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Consensus problems for multi-agent systems

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Abstract

In this paper, we study consensus problems in networks of dynamic agents with first-order, second-order and high-order dynamics, respectively. Several conditions are obtained to make all agents reach consensus. The detailed contents are as follows:

(1) We study guaranteed cost coordination in directed networks of agents with uncertainty. For convergence analysis of the networks, a class of Lyapunov functions are introduced as a measure of the disagreement dynamics. Using these Lyapunov functions, sufficient conditions are derived for state consensus of system with desired cost performance.

(2) We consider consensus control in directed networks of agents with double integrator dynamics. A sufficient and necessary condition is proved by using the eigenvector-eigenvalue method of finding solutions.

(3) We investigate consensus of high-order multi-agent systems. A new dynamic neighbor-based control law is proposed which contains two parts, one is the local feedback and the other is the distributed feedback of the first states of each agent. A sufficient condition is derived for state consensus of the system.

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Chapter 1

Introduction

A multi-agent system is a system composed of multiple interacting intelligent agents. Multi-agent systems can be used to solve problems which are difficult or impossible for an individual agent or monolithic system to solve. Due to recent technological advances in communication and computation, and important practical applications such as unmanned vehicles, automated highway systems and mini-satellites, distributed coordination of multi-agent systems has attracted more and more attention. Neighbor-based rules are widely applied in multi-agent systems, inspired originally by the aggregations of groups of individual agents in nature [1]. In contrast to conventional large-scale systems, where dominant centralized control is the core, multi-agent systems are concerned with both mobile individual dynamics and communication topologies (network structures for transmitting information). In distributed coordination of multi-agent systems, one critical problem is how to make all agents reach an agreement on certain quantities of interest. This problem is usually called the consensus problem.

Consensus problems were first studied by many researchers for first-order multi-agent systems [1]-[14]. For example, in [3], Olfati-Saber and Murray investigated a systematic framework of consensus problems with directed communication graphs or time-delays by a Lyapunov-based approach. Also, in [14], Lin *et al.* extended the results of [3] to the case of switching topology with time-delay and disturbances and presented conditions in terms of linear matrix inequalities for state consensus of the systems. Recently, more and more attention is paid to consensus related problems for second-order and high-order multi-agent systems [15]-[23]. For example, in [15, 18], Ren *et al.* gave several second-order and high-order control laws and derived sufficient conditions for the case of fixed

topology. Also, Qu [19] studied a class of nonlinear high-order multi-agent systems and showed consensus can be achieved even though the communication graph has no spanning trees.

In this paper, we study consensus control of multi-agent systems. First, we consider consensus problems of the multi-agent system with uncertainty on directed graphs, extending the work of [14]. The uncertainty is assumed to be norm-bounded. A quadratic cost function is proposed for the energy consumption of all agents. The analysis is performed by a Lyapunov-based approach. Since the closed-loop system matrix is singular and the final value of each agent might not be zero, it is hard to analyze the stability of the system directly using the existing approaches. For convergence analysis of the system, we introduce a new class of Lyapunov functions which filter out the agreement dynamics and is indeed a measure of the energy of the disagreement dynamics. Based on these Lyapunov functions, sufficient conditions are obtained for the state consensus of the system with desired cost performance. Second, we consider consensus problems of a class of second-order multi-agent systems with fixed topology. We introduce a simple but effective analysis method to handle the networks of second-order agents with fixed topology. This method can also be used to the general linear multi-agent systems and might shed light on the nature of the consensus behavior. Third, we consider consensus problems of high-order multi-agent systems in a way to extend the work of [15]. We introduce a new feedback dynamic neighbor-based control law which contains two parts, one is the local feedback and the other is the distributed feedback of the first states of each agent. Then we derive sufficient conditions are derived to make all agents reach consensus asymptotically. Different from the existing ones in [18, 19], our control law does not need any information except the relative information of the first states of agents.

Chapter 2

Preliminaries

Graph theory, Kronecker product and Lyapunov theory will be the main tools to study the stability of the protocols. In this section, we briefly introduce some basic concepts and properties about them (referring to [31, 25, 27] for more details).

2.1 Graph theory

Let $G(\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a directed graph of order n with the sets of nodes $\mathcal{V} = \{s_1, \dots, s_n\}$, the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $\mathcal{A} = [a_{ij}]$ with nonnegative elements. The node index is the element of a finite index set $\mathcal{I} = \{1, 2, \dots, n\}$. The edge is written as $e_{ij} = (s_i, s_j)$ with the first element s_i as the tail of the edge and the other s_j as the head. The set of neighbors of node s_i is denoted by $N_i = \{s_j \in \mathcal{V} : (s_i, s_j) \in \mathcal{E}\}$. The adjacency element a_{ij} ($i \neq j$) is positive if and only if $e_{ij} \in \mathcal{E}$, and a_{ij} is usually called the weight of the edge e_{ij} . In addition, it is assumed that $a_{ii} = 0$ for all $i \in \mathcal{I}$. For the directed graph, we define the Laplacian as $L = [l_{ij}]$, where $l_{ii} = \sum_{j=1}^n a_{ij}$ and $l_{ij} = -a_{ij}$, $i \neq j$. From the definition, we can see that an important fact of L is that all the row sums of L are zero and thus $\mathbf{1}_n = [1, 1, \dots, 1] \in \mathbb{R}^n$ is an eigenvector of L associated with the zero eigenvalue. Here, it should be noted that the definition of Laplacian L is a bit different from the definition in the traditional sense. If a directed graph has the property that $a_{ij} = a_{ji}$ for any $i \neq j$, the directed graph is called undirected graph. Then for the undirected graph, the Laplacian is symmetric since $a_{ij} = a_{ji}$.

A directed path is made up of a series of ordered edges: $(s_{i_1}, s_{i_2}), (s_{i_2}, s_{i_3}), \dots$, where $s_{i_j} \in \mathcal{V}$. If there is a directed path from every node to every other node, the graph is said to be strongly connected. Moreover, if there exists a node such that there is a directed path

from every other node to this node, the graph is said to have a spanning tree. Obviously, any undirected graph is strongly connected if and only if it has a spanning tree.

Lemma 1. [14, 31] If the graph G has a spanning tree, then the Laplacian L of the graph has the following properties:

(1) $\text{rank}(L) = n - 1$ and L has one simple eigenvalue at zero associated with the eigenvector $\mathbf{1}_n$, where $\mathbf{1}_n = [1, 1, \dots, 1]^T \in \mathbb{R}^n$.

(2) The rest $n - 1$ eigenvalues all have positive real-parts. Specially, if the graph G is undirected, then they are all positive and real.

Lemma 2. [14] Consider a directed graph G . Let D be the matrix with rows and columns indexed by the nodes and edges of G such that

$$D_{uf} = \begin{cases} 1, & \text{if the node } u \text{ is the tail of the edge } f, \\ 0, & \text{otherwise,} \end{cases}$$

and E be the 01-matrix with rows and columns indexed by the edges and nodes of G such that

$$E_{fu} = \begin{cases} 1, & \text{if the node } u \text{ is the head of the edge } f, \\ 0, & \text{otherwise.} \end{cases}$$

Then the Laplacian of G can be decomposed into $L = DW(D^T - E)$, where $W = \text{diag}\{w_1, w_2, \dots, w_{|\mathcal{E}|}\}$, w_i is the weight of the i th edge of G and $|\mathcal{E}|$ is the number of the edges.

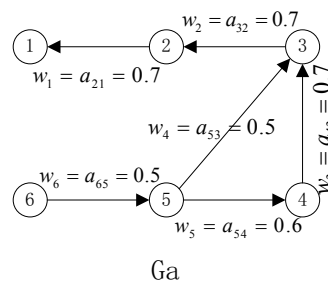


Fig.2.1 One example of directed graph that has spanning trees.

