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The Development of Vector Analysis, Differential Geometry and de Rham Cohomology

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A geometric odyssey

av

Malin Karlsson

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A geometric odyssey

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Abstract

Beginning with the discovery of Gauss's Theorema Egregium, the steps taken through the history of differential geometry are traced. The process of creating a calculus of vectors is followed as well. The theory of differential forms is compared to that of vector analysis, with illustrations of how the former can present a shorter and simpler way of doing calculations. From the differential forms, the path of differential geometry continues towards de Rham's theorems. This is a starting point for de Rham cohomology, which in three dimensions can be expressed either with vectors or differential forms.

Sammanfattning

Med början i Gauss upptäckt av Theorema Egregium så följer vi den historiska utvecklingen av differentialgeometri. Dessutom undersöks hur det gick till att skapa en kalkyl för vektorer. I jämförelse med vektoranalys illustrerar vi hur differentialformer kan bidra till kortare och enklare beräkningar. Från differentialformer så fortsätter utvecklingen mot de Rhams satser. Dessa är början till de Rham-kohomologi, som i tre dimensioner kan uttryckas med antingen vektorer eller differentialformer.

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Chapter 1

Introduction

When doing mathematics we use ideas, techniques and tricks that have been refined and improved through the combined efforts of many mathematicians. People with less sofisticated methods than we have now, have come up with clever ideas and have helped to develop the mathematics into what it is today. The purpose of this thesis is to follow the development of three branches of mathematics: that of vector analysis, differential geometry - especially the differential forms - and de Rham cohomology. We will follow the parallel development of vector analysis and differential geometry, and from thereon we will see how de Rham cohomology evolved.

The starting point of this history is the beginning of the 19th century when Carl Friedrich Gauss realized that bending a surface does not change its Gaussian curvature, which is, loosely, the two-dimensional curvature of a surface.

The path towards vector analysis began about the same time with the problem of expressing a complex number in the plane. It was followed by the question of how this could be done in three-dimensional space. Sir Rowan Hamilton found a functional apparatus for this in the quaternions, but the system had some flaws which the vector analysis set right at the end of the 19th century.

The other path, that of differential geometry, continued with a new algebra for geometrical objects, and this would eventually become an algebra for the differential forms. The idea of a manifold was presented by Bernhard Riemann in a famous lecture in the 1850s. At the turn of the 20th century, Élie Cartan properly defined the differential forms, which are useful when doing calculations on manifolds.

The third path involves Stokes's theorem which has been stated in many different ways. Together with ideas from Henri Poincaré, George de Rham used this theorem to generalize Poincaré's lemma. The theorems that he stated would lead to the de Rham cohomology in which differential forms play an important part.

To emphasize how the mathematics have improved, we will look at different proofs of Gauss's Theorema Egregium and different expressions for the Gaussian curvature.

Chapter 2

Gauss's Remarkable Theorem

Carl Friedrich Gauss made one of the first contributions to differential geometry with his *Theorema Egregium*, which is Latin for Remarkable Theorem. This theorem was written in his paper *Disquisitiones generales circa superficies curvas* (*General investigations of curved surfaces*) which was presented to the Royal Society of Sciences in Göttingen on October 8 1827 [12, pp.163-165], [16, p.iii], [24, p.7].

Theorema Egregium 2.0.1. *If a curved surface is developed upon any other surface whatever, the measure of curvature in each point remains unchanged.*

Gauss thought of developing one surface onto another as a special case of projecting one curved surface onto another, keeping similarity in the smallest parts [12, p.163]. In modern words we say that a surface $M \subseteq \mathbb{R}^3$ is mapped onto another surface $M' \subseteq \mathbb{R}^3$ preserving distances and angles between neighbouring points. That is, we are bending the surface M into another shape without stretching it. For example, a piece of paper can be turned into a cylinder if we let two opposite sides meet.

The remarkable thing about Theorema Egregium is that the Gaussian curvature is an **intrinsic** (inner) value, since it does not depend on the exterior, i.e. how the surface is situated in space.

To understand Gaussian curvature we begin by explaining the curvature of a circle. It is defined to be

$$\frac{1}{R}$$

where R is the radius of the circle. A small circle has large curvature and a large circle has small curvature. On a curve, the curvature at each point is defined to be the curvature of a circle approximating that curve. A curve that makes a narrow turn will have large curvature there since the radius of the approximated circle will be very small. A straight line is approximated by a circle with infinite radius and this has zero curvature. The normal of a curve is defined to point towards the center of the approximated circle.

On a surface we can obtain curves by cutting with planes through the surface. The cross section between the surface and the plane is a curve. A plane in which the normal of a surface lies, is called a normal plane. If we take a certain point on the surface with its corresponding surface normal then we can cut the surface with normal planes in all directions and obtain many curves with corresponding curvature. In particular, there will be a minimum and a maximum value of the curvatures. The two curves corresponding to these two values are always perpendicular according to a theorem by Euler (also see page 28). With R_1, R_2 as the two radii of the two extreme curvatures, **Gaussian curvature** K is defined to be

$$K = \pm \frac{1}{R_1 \cdot R_2}$$

at any point of the surface. The positive or negative sign depends on if the two normals of the two curves with corresponding curvatures $\frac{1}{R_1}$ and $\frac{1}{R_2}$ have the same direction or opposite direction. A sphere has $K > 0$ whereas a saddle surface has $K < 0$. A flat piece of paper has $K = 0$ because the two extreme radii of curvature are both infinite. The paper can be bent, without being deformed, into a cylinder or a cone and this looks different from being flat but the Gaussian curvature is the same since at least one of the extreme curves will be a straight line. A sphere can never be turned into a flat object since the Gaussian curvature differs between the two. A natural example of this is a map of the world in contrast to a globe. Some parts of the map always look a bit distorted since a spherical object has to be stretched in order to become flat.

2.1 Proof of Gauss's theorem

The following proof is essentially Gauss's own. Since vector analysis was developed during the end of the 19th century and we are still in the late 1820s, Gauss could not use vectors and the tools that came with them: scalar and vector product. He had to use what was at hand at the time and therefore this proof is given in coordinate notation. But as we will see, the possibility of using vectors almost shines through. Modern notation will be written within parentheses to facilitate the understanding of this.

Proof of Theorema Egregium. Let the coordinates of a point on an arbitrary surface in space be x, y, z. Let us assume that these can be expressed as functions of two variables such that

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v).$$

Thus we let $h(u,v) = (x(u,v), y(u,v), z(u,v))$ be a parameterization of the surface so that Gauss's definitions can be explained in a modern way.

Differentiating the point will result in

$$\begin{aligned}
dx &= a du + a' dv & &\left(= \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv\right)\\
dy &= b du + b' dv & &\left(= \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv\right)\\
dz &= c du + c' dv. & &\left(= \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv\right)
\end{aligned}$$

Gauss defines the relations

$$\begin{aligned}
A &= bc' - cb'\\
B &= ca' - ac'\\
C &= ab' - ba'.
\end{aligned}$$

In modern vector notation we see that A, B, C are the x, y, z-components of the surface normal,

$$\mathbf{n} = \frac{\partial h}{\partial u} \times \frac{\partial h}{\partial v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} bc' - cb' \\ ca' - ac' \\ ab' - ba' \end{pmatrix}.$$

Gauss also defines

$$\begin{aligned}
\alpha &= \frac{\partial^2 x}{\partial u^2} & \alpha' &= \frac{\partial^2 x}{\partial u \partial v} & \alpha'' &= \frac{\partial^2 x}{\partial v^2}\\
\beta &= \frac{\partial^2 y}{\partial u^2} & \beta' &= \frac{\partial^2 y}{\partial u \partial v} & \beta'' &= \frac{\partial^2 y}{\partial v^2}\\
\gamma &= \frac{\partial^2 z}{\partial u^2} & \gamma' &= \frac{\partial^2 z}{\partial u \partial v} & \gamma'' &= \frac{\partial^2 z}{\partial v^2}
\end{aligned}$$

and

$$D = A\alpha + B\beta + C\gamma \qquad \left(= \langle \mathbf{n}, \frac{\partial^2 h}{\partial u^2} \rangle\right) \tag{2.1}$$

$$D' = A\alpha' + B\beta' + C\gamma' \qquad \left(= \langle \mathbf{n}, \frac{\partial^2 h}{\partial u \partial v} \rangle\right) \tag{2.2}$$

$$D'' = A\alpha'' + B\beta'' + C\gamma''. \qquad \left(= \langle \mathbf{n}, \frac{\partial^2 h}{\partial v^2} \rangle\right) \tag{2.3}$$

D, D' and D'' are parts of what is known as the second fundamental form, but we use the unit normal $\nu = \frac{\mathbf{n}}{\sqrt{A^2+B^2+C^2}}$ instead of $\mathbf{n}$ and denote the parts by e, f, g such that

$$II = \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} \langle \nu, \frac{\partial^2 h}{\partial u^2} \rangle & \langle \nu, \frac{\partial^2 h}{\partial u \partial v} \rangle \\ \langle \nu, \frac{\partial^2 h}{\partial u \partial v} \rangle & \langle \nu, \frac{\partial^2 h}{\partial v^2} \rangle \end{pmatrix}.$$

Furthermore Gauss writes

$$\begin{aligned} E &= a^2 + b^2 + c^2 && \left(= \langle \frac{\partial h}{\partial u}, \frac{\partial h}{\partial u} \rangle \right) \\ F &= aa' + bb' + cc' && \left(= \langle \frac{\partial h}{\partial u}, \frac{\partial h}{\partial v} \rangle \right) \\ G &= a'^2 + b'^2 + c'^2 && \left(= \langle \frac{\partial h}{\partial v}, \frac{\partial h}{\partial v} \rangle \right) \end{aligned}$$

which are the inner products of the tangent vectors $\frac{\partial h}{\partial u}, \frac{\partial h}{\partial v}$, and these are part of the first fundamental form,

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

A, B, C are related to E, F, G by $A^2+B^2+C^2 = EG - F^2$ and Gauss decides to name it Δ:

$$A^2 + B^2 + C^2 = EG - F^2 = \Delta.$$

A modern way of defining Gaussian curvature is by

$$K = \frac{\det II}{\det I} = \frac{eg - f^2}{EG - F^2}. \tag{2.4}$$

Since $EG - F^2 > 0$ we can write K as

$$K = \det(II \cdot I^{-1}).$$

This is connected with the first definition of K since the extreme curvatures $\frac{1}{R_1}$ and $\frac{1}{R_2}$ are the eigenvalues of the matrix $II \cdot I^{-1}$. For our further calculations we will use equation (2.4) which in Gauss's own notation is

$$K = \frac{DD'' - D'^2}{(A^2 + B^2 + C^2)^2}.$$

Furthermore, Gauss defines

$$
\begin{aligned}
m &= a\alpha + b\beta + c\gamma & \left(= \langle \frac{\partial h}{\partial u}, \frac{\partial^2 h}{\partial u^2} \rangle\right) \quad (2.5)\\
m' &= a\alpha' + b\beta' + c\gamma' & \left(= \langle \frac{\partial h}{\partial u}, \frac{\partial^2 h}{\partial u \partial v} \rangle\right) \quad (2.6)\\
m'' &= a\alpha'' + b\beta'' + c\gamma'' & \left(= \langle \frac{\partial h}{\partial u}, \frac{\partial^2 h}{\partial v^2} \rangle\right) \quad (2.7)\\
n &= a'\alpha + b'\beta + c'\gamma & \left(= \langle \frac{\partial h}{\partial v}, \frac{\partial^2 h}{\partial u^2} \rangle\right) \quad (2.8)\\
n' &= a'\alpha' + b'\beta' + c'\gamma' & \left(= \langle \frac{\partial h}{\partial v}, \frac{\partial^2 h}{\partial u \partial v} \rangle\right) \quad (2.9)\\
n'' &= a'\alpha'' + b'\beta'' + c'\gamma''. & \left(= \langle \frac{\partial h}{\partial v}, \frac{\partial^2 h}{\partial v^2} \rangle\right) \quad (2.10)
\end{aligned}
$$

We want to show that Gaussian curvature is an intrinsic value, thus we need an expression that does not depend on the surface normal. We will obtain such an expression by eliminating A, B and C from D, D' and D''.

