysis exists in spherical harmonic expansions. This means that at the line of discontinuity an increase in oscillation can be observed in the approximating partial sums. Note that the difference f_{D14} for this function is much larger than in the approximation of the smooth function. In Figure 10b we see that for some points, the magnitude of f_{D14} is in the same order as the values of the original function.

The final graphical example is an approximation of the function $f(\theta, \varphi) = \theta^3 \cos(6\theta) \sin\varphi$. This function is chosen to examine the approximating function when the original function has a more complicated oscillatory behavior and is dependent on both θ and φ. Note that f is discontinuous at the south pole. The stereographic projection of f is shown in Figure 11 and the approximating function f_{A3} in Figure 12. From Figure 13b one can see that the largest discrepancies of the approximated function from f are near the point of discontinuity or where the function oscillates intensely. Hence the Gibbs phenomenon is present even though f only has a point discontinuity. Comparing Figures 10b and 13a it seems that the expansion converges faster to f than in the previous approximation. Hence it seems that functions with point discontinuity are more easily approximated than functions with line discontinuity, and that the expansions of smooth functions converge faster than expansions of discontinuous functions of any kind. This concludes our study about spherical harmonics.

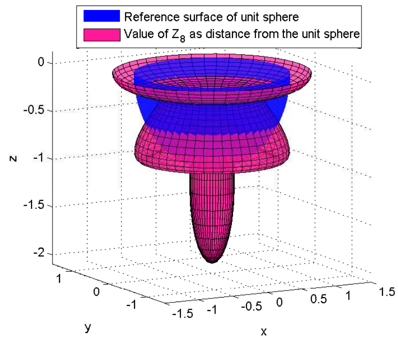


(a)

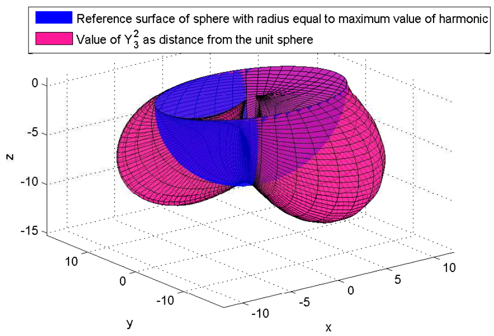


(b)

Figure 5: Stereographic projections.
a) $\mathcal{Z}_8(x)$ and b) $\tilde{Y}_3^2$.

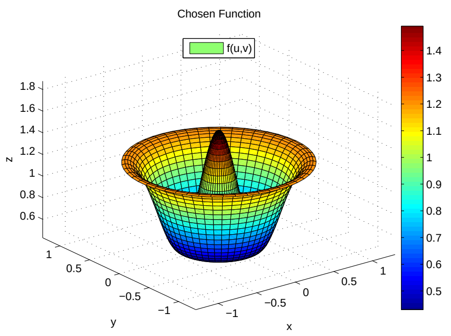


(a)

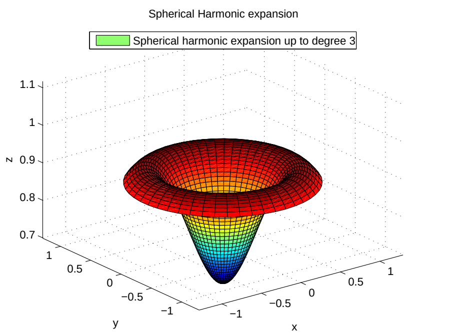


(b)

Figure 6: Functions on the part of the sphere used for stereographic projection.
a) $\mathcal{Z}_8(x)$ and b) $\tilde{Y}_3^2$.