To illustrate the concepts and results above, we give an example in Fig.2.1. In graph G_a , all other nodes have at least one directed path to the node 1. Therefore the graph G_a has spanning trees. The adjacency matrix of G_a is

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0.7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \end{bmatrix}$$

From G_a , we see that there are 6 edges. Without loss of generality, define the order of edge randomly as $e_{21}, e_{32}, e_{43}, e_{53}, e_{65}$. Then, the Laplacian of G_a can be expressed as

$$L = D_0 W_0 (D_0^T - E_0) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -0.7 & 0.7 & 0 & 0 & 0 & 0 \\ 0 & -0.7 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & -0.7 & 0.7 & 0 & 0 \\ 0 & 0 & -0.5 & -0.6 & 1.1 & 0 \\ 0 & 0 & 0 & 0 & -0.5 & 0.5 \end{bmatrix}$$

where

$$D_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, E_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$W_0 = \text{diag}\{w_1, w_2, \dots, w_n\} = \text{diag}\{a_{21}, a_{32}, a_{43}, a_{53}, a_{54}, a_{65}\} = \text{diag}\{0.7, 0.7, 0.7, 0.5, 0.6, 0.5\}.$$

2.2 Kronecker product

Definition 1. [25] Let $C = [c_{ij}] \in \mathbb{R}^{m \times l}$ and $F = [f_{ij}] \in \mathbb{R}^{p \times q}$. We say

$$C \otimes F = \begin{bmatrix} c_{11}F & c_{12}F & \cdots & c_{1l}F \\ c_{21}F & c_{22}F & \cdots & c_{2l}F \\ \vdots & \vdots & \vdots & \vdots \\ c_{m1}F & c_{m2}F & \cdots & c_{ml}F \end{bmatrix} \in \mathbb{R}^{mp \times lq}$$

is the Kronecker product of the matrices C and F .

Proposition 1. [25] For any $X_0, Y_0, Z_0, D_0 \in \mathbb{R}^{n \times n}$ and $a_0 \in \mathbb{R}$,

- (1) $(a_0 X_0) \otimes Y_0 = X_0 \otimes (a_0 Y_0) = a_0 (X_0 \otimes Y_0)$,
- (2) $(X_0 + Y_0) \otimes Z_0 = X_0 \otimes Z_0 + Y_0 \otimes Z_0$,
 $Z_0 \otimes (X_0 + Y_0) = Z_0 \otimes X_0 + Z_0 \otimes Y_0$,
- (3) $(X_0 \otimes Y_0)(Z_0 \otimes D_0) = (X_0 Z_0) \otimes (Y_0 D_0)$,
- (4) $(X_0 \otimes Y_0)^T = X_0^T \otimes Y_0^T$.

2.3 Lyapunov theory

Definition 2. (Class \mathbb{K} , \mathbb{KR} Functions.) [27] A function $\alpha(\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ belongs to class \mathbb{K} (denoted by $\alpha(\cdot) \in \mathbb{K}$) if it is continuous, strictly increasing and $\alpha(0) = 0$, where \mathbb{R}_+ denotes the set of all nonnegative real numbers. The function $\alpha(\cdot)$ is said to belong to class \mathbb{KR} if α is of class \mathbb{K} and in addition, $\alpha(p) \rightarrow \infty$ as $p \rightarrow \infty$.

Definition 3. (Positive Definite Functions.) [27] A continuous function $V(x, t) : \mathbb{R}^n \times \mathbb{R}_+ \mapsto \mathbb{R}$ is called a positive definite function, i.e. $V(x, t) > 0$, if for some $\alpha(\cdot)$ of class \mathbb{KR} ,

$$V(0, t) = 0 \quad \text{and} \quad V(x, t) \geq \alpha(|x|) \quad \forall x \in \mathbb{R}^n, \quad t \geq 0.$$

Lemma 3. [27] Consider a linear system given by

$$\dot{x}(t) = Fx(t) \quad x(0) = x_0, \tag{2.1}$$

where $x \in \mathbb{R}^n$ and $F \in \mathbb{R}^{n \times n}$ is a constant matrix. If there exists a positive definite function $V(x, t)$ such that $-\dot{V}(x, t)$ is positive definite, then $\lim_{t \rightarrow +\infty} V(t) = 0$ and the linear system (2.1) is asymptotically stable.

2.4 Some necessary lemmas

Before presenting the main results, we first introduce some necessary lemmas.

Lemma 4. (The Schur Complement) [26] For a given symmetric matrix S with the form $S = [S_{ij}]$, $S_{11} \in \mathbb{R}^{r \times r}$, $S_{12} \in \mathbb{R}^{r \times (n-r)}$, $S_{22} \in \mathbb{R}^{(n-r) \times (n-r)}$, then, $S < 0$ if and only if $S_{11} < 0$, $S_{22} - S_{21}S_{11}^{-1}S_{12} < 0$ or $S_{22} < 0$, $S_{11} - S_{12}S_{22}^{-1}S_{21} < 0$.

Lemma 5. [14] Consider the matrix given by

$$\Psi_n = \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{bmatrix}.$$

The following statements hold.

(1) The eigenvalues of Ψ_n are n with multiplicity $n-1$ and 0 with multiplicity 1 . The vectors $\mathbf{1}_n^T$ and $\mathbf{1}_n$ are the left and the right eigenvectors of Ψ_n associated with the zero eigenvalue, respectively.

(2) There exists an orthogonal matrix U such that $U^T \Psi_n U = \begin{bmatrix} nI_{n-1} & 0 \\ 0 & 0 \end{bmatrix}$ and the last column is $\frac{\mathbf{1}_n}{\sqrt{n}}$.

(3) Let $\Xi_1 \in \mathbb{R}^{n \times n}$ be the Laplacian of any directed graph, then $U^T \Xi_1 U = [\vartheta_1 \ 0]$, $\vartheta_1 \in \mathbb{R}^{n \times (n-1)}$.

For simplicity of the following analysis, denote

$$U = [U_1 \ \bar{U}_1],$$

where $\bar{U}_1 = \frac{\mathbf{1}_n}{\sqrt{n}}$ is the last column of U and U_1 is the rest part.

Chapter 2

Guaranteed cost consensus control of first-order multi-agent systems

3.1 Model

Suppose that the network system under consideration consists of n agents, e.g., birds, fishes, robots, etc, labeled 1 through n . Each agent is regarded as a node in a directed graph, G . Each edge $(s_j, s_i) \in \mathcal{E}$ corresponds to an available information link from i th agent to j th agent at time t . Moreover, each agent updates its current state based upon the information received from its neighbors.

Let $x_i \in \mathbb{R}$ be the i th agent's state that might be physical quantities including attitude, position, velocity, temperature, voltage and so on. Suppose the dynamics of each agent is a simple scalar continuous-time integrator:

$$\dot{x}_i(t) = u_i(t), \quad i = 1, 2, \dots, n \quad (3.1)$$

where $u_i(t)$ is the control input.

We say the consensus problem is solved if the states of agents satisfy

$$\lim_{t \rightarrow +\infty} (x_i(t) - x_j(t)) = 0$$

for all $i, j \in \mathcal{I}$.

To solve the above consensus problem, we use the following control law:

$$u_i(t) = \sum_{s_j \in N_i} (a_{ij} + \Delta a_{ij}(t)) [x_j(t) - x_i(t)], \quad (3.2)$$

where a_{ij} quantify the way the agents influence each other, $\Delta a_{ij}(t)$ denotes the uncertainty of a_{ij} with $|\Delta a_{ij}(t)| = \begin{cases} \leq \psi_{ij}, & a_{ij} \neq 0 \\ 0, & a_{ij} = 0 \end{cases}$ and ψ_{ij} is a specified constant for $i, j \in \mathcal{I}$. Here, we assume $\Delta a_{ij}(t)$ is a continuous function of time t .

Our objective is to find appropriate control laws to suppress the disturbances of the uncertainty and make all agents reach consensus.

By using control law (3.2), the network dynamics can be summarized as

$$\dot{x}(t) = -(L + \Delta L)x(t) \quad (3.3)$$

From (3.2), ΔL can be viewed as an uncertainty Laplacian. Then by Lemma 2, it can be decomposed into $\Delta L = E_1 \Sigma(t) E_2$, where E_1, E_2 are specified constant matrices and $\Sigma(t)$ is a diagonal matrix whose diagonal elements are the uncertainties of the edges, $\Delta a_{ij}(t)$. In terms of the decomposition of $L = DW(D^T - E)$ in Lemma 2, the form of E_1, E_2 correspond to D and $(D^T - E)$ respectively, whereas the form of $\Sigma(t)$ correspond to W . Specifically, in the example of Fig.2.1, the forms of E_1, E_2 and $\Sigma(t)$ are $E_1 = D_0, E_2 = D_0^T - E_0$ and $\Sigma(t) = \text{diag}\{\Delta a_{21}, \Delta a_{32}, \Delta a_{43}, \Delta a_{53}, \Delta a_{54}, \Delta a_{65}\}$. Since $|\Delta a_{ij}(t)| \leq \psi_{ij}$, for simplicity, we assume $\psi_{ij} = 1$, i.e., $\Sigma(t)^T \Sigma(t) = \Sigma(t)^2 \leq I_n$.