Take equations (2.1), (2.5) and (2.8)

$$
\begin{aligned}
D &= A\alpha + B\beta + C\gamma \\
m &= a\alpha + b\beta + c\gamma \\
n &= a'\alpha + b'\beta + c'\gamma,
\end{aligned}
$$

multiply by $bc' - cb', b'C - c'B$ and $cB - bC$ respectively and add them. Through this process, β and γ are eliminated,

$$
\begin{aligned}
&D(bc' - cb') + m(b'C - c'B) + n(cB - bC) = \\
&= \alpha(A(bc' - cb') + a(b'C - c'B) + a'(cB - bC)).
\end{aligned}
$$

We use the definitions of A, B, C, E, F and G on the left and right hand sides,

$$
\begin{aligned}
LHS &= D(bc' - cb') + (nc - mc')(ca' - ac') + (mb' - nb)(ab' - ba') \\
&= D(bc' - cb') + m(aa'^2 + ab'^2 + ac'^2 - a'aa' - a'bb' - a'cc') + \\
&\quad + n(a'a^2 + a'b^2 + a'c^2 - aaa' - abb' - acc') \\
&= DA + a(mG - nF) + a'(nE - mF),
\end{aligned}
$$

$$
\begin{aligned}
RHS &= \alpha(A(bc' - cb') + B(a'c - ac') + C(ab' - a'b)) \\
&= \alpha(A^2 + B^2 + C^2).
\end{aligned}
$$

Hence,

$$DA = \alpha\Delta + a(nF - mG) + a'(mF - nE).$$

In a similar manner with (2.1), (2.5) and (2.8) we will also get

$$\begin{aligned} DB &= \beta\Delta + b(nF - mG) + b'(mF - nE) \\ DC &= \gamma\Delta + c(nF - mG) + c'(mF - nE). \end{aligned}$$

Take the three equations that we have obtained, multiply by α'', β'' and γ'' respectively and add them,

$$LHS = DA\alpha'' + DB\beta'' + DC\gamma'' = D(A\alpha'' + B\beta'' + C\gamma'') = DD'',$$

$$\begin{aligned} RHS = &(\alpha\alpha'' + \beta\beta'' + \gamma\gamma'')\Delta + (a\alpha'' + b\beta'' + c\gamma'')(nF - mG) + \\ &+ (a'\alpha'' + b'\beta'' + c'\gamma'')(mF - nE) \\ = &(\alpha\alpha'' + \beta\beta'' + \gamma\gamma'')\Delta + m''(nF - mG) + n''(mF - nE) \\ = &(\alpha\alpha'' + \beta\beta'' + \gamma\gamma'')\Delta - nn''E + (nm'' + mn'')F - mm''G. \end{aligned}$$

Thus we have obtained an expression free of normal components:

$$DD'' = (\alpha\alpha'' + \beta\beta'' + \gamma\gamma'')\Delta - nn''E + (nm'' + mn'')F - mm''G.$$

The above process is repeated with equations (2.2), (2.6) and (2.9),

$$\begin{aligned} D' &= A\alpha' + B\beta' + C\gamma' \\ m' &= a\alpha' + b\beta' + c\gamma' \\ n' &= a'\alpha' + b'\beta' + c'\gamma'. \end{aligned}$$

This will result in

$$\begin{aligned} D'A &= \alpha'\Delta + a(n'F - m'G) + a'(m'F - n'E) \\ D'B &= \beta'\Delta + b(n'F - m'G) + b'(m'F - n'E) \\ D'C &= \gamma'\Delta + c(n'F - m'G) + c'(m'F - n'E). \end{aligned}$$

Take these equations, multiply by α', β' and γ' respectively and add them. Once again we obtain an expression free of normal components:

$$D'^2 = (\alpha'^2 + \beta'^2 + \gamma'^2)\Delta - n'^2E + 2m'n'F - m'^2G,$$

and we can compute

$$\begin{aligned} DD'' - D'^2 = &(\alpha\alpha'' + \beta\beta'' + \gamma\gamma'')\Delta - nn''E + (nm'' + mn'')F - mm''G - \\ &- ((\alpha'^2 + \beta'^2 + \gamma'^2)\Delta - n'^2E + 2m'n'F - m'^2G) \\ = &(\alpha\alpha'' + \beta\beta'' + \gamma\gamma'' - \alpha'^2 - \beta'^2 - \gamma'^2)\Delta + \\ &+ E(n'^2 - nn'') + F(nm'' - 2m'n' + mn'') + G(m'^2 - mm''). \end{aligned}$$

We would like to have an expression of the Gaussian curvature where we only use E, F, G and u, v. We recognize that

$$\begin{aligned}
\frac{\partial E}{\partial u} &= 2m & \frac{\partial F}{\partial u} &= m' + n & \frac{\partial G}{\partial u} &= 2n' \\
\frac{\partial E}{\partial v} &= 2m' & \frac{\partial F}{\partial v} &= m'' + n' & \frac{\partial G}{\partial v} &= 2n''
\end{aligned}$$

and rewrite it as

$$\begin{aligned}
m &= \frac{1}{2}\frac{\partial E}{\partial u} & n' &= \frac{1}{2}\frac{\partial G}{\partial u} \\
m' &= \frac{1}{2}\frac{\partial E}{\partial v} & n'' &= \frac{1}{2}\frac{\partial G}{\partial v} \\
m'' &= \frac{\partial F}{\partial v} - \frac{1}{2}\frac{\partial G}{\partial u} & n &= \frac{\partial F}{\partial u} - \frac{1}{2}\frac{\partial E}{\partial v}.
\end{aligned}$$

Also,

$$\begin{aligned}
(\alpha\alpha'' + \beta\beta'' + \gamma\gamma'' - \alpha'^2 - \beta'^2 - \gamma'^2) &= \frac{\partial}{\partial u}m'' - \frac{\partial}{\partial v}m' \\
&= \frac{\partial^2 F}{\partial u \partial v} - \frac{1}{2}\frac{\partial^2 E}{\partial v^2} - \frac{1}{2}\frac{\partial^2 G}{\partial u^2}.
\end{aligned}$$

Now we can put all the obtained expressions into the equation for Gaussian curvature

$$\Delta^2 K = DD'' - D'^2$$

and thus we have

$$\begin{aligned}
&4(EG - F^2)^2 K = \\
&= E\left(\frac{\partial E}{\partial v}\frac{\partial G}{\partial v} - 2\frac{\partial F}{\partial u}\frac{\partial G}{\partial v} + \left(\frac{\partial G}{\partial u}\right)^2\right) + \\
&\quad + F\left(\frac{\partial E}{\partial u}\frac{\partial G}{\partial v} - \frac{\partial E}{\partial v}\frac{\partial G}{\partial u} - 2\frac{\partial E}{\partial v}\frac{\partial F}{\partial v} + 4\frac{\partial F}{\partial u}\frac{\partial F}{\partial v} - 2\frac{\partial F}{\partial u}\frac{\partial G}{\partial u}\right) + \\
&\quad + G\left(\frac{\partial E}{\partial u}\frac{\partial G}{\partial u} - 2\frac{\partial E}{\partial u}\frac{\partial F}{\partial v} + \left(\frac{\partial E}{\partial v}\right)^2\right) - \\
&\quad - 2\big(EG - F^2\big)\left(\frac{\partial^2 E}{\partial v^2} - 2\frac{\partial^2 F}{\partial u \partial v} + \frac{\partial^2 G}{\partial u^2}\right).
\end{aligned}$$

In the above we can see that we have managed to write the equation for K using only E, F and G and their partial derivatives. A line element is an infinitesimal distance between two neighbouring points, and this can be expressed with E, F and G as parts,

$$\sqrt{dx^2 + dy^2 + dz^2} = \sqrt{Edu^2 + 2Fdudv + Gdv^2}.$$

Thus, in order to find K we only need to know the expression for a line element on the surface, since it contains all the necessary information to calculate K.

Now, suppose that the surface M is developed upon another surface M' and let every point x, y, z on M have a distinct corresponding point x', y', z' on M'. On this surface, we can assume that x', y' and z' are functions of u and v. The line element on M' can be expressed with E', F' and G' as parts, and these are also functions of u and v,

$$\sqrt{dx'^2 + dy'^2 + dz'^2} = \sqrt{E'du^2 + 2F'dudv + G'dv^2}.$$

When developing one surface upon another, the infinitesimal distances between points on M will be the same as for the corresponding points on M', that is, the line elements will be the same, and therefore

$$E = E',\ F = F',\ G = G'.$$

E, F, and G remain the same when we develop the surface upon another and therefore so does the Gaussian curvature. $\square$

Chapter 3

Vector analysis

3.1 Early vectors in the plane

Gauss could have simplified his calculations if he had known about vectors and the associated scalar and vector product. At the end of this section we will see this simplification when we state his proof in vector notation. Before that, we will see how the notion of a vector and vector analysis emerged.

At the turn of the 19th century Gauss had an idea that complex numbers can be represented geometrically. His idea was part of his proof of the fundamental theorem of algebra, in his doctoral dissertation of 1799. After the dissertation Gauss waited long before publishing anything more substantial on this idea. That was in 1831 and at that time five other men had already published more or less influential books and treatises on the subject. They were Caspar Wessel, Jean Robert Argand, Abbé Buée, John Warren and C.V. Mourey. Wessel, a Norwegian mathematician, was first when he in the same year as Gauss's doctoral dissertation, published *Om Directionens analytiske Betegning* (*On the analytical representation of direction*). It was written in Danish and unfortunately most of the European mathematicians did not see his work until it was published in a French version in 1897 [8, pp.5-6], [26, p.89], [39].

Wessel aimed at creating geometrical methods and the geometrical representation of complex numbers came as a part of this. We choose a line segment of a certain length and direction and define it to be the positive unit denoted by $+1$. We then take another line segment of unit length perpendicular to the positive unit. We let it have the same origin and denote it by $+\varepsilon$. The angle of direction at $+1$ is $0°$ and $+\varepsilon$ is $90°$. By taking line segments of unit length, oppositely directed to $+1$ and $+\varepsilon$ we obtain -1 and $-\varepsilon$ with corresponding angles $180°$ and $-90°$.

The segments of two coplanar lines, a and b, can be multiplied in the following way: The length of the resulting line segment c is the product of the lengths of a and b, $|c| = |a| \cdot |b|$. The resulting line segment lies in the same plane as a and b and the angle of direction of c is the sum of the angles of the two line segments. Since the length of a unit line is one, multiplication of unit lines is the same as adding angles.

$$
\begin{array}{lll}
(+1)(+1) = +1 & (+1)(+\varepsilon) = +\varepsilon & (+1)(-1) = -1 \\
(-1)(-1) = +1 & (-1)(-\varepsilon) = +\varepsilon & (+\varepsilon)(-\varepsilon) = +1 \\
(+\varepsilon)(+\varepsilon) = -1 & (+1)(-\varepsilon) = -\varepsilon & \\
(-\varepsilon)(-\varepsilon) = -1 & (-1)(+\varepsilon) = -\varepsilon &
\end{array}
$$

From these expression Wessel concluded that $\varepsilon = \sqrt{-1}$. We thus have a complex plane where any straight line can be represented by $x + \varepsilon y$ with real numbers x and y.

3.2 Quaternions

Knowing how to represent numbers in two dimensions, it is a natural step to ask how this can be done in three dimensions. Wessel answered this question by letting $x + \eta y + \varepsilon z$ represent any point in space, as r, ηr and εr are three mutually perpendicular radii of a sphere with radius r. But for multiplication of vectors in three dimensions Wessel presented a somewhat incomplete ad hoc method.

In an 1837 essay, William Rowan Hamilton showed that complex numbers can be represented as ordered pairs of real numbers (a, b). He also posed the question of how to represent three-dimensional numbers which he called a **Theory of Triplets**. Hamilton wanted these triplets to be associative and commutative as well as distributive. For two triplets u and v there should be exactly one triplet x such that $ux = v$. Moreover, if

$$(a_1 + b_1 i + c_1 j)(a_2 + b_2 i + c_2 j) = a_3 + b_3 i + c_3 j$$

then

$$(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2) = a_3^2 + b_3^2 + c_3^2$$

which is called the law of the moduli.

Hamilton pondered over this question for some years, and finally, on October 16, 1843, while walking with his wife alongside the Royal Canal toward Dublin, it struck him. What he needed was not three numbers, but four. The structure he had been searching for would be called **quaternions** and the solution to his problem was

$$i^2 = j^2 = k^2 = ijk = -1.$$

Stopping at a bridge called Brougham Bridge he took his knife and carved the insight into a stone [8, pp.23-32], [13, pp.375-376]. At this day the carving is no longer possible to see, but a stone plaque has been put there to commemorate the event.

With real numbers w, x, y, z and symbols i, j, k, quaternions are expressed in the form

$$q = w + ix + jy + kz.$$

The only part of the algebra that Hamilton had to give up was commutativity of multiplication. The following rule applies:

$$\begin{array}{lll} ij = k & jk = i & ki = j \\ ji = -k & kj = -i & ik = -j. \end{array}$$

The terms **vector** and **scalar** were defined by Hamilton as parts of a quaternion number q. The scalar part is

$$S.q = w$$

and the vector part is

$$V.q = ix + jy + kz.$$

Multiplication of two quaternions lacking scalar parts, $\alpha = ix + jy + kz$ and $\beta = ix' + jy' + kz'$, yields

$$S.\alpha\beta = -(xx' + yy' + zz')$$

$$V.\alpha\beta = i(yz' - zy') + j(zx' - xz') + k(xy' - yx').$$

Hamilton also introduced the operator $\triangleleft$,

$$\triangleleft = \frac{id}{dx} + \frac{jd}{dy} + \frac{kd}{dz}, \qquad -\triangleleft^2 = \left(\frac{d}{dx}\right)^2 + \left(\frac{d}{dy}\right)^2 + \left(\frac{d}{dz}\right)^2.$$

Later on this symbol would change into ∇ and be called nabla.

