SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Higher algebraic K-theory for the category of algebraic varieties

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2010 - No 10

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Självständigt arbete i matematik 30 högskolepoäng, avancerad nivå

Handledare: Torsten Ekedahl

2010

Abstract. In this paper we follow the methodology of Waldhausen and apply his construction of K-groups of Waldhausen categories to a non-Waldhausen case, namely the category of algebraic varieties. We then deduce some results such as $K_i(\mathrm{Var}_{\mathbf{k}}) = K_i(\mathrm{Sch}_{\mathbf{k}})$ and that the groups are almost always non-trivial when $\mathbf{k}$ is a finite field.

Acknowledgements

I owe my supervisor, Professor Torsten Ekedahl, a big thanks. He has shown me how big machinery can be developed from small and, sometimes, trivial steps. This has had a great impact on how I view mathematics and it embraces the philosophy of Alexander Grothendieck; everything should be divided into steps such that each step seems obvious, but when combined they make up a great theory ([9]). The evolution of mathematics, so to speak.

Contents

1. Prerequisites

In this chapter we will introduce the necessary mathematics that will be used throughout this paper.

Let $\mathcal{C}$ be a category. By a *simplicial object* F in $\mathcal{C}$ we mean a contravariant functor $F\colon \Delta \to \mathcal{C}$ where Δ is the category of finite ordinals. To be more precise, Δ is the category whose objects are ordered sets $\underline{n} = \{0 < 1 < \cdots < n\}$, n a non-negative integer, and whose morphisms are non-decreasing maps. We will use the notation $F_n := F(\underline{n})$.

Examples:

(i) If $\mathcal{C} = \mathbf{Set}$, the category of all sets, then we call a simplicial object in $\mathcal{C}$ a *simplicial set* for short.

(ii) If $\mathcal{C} = \mathbf{Cat}$, the category of all small categories, then we call a simplicial object in $\mathcal{C}$ a *simplicial category*.

To every (small) category we can associate a simplicial set, its *nerve*, viz., given the category $\mathcal{C}$ we define its nerve, $\mathcal{NC}$, to be the simplicial set whose *n-simplices*, i.e. elements in the set $\mathcal{N}(\mathcal{C})_n$, are diagrams in $\mathcal{C}$:

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} X_n \ ,$$

$X_i \in \mathrm{Ob}(\mathcal{C})$ and $f_i \in \mathrm{Mor}(\mathcal{C})$.

Suppose we are given a simplicial category $F \in \mathbf{Cat}^{\Delta^{op}}$. Consider the composition

$$\Delta^{op} \xrightarrow{F} \mathbf{Cat} \xrightarrow{\mathcal{N}(-)} Set^{\Delta^{op}} \ .$$

This means that we have associated to F a *simplicial simplicial set*

$$\mathcal{N}F\colon = \mathcal{N}(-) \circ F\colon \Delta^{op} \to \mathbf{Set}^{\Delta^{op}},$$

i.e. $\mathcal{N}F \in (\mathbf{Set}^{\Delta^{op}})^{\Delta^{op}}$. However, there is a natural isomorphism $(\mathbf{Set}^{\Delta^{op}})^{\Delta^{op}} \cong \mathbf{Set}^{\Delta^{op}\times\Delta^{op}}$ (cf. [7]) and so we can see $\mathcal{N}F$ as a *bisimplicial set*. By restricting ourselves to the diagonal of $\Delta^{op}\times\Delta^{op}$ we finally get a simplicial set $\mathcal{N}_\Delta F = \mathcal{N}\circ F\circ\Delta\colon \Delta^{op} \to \mathbf{Set}$, with $\Delta\colon \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ the canonical functor.

Suppose we are given a simplicial set $F\colon \Delta^{op} \to \mathbf{Set}$. The *geometric realisation*, $|F|$, of F is the topological space defined as follows.