Many recent studies [1, 2, 7, 19] have tried to explain, by appropriately modeling, the behavior of a group of animals, e.g., a flock of birds, whose velocities converge to a common value. In their models, it was always assumed that the edge weights, which describe the interactions between agents, are deterministic and unperturbed. In fact, the interactions between agents cannot be measured precisely without any error. Considering this situation, in the model (3.3), the uncertainty is included.

In nature, groups of animals are often moving in a most labor-saving way, e.g., flocks of geese often fly in a V-shaped formation. So, it is meaningful to find energy-efficient control laws for state consensus of the systems. To do this, we define the following integral

quadratic cost function as a measure of energy consumption of all agents:

$$J = \int_0^{+\infty} u^T(t) R_u u(t) dt,$$

where $u(t) = [u_1(t), u_2(t), \dots, u_n(t)]^T \in \mathbb{R}^n$ and $R_u \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix.

In this paper, our objective is to find design rules for the parameters a_{ij} to minimize the performance index J .

3.2 Main results

In this section, we will perform the analysis for the system (3.3) respectively and present conditions which make all agents reach consensus.

Theorem 1. For the system (3.3), if there exists a positive definite matrix $\bar{P} \in \mathbb{R}^{(n-1) \times (n-1)}$ and positive scalars $\epsilon_1, \epsilon_2, \epsilon_3$, for all $\Sigma^T(t)\Sigma(t) \leq I_n$, such that

$$\Gamma = \begin{bmatrix} \zeta_1 & 0 & \bar{P}U_1^T E_1 & U_1^T L^T & U_1^T L^T R_u E_1 \\ * & \zeta_2 & 0 & 0 & 0 \\ * & * & -\epsilon_1 I & 0 & 0 \\ * & * & * & -R_u^{-1} & 0 \\ * & * & * & * & -\epsilon_2 I \end{bmatrix} < 0, \quad (3.4)$$

where $'*'$ denotes the symmetric term of Γ , $\zeta_1 = -\bar{L}^T \bar{P} - \bar{P} \bar{L} + (\epsilon_1 + \epsilon_2 + \epsilon_3) \bar{E}$, $\zeta_2 = -R_u^{-1} + \frac{1}{\epsilon_3} E_1 E_1^T$, $\bar{E} = U_1^T E_2^T E_2 U_1$ and $\bar{L} = U_1^T L U_1$. Then, consensus can be achieved and an upper bound for the cost function J is $J^* = x^T(0) U_1 \bar{P} U_1^T x(0)$.

Proof: Let

$$\delta(t) = U_1^T x(t), \quad \bar{\delta}(t) = \bar{U}_1^T x(t).$$

Then

$$\begin{bmatrix} \delta(t) \\ \bar{\delta}(t) \end{bmatrix} = U^T x(t)$$

and

$$x(t) = U_1\delta(t) + \bar{U}_1\bar{\delta}(t).$$

Since $L\mathbf{1}_n = 0$ and $\Delta L\mathbf{1}_n = 0$, it follows that $\mathbf{1}_n^T(L + \Delta L)^T R_u(L + \Delta L) = 0$ and $(L + \Delta L)^T R_u(L + \Delta L)\mathbf{1}_n = 0$. Consequently,

$$\begin{aligned} J &= \int_0^{+\infty} x^T(t)UU^T(L + \Delta L)^T R_u(L + \Delta L)UU^T x(t)dt \\ &= \int_0^{+\infty} \delta^T(t)U_1^T(L + \Delta L)^T R_u(L + \Delta L)U_1\delta(t)dt \end{aligned}$$

Now, we shall construct a Lyapunov function for the system (3.3):

$$V = x^T(t)Px(t),$$

where $P = P^T \geq 0$ satisfying $\text{rank}(P) = n - 1$ and $P\mathbf{1}_n = 0$.

Let $\bar{P} = U_1^T P U_1$. Then

$$\begin{aligned} V &= x^T(t)Px(t) \\ &= x^T(t)UU^T P U U^T x(t) \\ &= \begin{bmatrix} \delta(t) \\ \bar{\delta}(t) \end{bmatrix}^T \begin{bmatrix} \bar{P} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta(t) \\ \bar{\delta}(t) \end{bmatrix} \\ &= \delta^T(t)\bar{P}\delta(t) > 0. \end{aligned} \tag{3.5}$$

Calculating \dot{V} , we have

$$\begin{aligned} \dot{V} &= 2\delta^T(t)\bar{P}\dot{\delta}^T(t) \\ &= 2\delta^T(t)\bar{P}U_1^T[-(L + \Delta L)UU^T x(t)] \\ &= -2\delta^T(t)\bar{P}U_1^T[LU_1 + \Delta LU_1, 0_{n \times 1}] \begin{bmatrix} \delta(t) \\ \bar{\delta}(t) \end{bmatrix} \\ &= -\delta^T(t)[(\bar{L} + \overline{\Delta L}(t))^T \bar{P} + \bar{P}(\bar{L} + \overline{\Delta L}(t))]\delta(t) \end{aligned} \tag{3.6}$$

where $\overline{\Delta L} = U_1^T \Delta L U_1$.

For any given matrices $Y_1, Y_2 \in \mathbb{R}^{n \times n}$ and any matrix $S \in \mathbb{R}^{n \times n}$ satisfying $S^T S \leq I_n$,

$$Y_1^T S Y_2 + Y_2^T S^T Y_1 \leq a^{-1} Y_1^T Y_1 + a Y_2^T Y_2 \quad (3.7)$$

where a is a positive scalar.

By (3.7), we have

$$\begin{aligned} & -\delta^T(t)[\bar{P}\bar{\Delta L}(t) + \bar{\Delta L}(t)^T \bar{P}]\delta(t) \\ = & -\delta^T(t)[\bar{P}U_1^T E_1 \Sigma(t) E_2 U_1 + (U_1^T E_1 \Sigma(t) E_2 U_1)^T \bar{P}]\delta(t) \\ \leq & \frac{1}{\epsilon_1} \delta^T(t) \bar{P} U_1^T E_1 E_1^T U_1 \bar{P} \delta(t) + \epsilon_1 \delta^T(t) U_1^T E_2^T E_2 U_1 \delta(t) \\ \leq & \delta^T(t) (\frac{1}{\epsilon_1} \bar{P} U_1^T E_1 E_1^T U_1 \bar{P} + \epsilon_1 \bar{E}) \delta(t) \end{aligned}$$

where $\epsilon_1 > 0$. Similarly,

$$\begin{aligned} & \delta^T(t) U_1^T L^T R_u \Delta L U_1 \delta(t) + \delta^T(t) U_1^T \Delta L^T R_u L U_1 \delta(t) \\ \leq & \delta^T(t) [\frac{1}{\epsilon_2} U_1^T L^T R_u E_1 E_1^T R_u L U_1 + \epsilon_2 \bar{E}] \delta(t) \end{aligned}$$

where $\epsilon_2 > 0$. Therefore,

$$\begin{aligned} \dot{V} & \leq \delta^T(t) (-\bar{L}^T \bar{P} - \bar{P} \bar{L} + \frac{1}{\epsilon_1} \bar{P} U_1^T E_1 E_1^T U_1 \bar{P} + \epsilon_1 \bar{E}) \delta(t) \\ & \leq \delta^T(t) U_1^T (L + \Delta L)^T R_u (L + \Delta L) U_1 \delta(t) \\ & + \delta^T(t) (-\bar{L}^T \bar{P} - \bar{P} \bar{L} + \frac{1}{\epsilon_1} \bar{P} U_1^T E_1 E_1^T U_1 \bar{P} + \epsilon_1 \bar{E}) \delta(t) \\ & \leq \delta^T(t) \Theta \delta(t) \end{aligned} \quad (3.8)$$

where

$$\Theta = \phi_1 + U_1^T \Delta L^T R_u \Delta L U_1 = \phi_1 + U_1^T E_2^T \Sigma^T E_1^T R_u E_1 \Sigma E_2 U_1$$

with

$$\begin{aligned} \phi_1 & = -\bar{L}^T \bar{P} - \bar{P} \bar{L} + (\epsilon_1 + \epsilon_2) \bar{E} + \frac{1}{\epsilon_1} \bar{P} U_1^T E_1 E_1^T U_1 \bar{P} \\ & + U_1^T L^T R_u L U_1 + \frac{1}{\epsilon_2} U_1^T L^T R_u E_1 E_1^T R_u L U_1. \end{aligned}$$