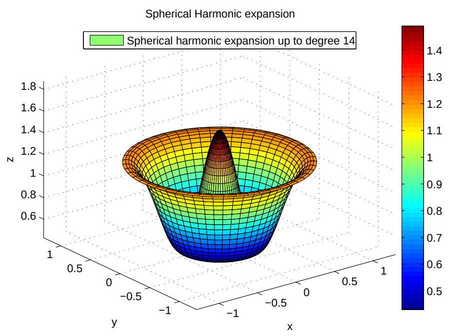


(a)

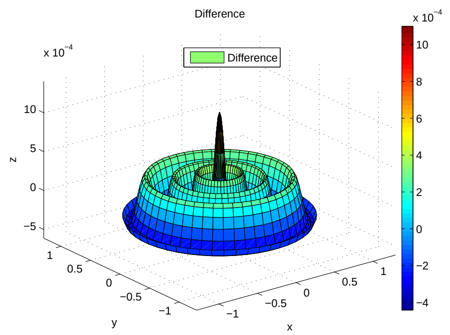


(b)

Figure 7: a) Stereographic projection of $f = (\theta, \varphi) = \frac{\cos 4\theta}{2} + \sin \frac{\theta}{2}$ and b) The approximation f_{A3}.

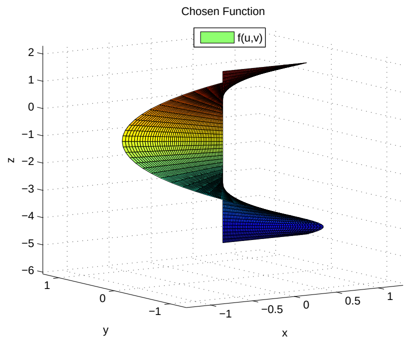


(a)

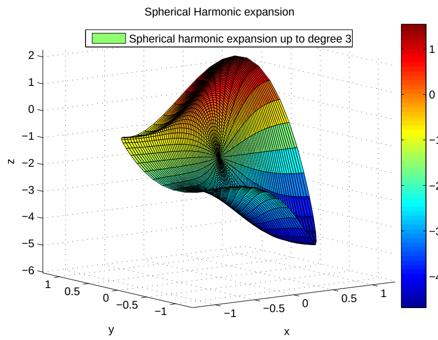


(b)

Figure 8: a) The approximation f_{A14} and b) the difference f_{D14}.

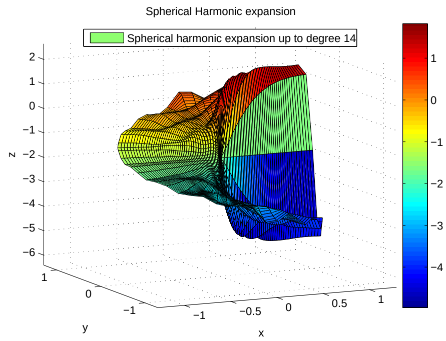


(a)

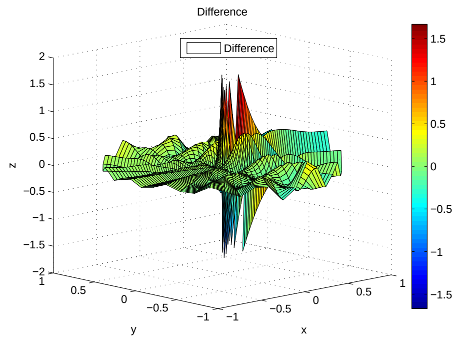


(b)

Figure 9: a) Stereographic projection of $f = \frac{\pi}{2} - \varphi$ and b) The approximation f_{A3}.



(a)



(b)

Figure 10: a) The approximation f_{A14} and b) the difference f_{D14}.