Remark 3.2.1. In modern language we may think of $\{1, i, j, k\}$ as a basis for the space of quaternions.

3.3 Vector analysis by Gibbs and Heaviside

In the quaternions we can see many similarities to vector analysis and hence it may not come as a surprise that vector analysis was developed from the quaternions. Josiah Willard Gibbs and Oliver Heaviside did this almost simultaneously and in the same manner, but they did not know about each other's existence until Heaviside received a copy of Gibbs's pamphlet *Elements of vector analysis* in 1888 [8, pp.151-168].

Both of them had an interest in electricity and magnetism which led them to read *A treatise on electricity and magnetism* from 1873 by James Clerk Maxwell. In some of the calculations Maxwell had used quaternions. Since neither Gibbs nor Heaviside knew anything about quaternions they felt a need to study that too. At that time Peter Guthrie Tait was an influential figure in this area, so a natural step was to read his work. With a little more knowledge at hand, both Gibbs and Heaviside realized that although liking the idea of the quaternions, they did not think that quaternion methods were natural in physical applications.

In vector analysis the quaternion is divided into two independent pieces and the scalar part is changed to be positive. It is, for example, more natural to think of the length of a vector as positive. A vector is now a quaternion without scalar part. The notational style of Gibbs is very similar to those of Tait and Hamilton. Gibbs called the scalar product $\alpha.\beta$, direct product, which is the same as Tait's $-S\alpha\beta$ or Hamilton's $-S.\alpha\beta$. The vectors (or quaternions) α and β may be interchanged as follows:

$$\begin{aligned} \alpha.\beta &= \beta.\alpha \\ S\alpha\beta &= S\beta\alpha. \end{aligned}$$

The vector product $\alpha \times \beta$, which Gibbs called skew product, is the same as Tait's $V\alpha\beta$ or Hamilton's $V.\alpha\beta$. Vector multiplication is anti-commutative, as is the vector part of quaternion multiplication of α and β:

$$\begin{aligned} \alpha \times \beta &= -\beta \times \alpha \\ V\alpha\beta &= -V\beta\alpha. \end{aligned}$$

From one type of multiplication with quaternions

$$\alpha\beta = -(xx' + yy' + zz') + (yz' - zy')i + (zx' - xz')j + (xy' - yx')k,$$

we will get the two

$$\alpha.\beta = xx' + yy' + zz'$$

and

$$\alpha \times \beta = (yz' - zy')i + (zx' - xz')j + (xy' - yx')k.$$

3.4 A shorter proof

This section is concluded with another proof of Gauss's Theorema Egregium using vectors.

Proof of Theorema Egregium. As in the previous proof we let

$$h(u, v) = (x(u, v), y(u, v), z(u, v))$$

be a parameterization of a surface, and let h_i and h_{ij} denote the first and second partial derivatives $\frac{\partial h}{\partial i}$ and $\frac{\partial^2 h}{\partial i \partial j}$. The first and second fundamental forms are thus

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \langle h_u, h_u \rangle & \langle h_u, h_v \rangle \\ \langle h_u, h_v \rangle & \langle h_v, h_v \rangle \end{pmatrix}$$

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} \langle h_{uu}, \frac{h_u \times h_v}{\sqrt{EG-F^2}} \rangle & \langle h_{uv}, \frac{h_u \times h_v}{\sqrt{EG-F^2}} \rangle \\ \langle h_{uv}, \frac{h_u \times h_v}{\sqrt{EG-F^2}} \rangle & \langle h_{vv}, \frac{h_u \times h_v}{\sqrt{EG-F^2}} \rangle \end{pmatrix}.$$

We use the formula for Gaussian curvature,

$$K = \frac{eg - f^2}{EG - F^2},$$

and rewrite it as

$$\begin{aligned} K(EG - F^2) &= eg - f^2 \\ &= \langle h_{uu}, \frac{h_u \times h_v}{\sqrt{EG - F^2}} \rangle \langle h_{vv}, \frac{h_u \times h_v}{\sqrt{EG - F^2}} \rangle - \langle h_{uv}, \frac{h_u \times h_v}{\sqrt{EG - F^2}} \rangle^2 \\ &= \frac{1}{EG - F^2} \left(\langle h_{uu}, h_u \times h_v \rangle \langle h_{vv}, h_u \times h_v \rangle - \langle h_{uv}, h_u \times h_v \rangle^2 \right). \end{aligned}$$

Since

$$\langle a, b \times c \rangle = \det(a^t, b^t, c^t)$$

for vectors a, b, c, where each vector is a row-vector, we have

$$\begin{aligned} & K(EG - F^2)^2 = \\ & = \det(h_{uu}^t, h_u^t, h_v^t) \cdot \det(h_{vv}^t, h_u^t, h_v^t) - \det(h_{uv}^t, h_u^t, h_v^t) \cdot \det(h_{uv}^t, h_u^t, h_v^t) \\ & = \det\left(\begin{pmatrix} h_{uu} \\ h_u \\ h_v \end{pmatrix} \cdot (h_{vv}^t, h_u^t, h_v^t) \right) - \det\left(\begin{pmatrix} h_{uv} \\ h_u \\ h_v \end{pmatrix} \cdot (h_{uv}^t, h_u^t, h_v^t) \right) \\ & = \det \begin{pmatrix} \langle h_{uu}, h_{vv} \rangle & \langle h_{uu}, h_u \rangle & \langle h_{uu}, h_v \rangle \\ \langle h_u, h_{vv} \rangle & E & F \\ \langle h_v, h_{vv} \rangle & F & G \end{pmatrix} - \\ & \quad - \det \begin{pmatrix} \langle h_{uv}, h_{uv} \rangle & \langle h_{uv}, h_u \rangle & \langle h_{uv}, h_v \rangle \\ \langle h_u, h_{uv} \rangle & E & F \\ \langle h_v, h_{uv} \rangle & F & G \end{pmatrix} \\ & = \langle h_{uu}, h_{vv} \rangle \cdot \det \begin{pmatrix} E & F \\ F & G \end{pmatrix} + \det \begin{pmatrix} 0 & \langle h_{uu}, h_u \rangle & \langle h_{uu}, h_v \rangle \\ \langle h_u, h_{vv} \rangle & E & F \\ \langle h_v, h_{vv} \rangle & F & G \end{pmatrix} - \\ & \quad - \langle h_{uv}, h_{uv} \rangle \cdot \det \begin{pmatrix} E & F \\ F & G \end{pmatrix} - \det \begin{pmatrix} 0 & \langle h_{uv}, h_u \rangle & \langle h_{uv}, h_v \rangle \\ \langle h_u, h_{uv} \rangle & E & F \\ \langle h_v, h_{uv} \rangle & F & G \end{pmatrix} \end{aligned}$$

$$= \det \begin{pmatrix} \langle h_{uu}, h_{vv}\rangle - \langle h_{uv}, h_{uv}\rangle & \langle h_{uu}, h_u\rangle & \langle h_{uu}, h_v\rangle \\ \langle h_u, h_{vv}\rangle & E & F \\ \langle h_v, h_{vv}\rangle & F & G \end{pmatrix} -$$

$$- \det \begin{pmatrix} 0 & \langle h_{uv}, h_u\rangle & \langle h_{uv}, h_v\rangle \\ \langle h_u, h_{uv}\rangle & E & F \\ \langle h_v, h_{uv}\rangle & F & G \end{pmatrix}.$$

The following part is close to Gauss's treatment and the relations are the same as Gauss's m, m', m'', n, n', n'' on page 9, which we deduce by differentiation of E, F and G.

$$\langle h_{uu}, h_u\rangle = \frac{1}{2}E_u \qquad \langle h_{uv}, h_v\rangle = \frac{1}{2}G_u$$
$$\langle h_{uv}, h_u\rangle = \frac{1}{2}E_v \qquad \langle h_{vv}, h_v\rangle = \frac{1}{2}G_v$$
$$\langle h_{vv}, h_u\rangle = F_v - \frac{1}{2}G_u \qquad \langle h_{uu}, h_v\rangle = F_u - \frac{1}{2}E_v$$

With

$$\frac{1}{2}G_{uu} = \langle h_{uuv}, h_v\rangle + \langle h_{uv}, h_{uv}\rangle$$
$$F_{uv} - \frac{1}{2}E_{vv} = \langle h_{uuv}, h_v\rangle + \langle h_{uu}, h_{vv}\rangle$$

we also have

$$\langle h_{uu}, h_{vv}\rangle - \langle h_{uv}, h_{uv}\rangle = F_{uv} - \frac{1}{2}E_{vv} - \frac{1}{2}G_{uu}.$$

We put the above into the eqation for Gaussian curvature and get

$$K(EG - F^2)^2 =$$
$$= \det \begin{pmatrix} F_{uv} - \frac{1}{2}G_{uu} - \frac{1}{2}E_{vv} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{pmatrix} -$$
$$- \det \begin{pmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{pmatrix}.$$

We have thus obtained an equation only depending on intrinsic values, that is, E, F and G, and the rest of the theorem follows. □

Chapter 4

Differential forms

4.1 Theory of Extension

Hamilton published his first paper on quaternions in 1844. In the same year, Hermann Grassmann, a teacher from Stettin, Pomerania (today in Poland), published a book with the long name *Die lineale Ausdehnungslehre, ein neuer Zweig der Mathematik dargestellt und durch Anwendungen auf die übrigen Zweige der Mathematik, wie auch auf die Statik, Mechanik, die Lehre vom Magnetismus und die Krystallonomie erläutert*, or in short *Theory of extension*. The long title may be a pointer to how the rest of the book was written, at least in the eyes of contemporary mathematicians. In general it was considered to be too cumbersome to read and therefore it did not gain much popularity at first.

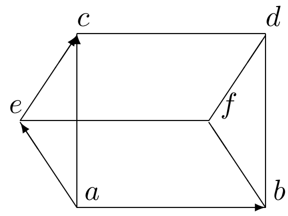
Grassmann's idea was to develop a theory that would work in dimensions of arbitrary size, and he could have competed with Hamilton's quaternions in becoming the forerunner of vector analysis. But unfortunately, the influence of Grassmann's ideas was weak since his contemporaries had difficulties with understanding what he had written [8, pp.47-77][13, p.362].

Grassmann introduced something he calls **forms**. A form can be a point, a directed line segment (*Strecken*), an oriented area, et cetera. A point has order zero and if we let the point move in one direction we will obtain a line. This is a first order system. If the line is moved in a rectilinear direction, a plane is produced. This is a second order system. The procedure can be continued to obtain systems of higher order.

Forms can be joined by connections to produce new forms. The connections can be addition and subtraction as well as multiplication and division. If two forms are connected by multiplication, then in general a form of higher order will be obtained. Multiplication in the eyes of Grassmann was any distributive operation.

Grassmann introduced a type of multiplication which was called **outer**

multiplication (or exterior multiplication). Outer multiplication can be illustrated by multiplication of two directed line segments in a plane. This can be seen as letting the directed line segment ab move along the directed line segment ac to produce an oriented area in the plane, the parallelogram $abdc$.



This oriented area has been called a **bivector** by later authors [3, 32] since it by moderns terms is made of two vectors. The orientation can be understood with the help of vector analysis. The cross product of two vectors is a vector perpendicular to the two first. The third vector will point in a direction according to the right hand rule. If any of the two multiplied vectors point in opposite direction, then the vector produced will have a direction opposite to the vector produced in the first case.

Since Grassmann did not have the tools of vector analysis at hand, he probably did not think of orientation in terms of a perpendicular vector pointing in one direction or the other. It is more likely that he thought of orientation as taking a walk around the perimeter of the parallelogram in one direction or the other. For example, if a positive orientation is given by walking around the perimeter in the order $a - b - d - c - a$, then a negative orientation is obtained by walking around the perimeter in the order $a - c - d - b - a$. Thus the oriented area produced by letting ab move along ae differs from the one where ab move along $ea = -ae$. One is positively oriented whereas the other is negatively oriented.

If we try to move ab along ef we realize that no parallelogram is produced. That is, the product of two parallel line segments is zero.

The distributive rule is exemplified by moving ab along ae and then ec. The sum of the two oriented areas obtained equals the oriented area obtained when ab move along $ac = ae + ec$. Moreover, the oriented area of ab along ae, is the same as the oriented area we obtain by first letting ab move along ac and then $ce = -ec$. The rules for outer multiplication of forms would eventually become algebraic rules for calculus on differential forms.

4.2 Manifolds

At the age of 27, Bernhard Riemann took a major step in the development of differential geometry. Riemann wanted to obtain a lecturing position at the University of Göttingen and for this he had to hold a lecture. Gauss who was a professor at Göttingen had chosen the topic of the lecture out of three that Riemann had proposed. This was the only topic that Riemann had not prepared and it took him almost two months to finish it. The lecture is called *Über die Hypothesen, welche der Geometrie zu Grunde liegen* (*On the hypotheses which underlie geometry*) and was given on June 10 1854 [13, p. 650].