By the Schur Complement Lemma, we have

$$\begin{aligned}\Theta < 0 \Leftrightarrow \phi &= \begin{bmatrix} \phi_1 & U_1^T E_2^T \Sigma^T E_1^T \\ E_1 \Sigma E_2 U_1 & -R_u^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \phi_1 & 0 \\ 0 & -R_u^{-1} \end{bmatrix} + \begin{bmatrix} U_1^T E_2^T \\ 0 \end{bmatrix} \Sigma^T \begin{bmatrix} 0 \\ E_1 \end{bmatrix}^T + \begin{bmatrix} 0 \\ E_1 \end{bmatrix} \Sigma \begin{bmatrix} U_1^T E_2 \\ 0 \end{bmatrix}^T < 0.\end{aligned}$$

Again, by (3.7), we have

$$\begin{aligned}\phi &\leq \begin{bmatrix} \phi_1 & 0 \\ 0 & -R_u^{-1} \end{bmatrix} + \epsilon_3 \begin{bmatrix} U_1^T E_2^T \\ 0 \end{bmatrix} \begin{bmatrix} U_1^T E_2 \\ 0 \end{bmatrix}^T + \frac{1}{\epsilon_3} \begin{bmatrix} 0 \\ E_1 \end{bmatrix} \begin{bmatrix} 0 \\ E_1 \end{bmatrix}^T \\ &= \begin{bmatrix} \phi_1 + \epsilon_3 \bar{E} & 0 \\ 0 & -R_u^{-1} + \frac{1}{\epsilon_3} E_1 E_1^T \end{bmatrix} \\ &\triangleq \bar{\phi}\end{aligned}$$

with $\epsilon_3 > 0$. Note that

$$\begin{aligned}\bar{\phi} &= \begin{bmatrix} -\bar{L}^T \bar{P} - \bar{P} \bar{L} + (\epsilon_1 + \epsilon_2 + \epsilon_3) \bar{E} & 0 \\ 0 & -R_u^{-1} + \frac{1}{\epsilon_3} E_1 E_1^T \end{bmatrix} \\ &+ \begin{bmatrix} \bar{P} U_1^T E_1 & U_1^T L^T & U_1^T L^T R_u E_1 \\ 0 & 0 & 0 \end{bmatrix} \\ &\times \text{diag}\{\frac{1}{\epsilon_1} I_n, R_u, \frac{1}{\epsilon_2} I_n\} \begin{bmatrix} \bar{P} U_1^T E_1 & U_1^T L^T & U_1^T L^T R_u E_1 \\ 0 & 0 & 0 \end{bmatrix}^T.\end{aligned}$$

Then, by the Schur Complement Lemma, we have $\bar{\phi} < 0$ is equivalent to $\Gamma < 0$. Then the condition $\Gamma < 0$ guarantees $\Theta < 0$ and hence $\dot{V} < 0$ from (3.8). It follows from Lemma 3 that $V(+\infty) = \lim_{t \rightarrow +\infty} V(t) = 0$ and hence $\lim_{t \rightarrow +\infty} \delta(t) = 0$. Note that $x(t) = U_1 \delta(t) + \bar{U}_1 \bar{\delta}(t)$, where $\bar{U}_1 = \frac{\mathbf{1}_n}{\sqrt{n}}$ and $\bar{\delta}(t) \in \mathbb{R}$. Then $\lim_{t \rightarrow +\infty} [x_i(t) - x_j(t)] = \lim_{t \rightarrow +\infty} [(x_i(t) - \frac{1}{\sqrt{n}} \bar{\delta}(t)) - (x_j(t) - \frac{1}{\sqrt{n}} \bar{\delta}(t))] = 0$. That is, all agents can reach consensus under the condition $\Gamma < 0$.

In addition, from (3.6) and (3.8), $\Theta < 0$ implies that

$$\begin{aligned}\dot{V} &= -\delta^T(t) (\bar{L} + \overline{\Delta L}(t))^T \bar{P} \delta(t) - \delta^T(t) \bar{P} (\bar{L} + \overline{\Delta L}(t)) \delta(t) \\ &+ \delta^T(t) U_1^T (L + \Delta L)^T R_u (L + \Delta L) U_1 \delta(t) \\ &< \delta^T(t) \Theta \delta(t).\end{aligned}$$

Thus,

$$\dot{V} < -\delta^T(t)U_1^T(L + \Delta L)^T R_u(L + \Delta L)U_1\delta(t)$$

and hence

$$\begin{aligned} -V(0) &= V(+\infty) - V(0) = \int_0^{+\infty} \dot{V} dt \\ &< -\int_0^{+\infty} \delta^T(t)U_1^T(L + \Delta L)^T R_u(L + \Delta L)U_1\delta(t)dt. \end{aligned}$$

That is,

$$\int_0^{+\infty} \delta^T(t)U_1^T(L + \Delta L)^T R_u(L + \Delta L)U_1\delta(t)dt < V(0) = \delta^T(0)\bar{P}\delta(0) = x^T(0)U_1\bar{P}U_1^T x(0).$$

This completes the proof.

Remark 1. In Theorem 1, we only discuss the fixed topology case. Since the analysis is performed based on the Lyapunov theory, the results can be extended to the case where the edge weights are time-varying and the links between agents are dynamically changing, if there exists a common Lyapunov function for all possible topology graphs.

Remark 2. In [24], V. Gupta *et al.* tried to set up the LQR problem for the problem of controlling a discrete-time network of agents with fixed topology. The cost function is taken as

$$J = \sum_{k=0}^{\infty} \{x^T(k)Qx(k) + u^T(k)R_u u(k)\},$$

where $Q > 0$ and $R_u \geq 0$. This cost performance index is invalid for our model, because all agents might converge to a nonzero common value and J might tend to infinity as time goes on. Actually, to measure the disagreement dynamics of the networks, we can use the following cost function:

$$J = \int_0^{+\infty} \{x^T(s)Qx(s) + u^T(s)R_u u(s)\}ds$$

where Q is a symmetric positive semi-definite matrix, $Q\mathbf{1}_n = 0$ and R_u is a positive definite matrix.

3.3 Simulations

Here numerical simulations will be given to illustrate the theoretical results obtained in the previous sections. These simulations are performed with four agents. Fig.3.1 shows the communication topology graph which has a spanning tree.

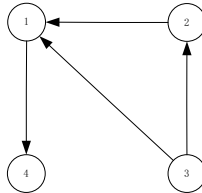


Fig.3.1 The communication topology.

Suppose the uncertainty matrices for the network as shown in Fig.3.1 are

$$E_1 = \begin{bmatrix} 0.3 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.3 & 0.3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0.3 & 0 & 0 & -0.3 \\ -0.3 & 0.3 & 0 & 0 \\ -0.3 & -0.3 & 0.3 & 0 \\ 0 & 0 & 0.3 & 0 \end{bmatrix}.$$

Then, applying Theorem 1 and taking $R_u = I$, it is solved that a feasible solution is $\bar{P} = I$ and

$$L = \begin{bmatrix} 0.5419 & 0 & 0 & -0.5419 \\ -0.3255 & 0.3255 & 0 & 0 \\ -0.2765 & -0.2576 & 0.5341 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

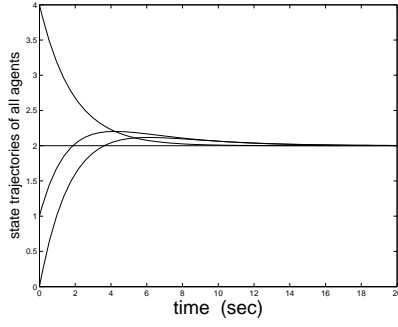


Fig.3.2 State trajectories of the network of agents.

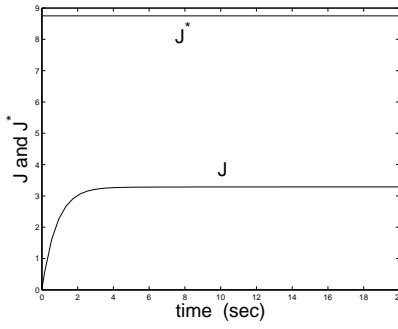


Fig.3.3 The cost function J of the network and one of its upper bounds

$$J^* = x^T(0)U_1\bar{P}U_1^T x(0) = x^T(0)U_1U_1^T x(0).$$

The state trajectories of all agents and the corresponding cost function J of the network are shown in Fig.3.2 and Fig.3.3, respectively. It is clear from Fig.3.2 that all agents asymptotically reach consensus whereas from Fig.3.3 the cost performance index J is smaller than the bound $J^* = x^T(0)U_1\bar{P}U_1^T x(0) = x^T(0)U_1U_1^T x(0)$ which is consistent with Theorem 1.