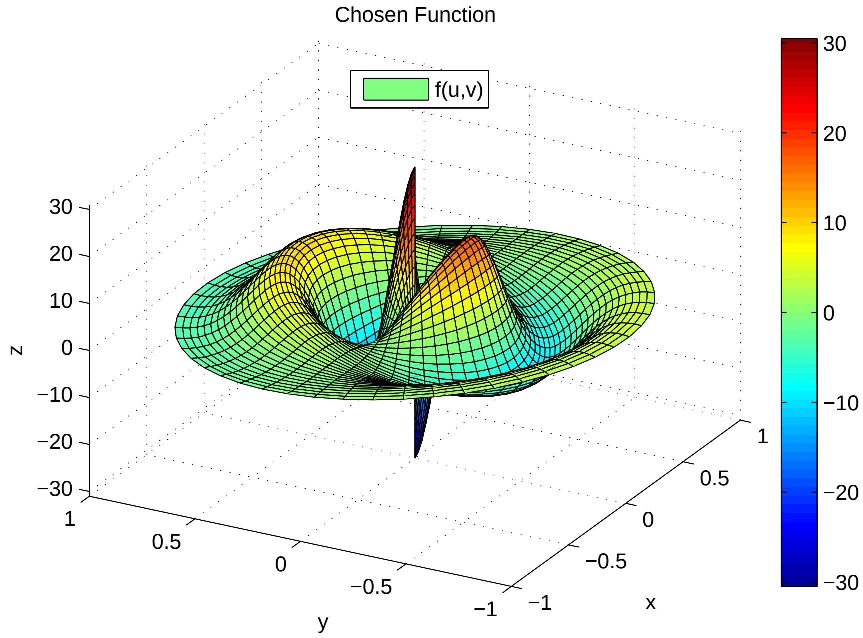


Figure 11: Stereographic projection of $f(\theta, \varphi) = \theta^3 \cos 6\theta \sin \varphi$.