The lecture is almost completely free from mathematical formalism and the focus lies on reasoning and explaining concepts and ideas. Riemann probably did like this as he wanted all members of the faculty to understand the lecture, even those who where not familiar with mathematics [37, p.133]. One thing that Riemann introduced was making a distinction between metric and topological properties. Making this distinction made it possible to explore the general notion of a manifold.

A manifold is essentially a set of objects (*Bestimmungsweisen*). It can be discrete or continuous depending on if there is a discrete transition between objects or a continuous transition. In the former case the objects are called elements and in the latter points. Riemann proposes that an example of a continuous manifold in three-dimensional space is color. Goethe's *Theory of colours* was published in 1810 [18] and with that as a background the example of color may very well have been an example that was well understandable by a 19th century person. In a rainbow the colors span from red to violet in such a way that it is not possible to point out exactly where red turns orange and so on. This is a one-dimensional continuous manifold. Lightness (from black to white) and saturation (from greyish to clear) add two more dimensions to the continuous manifold.

Using induction, an n-dimensional manifold can be created. We take objects that can form a continuous manifold and move from one point to another in a well determined way. The objects we pass form a curve, or as Riemann also called it, a simply extended manifold. The only possible directions are forwards and backwards. This curve can also move in a well determined way to another curve, such that every point on the first curve moves to a corresponding point on the second curve. These objects form a surface, or a doubly extended manifold. A triply extended manifold is obtained by taking this doubly extended manifold and letting it move in a well determined way to another doubly extended manifold. By continuing this procedure we will get manifolds of higher dimensions.

Remark 4.2.1. Grassmann's nth order systems are created in almost the same way as Riemann's n-dimensional manifolds. So it seems that Riemann could have gotten his idea from Grassmann, but Riemann claims in his lecture that he had only been influenced by Gauss and Herbart, a German philosopher [37, p.136]. Considering that few people had read Grassmann's Theory of Extension, this seems probable.

Following Riemann, many mathematicians contributed to the clarification of the definition of a manifold. Hermann Weyl made the first intrinsic definition of a manifold in 1912. In 1936 Hassler Whitney made the first modern statement of a manifold. Before him, there were both intrinsic and extrinsic definitions. Whitney's embedding theorem linked intrinsic and extrinsic definitions, stating that any differentiable manifold can be embedded in $\mathbb{R}^{2m+1}$ [1, p.144, 161], [40].

Definition 4.2.2. An n-dimensional **differentiable manifold** M is a topological space where every point has an open neighbourhood M_i homeomorphic to an open set in $\mathbb{R}^n$, i.e. there is a continuous map $\phi_i : M_i \to \mathbb{R}^n$ which has a continuous inverse ϕ_i^{-1}.

Furthermore, for two connected subsets M_i and M_j of M, the coordinate change $\phi_j \circ \phi_i^{-1} : \phi_i(M_i \cap M_j) \to \phi_j(M_i \cap M_j)$ is differentiable.

An example of a manifold is Euclidean space itself.

Definition 4.2.3. A **manifold-with-boundary** M is as in the above definition except that every point has an open neighbourhood homeomorphic to either $\mathbb{R}^n$ or $\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n \geq 0\}$. The set of all points of the latter type is called the **boundary** of M, denoted by ∂M, and can be seen to form a differentiable manifold of dimension $n-1$.

We will assume that every manifold satisfies the **Hausdorff** separation axiom, that every two distinct points have disjoint open neighbourhoods.

In the proofs of Theorema Egregium we denoted the tangent vectors by $\frac{\partial h}{\partial u}$ and $\frac{\partial h}{\partial v}$ for a surface parameterized by $h(u, v)$. In a modern point of view it is convenient to consider the differential operator $\frac{\partial}{\partial u}$ a tangent vector. Thus $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ constitutes a basis for all tangent vectors on the surface.

The space of tangent vectors at a point p on a manifold M is called a **tangent space**, denoted by T_pM. The set of all tangent vectors $\frac{\partial}{\partial u_i}$ thus constitutes a basis for the tangent space.

We define a **vector field** X to be a function that assigns a vector to each point p in M,

$$X(p) = \sum_{i=1}^{n} a_i(p) \left. \frac{\partial}{\partial u_i} \right|_p$$

and it is said to be differentiable if all the a_i's are differentiable.

Definition 4.2.4. Given a differentiable manifold M, a **Riemannian metric** g on M is a mapping such that with each point $p \in M$ we associate an inner product $g_p : T_pM \times T_pM \to \mathbb{R}$. This inner product satisfies the following property: If U is any open set in M and X, Y are differentiable vector fields on U, then the function $g(X, Y) : U \to \mathbb{R}$ given by

$$g(X, Y)(p) = g_p(X|_p, Y|_p)$$

is differentiable on U.

A **Riemannian manifold** M is a differentiable manifold with a Riemannian metric.

4.3 Defining differential forms

Grassmann's ideas on geometry were spread to a wider audience with the aid of two Italian mathematicians, Giuseppe Peano and his assistant Cesare Burali-Forti. Peano's book on geometric calculus *Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann* was published in 1888 and in 1897 Burali-Forti followed with his *Introduction à la géométrie différentielle, suivant la méthode de H. Grassmann* (*Introduction to differential geometry*). Peano and Burali-Forti concretized Grassmann's abstract ideas and used his methods for calculations on geometric objects in three-dimensional space. Partly inspired by Burali-Forti, the French mathematician Élie Cartan used these ideas and applied them on a more general setting with differential forms on manifolds.

Differential forms, or differential expressions, where properly defined for the first time in 1899 by Cartan in his *Sur certaines expressions différentielles et sur le problème de Pfaff* (*On certain differential expressions and the Pfaff problem*). They were introduced as a part of solving the Pfaff problem, meaning solving systems of first order differential equations. Differential forms before Cartan's definition had foremost been seen as those things that appear under integral sign [4, 21].

Definition 4.3.1. Given n variables $x_1, x_2, \ldots, x_n$, **differential forms** are homogeneous expressions ω formed by a finite number of additions and multiplications of n differentials $dx_1, dx_2, \ldots, dx_n$ and certain differentiable coefficient functions of $x_1, x_2, \ldots, x_n$.

In general, a differential form of order p, a **p-form** ω, can be written as

$$\omega = \sum f_{i_1,\ldots,i_p}\, dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

where $i_1, \ldots, i_p$ range from 1 to n. The symbol $\wedge$, called **wedge**, is the operator for exterior multiplication. For a manifold M, we let $\Omega^p(M)$ denote the set of all p-forms on M. A 0-form is defined to be a differentiable function. Henceforth we will assume that all p-forms are C^∞.

Cartan closely followed his predecessors in notational style. As an example, Burali-Forti defined first order forms to be of the type

$$x_1 P_1 + x_2 P_2 + \ldots + x_n P_n$$

where $x_1, \ldots, x_n$ are real numbers and $P_1, \ldots, P_n$ represent points. A second order form is

$$x_1 P_1 Q_1 + x_2 P_2 Q_2 + \ldots + x_n P_n Q_n$$

where $P_i Q_i$ is a line segment between the points P_i and Q_i. A third order form is

$$x_1 P_1 Q_1 R_1 + x_2 P_2 Q_2 R_2 + \ldots + x_n P_n Q_n R_n$$

where $P_i Q_i R_i$ is a triangle with the points P_i, Q_i and R_i as vertices. Forms of higher order are obtained in a similar manner. Cartan's first order differential forms are of the type

$$A_1\, dx_1 + A_2\, dx_2 + \ldots + A_n\, dx_n$$

where A_i are functions of $x_1, \ldots, x_n$ and $dx_1, \ldots, dx_n$ are differentials. A second order differential form is

$$A_1\, dx_1 \wedge dx_2 + A_2\, dx_2 \wedge dx_3 + \ldots + A_n\, dx_n \wedge dx_1$$

where each $dx_i \wedge dx_j$ is a differential 2-form.

Remark 4.3.2. A classical notion of dx_i is that it is an infinitesimal change of x_i. In a sense we can think of it as an infinitesimal line segment with direction. Burali-Forti's second order forms and Cartan's first order differential forms thus share similarities in the sense that they are sums of line segments with coefficients. Although, in a modern sense, with the differential forms Cartan uses the dual of a vector, which is also known as a covariant vector.

The rules for exterior multiplication, which originated from Grassmann and was exemplified with an oriented area on page 18, can be applied to differential forms. We use the 1-forms dx, dy, dz for an illustration.

The oriented area made of ab and ac have the same area as the oriented area of ab and $ca = -ac$ except for a difference in signs. In the same way differential forms change sign if two forms are interchanged, that is, the wedge product is anti-commutative,

$$dx \wedge dy = -\, dy \wedge dx\,.$$

The parallelogram of two parallel line segments has zero area. For differential forms we interpret this as

$$dx \wedge dx = 0.$$

This can also be deduced from the first relation since $dx \wedge dx = -\, dx \wedge dx$ must mean that the product is zero. The distributive law was shown by letting ab move along ae and then ec which is the same as letting ab move along ac. In differential forms we write it as

$$dx \wedge (dy + dz) = dx \wedge dy + dx \wedge dz\,.$$

The wedge product is bilinear, so for p-forms ω, q-forms θ and 0-forms f, the following rules apply:

$$\begin{gathered}\omega \wedge (\theta_1 + \theta_2) = \omega \wedge \theta_1 + \omega \wedge \theta_2 \\ (\omega_1 + \omega_2) \wedge \theta = \omega_1 \wedge \theta + \omega_2 \wedge \theta \\ f\omega \wedge \theta = \omega \wedge f\theta = f \cdot (\omega \wedge \theta).\end{gathered}$$

The product of ω and θ is a $(p+q)$-form $\omega \wedge \theta$. Since the wedge product is anti-commutative, changing the order of multiplication will result in

$$\omega \wedge \theta = (-1)^{pq} \theta \wedge \omega.$$

Finally, the wedge product is associative:

$$\omega \wedge (\theta \wedge \eta) = (\omega \wedge \theta) \wedge \eta.$$

4.4 The differential in differential forms

The term differential form indicates that it should be possible to differentiate a form. Cartan's first attempt was after having defined differential forms. He called it a derived expression (*expréssion dérivée*) [4].

For a 1-form

$$\omega = A_1\, dx_1 + A_2\, dx_2 + \ldots + A_n\, dx_n$$

the derived expression ω' is a 2-form

$$\omega' = dA_1\, dx_1 + dA_2\, dx_2 + \ldots + dA_n\, dx_n\, .$$

Cartan also introduced higher order derivatives as products of ω and ω'. For example $\omega'' = \omega\omega'$ and $\omega''' = \frac{1}{2}\omega'^2$. The derived expressions did not prove to be useful, and in 1901 he made a more general definition of the differential [20].

Definition 4.4.1. For a p-form $\omega = \sum_I f_I dx_I$ with $I = (i_1, \ldots, i_p)$ and $dx_I = dx_{i_1} \wedge \ldots \wedge dx_{i_p}$, the **exterior differential** is

$$d\omega = \sum_I (df_I) \wedge dx_I.$$

The form $d\omega$ is of order $(p+1)$. The exterior derivative of a function f is a 1-form,

$$df = \sum_i \frac{\partial f}{\partial x_i}\, dx_i\, .$$

The exterior differential is a linear operator,

$$d(\omega + \theta) = d\omega + d\theta.$$

If ω is a p-form, then the exterior derivative of $\omega \wedge \theta$ is

$$d(\omega \wedge \theta) = (d\omega) \wedge \theta + (-1)^p \omega \wedge d\theta.$$

4.5 Moving frames and Gaussian curvature

In general a manifold does not naturally have a tangent space. Therefore it needs to be equipped with one for each point of the manifold. The union of all tangent spaces of a manifold is called a **tangent bundle**. Comparing tangent vectors at different points of the manifold can be tricky, but with Cartan's moving frames we can overcome this difficulty.

A **moving frame** is a function that assigns an ordered basis of vectors, i.e. a frame, to each tangent space of the manifold M at p,

$$p \mapsto (V_1(p), \ldots, V_n(p)).$$

A **1-form on** M is a real valued function on the tangent bundle of M, and at each point this function is linear. This means that, for each tangent vector $V(p)$ of M, a 1-form ω defines a real number $\omega(V)$, and for each point $p \in M$, $\omega_p : T_pM \to \mathbb{R}$ is a linear function. We say that ω_p is an element of the dual space of T_pM. Thus 1-forms are duals of vector fields.

For an orthonormal basis $(E_1(p), \ldots, E_n(p))$ we let $(\omega_1, \ldots, \omega_n)$ be a dual basis such that

$$\omega_i(E_j) = \delta_{ij}.$$

With the help of moving frames we can in yet another way prove Gauss's Remarkable Theorem, but for this we will need an equation for Gaussian curvature in a more modern dressing.