Chapter 4

Consensus control of second-order multi-agent systems

4.1 Model

We assume that each agent is a node in a directed graph, G . Each edge $(s_j, s_i) \in \mathcal{E}$ corresponds to an available information link from i th agent to j th agent. Moreover, each agent updates its current state based upon the information received from its neighbors. Let x_i be the position state of the i th agent, v_i be the speed. Suppose each agent has the dynamics as follows

$$\dot{x}_i = v_i \tag{4.1}$$

$$\dot{v}_i = u_i$$

where $x_i(t) \in \mathbb{R}$ is the position state, $v_i(t) \in \mathbb{R}$ is the velocity state, and $u_i(t) \in \mathbb{R}$ is the control input.

We say that the control law $u_i(t)$ solves the consensus problem if the states of agents satisfy $\lim_{t \rightarrow +\infty} [x_i(t) - x_j(t)] = 0$, and $\lim_{t \rightarrow +\infty} v_i(t) = 0$, for all $i, j \in \mathcal{I}$. Furthermore, if $\lim_{t \rightarrow +\infty} x_i(t) = \frac{1}{n} \sum_{j=1}^n x_j(0)$, we say the control law u_i solves the average consensus problem.

To solve the above consensus problems is a challenging task. One needs to find suitable distributed state feedback controller for each agent not only to solve the agreement of the position states of network but also to stabilize the speeds of the network. To solve the consensus problem, we use the following control law:

$$u_i(t) = -2k_1 v_i + k_2 \sum_{s_j \in N_i} a_{ij} (x_j(t) - x_i(t)). \tag{4.2}$$

Let $\bar{v}_i = \frac{v_i}{k_1} + x_i$. Then it follows that

$$\begin{aligned}\dot{x}_i &= -k_1 x_i + k_1 \bar{v}_i \\ \dot{\bar{v}}_i &= v_i + \frac{\dot{v}_i}{k_1} = -v_i + \frac{k_2}{k_1} \sum_{s_j \in N_i} a_{ij} (x_j(t) - x_i(t)) \\ &= k_1 x_i - k_1 \bar{v}_i + \frac{k_2}{k_1} \sum_{s_j \in N_i} a_{ij} (x_j(t) - x_i(t))\end{aligned}$$

Denote

$$\xi = [x_1, \bar{v}_1, \dots, x_n, \bar{v}_n]^T, \quad A = \begin{bmatrix} -k_1 & k_1 \\ k_1 & -k_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ \frac{k_2}{k_1} & 0 \end{bmatrix}.$$

Using the control law (4.2) the network dynamics can be summarized as

$$\dot{\xi} = \Phi \xi \tag{4.3}$$

where $\Phi = I_n \otimes A - L \otimes B$ and L is the Laplacian of the graph G .

4.2 Main Results

Lemma 6. Consider the equation

$$x^2 + 2c_1 x + c_2(a + bi) = 0 \tag{4.4}$$

where $a > 0, c_1, c_2, a, b \in \mathbb{R}$ and i denotes the imaginary unit. The zeros of (4.4) are on the open left-half-plane(LHP) if and only if $\frac{c_1^2}{c_2} > \frac{b^2}{4a}, c_1 > 0, c_2 > 0$.

Proof: If $c_1 \leq 0$, there is at least one root of (4.4) not located on the open LHP according to the Theory of Vieta.

Then, let $x = \sigma + w\iota$, and we get

$$\sigma^2 - w^2 + 2c_1 \sigma + c_2 a = 0 \tag{4.5}$$

$$2\sigma w + 2c_1 w + c_2 b = 0 \tag{4.6}$$

It can be checked that $\sigma = 0$ and $w = 0$ when $c_2 = 0$, and $\sigma = 0$ and $w = -\frac{c_2 b}{2c_1}$ when $\frac{c_1^2}{c_2} = \frac{b^2}{4a}$ ($c_2 \neq 0$). Namely, the equation (4.4) has at least one root on the imaginary axis if $\frac{c_1^2}{c_2} = \frac{b^2}{4a}$ ($c_2 \neq 0$) or $c_2 = 0$.

Now, we will prove that the zeros of (4.4) are on the open LHP if and only if $\frac{c_1^2}{c_2} > \frac{b^2}{4a}$, $c_1 > 0$, $c_2 > 0$ by contradiction.

Sufficiency From(4.5)(4.6), for any $\sigma > 0$ and any $c_1 > 0$, we have

$$\sigma^2 - \left(\frac{c_2 b}{2\sigma + 2c_1}\right)^2 + 2c_1\sigma + c_2 a = 0 \quad (4.7)$$

Note that $a > 0$, $\frac{c_1^2}{c_2} > \frac{b^2}{4a}$, $c_2 > 0$, and hence

$$\begin{aligned} & \sigma^2 - \left(\frac{c_2 b}{2\sigma + 2c_1}\right)^2 + 2c_1\sigma + c_2 a \\ & > \sigma^2 + 2c_1\sigma + c_2 a - \left(\frac{c_2 b}{2c_1}\right)^2 \\ & > \sigma^2 + 2c_1\sigma > 0 \end{aligned} \quad (4.8)$$

Clearly, (4.8) contradicts with (4.7). It implies that the roots of (4.4) are on the open LHP if $\frac{c_1^2}{c_2} > \frac{b^2}{4a}$, $c_1 > 0$, $c_2 > 0$.

Necessity Suppose that both roots of (4.4) are on the LHP with $\frac{c_1^2}{c_2} < \frac{b^2}{4a}$, ($c_2 \neq 0$). Then, we have $\sigma < 0$ and $|\sigma| < 2c_1$ according to the Theory of Vieta again. It follows that

$$\begin{aligned} 0 & = \sigma^2 - \left(\frac{c_2 b}{2\sigma + 2c_1}\right)^2 + 2c_1\sigma + c_2 a \\ & < \sigma^2 + 2c_1\sigma + c_2 a - \left(\frac{c_2 b}{2c_1}\right)^2 < 0 \end{aligned} \quad (4.9)$$

This yields a contradiction.

Lemma 7. [29] Consider a linear system given by

$$\dot{x}(t) = Fx(t) \quad x(0) = x_0 \quad (4.10)$$

where $x(t) \in \mathbb{R}^n$ is the state, and $x(0) \in \mathbb{R}^n$ is the initial condition. Suppose that the characteristic polynomial of F can be written as

$$f(s) = (s - \mu_1)^{\sigma_1} (s - \mu_2)^{\sigma_2} \cdots (s - \mu_l)^{\sigma_l}$$

where $\mu_i, i = 1, 2, \dots, l$ are all different eigenvalues of F . Then the solution of (4.10) can be given by

$$x(t) = \sum_{j=1}^l e^{\mu_j t} x_{0j} \left[\sum_{i=0}^{\sigma_j-1} \frac{t^i (A - \mu_j I_n)^i}{i!} \right]$$

where $x_{0j} \in U_j$ and $\sum_{j=1}^l x_{0j} = x(0)$ with $\mathbb{R}^n = U_1 \oplus U_2 \oplus \cdots \oplus U_l$, $U_i = \{\xi | (\mu_j I_n - A)^{\sigma_i} \xi = 0\}$ and ' \oplus ' denotes direct sum.

Denote

$$k_0 = \max_{|\lambda_i| \neq 0} \left\{ \frac{[\text{Im}(\lambda_i)]^2}{4\text{Re}(\lambda_i)} \right\}$$

where λ_i is the eigenvalue of the Laplacian L , and $\text{Im}(\lambda_i), \text{Re}(\lambda_i)$ are the imaginary part and real part of λ_i respectively.

Theorem 2. Consider a directed network of agents with fixed communication topology G that has a spanning tree. Then, under the control law (4.2) the multi-agent system (4.1) can reach consensus if and only if $k_1 > 0, k_2 > 0$ and $\frac{k_1^2}{k_2} > k_0$.

Proof: Firstly, we will prove that Φ has only one eigenvalue at zero, and its other $2n - 1$ eigenvalues located on the open LHP if and only if $k_1 > 0, k_2 > 0$ and $\frac{k_1^2}{k_2} > k_0$.