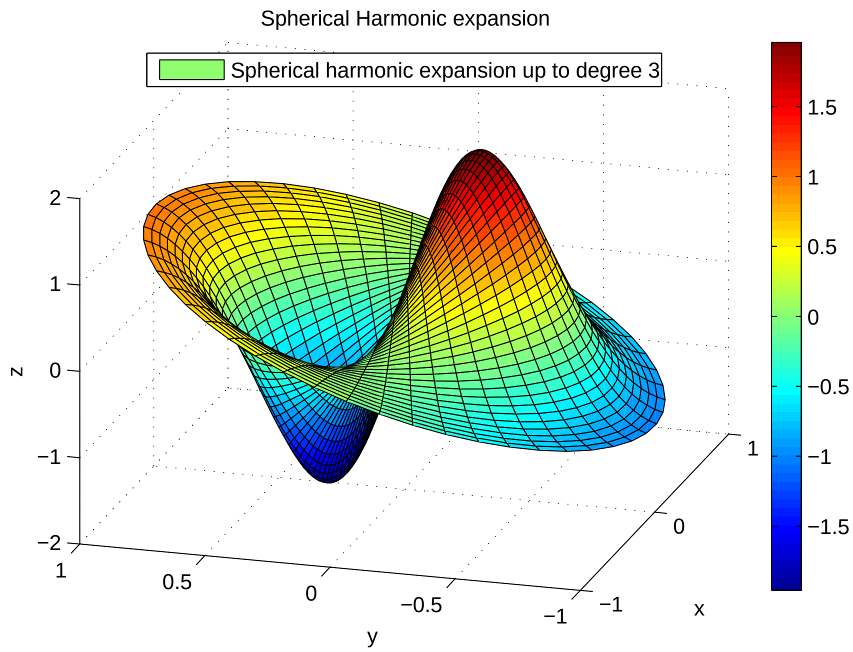
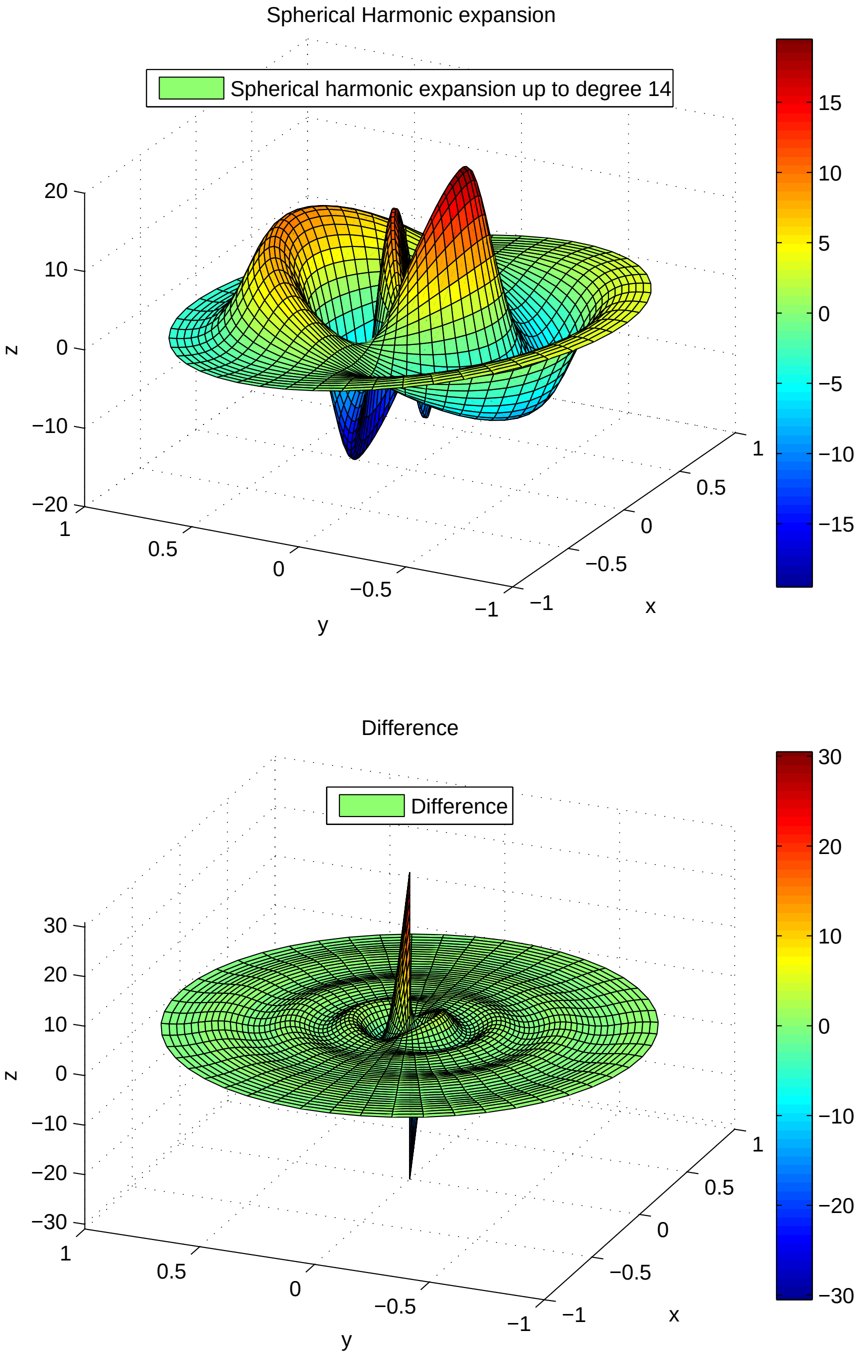


Figure 12: Stereographic projection of the spherical harmonic expansion up to degree 3 of $f(\theta, \varphi) = \theta^3 \cos 6\theta \sin \varphi$.



(a)

(b)

Figure 13: Stereographic projection of the spherical harmonic expansion up to degree 14 of $f(\theta, \varphi) = \theta^3 \cos 6\theta \sin \varphi$ and the difference between the two.

A Appendix A

A.1 The Laplace operator in Spherical Coordinates

A problem can sometimes be simplified if one changes coordinates from the regular Cartesian xyz-system to a more suitable way of describing the position of any given point. The most common examples of such systems are *cylindrical* and *spherical coordinates* (for a thorough discussion about the Laplace operator in spherical coordinates, see [3], p. 66-67).