Gaussian curvature in vector notation

For a surface M in $\mathbb{R}^3$, the **normal map**, or **Gauss map**, $\nu : M \to S^2$ maps the direction of the surface normal to the unit sphere. With this, Gauss could define Gaussian curvature at a point $p \in M$ as

$$K = \frac{\text{area of } \nu(A)}{\text{area of } A}$$

where $A \subseteq M$ is an infinitely small area element at p. This equation can be equally well written in vector notation. We let $h = h(u, v)$ be a parameterization of M and choose a coordinate system (u, v) such that a point p on M and its corresponding point $\nu(p)$ on S^2 have the same u, v. The vectors

$$h_u = \frac{\partial h}{\partial u}, \quad h_v = \frac{\partial h}{\partial v}$$

span the tangent plane at p. Both M and S^2 have the same normal, so their tangent planes are parallell, and to make it easier for ourselves, we say that the tangent vectors lie in the same plane. Thus ν_u and ν_v can be expressed as linear combinations of h_u and h_v,

$$\begin{aligned}\nu_u &= ph_u + qh_v\\ \nu_v &= q'h_u + rh_v.\end{aligned}$$

The following will show us that

$$\nu_u \times \nu_v = K\, h_u \times h_v.$$

The vector product of ν_u and ν_v is

$$\begin{aligned}\nu_u \times \nu_v &= (ph_u + qh_v) \times (q'h_u + rh_v)\\ &= pr(h_u \times h_v) + qq'(h_v \times h_u)\\ &= (pr - qq')h_u \times h_v.\end{aligned}$$

By scalar multiplication with h_u and h_v of the linear combinations for ν_u and ν_v we obtain expressions with coefficients of the first and second fundamental form,

$$\begin{aligned}-e &= \langle h_u, \nu_u\rangle = \langle h_u, ph_u + qh_v\rangle = pE + qF\\ -f &= \langle h_v, \nu_u\rangle = \langle h_v, ph_u + qh_v\rangle = pF + qG\\ -f &= \langle h_u, \nu_v\rangle = \langle h_u, q'h_u + rh_v\rangle = q'E + rF\\ -g &= \langle h_v, \nu_v\rangle = \langle h_v, q'h_u + rh_v\rangle = q'F + rG.\end{aligned}$$

From $0 = \frac{\partial}{\partial i}\langle \nu, h_j\rangle = \langle \nu_i, h_j\rangle + \langle \nu, \frac{\partial^2 h}{\partial i \partial j}\rangle$ we see that e, f and g are the same as on page 6. We put the above in matrix form

$$\begin{pmatrix} -e & -f\\ -f & -g\end{pmatrix} = \begin{pmatrix} p & q\\ q' & r\end{pmatrix}\begin{pmatrix} E & F\\ F & G\end{pmatrix}$$

and the determinant is

$$eg - f^2 = (pr - qq')(EG - F^2).$$

Since $K = \dfrac{eg - f^2}{EG - F^2}$, we see that $\nu_u \times \nu_v = K\, h_u \times h_v$.

Gaussian curvature with differential forms

From linear algebra, an area element on the surface M is given by

$$|h_u \times h_v|\, du \wedge dv$$

and on the unit sphere by

$$|\nu_u \times \nu_v|\, du \wedge dv\,.$$

Thinking in terms of comparing area elements on M and S^2, we can write the equation for Gaussian curvature as

$$|\nu_u \times \nu_v|\, du \wedge dv = K\, |h_u \times h_v|\, du \wedge dv\,.$$

This will lead us to an expression for K in differential form. First, we let $\mathbf{h} = (x(u,v), y(u,v), z(u,v))$ be the position vector of the point $p = h(u,v)$. Using exterior differentiation we get a vector with 1-forms as coefficients,

$$d\mathbf{h} = (dx, dy, dz)$$

and if we use the parameterization we can write this as

$$d\mathbf{h} = h_u\, du + h_v\, dv\,.$$

We let (E_1, E_2, E_3) be a right-hand orthonormal frame for the surface with E_1, E_2 spanning the tangent plane and E_3 being normal to M. We also let the differential 1-forms $\theta_1, \theta_2, \theta_3$ be the duals of the vector fields E_1, E_2, E_3 such that $\theta_i(E_j) = \delta_{ij}$. For every tangent vector $V \in T_pM$,

$$\theta_i(V) = \langle V, E_i(p)\rangle, \quad i = 1, 2, 3.$$

The dual form $\theta_3(V)$ is zero since E_3 is orthogonal to the tangent plane. Now we can write $d\mathbf{h}$ in the basis of the moving frame as

$$d\mathbf{h} = \theta_1 E_1 + \theta_2 E_2,$$

here simplifying the writing by omitting V and p. The vector product is

$$\begin{aligned} d\mathbf{h} \times d\mathbf{h} &= (\theta_1 E_1 + \theta_2 E_2) \times (\theta_1 E_1 + \theta_2 E_2) = \\ &= (\theta_1 \wedge \theta_2)(E_1 \times E_2) + (\theta_2 \wedge \theta_1)(E_2 \times E_1). \end{aligned}$$

For tangent vectors V and W

$$\begin{aligned} \theta_1 \wedge \theta_2(V, W) &= \theta_1(V)\theta_2(W) - \theta_1(W)\theta_2(V) \\ &= -\big(\theta_2(V)\theta_1(W) - \theta_2(W)\theta_1(V)\big) \\ &= -\theta_2 \wedge \theta_1(V, W) \end{aligned}$$

and with

$$(\theta_2 \wedge \theta_1)(E_2 \times E_1) = (-\,\theta_1 \wedge \theta_2)(-E_1 \times E_2)$$

we obtain

$$d\mathbf{h} \times d\mathbf{h} = 2(\theta_1 \wedge \theta_2)E_3.$$

The 2-form $\theta_1 \wedge \theta_2$ is called the **area form**. An area form $\theta_1 \wedge \theta_2(V, W)$ is the oriented area of the parallellogram spanned by V and W. We can also express this with h_u and h_v,

$$\begin{aligned} 2(\theta_1 \wedge \theta_2)E_3 &= d\mathbf{h} \times d\mathbf{h} = (h_u\, du + h_v\, dv) \times (h_u\, du + h_v\, dv) = \\ &= (h_u \times h_v)(du \wedge dv) + (h_v \times h_u)(dv \wedge du) = 2(h_u \times h_v)(du \wedge dv). \end{aligned}$$

The vector $h_u \times h_v$ has the same direction as the surface normal E_3 and it follows that

$$\theta_1 \wedge \theta_2 = |h_u \times h_v| \, du \wedge dv \, .$$

By differentiating $E_i(p)$, $i = 1, 2, 3$, we have the directional derivative of $E_i(p)$ as a point p moves along a curve on M with direction V,

$$dE_i(p) = \omega_{i1}(V) \, E_1(p) + \omega_{i2}(V) \, E_2(p) + \omega_{i3}(V) \, E_3(p) \, .$$

We call ω_{ij} the **connection form** of the frame field and it can be computed by $\omega_{ij}(V) = \langle dE_i(p), E_j(p) \rangle$. It states the initial rate at which E_i rotates toward E_j as p moves in the direction of V on M. From the dE_i we will know how the frame rotates if we move it to another point. Since the vectors in the frame are orthogonal, with $\langle E_i, E_j \rangle = \delta_{ij}$, we have

$$0 = d\langle E_i, E_j \rangle = \langle dE_i, E_j \rangle + \langle E_i, dE_j \rangle$$

which is

$$\omega_{ij} + \omega_{ji} = 0$$

and especially,

$$\omega_{ii} = 0.$$

We have defined E_3 as being normal to M. Thus it is a normal map from M to S^2, mapping points p in M to points $E_3(p)$ in S^2. Since E_3 is orthogonal to both surfaces, the tangent planes are spanned by E_1 and E_2. We obtain a tangent vector on S^2 by differentiating E_3:

$$dE_3 = \omega_{31} \, E_1 + \omega_{32} \, E_2 \, .$$

We can obtain an area form on S^2 in the same way as we obtained an area form $\theta_1 \wedge \theta_2$ on M, namely $\omega_{31} \wedge \omega_{32} = |\nu_u \times \nu_v| \, du \wedge dv$. Now we can express Gaussian curvature as

$$\omega_{31} \wedge \omega_{32} = K\theta_1 \wedge \theta_2.$$

The connection between different expressions for Gaussian curvature

Our first definition of Gaussian curvature, on page 4, was

$$K = \pm \frac{1}{R_1 \cdot R_2}.$$

With moving frames we can see how this expression is connected with the above differential expression. We begin by differentiating the vector $d\mathbf{h}$ a second time and this is always zero.

$$0 = d(d\mathbf{h}) = d(\theta_1 E_1 + \theta_2 E_2) = (d\theta_1)E_1 - \theta_1 \wedge dE_1 + (d\theta_2)E_2 - \theta_2 \wedge dE_2$$

$$\Longleftrightarrow$$

$$\begin{aligned} (d\theta_1)E_1 + (d\theta_2)E_2 &= \theta_1 \wedge dE_1 + \theta_2 \wedge dE_2 = \\ &= \theta_1 \wedge (\omega_{12}E_2 + \omega_{13}E_3) + \theta_2 \wedge (\omega_{21}E_1 + \omega_{23}E_3) \end{aligned}$$

What we have obtained here is **Cartan's first structural equation** of Euclidean space,

$$d\theta_j = \sum_i \theta_i \wedge \omega_{ij}.$$

For our further calculations we are interested in $d\theta_3 = 0$, which is

$$\theta_1 \wedge \omega_{13} + \theta_2 \wedge \omega_{23} = 0. \tag{4.1}$$

We multiply this with θ_2 to obtain

$$\theta_2 \wedge \theta_1 \wedge \omega_{13} = 0.$$

The forms θ_1, θ_2 and ω_{13} are linearly dependent since the equation is zero. From the definition of $\theta_i(V)$ we know that θ_1 and θ_2 are linearly independent and therefore ω_{13} must be a linear combination of those two, such that $\omega_{13} = a\theta_1 + b'\theta_2$. In a similar manner we have $\omega_{23} = b\theta_1 + c\theta_2$. If we put these into equation (4.1)

$$\theta_1 \wedge (a\theta_1 + b'\theta_2) + \theta_2 \wedge (b\theta_1 + c\theta_2) = b'\theta_1 \wedge \theta_2 + b\theta_2 \wedge \theta_1 = 0,$$

we can see that $b' = b$. Thus we have the expressions

$$\begin{aligned} \omega_{13} &= a\theta_1 + b\theta_2 \\ \omega_{23} &= b\theta_1 + c\theta_2 \end{aligned} \tag{4.2}$$

where a, b, c are functions of the chosen frame $E = (E_1, E_2, E_3)$.

Let E_1 and E_2 be the **principal directions** in which the normal curvatures (the curvatures obtained by cutting the surface by normal planes as explained on page 4) take their minimum and maximum values, $\frac{1}{R_1}$ and $\frac{1}{R_2}$. As before mentioned, the connection form ω_{23} states the initial rate as E_2 rotates toward E_3 when we move along a curve on M from a point p in the direction V. This direction is tangent to M and when $E_1 = V$ the frame (E_1, E_2, E_3) is called a **Darboux frame**. The Darboux frame is a forerunner to Cartan's moving frame named after Gaston Darboux. According to a theorem by Euler, the normal curvature in a direction making an angle φ with E_1 is

$$\frac{1}{R_1}\cos^2\varphi + \frac{1}{R_2}\sin^2\varphi.$$

If we rotate the frame around E_3 we obtain a new frame $E' = (E'_1, E'_2, E'_3)$ such that E'_1 makes an angle φ with E_1, E'_2 makes an angle $+\frac{\pi}{2}$ with E'_1 and $E'_3 = E_3$. The normal curvatures in the directions E'_1 and E'_2 are

$$\begin{aligned} a(E') &= \frac{1}{R_1}\cos^2\varphi + \frac{1}{R_2}\sin^2\varphi, \\ c(E') &= \frac{1}{R_1}\sin^2\varphi + \frac{1}{R_2}\cos^2\varphi. \end{aligned}$$

We define ω_{23} to be the **geodesic torsion**. In the principal directions the geodesic torsion is zero. The geodesic torsion in the direction of E_1' is denoted by $b(E')$. We can obtain this from the coefficient for E_2' in dE_3,

$$dE_3 = \omega_{31}' E_1' + \omega_{32}' E_2'.$$

We write E_1 and E_2 as linear combinations of E_1' and E_2',

$$\begin{aligned} E_1 &= \cos\varphi\, E_1' - \sin\varphi\, E_2' \\ E_2 &= \sin\varphi\, E_1' + \cos\varphi\, E_2' \end{aligned}$$

and put it into

$$\begin{aligned} dE_3 &= \omega_{31}\, E_1 + \omega_{32}\, E_2 \\ &= \omega_{31}(\cos\varphi\, E_1' - \sin\varphi\, E_2') + \omega_{32}(\sin\varphi\, E_1' + \cos\varphi\, E_2'). \end{aligned}$$

We group the coefficients for E_2' and use the linear combination of ω_{13} and ω_{23} from (4.2),

$$\begin{aligned} \omega_{32}' &= -\omega_{31}\sin\varphi + \omega_{32}\cos\varphi \\ &= \omega_{13}\sin\varphi - \omega_{23}\cos\varphi \\ &= (a\theta_1 + b\theta_2)\sin\varphi - (b\theta_1 + c\theta_2)\cos\varphi \\ &= (a\sin\varphi - b\cos\varphi)\theta_1 + (b\sin\varphi - c\cos\varphi)\theta_2. \end{aligned}$$