Since the graph has a spanning tree, according to Lemma 1 there exists a nonsingular matrix W such that

$$M = W^{-1} L W = \begin{bmatrix} 0 & & & \\ & J_1 & & \\ & & \ddots & \\ & & & J_s \end{bmatrix}$$

where J_2, \dots, J_s are Jordan blocks and the eigenvalue of J_i has positive real part. Then,

$$\begin{aligned}
& (W^{-1} \otimes I_2)\Phi(W \otimes I_2) \\
&= (W^{-1} \otimes I_2)(I_n \otimes A - L \otimes B)(W \otimes I_2) \\
&= (W^{-1} \otimes A)(W \otimes I_2) - (W^{-1}L) \otimes B(W \otimes I_2) \\
&= I_n \otimes A - M \otimes B \\
&= \begin{bmatrix} A & & & \\ & A - \lambda_2 B & * & \\ & & \ddots & * \\ & & & A - \lambda_n B \end{bmatrix}
\end{aligned}$$

where $\lambda_2, \dots, \lambda_n$ are the nonzero eigenvalues of L .

Consider the characteristic polynomial of Φ , we have

$$\begin{aligned}
& \det(\Phi - sI_{2n}) \\
&= \det(A - sI_2) \prod_{i=2}^n \det(A - \lambda_i B - sI_2) \\
&= s(s + 2k_1) \prod_{i=2}^n (s^2 + 2k_1 s + k_2 \lambda_i)
\end{aligned}$$

From Lemma 1, $\text{Re}(\lambda_i) > 0, i = 2, \dots, n$. Then by Lemma 6, the roots of $s^2 + 2k_1 s + k_2 \lambda_i$ are on the open LHP if and only if $k_1 > 0, k_2 > 0, \frac{k_1^2}{k_2} > k_0$. Thus, all the eigenvalues of Φ have negative real-parts except one at zero if and only if $k_1 > 0, k_2 > 0, \frac{k_1^2}{k_2} > k_0$.

Noting that

$$\begin{aligned}
& \Phi \mathbf{1}_n \otimes [1 \quad 1]^T \\
&= (I_n \mathbf{1}_n) \otimes (A [1 \quad 1]^T) - (L \mathbf{1}_n) \otimes (B [1 \quad 1]^T) \\
&= \mathbf{0}_{2n},
\end{aligned}$$

we have the vector $\mathbf{1}_{2n}$ is the eigenvector of Φ associated with the zero eigenvalue. Therefore, from Lemma 7,

$$\lim_{t \rightarrow +\infty} \bar{v}_j(t) = \lim_{t \rightarrow +\infty} \bar{v}_i(t) = \lim_{t \rightarrow +\infty} x_i(t) = \lim_{t \rightarrow +\infty} x_j(t) = \frac{1}{n} \sum_{i=1}^n (x_i(0) + \frac{v_i(0)}{2k_1})$$

$$\lim_{t \rightarrow +\infty} v_i(t) = 0, i = 1, 2, \dots, n, j = 1, 2, \dots, n.$$

This implies that consensus is achieved.

Corollary 1. Consider an undirected network of agents with fixed communication topology G that is connected. Then, under the control law (4.2), $\lim_{t \rightarrow \infty} x_i(t) = \frac{1}{n} \sum_i (x_i(0) + \frac{v_i(0)}{2k_1})$ and $\lim_{t \rightarrow \infty} v_i(t) = 0$ for any $i \in \mathcal{I}$ if and only if $k_1 > 0, k_2 > 0$. Furthermore, if the initial speeds satisfy $\sum_{i=1}^n v_i(0) = 0$, average-consensus will be achieved.

Proof: Note that

$$\begin{aligned} & \mathbf{1}_n^T \otimes [1 \quad 1] \Phi \\ &= \mathbf{1}_n^T \otimes [1 \quad 1] (I_n \otimes A - L \otimes B) \\ &= (\mathbf{1}_n^T I_n) \otimes [[1 \quad 1] A - \mathbf{1}_n^T L \otimes [1 \quad 1] B] \\ &= \mathbf{0}_{2n} \end{aligned} \tag{4.11}$$

Then, $\sum_i \dot{x}_i + \dot{v}_i = 0$. Thus, $\sum_i x_i + \bar{v}_i$ is an invariant quantity and $\sum_i [x_i(t) + \bar{v}_i(t)] = \sum_i [x_i(0) + \bar{v}_i(0)]$ for any $t \geq 0$. Since the graph is undirected, then $k_0 = 0$. Thus, according to Theorem 2, we have

$$\lim_{t \rightarrow \infty} x_i(t) = \lim_{t \rightarrow \infty} \bar{v}_i(t) = \frac{1}{2n} \sum_i [x_i(0) + \bar{v}_i(0)] = \frac{1}{n} \sum_i (x_i(0) + \frac{v_i(0)}{2k_1})$$

and

$$\lim_{t \rightarrow \infty} v_i(t) = 0$$

for any $i \in \mathcal{I}$, if and only if $k_1 > 0, k_2 > 0$.

4.3 Simulation results

In this section, we will present some numerical simulations to illustrate the theoretical results obtained in the previous sections. These simulations are performed with 6 agents,

whose initial conditions are set randomly. Fig.4.1 denotes topology structure of the multi-agent network. The weight of each edge is 0.5 and by simple computations it is solved that $k_0 = 0.375$. Fig.4.2-4.4 show the state trajectories. It is clear that the multi-agent system reaches consensus when $\frac{k_1^2}{k_2} > k_0$, oscillates when $\frac{k_1^2}{k_2} = k_0$ and diverges when $\frac{k_1^2}{k_2} < k_0$. This is consistent with Theorem 2.

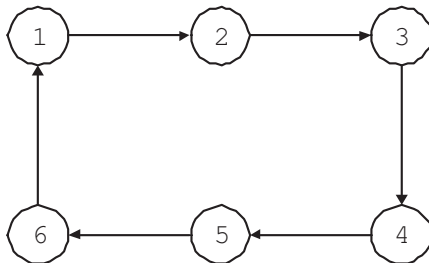


Fig.4.1 Topology

Case 1 $\frac{k_1^2}{k_2} = 0.5 > k_0$

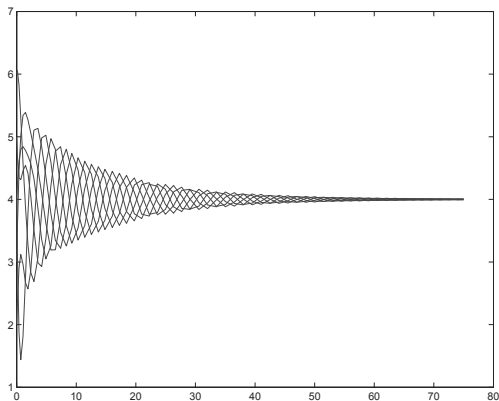


Fig.4.2(a) Position trajectories of the network

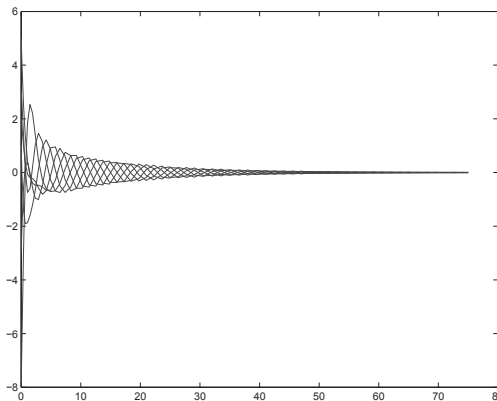


Fig.4.2(b) Velocity trajectories of the network

Case 2 $\frac{k_1^2}{k_2} = 0.375 = k_0$.