In cylindrical coordinates a point in space is described in the same way as in polar coordinates, with the addition of the z-axis having the same role as in the Cartesian coordinates. In this way, any point in Cartesian coordinates can be transformed to cylindrical coordinates with the relations

$$x = \rho \cos \varphi, \qquad y = \rho \sin \varphi, \qquad z = z, \tag{29}$$

and

$$\rho = \sqrt{x^2 + y^2}, \qquad \varphi = \tan^{-1} \frac{y}{x}, \qquad z = z, \tag{30}$$

where ρ is the distance from the origin in the xy-plane and θ is the angle between the x-axis and $\hat{\rho}$. Similarly one can derive the relations between Cartesian and spherical coordinates,

$$x = r \sin \theta \cos \varphi, \qquad y = r \sin \theta \sin \varphi, \qquad z = r \cos \theta, \tag{31}$$

where r is the distance from the origin, φ is the angle between the x-axis and the projection of $\hat{r}$ in the xy-plane and θ is the angle between the z-axis and $\hat{r}$. Furthermore, from equation (29) and equation (31) one can deduce the relations between cylindrical and spherical coordinates to be

$$z = r \cos \theta, \qquad \rho = r \sin \theta, \qquad \varphi = \varphi. \tag{32}$$

Note the similarity between equation (29) and equation (32). This relation will be useful in significantly shortening the derivation of the Laplace operator. To be able to determine whether a given function $f(r, \varphi, \theta)$ is harmonic one must be able to express Laplace's equation

$$f_{xx} + f_{yy} + f_{zz} = 0$$

as an operator involving r, φ and θ only. The most straightforward way of accomplishing this is to use the relations in (31) together with the chain rule. However, this process is long and it does not enlighten the reader in any special way. Therefore, an approach using the laplacian in cylindrical coordinates and the aforementioned similarity between equation (29) and equation (32), that hopefully highlights the process of using the chain rule, will be opted here.

Using the chain rule in cylindrical coordinates is fairly straightforward, and results in the laplacian in cylindrical coordinates

$$\Delta f = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial f}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2} \tag{33}$$

for the function $f(\rho, \varphi, z)$.

To derive the laplacian in spherical coordinates, one can use that the expressions (29) and (32) are identical except for notation. Motivated by this a change of notation ($y \to \rho$, $\rho \to r$, $\varphi \to \theta$ and $x \to z$) the relations

$$\frac{\partial f}{\partial \rho} = \sin\varphi \frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta} \tag{34}$$

and

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} \tag{35}$$

can be derived. When comparing the laplacian in cylindrical coordinates in equation (33) and equation (35), one can see that two of the terms are represented in both expressions. Furthermore, the relations in (32) together with equation (34) suggest that the remaining terms can be expressed as

$$\frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \varphi^2} = \frac{1}{r}\frac{\partial f}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial f}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}. \tag{36}$$

Adding equation (35) and equation (36) covers all of the the terms in the expression for the cylindrical laplacian. Hence the sum of these two expressions give the laplacian for spherical coordinates

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial f}{\partial \theta} \tag{37}$$

for a function $f(r, \varphi, \theta)$. This can also be written as

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right). \tag{38}$$

For the purpose of spherical harmonics it will be convenient to define the *spherical Laplace operator* Δ_s as the part of equation (38) that is independent of r, that is

$$\Delta_s = \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}. \tag{39}$$

A.1.1 The spherical Laplace operator in stereographic coordinates

Consider the spherical Laplace operator, as defined in equation (39) for a function $f(\theta, \varphi)$. The objective of this section is to calculate $\Delta_s f$ for a function $f(\rho, \varphi)$ in stereographic coordinates, where a change to the variables ρ and φ has been made according to equation (28). Since only the partial derivatives of f with respect to θ are affected, we start by calculating these. Using equation (28), a few relations that will be helpful in the forthcoming calculations can be derived:

$$\frac{d\rho}{d\theta} = \frac{1}{\frac{d\theta}{d\rho}} = -\frac{(\rho^2+1)}{2} \tag{40}$$

and

$$\begin{aligned}\frac{\rho^2+1}{2} &= \frac{1}{2}\left(\frac{\sin^2\theta}{(1-\cos\theta)^2} + 1\right) = \frac{1}{2}\left(\frac{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}{(1-\cos\theta)^2}\right) \\ &= \frac{1}{2}\frac{2(1-\cos\theta)}{(1-\cos\theta)^2} = \frac{1}{1-\cos\theta}.\end{aligned} \tag{41}$$