The duals θ_1 and θ_2 are

$$\theta_1(E_1') = \langle E_1', E_1\rangle = \cos\varphi, \quad \theta_2(E_1') = \langle E_1', E_2\rangle = \sin\varphi$$

and we have

$$\begin{aligned} \omega_{32}' &= (a\sin\varphi - b\cos\varphi)\cos\varphi + (b\sin\varphi - c\cos\varphi)\sin\varphi \\ &= (a-c)\sin\varphi\cos\varphi - b(\cos^2\varphi - \sin^2\varphi). \end{aligned}$$

Here a, b, c are functions of the frame E, where the geodesic torsion $b(E) = 0$ and $a(E) = \frac{1}{R_1}$, $c(E) = \frac{1}{R_2}$. Thus,

$$\omega_{23}' = (c-a)\sin\varphi\cos\varphi = \left(\frac{1}{R_2} - \frac{1}{R_1}\right)\sin\varphi\cos\varphi$$

and we have

$$b(E') = \left(\frac{1}{R_2} - \frac{1}{R_1}\right)\sin\varphi\cos\varphi.$$

Now we can calculate the wedge product of ω_{13} and ω_{23},

$$\begin{aligned} \omega_{13}\wedge\omega_{23} &= (a\,\theta_1 + b\,\theta_2)\wedge(b\,\theta_1 + c\,\theta_2) \\ &= ac\,\theta_1\wedge\theta_2 + b^2\,\theta_2\wedge\theta_1 \\ &= (ac - b^2)\,\theta_1\wedge\theta_2, \end{aligned}$$

and since $\omega_{13} \wedge \omega_{23} = -\omega_{31} \wedge -\omega_{32} = \omega_{31} \wedge \omega_{32}$, we see that $K = ac - b^2$, which is also

$$\begin{aligned}
ac - b^2 &= \left(\frac{1}{(R_1)^2} + \frac{1}{(R_2)^2}\right)\sin^2\varphi\cos^2\varphi + \frac{1}{R_1R_2}(\sin^4\varphi + \cos^4\varphi) - \\
&\quad - \left(\frac{1}{R_2} - \frac{1}{R_1}\right)^2 \sin^2\varphi\cos^2\varphi \\
&= \frac{1}{R_1R_2}(\sin^4\varphi + \cos^4\varphi + 2\sin^2\varphi\cos^2\varphi) \\
&= \frac{1}{R_1R_2}(\sin^2\varphi + \cos^2\varphi)^2 \\
&= \frac{1}{R_1R_2}.
\end{aligned}$$

A second expression for Gaussian curvature with differential forms

Our last expression for K will be obtained by first differentiating $dE_1 = \omega_{12}\,E_2 + \omega_{13}\,E_3$ a second time,

$$\begin{aligned}
0 &= d(dE_1) \\
&= d(\omega_{12}E_2 + \omega_{13}E_3) \\
&= (d\omega_{12})E_2 - \omega_{12} \wedge dE_2 + (d\omega_{13})E_3 - \omega_{13} \wedge dE_3 \\
&= (d\omega_{12})E_2 + (d\omega_{13})E_3 - \omega_{13} \wedge \omega_{32}\,E_2 - \omega_{12} \wedge \omega_{31}\,E_3\,.
\end{aligned}$$

The relation

$$d\omega_{12} = \omega_{13} \wedge \omega_{32}$$

from the coefficients for E_2, is a special case of **Cartan's second structural equation** of Euclidean space,

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}.$$

Since

$$\omega_{13} \wedge \omega_{32} = -\omega_{31} \wedge \omega_{32}$$

we can write our last equation for Gaussian curvature as

$$d\omega_{12} = -K\,\theta_1 \wedge \theta_2\,.$$

4.6 A short proof of the Remarkable Theorem

Theorema Egregium will now be proven in a third way, with differential forms. We begin by restating the theorem, where we will use the term isometry. An **isometry** is a distance preserving mapping, that is, one that does not stretch the surface.

Theorema Egregium 4.6.1. *If* f *is an isometry such that* $f : M \to M'$*, then* $K(p) = K'(f(p))$*, for all* $p \in M$*.*

What we are saying is that if we have a mapping that preserves distances then the Gaussian curvature remains the same.

In this proof we will use pushforward and pullback of forms and for that we will use a C^∞ map $f : M \to M'$. The **pushforward**

$$f_* : T_pM \to T_{f(p)}M'$$

takes a tangent vector originating at the point p on M and moves it - pushes it forward - so that we obtain a corresponding tangent vector at the point $f(p)$ on M'. The **pullback**

$$f^* : \Omega^k_{f(p)}(M') \to \Omega^k_p(M)$$

is a linear map that takes a C^∞ k-form defined in a neighbourhood of $f(p)$ on M' and moves it - pulls it back - so that we obtain a smooth k-form in a neighbourhood of p on M. The pushforward and the pullback are connected by

$$f^*\omega(V_1, \ldots, V_k) = \omega(f_*V_1, \ldots, f_*V_k)$$

for a k-form $\omega \in \Omega^k_{f(p)}(M')$ and $V_1, \ldots, V_k \in T_pM$. Properties of the pullback that we will need, with k-forms ω and θ, are

$$\begin{aligned} f^*(\omega \wedge \theta) &= (f^*\omega) \wedge (f^*\theta) \\ f^*(d\omega) &= d(f^*\omega) \\ f^*g &= g \circ f, \quad g \in \Omega^0_{f(p)}(M'). \end{aligned}$$

Proof of Theorema Egregium with differential forms. For a point p in M we choose a frame (E_1, E_2), that is tangent to M, on some neighbourhood of p. With pushforward we can obtain a corresponding tangent frame (E'_1, E'_2) around $f(p)$ in M' with

$$E'_i(f(p)) = f_*(E_i(p)), \quad i = 1, 2, \quad p \in M.$$

This will provide us with the ability to compare M and M'. We let θ_1, θ_2 be the duals of E_1, E_2 and θ'_1, θ'_2 be the corresponding duals of E'_1, E'_2 with

$$f^*\theta'_i(E_j) = \theta'_i(f_*E_j) = \theta'_i(E'_j) = \delta_{ij} = \theta_i(E_j).$$

Using pullback we can transfer the connection form ω'_{12} (which measures the rate of change as E'_1 rotates towards E'_2) from M' to M such that

$$f^*\omega'_{12} = \omega_{12}.$$

We will begin in M' with the formula for Gaussian curvature

$$d\omega'_{12} = -K'\theta'_1 \wedge \theta'_2.$$

To be able to compare Gaussian curvature on M and on M' we use pullback on $d\omega'_{12}$ to transfer it from M' to M,

$$\begin{aligned} f^*(d\omega'_{12}) &= f^*(-K'\theta'_1 \wedge \theta'_2) \\ &= -f^*(K')f^*(\theta'_1) \wedge f^*(\theta'_2) \\ &= -K'(f)\,\theta_1 \wedge \theta_2\,. \end{aligned}$$

Since $f^*(d\omega'{}_{12}) = d(f^*\omega'{}_{12}) = d\omega_{12}$ we have

$$d\omega_{12} = -K'(f)\,\theta_1 \wedge \theta_2\,.$$

Comparing this with

$$d\omega_{12} = -K\,\theta_1 \wedge \theta_2$$

we see that $K = K'(f)$, and especially $K(p) = K'(f(p))$. □

4.7 Integration of differential forms

There are two terms that are particularly useful when integrating differential forms. These are the notions of closed and exact forms.

Definition 4.7.1. A p-form ω is said to be **exact** if there is a $(p-1)$-form θ such that $\omega = d\theta$. If $d\omega = 0$ then ω is said to be **closed**.

Proposition 4.7.2. *An exact form is always closed, that is,* $d^2\omega = 0$.

Proof. The exterior differential is a linear operator, so it is sufficient to consider a p-form $\omega = f_I\,dx_I$ with exterior derivative

$$d\omega = (df_I) \wedge dx_I = \sum_j \frac{\partial f_I}{\partial x_j}\,dx_j \wedge dx_I\,.$$

It follows that

$$d^2\omega = d(d\omega) = d\left(\sum_j \frac{\partial f_I}{\partial x_j}\,dx_j \wedge dx_I\right) =$$

$$= \sum_j d\left(\frac{\partial f_I}{\partial x_j}\right) \wedge (dx_j \wedge dx_I) = \sum_{i,j} \frac{\partial^2 f_I}{\partial x_i \partial x_j}\,dx_i \wedge dx_j \wedge dx_I$$

This sum can be separated into three parts: $i < j$, $i = j$ and $i > j$. For $i = j$ the sum is zero, and since $\dfrac{\partial^2 f}{\partial x_i \partial x_j} = \dfrac{\partial^2 f}{\partial x_j \partial x_i}$ and $dx_i \wedge dx_j \wedge dx_I = -dx_j \wedge dx_i \wedge dx_I$ for $i \neq j$ we have

$$d^2\omega = \sum_{i<j} \left(\frac{\partial^2 f_I}{\partial x_i \partial x_j} - \frac{\partial^2 f_I}{\partial x_i \partial x_j}\right) dx_i \wedge dx_j \wedge dx_I = 0. \qquad \square$$

4.8 Vector analysis analogy

In vector analysis a scalar field ϕ is said to be a **potential** of a vector field F if F can be written as the gradient of ϕ,

$$F = \nabla\phi.$$

F is said to be a **vector potential** of a vector field G if

$$G = \nabla \times F.$$

These two can be compared with the analogous exact form. Analogues to the closed form are

$$\nabla \times F = 0$$

where F is said to be **irrotational**, or

$$\nabla \cdot F = 0$$

where F is said to be **solenoidal**. Just as $d^2 = 0$ for differential forms, an easy calculation shows that

$$\nabla \times (\nabla\phi) = 0$$

and

$$\nabla \cdot (\nabla \times F) = 0.$$

From this we can see that a vector field that has a potential is irrotational and a vector field with a vector potential is solenoidal. The converse though, that an irrotational vector field has a potential or a solenoidal field has a vector potential, is not true for all domains.

Example 4.8.1. Take for example the function

$$F(x, y, z) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0 \right)$$

in $\mathbb{R}^3$ with the z-axis removed. This function is irrotational, or closed, since $\nabla \times F = 0$. In order for this function to have a potential, or be exact, we will need a function $\phi = \phi(x, y, z)$ such that

$$F = \nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} \right).$$

To check whether this is true, we integrate F along a curve γ which goes around the origin in the x-y-plane.

$$\gamma = (r\cos\theta, r\sin\theta, 0), \quad 0 \le \theta < 2\pi, \quad r \text{ positive and fixed}$$

We change the coordinates of F to $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, such that

$$F = \left(\frac{-\sin\theta}{r}, \frac{\cos\theta}{r}, 0 \right).$$

Integrating F along γ yields

$$\begin{aligned} \int_\gamma F \, d\gamma &= \int_0^{2\pi} F \cdot \frac{d\gamma}{d\theta} \, d\theta = \int_0^{2\pi} F \cdot (-r\sin\theta, r\cos\theta, 0) \, d\theta = \\ &= \int_0^{2\pi} (\sin^2\theta + \cos^2\theta) \, d\theta = \int_0^{2\pi} d\theta = 2\pi. \end{aligned}$$

By the chain rule we have

$$\frac{d}{d\theta}\phi(\cos\theta, \sin\theta, z) = \frac{\partial\phi}{\partial x} \cdot \frac{dx}{d\theta} + \frac{\partial\phi}{\partial y} \cdot \frac{dy}{d\theta} + \frac{\partial\phi}{\partial z} \cdot \frac{dz}{d\theta} = \nabla\phi \cdot \frac{d\gamma}{d\theta},$$

and in the integration along γ, we exchange F for $\nabla\phi$,

$$\begin{aligned} \int_\gamma F \, d\gamma &= \int_0^{2\pi} F \cdot \frac{d\gamma}{d\theta} \, d\theta = \int_0^{2\pi} \nabla\phi \cdot \frac{d\gamma}{d\theta} \, d\theta = \\ &= \int_0^{2\pi} \frac{d\phi}{d\theta} \, d\theta = \int_0^{2\pi} d\phi = \phi(1,0,0) - \phi(1,0,0) = 0. \end{aligned}$$

The integration of F along γ have resulted in both 2π and 0, so the assumption that F has a potential ϕ is wrong.

Thus we see that in the punctured plane, an irrotational vector field does not always have a potential. This is also true for the analogous differential forms. But, in a star-shaped domain, all closed forms are exact.

Definition 4.8.2. A subset $U \subseteq \mathbb{R}^n$ is said to be **star-shaped** with respect to a point $x \in U$, if for each $y \in U$ the line segment

$$\lambda x + (1-\lambda)y, \quad 0 \le \lambda \le 1$$

is contained in U.

A first proof of a sufficient condition for a closed form to be exact was stated by Vito Volterra in 1889 [21], although it is more commonly named after Henri Poincaré.