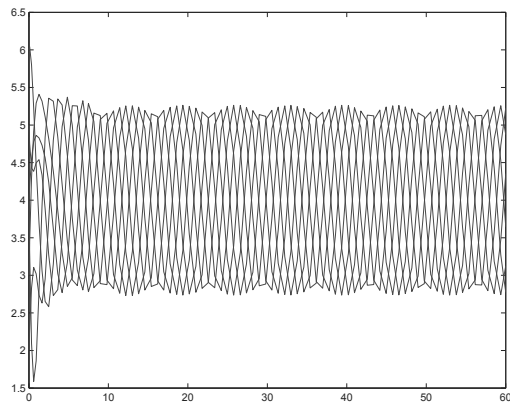


Fig.4.3(a) Position trajectories of the network

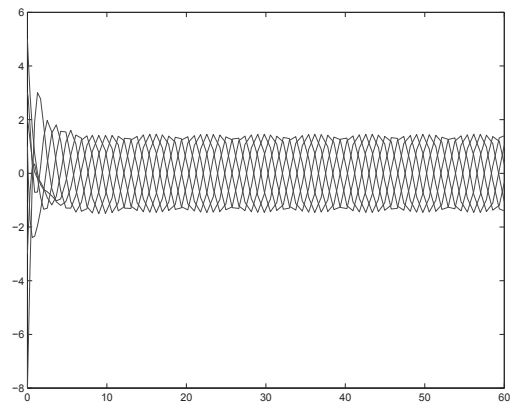


Fig.4.3(b) Velocity trajectories of the network

Case 3 $\frac{k_1^2}{k_2} = 0.2 < k_0$

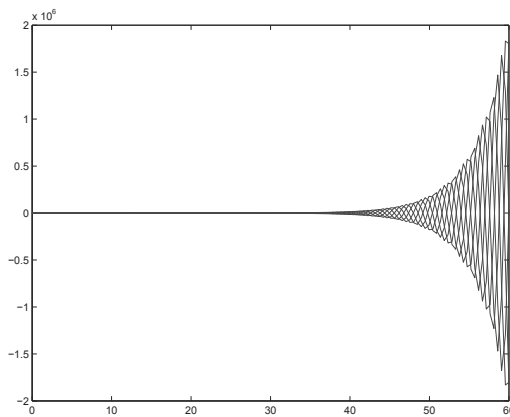


Fig.4.4(a) Position trajectories of the network

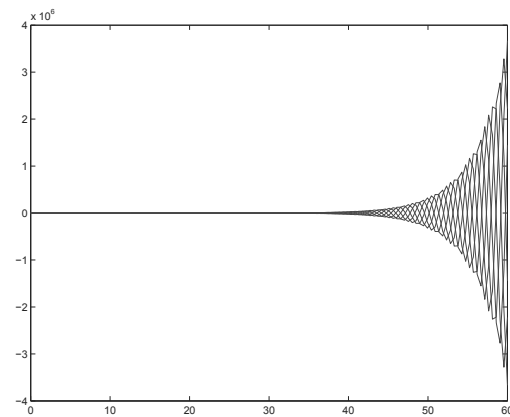


Fig.4.4(b) Velocity trajectories of the network

Chapter 5

Dynamic consensus control of high-order multi-agent systems

5.1 Model

Suppose that the multi-agent system under consideration consists of n agents. Each agent is regarded as a node in an undirected graph G . Suppose the dynamics of the i th agent ($i \in \mathcal{I}$) is

$$\begin{aligned}\dot{\xi}_i^{(0)}(t) &= \xi_i^{(1)}(t) \\ &\vdots \\ \dot{\xi}_i^{(l-2)}(t) &= \xi_i^{(l-1)}(t) \\ \dot{\xi}_i^{(l-1)}(t) &= u_i(t) \\ y_i(t) &= \xi_i^{(0)}(t)\end{aligned}\tag{5.1}$$

where $\xi_i^{(j)} \in \mathbb{R}$ is the j th variable of ξ_i , $j = 0, 1, \dots, l-1$, $u_i(t) \in \mathbb{R}$ is the control input and $y_i(t)$ is output function of each agent. Here, each agent can only get the output function $y_i(t)$ information from its neighbors.

We say the control law u_i asymptotically solves the consensus problem, if the states of agents satisfy

$$\lim_{k \rightarrow +\infty} [\xi_i(t) - \xi_j(t)] = 0,\tag{5.2}$$

for all $i, j \in \mathcal{I}$.

In order to solve the consensus problem, we use the following consensus control law:

$$\begin{aligned} \dot{p}_i &= -\gamma p_i + \sum_{s_j \in N_i(t)} a_{ij} (\xi_j^{(0)}(t) - \xi_i^{(0)}(t)) \\ u_i &= -\sum_{j=1}^{l-1} k_j \xi_i^{(j)}(t) - \sum_{s_j \in N_i(t)} a_{ij} (\xi_j^{(0)}(t) - \xi_i^{(0)}(t)) + p_i \end{aligned} \quad (5.3)$$

for any $i \in \mathcal{I}$, where $p_i(0) = 0$, $k_j > 0$ and $\gamma > 0$ are specified parameters.

Let $\psi_i(t) = [\xi_i^T(t), p_i^T(t)]^T$ and $\psi = [\psi_1(t)^T, \psi_2(t)^T, \dots, \psi_n(t)^T]^T$. Then under the control law (5.3), the network dynamics of the multi-agent system is

$$\dot{\psi}(t) = (I_n \otimes A)\psi(t) - (L \otimes B)\psi(t). \quad (5.4)$$

where L is the Laplacian of the graph G ,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & -k_1 & \cdots & -k_{l-1} & 1 \\ 0 & \cdots & \cdots & 0 & -\gamma \end{bmatrix} \in \mathbb{R}^{(l+1) \times (l+1)},$$

and

$$B = \begin{bmatrix} 0_{l-1} & 0_{(l-1) \times l} \\ 1 & 0_{1 \times l} \\ 1 & 0_{1 \times l} \end{bmatrix} \in \mathbb{R}^{(l+1) \times (l+1)}.$$

5.2 Main Results

In this section, we first perform a model transformation and turn the original systems into equivalent ones that will be used in the following analysis.

Lemma 8. Let $\beta(t) = e^{\bar{A}t} \frac{1}{n} \sum_{i=1}^n \xi_i(0)$ and $\delta(t) = \psi(t) - \mathbf{1}_n \otimes [\beta(t)^T, 0_{1 \times l}]^T$, where

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & -k_1 & \cdots & -k_{l-2} & -k_{l-1} \end{bmatrix} \in \mathbb{R}^{l \times l}.$$

Then $(\mathbf{1}_n^T \otimes I_{l+1})\delta(t) = 0$. Moreover, the system (5.4) is equivalent to

$$\dot{\delta}(t) = (I_n \otimes A - L \otimes B)\delta(t). \quad (5.5)$$

Proof: Since G is undirected, it follows from (5.3) that $\sum_{i=1}^n \dot{p}_i(t) = -\gamma \sum_{i=1}^n p_i(t)$. Since $p_i(0) = 0$ for any $i \in \mathcal{I}$, we have $\sum_{i=1}^n p_i(t) = e^{-\gamma t} \sum_{i=1}^n p_i(0) = 0$. Similarly, it can be obtained that $\frac{1}{n} \sum_{i=1}^n \dot{\xi}_i(t) = \frac{1}{n} \bar{A} \sum_{i=1}^n \xi_i(t)$ and thus $\frac{1}{n} \sum_{i=1}^n \xi_i(t) = e^{\bar{A}t} \frac{1}{n} \sum_{i=1}^n \xi_i(0)$. Evidently, $(\mathbf{1}_n^T \otimes I_{l+1})\delta(t) = 0$.

$$\begin{aligned} \dot{\delta}(t) &= \dot{\psi}(t) - \mathbf{1}_n \otimes [\dot{\beta}(t)^T, 0_{1 \times 1}]^T \\ &= (I_n \otimes A - L \otimes B)(\delta(t) + \mathbf{1}_n \otimes [\dot{\beta}(t)^T, 0_{1 \times 1}]^T) - \mathbf{1}_n \otimes [\bar{A}\beta(t)^T, 0_{1 \times 1}]^T \\ &= (I_n \otimes A - L \otimes B)\delta(t) + \mathbf{1}_n \otimes [\bar{A}\beta(t)^T, 0_{1 \times 1}]^T - \mathbf{1}_n \otimes [\bar{A}\beta(t)^T, 0_{1 \times 1}]^T \\ &= (I_n \otimes A - L \otimes B)\delta(t). \end{aligned}$$

Clearly, the system (5.4) is equivalent to the system (5.5). \square

Lemma 9. [30] Consider an equation given by

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0, \quad (5.6)$$

where $a_0, a_1, \dots, a_n \in \mathbb{R}$. The roots of (5.6) all have negative real parts if all a_i are positive and

$$a_{i-1} a_{i+2} < 0.4655 a_i a_{i+1} \quad (5.7)$$

for $i = 1, 2, \dots, n-2$.