Given the relations in (40) and (41), the first and second partial derivatives for a function $f(\rho,\varphi)$ can be expressed as

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial \rho}\frac{d\rho}{d\theta} = -\frac{\partial f}{\partial \rho}\frac{1}{1-\cos\theta} \quad \text{and}$$
$$\frac{\partial^2 f}{\partial \theta^2} = \frac{\partial^2 f}{\partial \rho^2}\frac{1}{(1-\cos\theta)^2} + \frac{\partial f}{\partial \rho}\frac{\sin\theta}{(1-\cos\theta)^2}.$$

With the appropriate variable change from $(\theta,\varphi) \to (\rho,\varphi)$, using the chain rule for composite functions in the expression for the spherical Laplace operator for a function $f(\rho,\varphi)$ gives

$$\begin{aligned}
\Delta_s f &= \frac{\cos\theta}{\sin\theta}\left(-\frac{\partial f}{\partial \rho}\frac{1}{1-\cos\theta}\right) + \left(\frac{\partial^2 f}{\partial \rho^2}\frac{1}{(1-\cos\theta)^2} + \frac{\partial f}{\partial \rho}\frac{\sin\theta}{(1-\cos\theta)^2}\right) \\
&+ \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2} \\
&= \frac{\partial f}{\partial \rho}\frac{1}{\sin\theta(1-\cos\theta)} + \frac{\partial^2 f}{\partial \rho^2}\frac{1}{(1-\cos\theta)^2} + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2} \\
&= \frac{1}{(1-\cos\theta)^2}\left(\frac{\partial f}{\partial \rho}\frac{1-\cos\theta}{\sin\theta} + \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\sin^2\theta}\frac{\partial f}{\partial^2 \varphi}\right) \\
&= \frac{(\rho^2+1)^2}{4}\left(\frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial^2 \varphi}\right) \\
&= \frac{(\rho^2+1)^2}{4\rho^2}\left(\rho\frac{\partial f}{\partial \rho} + \rho^2\frac{\partial^2 f}{\partial \rho^2} + \frac{\partial^2 f}{\partial^2 \varphi}\right).
\end{aligned} \tag{42}$$

Note that the expression in brackets in equation (42) is the laplacian in polar coordinates (see Appendix A.1). Hence the laplacian in stereographic coordinates, regarding functions defined on the unit sphere, is a function of ρ times the laplacian in polar coordinates.

B Appendix B

For this study, MATLAB was used to plot all the figures in section 5. A separate code was written for every type of function. However, since the script for approximating functions contains most of the code written, only this script will be included to not overburden the appendix. The script can be copied into the MATLAB editor and used without any processing.