Theorem 4.8.3 (Poincaré's lemma)**.** *Let U be an open and star-shaped subset of $\mathbb{R}^n$. For $p \ge 1$, if ω is a closed p-form in U then it is exact.*

4.9 Stokes's general theorem

A particularly useful theorem concerning integration on manifolds is Stokes's general theorem.

Theorem 4.9.1. *Let ω be a p-form on a $(p+1)$-dimensional orientable manifold M with p-dimensional boundary ∂M. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

This theorem has appeared in many different shapes. Classical examples are Gauss's, Green's and Stokes's theorems. In differential form notation we see them as a 2-form in three-dimensional space, a 1-form in two-dimensional space and a 1-form in three-dimensional space respectively.

Gauss's theorem

Considering the usefulness of these theorems in electromagnetic theory and other physical applications it is no wonder that they would be written in vector form. In an 1882-1883 paper, *The Relations between Magnetic Force and Electric Current*, Heaviside stated and proved Gauss's and Stokes's theorems [8, pp.163-164], [19]. Heaviside called Gauss's theorem the Theorem of Divergence.

We have a closed surface S forming a boundary of a volume V. With $A = (X, Y, Z)$ being a force, for example electric force, dS the surface element, dV the volume element and ε the angle between A and the outward normal n, we write

$$\iint A\cos\varepsilon\, dS = \iiint \left(\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}\right) dV\,.$$

The left hand integral is taken over the surface of V, and the right hand integral is taken over the volume. A more modern way of writing this is

$$\oiint_S A \cdot n\, dS = \iiint_V \nabla \cdot A\, dV\,.$$

We leave Heaviside for a while and turn to Gauss. In 1813 he stated and proved three special cases of the divergence theorem and it has therefore been named after him [15, 20].

Gauss begins his proof by considering a surface in space which bounds a solid body. We let P be a point in an infinitesimal surface element ds of the surface and PQ the exterior normal vector to the surface at P. The angles that PQ makes with the positive x-, y- and z-axes we denote by QX, QY and QZ. In the y-z-plane we consider an infinitesimal element $d\Sigma$. From $d\Sigma$ a cylinder is erected. This cylinder passes through the surface and an even number of surface elements $ds', ds'', ds''', \ldots$ is made. We want the relation between $d\Sigma$ in the y-z-plane and any ds on the surface and recognize that

$$d\Sigma = \pm ds' \cos QX' = \pm ds'' \cos QX'' = \pm ds''' \cos QX''' = \ldots$$

The sign is positive if the angle QX is acute and negative if the angle is obtuse. If the cylinder enters where the angle is obtuse, then it will exit where the angle is acute. Therefore,

$$d\Sigma = -ds' \cos QX' = +ds'' \cos QX'' = -ds'' \cos QX''' = \ldots$$

Summation leads to the first conclusion: if the integral $\int ds \cos QX$ is taken over the whole surface it is equal to zero. Secondly, if T, U, V are rational functions such that $T = T(y, z)$, $U = U(x, z)$ and $V = V(x, y)$, then

$$\int (T \cos QX + U \cos QY + V \cos QZ)\, ds = 0.$$

The third case is when we take cylinders of cross sectional area $d\Sigma$ and length x to approximate the volume of the body. It follows that the volume of the body is expressed by $\int ds.x \cos QX$ *(expression in Gauss's own notation)* over the whole surface.

A general proof of Gauss's theorem was given by the Russian Michael Ostrogradsky in 1826 [20] and the theorem is sometimes attributed to him.

Given a solid which is bounded by a surface, we let ε and ω be the surface element and volume element respectively. We denote the angles QX, QY, QZ by α, β, γ. We let a, b, c be constants and denote by p, q, r differentiable functions of x, y, z. The right hand integral is taken over a solid and the left hand integral over its boundary surface. We write the equation as

$$\int (ap \cos \alpha + bq \cos \beta + cr \cos \gamma)\varepsilon = \int \left(a\frac{\partial p}{\partial x} + b\frac{\partial q}{\partial y} + c\frac{\partial r}{\partial z} \right) \omega.$$

By letting $a = b = c = 1$ we can deduce Gauss's three special cases. We let $p = 1$ and $q = r = 0$ for the first case. In the second case we let $\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y} = \frac{\partial r}{\partial z} = 0$ and for the third case we let $p = x$ and $q = r = 0$.

Example 4.9.2. An example of Gauss's theorem is the flow of an electric field $E(r)$ through a closed surface S. This is proportional with constant k to the electric charge Q that is bounded by the surface,

$$\oiint_S E \cdot n \, dS = kQ.$$

Q is connected with the charge density $\rho(r)$ by the equation

$$Q = \iiint_V \rho \, dV$$

where V is the volume bounded by S. With Gauss's theorem we thus have

$$\oiint_S E \cdot n \, dS = \iiint_V \nabla \cdot E \, dV = k \iiint_V \rho \, dV$$
$$\Longleftrightarrow$$
$$\iiint_V (\nabla \cdot E - k\rho) \, dV = 0$$
$$\Longleftrightarrow$$
$$\nabla \cdot E = k\rho.$$

This leads us to Maxwell's equations.

Example 4.9.3. Maxwell's equations are often written in vector notation as

$$\begin{aligned}
\nabla \cdot E &= 4\pi\rho \\
\nabla \cdot B &= 0 \\
\nabla \times E &= -\frac{1}{c}\frac{\partial B}{\partial t} \\
\nabla \times B &= \frac{4\pi}{c}J + \frac{1}{c}\frac{\partial E}{\partial t}
\end{aligned}$$

where B is magnetic field, E electric field, J electric current density and ρ charge density. The constant c is the speed of light.

Remark 4.9.4. In order to fit with quaternion notation Maxwell defined that the **convergence** of E equals $-4\pi\rho$. This was changed in 1878 by William Kingdon Clifford in his *Elements of Dynamic*. There he used the negative of Maxwell's convergence and called it **divergence** [8, p.142].

In free space where $J = 0$ and $\rho = 0$, Maxwell's equations are reduced to

$$\nabla \cdot E = 0 \tag{4.3}$$
$$\nabla \cdot B = 0 \tag{4.4}$$
$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t} \tag{4.5}$$
$$\nabla \times B = \frac{1}{c}\frac{\partial E}{\partial t}. \tag{4.6}$$

With differential forms the equations can be reduced further. We begin by letting $B = (B_x, B_y, B_z)$, $E = (E_x, E_y, E_z)$ and construct the 2-form

$$\begin{aligned}
\omega = B_x \, dy \wedge dz + B_y \, dz \wedge dx + B_z \, dx \wedge dy + \\
+ (E_x \, dx + E_y \, dy + E_z \, dz) \wedge c \, dt \, .
\end{aligned}$$

With the convention that $B_{ij} = \frac{\partial B_i}{\partial j}$, $i, j = x, y, z, t$ we calculate $d\omega$,

$$
\begin{aligned}
d\omega &= d\big(B_x\, dy \wedge dz + B_y\, dz \wedge dx + B_z\, dx \wedge dy + \\
&\quad + (E_x\, dx + E_y\, dy + E_z\, dz) \wedge c\, dt\big) \\
&= (B_{xx} + B_{yy} + B_{zz})\, dx \wedge dy \wedge dz + \\
&\quad + \big((B_{xt} + cE_{zy} - cE_{yz})\, dy \wedge dz + \\
&\quad + (B_{yt} + cE_{xz} - cE_{zx})\, dz \wedge dx + \\
&\quad + (B_{zt} + cE_{yx} - cE_{xy})\, dx \wedge dy\big) \wedge dt\,.
\end{aligned}
$$

$B_{xx} + B_{yy} + B_{zz}$ in the above equation can be recognized from equation (4.4),

$$B_{xx} + B_{yy} + B_{zz} = \nabla \cdot B = 0.$$

The remaining part is a reformulation of (4.5),

$$\frac{\partial B}{\partial t} + \nabla \times cE = 0.$$

Hence,

$$d\omega = 0.$$

We continue by constructing another 2-form from B and E,

$$
\begin{aligned}
\theta &= -(B_x\, dx + B_y\, dy + B_z\, dz) \wedge c\, dt + \\
&\quad + E_x\, dy \wedge dz + E_y\, dz \wedge dx + E_z\, dx \wedge dy\,.
\end{aligned}
$$

The exterior derivative is

$$
\begin{aligned}
d\theta &= d\big(-(B_x\, dx + B_y\, dy + B_z\, dz) \wedge c\, dt + \\
&\quad + E_x\, dy \wedge dz + E_y\, dz \wedge dx + E_z\, dx \wedge dy\big) \\
&= (E_{xx} + E_{yy} + E_{zz})\, dx \wedge dy \wedge dz + \\
&\quad + \big((E_{xt} - cB_{zy} + cB_{yz})\, dy \wedge dz + \\
&\quad + (E_{yt} - cB_{xz} + cB_{zx})\, dz \wedge dx + \\
&\quad + (E_{zt} - cB_{yx} + cB_{xy})\, dx \wedge dy\big) \wedge dt\,.
\end{aligned}
$$

The above equation corresponds to (4.3) and (4.6) and we can conclude that

$$\nabla \cdot E + \frac{\partial E}{\partial t} - \nabla \times cB = 0.$$

This means that also

$$d\theta = 0.$$

The **Hodge star operator** $*$ is a linear operator such that given an orthonormal basis $e_1, \ldots, e_n$

$$*(e_1 \wedge \cdots \wedge e_p) = e_{p+1} \wedge \cdots \wedge e_n$$

with $0 \leq p \leq n$. Applying the Hodge star operator to ω will result in

$$\begin{aligned}
*\omega &= *\big(B_x\, dy \wedge dz + B_y\, dz \wedge dx + B_z\, dx \wedge dy + \\
&\quad + (E_x\, dx + E_y\, dy + E_z\, dz) \wedge c\, dt\big) \\
&= -(B_x\, dx + B_y\, dy + B_z\, dz) \wedge c\, dt + \\
&\quad + E_x\, dy \wedge dz + E_y\, dz \wedge dx + E_z\, dx \wedge dy \\
&= \theta.
\end{aligned}$$

Maxwell's equations can thus be expressed by two short equations,

$$\begin{aligned}
d\omega &= 0 \\
d{*}\omega &= 0.
\end{aligned}$$

Here we have calculated our way from expressions dependent on the coordinate system (x, y, z, t) to the two equations above that are independent of any coordinate system.

Stokes's theorem

We now turn to Stokes's theorem, referred to by Heaviside as the Theorem of Version [19].

We let a surface be bounded by a closed curve. A is any vector, for example representing magnetic force. A_1 is a vector (X_1, Y_1, Z_1) and for X_1 we have the relation

$$\left(\frac{dZ}{dy} - \frac{dY}{dz}\right) dy\, dz = X_1\, dy\, dz,$$

with analogous formulation for Y_1 and Z_1. The angle between A and the curve element dr is ε. The angle between A_1 and the outward normal of the surface element dS is ε_1. The equation states that the line integral of A equals the surface integral of A_1,

$$\int A \cos\varepsilon\, dr = \iint A_1 \cos\varepsilon_1\, dS\,.$$

Since $A_1 = \nabla \times A$, a more modern way of writing this is

$$\oint_C A \cdot d\mathbf{r} = \iint_S (\nabla \times A) \cdot n\, dS\,.$$

Stokes's theorem was first seen in print in 1854. It appeared as a question in the Smith's Prize Exam at Cambridge which George Stokes had set. He had recieved this equation in a letter four years earlier from William Thomson (also known as Lord Kelvin) [20].

We let dS be an element of a bounded surface and ds an element of its boundary line. The cosines of the angles QX, QY, QZ are denoted by

l, m, n and X, Y, Z are functions of x, y, z. The integrals are taken over the surface and its boundary respectively and the question was to prove that the following equality holds,

$$\int \left(X \frac{\partial x}{\partial s} + Y \frac{\partial y}{\partial s} + Z \frac{\partial z}{\partial s} \right) ds =$$
$$= \iint \left\{ l \left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z} \right) + m \left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x} \right) + n \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) \right\} dS \,.$$

By letting the surface lie in a plane and $Z = 0$ we arrive at Green's theorem,

$$\int \left(X \frac{\partial x}{\partial s} + Y \frac{\partial y}{\partial s} \right) ds = \iint \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) dS \,.$$

Here $n = 1$ since the normal of the plane is parallell to the z-axis. Green's theorem was proved by Riemann in his doctoral thesis of 1851 [20].

Gauss's, Green's and Stokes's theorems are all special cases of the general Stokes's theorem. This general theorem has been stated by many mathematicians and among them Poincaré. In 1899 he stated that for an r-dimensional manifold in n-dimensional space,

$$\int \sum A \, d\omega = \int \sum \sum_k \pm \frac{dA}{dx_k} \, dx_k \, d\omega \,.$$

The left hand integral is taken over the $(r-1)$-dimensional boundary of the manifold and the right hand integral is taken over the entire manifold. A is a function of n variables and $d\omega$ is a product of $(r-1)$ of the dx_i's with the sum being taken over all such distinct products.