Theorem 3. Let $\alpha_{l+1} = \lambda_{\max}(\gamma + 1)$, $\alpha_l^0 = \gamma k_1 + \lambda_{\min}$, $\alpha_l^1 = \gamma k_1 + \lambda_{\max}$, $\alpha_{l-1} = k_2 \gamma + k_1$, $\alpha_{l-2} = k_3 \gamma + k_2$, \dots , $\alpha_2 = k_{l-1} \gamma + k_{l-2}$, $\alpha_1 = k_{l-1} + \gamma$, $\alpha_0 = 1$, where λ_{\min} denotes the smallest nonzero eigenvalue of L and λ_{\max} denotes the largest eigenvalue of L . Consider a network of high-order agents with a fixed topology G that is connected. Then the multi-agent system (5.4) can reach consensus if $\alpha_i > 0$, $i = 0, 1, \dots, n$, $\alpha_{i-1} \alpha_{i+2} < 0.4655 \alpha_i \alpha_{i+1}$, $i = 1, 2, \dots, l-3$, $\alpha_{l-3} \alpha_l^1 < 0.4655 \alpha_{l-2} \alpha_{l-1}$ and $\alpha_{l-2} \alpha_{l+1} < 0.4655 \alpha_{l-1} \alpha_l^0$.

Proof: By Lemma 1, we can denote the eigenvalues of L as $0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_n$. There exists an orthogonal matrix $W \in \mathbb{R}^{n \times n}$ such that $W^T L W = \text{diag}\{0, \lambda_2, \dots, \lambda_n\}$. It follows that

$$\begin{aligned} & (W^T \otimes I_{l+1})(I_n \otimes A - L \otimes B)(W \otimes I_{l+1}) \\ &= \text{diag}\{A, A - \lambda_2 B, \dots, A - \lambda_n B\}. \end{aligned} \quad (5.8)$$

By Lemma 1 again, we see that the first column of W is $\frac{1}{\sqrt{n}}$. Let \bar{W} denote the rest $n - 1$ columns of W and $\bar{\delta}(t) = (\bar{W} \otimes I_{l+1})^T \delta(t)$. By Lemma 5, $\delta^T(t)(W \otimes I_{l+1}) = [0_{1 \times (l+1)} \quad \bar{\delta}^T(t)]$. Thus,

$$(W \otimes I_{l+1})^T \dot{\delta}(t) = \text{diag}\{A, A - \lambda_2 B, \dots, A - \lambda_n B\}(W \otimes I_{l+1})^T \delta(t). \quad (5.9)$$

It follows that the system (5.5) is equivalent to

$$\dot{\bar{\delta}}(t) = \Phi \bar{\delta}(t),$$

where $\Phi = \text{diag}\{A - \lambda_2 B, \dots, A - \lambda_n B\}$. Calculating the characteristic polynomial of $A - \lambda_i B$, we have

$$\begin{aligned} \det(sI - A + \lambda_i B) &= (s + \gamma)(\lambda_i + s^l + \sum_{j=1}^{l-1} k_j s^j) + \lambda_i \\ &= \alpha_0 s^{l+1} + \alpha_1 s^l + \dots + \alpha_{l-1} s^2 + (\gamma k_1 + \lambda_i) s + \lambda_i(\gamma + 1) \\ &= 0. \end{aligned}$$

Note that $\lambda_i(\gamma + 1)\alpha_{l-2} \leq \alpha_{l+1}\alpha_{l-2} < 0.4655\alpha_l^0\alpha_{l-1} \leq 0.4655(\gamma k_1 + \lambda_i)\alpha_{l-1}$ and $(\gamma k_1 + \lambda_i)\alpha_{l-3} \leq \alpha_l^1\alpha_{l-3} < 0.4655\alpha_{l-2}\alpha_{l-1}$. Then by Lemma 9, we have all the eigenvalues of $A - \lambda_i B$, $i = 2, \dots, l$, have negative real parts under the condition given by Theorem 3.

It follows that

$$\lim_{t \rightarrow +\infty} \bar{\delta}(t) = 0$$

and

$$\lim_{t \rightarrow +\infty} \delta(t) = 0.$$

Thus,

$$\lim_{t \rightarrow +\infty} \xi_i(t) = 0$$

for any $i \in \mathcal{I}$. This implies that the multi-agent system reach consensus. This completes the proof. \square

5.3 Simulations

Numerical simulations are given to illustrate the theoretical results obtained in the previous sections. Fig.5.1 shows a graph with $n = 4$ nodes. Suppose that the weight of each edge is 1 and each agent has three-order dynamics. By computation, we have the largest eigenvalue of the Laplacian of the graph in Fig.5.1 is 4 and the smallest nonzero eigenvalue is 2. The initial condition is set as $\psi(0) = [1 \ 0 \ 0 \ 0 \ 2 \ 0 \ 0 \ 0 \ -2 \ 0 \ 0 \ 0 \ -4 \ 0 \ 0 \ 0]^T$ and the parameters are taken as $(\gamma, k_1, k_2) = (1, 6, 2)$. Therefore, corresponding to Theorem 3, $\alpha_0 = 1$, $\alpha_1 = 3$, $\alpha_2 = 8$, $\alpha_3^0 = 8$, $\alpha_3^1 = 10$ and $\alpha_4 = 8$. Clearly, $\alpha_0 \alpha_3^1 < 0.4655 \alpha_1 \alpha_2$ and $\alpha_1 \alpha_4 < 0.4655 \alpha_2 \alpha_3^0$.

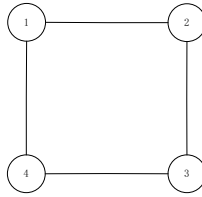


Fig.5.1 The network topology.

Figs.5.2, 5.3 and 5.4 show the state trajectories of all agents. It is clear that all agents reach consensus, which verifies Theorem 3.

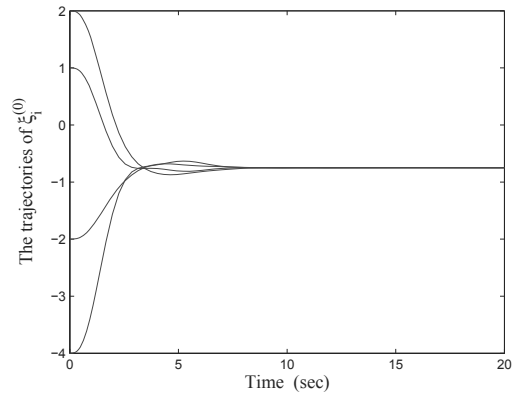


Fig.5.2 The trajectories of the first variables of all agents.

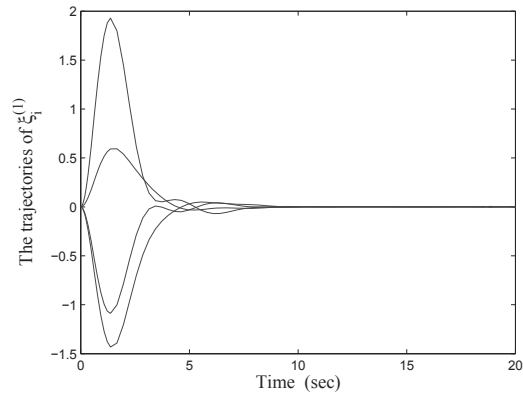


Fig.5.3 The trajectories of the second variables of all agents.

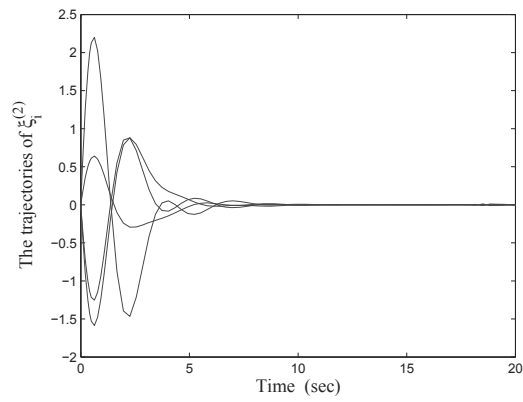


Fig.5.4 The trajectories of the third variables of all agents.

Chapter 6

Conclusions

In this paper, we first investigate guaranteed cost coordination in directed networks of agents with norm-bounded uncertainty, where each agent updates its state based on a simple neighbor rule. The analysis is performed by a Lyapunov-based approach. A class of Lyapunov functions are introduced as a measure of the disagreement dynamics. Using these Lyapunov functions, sufficient conditions are derived which make all agents reach consensus asymptotically while satisfying desired cost performance. Second, we consider consensus control for networks of agents with double integrator dynamics. To solve the consensus problem, a control law is adopted which contains two aspects, the agreement of the position states and the convergence to zero of the speed states. The corresponding convergence analysis was provided. A sufficient and necessary condition was established by using the eigenvector-eigenvalue method of finding solutions. Third, we investigate consensus of high-order multi-agent systems. A new dynamic neighbor-based control law is proposed which contains two parts, one is the local feedback and the other is the distributed feedback of the first states of each agent. Sufficient conditions are derived for state consensus of the system.

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