The script below produces the approximating sum of a chosen function that is defined on the sphere. Furthermore, it uses stereographic projection to plot the chosen function, approximating function and their difference. Calculations in the script can be used to plot for instance zonal or spherical harmonics, but a specification how to graphically represent these would have to be added in the code. One known problem with this source code, is that for too large n (if the degree of spherical harmonics included in the sum is around 30 or larger, depending on the function approximated), MATLAB cannot process the calculations due to a too large directory name.

```
function y = SpHarmonicapprx(func,n)

%SPHARMONICAPPRX(func,n) approximates a given function of two variables
%(u,v)=(theta,phi),that is defined on the unit sphere, by expanding it into
```

```
%spherical harmonics. The variables u,v must be symbolic and n refers to the
%first summation index, thus producing the first (n)*(n+1)/2 terms of the
%series expansion.

%Error statement regarding the number of input arguments
if nargin > 2 || nargin < 2
   error(['Wrong number of input arguments. Arguments must contain the number'...
       'of terms in expansion (n) and the function to be approximated.'])
end

%Error statement regarding the allowed values of n
if numel(n) > 1 || ~isreal(n) || n ~= round(n)
    error('n must be a positive scalar integer')
end

%Introducing symbolic variables
syms x k q c u v;
C = c ;

%Computing the Zonal Harmonics with its norm squared (pi factor omitted
%in these calculations for clarity in output),ZNn,up to degree n and sorting
%them in a 1x(n+1) matrix = [ZN0 ZN1 ZN2 ... ZNn]. This is used when computing the
%constants A0. Also computes the Zonal Harmonics for the series and
%arranges them in a matrix [Z0 Z1 ... Zn].
ZNtot = [];
Ztot = [];
for i = 0:n
    s = floor(i/2);
    f = ((-1)^k/sym('k!'))*sym('(2*q - 2*k)!')/(sym('(q -2*k)!')*sym('(q - k)!'))*x^(q - 2*k);
    Pol = (1/2^q)*symsum(f,k,0,s);
    Legendre = subs(Pol,q,i);
    Legsimple = simplify(Legendre);
    P = expand(Legsimple);
    ZN = (2*i+1)/(4)*subs(P,x,cos(u));
    ZNtot = [ZNtot ZN];
    Z = subs(P,x,cos(u));
    Ztot = [Ztot Z];
end
Ztot = Ztot;
ZNtot = ZNtot;

%Computing every order of the Associated Legendre function with its norm squared
%(pi factor omitted in these calculations for clarity in output),PNmn, up to degree n,
%sorting them in two 1x(n(n+1)/2) -matrixes with the corresponding sine or cosine
%function [cos(v)*PN11 cos(v)*PN12 cos(2v)*PN22 cos(v)*PN31... cos(n-1)*PN(n-1)n
%cos(nv)*PNnn] and the same for sine. Also computes the Spherical Harmonics
%for the series expansion and arranges them in two matrixes [cos(v)*P11 ...
%cos(nv)*Pnn] and [sin(v)*P11... ]
Ycos = [];
YNcos = [];
for i = 1:n;
    s = floor(i/2);
    f = ((-1)^k/sym('k!'))*sym('(2*q - 2*k)!')/(sym('(q -2*k)!')*sym('(q - k)!'))*x^(q - 2*k);
    Pol = (1/2^q)*symsum(f,k,0,s);
    Legendre = subs(Pol,q,i);
    Legsimple = simplify(Legendre);
    P = expand(Legsimple);
    Yleg = [];
    Pleg = [];
    for j = 1:i
        Pmnx = diff(P,j);
        YNmncos = (2*i+1)/(2)*factorial(i-j)/factorial(i+j)*cos(j*v)*sin(u)^j*subs(Pmnx,x,cos(u));
        Yleg = [Yleg YNmncos];
        Pmn = cos(j*v)*sin(u)^j*subs(Pmnx,x,cos(u));
        Pleg = [Pleg Pmn];
    end
    YNcos = [YNcos Yleg];
    Ycos = [Ycos Pleg];
end
YNcos = YNcos;
Ycos = Ycos;

Ysin = [];