Cartan presented the general Stokes's theorem in a course he held in Paris 1936-1937 [20]. For any oriented $(p+1)$-dimensional domain A with p-dimensional boundary C we have

$$\int_C \omega = \int_A d\omega.$$

Chapter 5

de Rham cohomology

5.1 Two theorems by de Rham

In 1895, an article named *Analysis situs* by Poincaré was published. The expression analysis situs is the old term for topology and this article can be considered a starting point for modern topology [11, p.15], [35, p.123]. With inspiration from Poincaré's ideas in Analysis situs, Cartan and the Swiss mathematician Georges de Rham some years later worked on a way to generalize Poincaré's lemma. In 1929 Cartan stated three conjectures which de Rham proved in his doctoral thesis *Sur l'analysis situs des variétés a n-dimension* in 1931 [21]. We will look at two of the theorems which are commonly referred to as *de Rham's theorems* and for that we need some knowledge of simplices and chains.

Definition 5.1.1. A k-**simplex** $[P_0P_1 \dots P_k]$ in Euclidean space, is a k-dimensional analogue of a triangle determined by $k+1$ points.

A 0-simplex is a point $[P_0]$, a 1-simplex is a directed line segment $[P_0P_1]$, a 2-simplex is a triangle $[P_0P_1P_2]$, a 3-simplex is a tetraeder $[P_0P_1P_2P_3]$ and so on. If two points are interchanged then the orientation of the simplex will change. For example, the 2-simplex $[P_0P_1]$ is the same as $-[P_1P_0]$. A k-simplex on a manifold M can be obtained from the C^∞ bijective image of a k-simplex in $\mathbb{R}^n$.

Definition 5.1.2. A k-**chain** C^k is a linear combination of k-simplices a_i^k with integer coefficients λ_i, such that

$$C^k = \sum \lambda_i a_i^k.$$

New k-chains can be created by adding and subtracting k-chains and multiplying them by integers.

The boundary of a point $[P_0]$ is the empty set, $\partial[P_0] = \varnothing$. The boundary of a directed line segment $[P_0P_1]$ is its end point subtracted by its starting point,

$$\partial[P_0P_1] = P_1 - P_0.$$

The boundary of the sum of two directed line segments $[P_0Q] + [QP_1]$ is

$$\partial([P_0, Q] + [Q, P_1]) = (Q - P_0) + (P_1 - Q) = P_1 - P_0.$$

The boundary of an oriented triangle $[P_0P_1P_2]$, a 2-chain, is the 1-chain consisting of the directed line segments on the boundary of that triangle. That is, if $[P_0P_1]$, $[P_1P_2]$ and $[P_2P_0]$ form the boundary of that triangle, then

$$\partial[P_0P_1P_2] = [P_0P_1] + [P_1P_2] + [P_2P_0].$$

A plus sign indicates that the direction of the line segment is consistent with the direction of the triangle and a minus sign indicates that the line segment and the triangle have opposite directions. Thus, with directed line segments $[P_0P_1]$, $[P_1P_2]$ and $[P_0P_2]$, the boundary will be

$$\partial[P_0P_1P_2] = [P_0P_1] + [P_1P_2] - [P_0P_2].$$

Definition 5.1.3. A k-chain C is the **boundary** of a $(k+1)$-chain D if $C = \partial D$. A **k-cycle** is a k-chain C with $\partial C = \varnothing$.

The boundary of a boundary is the empty set, $\partial \circ \partial = \varnothing$, as we can see in the above example with the boundary of a triangle,

$$\begin{aligned}
\partial(\partial[P_0P_1P_2]) &= \partial([P_0P_1] + [P_1P_2] + [P_2P_0]) = \\
&= (P_1 - P_0) + (P_2 - P_1) + (P_0 - P_2) = \\
&= \varnothing.
\end{aligned}$$

Boundaries and cycles have a neat analogue in the exact form $\omega = d\theta$ and the closed form $d\omega = 0$ for k and $(k-1)$-forms ω and θ. We can also compare $\partial \circ \partial = \varnothing$ with $d^2 = 0$.

Now, for some manifold D, we let $C_1, \ldots, C_m$ be cycles whose linear combination forms a boundary of that manifold, that is,

$$\sum_{i=1}^{m} a_i C_i = \partial D.$$

Remark 5.1.4. Poincaré defined this in terms of linear combinations of independent closed manifolds instead of cycles. If we say that each C_i is instead a manifold M_i and $a_i > 0$ in the above linear combination, then each a_iM_i is the sum of a_i distinct manifolds obtained from M_i by slight deformations [11, p.19].

The value of the integral of a form ω over the boundary of D is equal to the linear combination of the values of the integrals taken over the independent cycles C_i,

$$\int_{\partial D} \omega = \int_{\sum a_i C_i} \omega = \sum_{i=1}^{m} a_i \int_{C_i} \omega.$$

For a closed k-form ω and k-cycles C, Poincaré called these values **periods of the integral**, denoted by

$$\int_C \omega.$$

Stokes's theorem gives us that there are two cases when the period is zero: if ω is exact or if C is a boundary. If ω is exact, there is a θ such that $\omega = d\theta$ and

$$\int_C \omega = \int_C d\theta = \int_{\partial C} \theta = \int_{\varnothing} \theta = 0.$$

We state the first theorem remembering that Poincaré's lemma tells us that on a star-shaped domain in $\mathbb{R}^n$ a closed form is exact.

Theorem 5.1.5 (First theorem of de Rham)**.** *A closed form ω is exact if all of its periods are zero, that is if*

$$\int_C \omega = 0 \quad \textit{for all } C.$$

The second case with the period being zero, is when C is a boundary of D such that $C = \partial D$ and

$$\int_C \omega = \int_{\partial D} \omega = \int_D d\omega = \int_D 0 = 0.$$

This means that

$$\sum_{i=1}^{m} a_i \int_{C_i} \omega = 0.$$

Since we have m independent k-cycles C_i, the maximum number of linearly independent periods is m. Poincaré claimed that there would always be integrals for which we could obtain the maximum number of periods and this was proven by de Rham.

Theorem 5.1.6 (Second theorem of de Rham)**.** *Suppose that all independent k-cycles C_i on a manifold M are given distinct values,* $\mathrm{per}(C_i)$*, with the condition that when $\sum_i a_i C_i$ is a boundary then $\sum_i a_i \,\mathrm{per}(C_i) = 0$.*

Then there is a closed k-form ω on M which, integrated over each C_i, takes the given values

$$\int_{C_i} \omega = \mathrm{per}(C_i) \quad \textit{for all } C_i.$$

5.2 The de Rham cohomology

The content of de Rham's thesis was to be the starting point of the de Rham cohomology. However, it was not called cohomology until the Moscow conference of 1935 where the topic of cohomology was presented. Before that everything was defined using homology, which is alike but with chains instead of forms [21].

For a general manifold M, $\Omega^p(M)$ is the set of all p-forms on M. We let $B^p(M)$ be the set of all exact p-forms on M such that

$$B^p(M) = \{d\beta : \beta \in \Omega^{p-1}(M)\},$$

and $Z^p(M)$ the set of all closed p-forms on M such that

$$Z^p(M) = \{\alpha \in \Omega^p(M) : d\alpha = 0\}.$$

Definition 5.2.1. The pth **de Rham cohomology** of M is

$$H^p(M) = \frac{Z^p(M)}{B^p(M)}.$$

These quotient spaces turn out to represent certain topological aspects of the manifold. In a sense they can be thought of as measuring the number of p-dimensional holes in M that prevents it from being able to shrink to a point. For $H^0(M)$ this is seen as the number of connected components in M. If M has k connected components, then

$$H^0(M) = \mathbb{R}^k.$$

A star-shaped manifold M has one connected component and thus $H^0(M) = \mathbb{R}$. Two disjoint spheres S_1, S_2 will have $H^0(S_1 \cup S_2) = \mathbb{R}^2$.

$H^1(M)$ can be seen as the number of holes that prevents a circle from shrinking to a point. The punctured plane has one hole in the origin and therefore

$$H^1(\mathbb{R}^2 \backslash \{0\}) = \mathbb{R}.$$

The plane on the other hand, can be contracted to one point so

$$H^1(\mathbb{R}^2) = 0.$$

$H^2(M)$ measures the number of hollow cavities. A sphere, S^2, has exactly one hollow cavity and

$$H^2(S^2) = \mathbb{R}.$$

For an n-dimensional sphere the pth de Rham cohomology is

$$H^p(S^n) \cong \begin{cases} \mathbb{R} & p = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

In three dimensions we can use vector analysis to express the de Rham cohomology. When using differential forms we define the de Rham cohomology to be the quotient space of closed forms modulo exact forms. In the same way we can say that the first de Rham cohomology of an open subset $U \subseteq \mathbb{R}^3$ is the irrotational vector fields modulo the vector fields that has a potential,

$$H^1(U) = \frac{\{F \in C^\infty(U, \mathbb{R}^3) : \nabla \times F = 0\}}{\{\nabla\phi : \phi \in C^\infty(U, \mathbb{R})\}}.$$

The second de Rham cohomology of U is the solenoidal vector fields modulo the vector fields that has a vector potential,

$$H^2(U) = \frac{\{F \in C^\infty(U, \mathbb{R}^3) : \nabla \cdot F = 0\}}{\{\nabla \times F : F \in C^\infty(U, \mathbb{R}^3)\}}.$$

Following, H^0 is defined to be the scalar fields where the gradient equals zero

$$H^0(U) = \{\phi \in C^\infty(U, \mathbb{R}) : \nabla\phi = 0\}.$$

For the plane we know that $H^1(\mathbb{R}^2) = 0$ and this is the same as saying that all irrotational vector fields that has potential are conservative. That is, any line integral of F along a closed curve is zero. In the example preceding Poincaré's lemma we learned that in a plane with the origin removed, a line integral around the origin is not zero. The corresponding cohomology is as above, $H^1(\mathbb{R}^2 \backslash \{0\}) = \mathbb{R}$.

Chapter 6

Conclusion

A milestone for the differential geometry was Gauss's realization in the Theorema Egregium that Gaussian curvature is an intrinsic property. The knowledge that extrinsic coordinates were not necessary for the computation of the curvature, opened a way for the idea of doing calculations independent of an extrinsic coordinate system. This was at a time when the idea of a vectorial system was still new. Wessel had found a representation for directed line segments in the complex plane, and expressed them as $x + \varepsilon y$. Hamilton followed with his four-dimensional quaternions $w + ix + jy + kz$. He coined the term vector for the quaternion part $ix + jy + kz$. Gibbs and Heaviside then improved the theory of quaternions and this resulted in the vector analysis.

When Hamilton was busy with his quaternions, Grassmann developed the idea of doing computations with geometrical objects, which he called forms. One of the multiplications he defined was the exterior multiplication; this was explained by multiplying directed line segments producing oriented areas. The ideas of Grassmann were brought forward by Peano and Burali-Forti to Cartan, who realized that this kind of multiplication could be used on differential forms. The differential forms are in a sense duals of geometrical objects, and they have the advantage of being very convenient for working in a coordinate-free setting.

Riemann constructed the geometrical framework for a generalization of space which he called manifolds. Grassmann had made a similar construction, but his was not as general as Riemann's since it merely described Euclidean space. It can be very unnatural to perceive a manifold as being defined by means of an exterior coordinate system, but since each point in a manifold has a neighbourhood which locally resembles Euclidean space we can instead switch to an interior point of view. Differential forms are functions of one or several tangent vectors. These can be intrinsically defined. Since the differential forms only need intrinsic information, they are very convenient to use when we want to do calculations on manifolds.

We have seen that the calculus of differential forms have analogues in vector analysis. The exterior derivative of a 0-form, 1-form and 2-form corresponds to the gradient, curl and divergence. In vector analysis each kind of differentiation has to be treated differently, whereas with differential forms we only need the d-operator. There is a similar simplification in the scalar and vector multiplication, since the differential forms only have exterior multiplication. In three-dimensional space the exterior product of a 1-form and a 2-form corresponds to the scalar product, and the exterior product of two 1-forms corresponds to the vector product.

Vector analysis has the advantage that the use of a fixed coordinate system makes it easier for us to picture geometrically what happens when we do computations. The downside is that we have to use different kinds of differentiations and multiplications, and a wrong choice of coordinate system can lead to cumbersome calculations. However, as we saw in the proof of Theorema Egregium, the introduction of vector notation did simplify calculations. But, changing to differential forms can bring the simplification even further, and it has the advantage that we do not need to rely on any coordinate system.

Poincaré's lemma showed us that closed forms are exact in star-shaped domains. Cartan continued on this idea and stated conjectures which de Rham later proved. In his two theorems, de Rham used Stokes's theorem to create a link between chains and differential forms. This would later become a link that connects homology and cohomology.

There is an advantage of using cohomology over homology. When we construct a de Rham cohomology we use the exterior derivative. A homology is constructed in a similar way by using the boundary. To know the boundary of a manifold, we need to have certain information about the whole manifold, that is, we need global information. The exterior derivative only needs local information. Thus, changing from homology to cohomology means changing from using global information to using local information.

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