YNsin = [];
```

```
for i = 1:n;
    s = floor(i/2);
    f = ((-1)^k/sym('k!'))*sym('(2*q - 2*k)!')/(sym('(q -2*k)!')*sym('(q - k)!'))*x^(q - 2*k);
    Pol = (1/2^q)*symsum(f,k,0,s);
    Legendre = subs(Pol,q,i);
    Legsimple = simplify(Legendre);
    P = expand(Legsimple);
    Yleg = [];
    Pleg = [];
    for j = 1:i
        Pmnx = diff(P,j);
        YNmnsin = (2*i+1)/(2)*factorial(i-j)/factorial(i+j)*sin(j*v)*sin(u)^j*subs(Pmnx,x,cos(u));
        Yleg = [Yleg YNmnsin];
        Pmn = sin(j*v)*sin(u)^j*subs(Pmnx,x,cos(u));
        Pleg = [Pleg Pmn];
    end
    YNsin = [YNsin Yleg];
    Ysin = [Ysin Pleg];
end
YNsin = YNsin;
Ysin = Ysin;


%Calculating the coefficients A0n (for Zonal Harmonics) and Anm (for cos*Pmn) and
%Bnm (for sin*Pmn) of the series of Spherical Harmonics.
A0tot = [];
a0 = numel(ZNtot);
F = func*sin(u);
for i = 1:a0;
    Z = ZNtot(1,i);
    A0 = int(int(F*Z,v,0,2*pi),u,0,pi);
    A0tot = [A0tot A0];
end
A0tot = A0tot;

a = numel(YNcos);
Atot = [];
for i = 1:a;
    Pcos = YNcos(1,i);
    A = int(int(F*(Pcos),v,0,2*pi),u,0,pi);
    Atot = [Atot A];
end
Atot = Atot;

b = numel(YNsin);
Btot = [];
for i = 1:b;
    Psin = YNsin(1,i);
    B = int(int(F*(Psin),v,0,2*pi),u,0,pi);
    Btot = [Btot B];
end
Btot = Btot;

%The final expression in terms of the appropriate constants times the
%spherical harmonics.

S = Ztot*(A0tot)' + Ycos*(Atot)' + Ysin*(Btot)';
y1 = S/pi;
y = simplify(y1);
y2 = vpa(y);                                %Used for expansions containing many terms,
                                            %as MATLAB cannot handle too long function Directory names


%Projecting the Spherical Harmonic expansion onto the disc with radius 1
%with the aid of Stereographic Coordinates. Only the bottom part of the
%sphere (pi/2 < theta > pi) is projected to avoid the dicontinuities in the
%projection point, (0,0,1).
figure('Name','Spherical Harmonic expansion');
ezsurf(sin(u)*cos(v)/(1 - cos(u)),sin(u)*sin(v)/(1 - cos(u)),y2,[pi/2,pi,0,2*pi])
%colormap([1 .1 .6])
legend(['Spherical harmonic expansion up to degree ' int2str(subs(C,c,n))],'Location','Best')
set(gca,'Title',text('String','Spherical Harmonic expansion'));

%Projecting the chosen function, defined on the sphere, onto the disc of radius 1.
```

```
figure('Name','Stereographical projection of chosen function for reference');
ezsurf(sin(u)*cos(v)/(1 - cos(u)),sin(u)*sin(v)/(1 - cos(u)),func,[pi/2,pi,0,2*pi])
%colormap([0 0 1])
legend('f(u,v)')
set(gca,'Title',text('String','Chosen Function'));

%A surface plot of the difference between the chosen function and its
%Spherical harmonic approximation of degree n
if simple(y-func)==0
    figure1 = figure(...
  'Color',[1 1 1],...
  'PaperPosition',[0.6345 6.345 20.3 15.23],...
  'PaperSize',[20.98 29.68]);
    annotation1 = annotation(...
    figure1,'textbox',...
  'Position',[0.08214 0.3452 0.8143 0.5669],...
  'LineStyle','none',...
  'Color',[0 0.749 0.749],...
  'FitHeightToText','off',...
  'FontName','Bell MT',...
  'FontSize',35,...
  'HorizontalAlignment','center',...
  'String',{'The chosen function','and the Spherical','Harmonic expansion','are identical!'});
else
    figure('Name','Difference between function and approximation');
    ezsurf(sin(u)*cos(v)/(1 - cos(u)),sin(u)*sin(v)/(1 - cos(u)),func-y2,[pi/2,pi,0,2*pi]);
    legend('Difference')
end
set(gca,'Title',text('String','Difference'))
```

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