Definition 1.1. Given a morphism of ordered sets $g\colon \underline{n} \to \underline{m}$, we define

$$\begin{array}{ccc} g\colon \Delta_n & \to & \Delta_m \\ (x_0,\ldots,x_n) & \mapsto & (y_0,\ldots,y_m) \end{array},$$

where

$$y_i = \sum_{\substack{0\le j\le n \\ g(j)=i}} x_j.$$

Let

$$f\colon \coprod_{n\ge 0} F_n \times \Delta_n \to \left(\coprod_{n\ge 0} F_n \times \Delta_n\right) / \sim$$

be the canonical map. Equip F_n with the discrete topology. The standard n-simplex Δ_n is given the usual topology, namely, the subspace topology in $\mathbb{R}^{n+1}$. The relation $\sim$ is defined such that, for $(x,t) \in F_n \times \Delta_n$ and $(y,s) \in F_m \times \Delta_m$,

$$(x,t) \sim (y,s) \iff \exists\ g\colon \underline{n} \to \underline{m} \text{ s.t. } (x, g(t)) = (F(g)(y), s).$$

Using this construction we define $|F| = \left(\coprod_{n\geq 0} F_n \times \Delta_n\right) / \sim$ with the quotient topology under f.

By the geometric realisation of a simplicial category we mean the geometric realisation of the associated simplicial set. Given a (small) category $\mathcal{C}$, its *classifying space*, $\mathcal{B}(\mathcal{C})$, will be the geometric realisation of its nerve.

Definition 1.2. Let F and G be two simplicial sets. Then a *map of simplicial sets* $f\colon F \to G$ is a collection of maps $f = \{f_n\}_{n\geq 0}$, $f_n\colon F_n \to G_n$, such that the diagrams

$$\begin{tikzcd} F_n \arrow[r, "\partial_k^F"] \arrow[d, "f_n"'] & F_{n-1} \arrow[d, "f_{n-1}"] \\ G_n \arrow[r, "\partial_k^G"] & G_{n-1} \end{tikzcd}$$

and

$$\begin{tikzcd} F_{n-1} \arrow[r, "s_k"] \arrow[d, "f_{n-1}"'] & F_n \arrow[d, "f_n"] \\ G_{n-1} \arrow[r, "s_k"] & G_n \end{tikzcd}$$

commute, where, for $0 \leq k \leq n$, ∂_k^F and ∂_k^G are the k^{th} face maps and s_k^F, s_k^G are the k^{th} degeneracy maps.

One important property that we have is that maps between simplicial sets induce continuous functions between their geometric realisations. To see this, consider the map $f\colon F \to G$ of simplicial sets. We then have a map (cf. [8, 10, 13]):

$$\begin{array}{rcl} |f|\colon\ |F| & \to & |G| \\ (x,t) & \mapsto & (f(x),t). \end{array}$$

As usual, let F be a simplicial set. By F^{nd} we will mean the contravariant functor associated to F, whose simplices are non-degenerates, i.e.,

$$F_n^{\text{nd}}\colon = F_n \setminus \bigcup \text{Im}(s_i),$$

where $s_i\colon F_{n-1} \to F_n$, $0 \leq i \leq n-1$, are the degeneracy maps. Note that F^{nd} is not a simplicial set as the face of a non-degenerate simplex does not actually have to be non-degenerate. Consider the obvious sequence

$$\begin{tikzcd} \coprod F_n^{\text{nd}} \times \Delta_n \arrow[r, hook] \arrow[rr, bend right, "f"'] & \coprod F_n \times \Delta_n \arrow[r] & |F|\,, \end{tikzcd},$$

where f is the composition. Let $(x,t) \in |F|$. If x is non-degenerate, then we have that $(x,t) \in \coprod F_n^{\text{nd}} \times \Delta_n$. However, if x is a degenerate there exists a $y \in F_{n-1}$ such that $x = s_i(y)$ for some degenerate map s_i. But the topology of $|F|$ is such that $(x,t) = (s_i(y),t) = (y, s_i(t))$. If y is non-degenerate we have that $f(y, s_i(t)) = (x,t)$. If not, we continue this process which eventually must terminate. Therefore we end up with a non-degenerate after at most n steps. This means that f is surjective but also bijective on interiors.

The structure on $|F|$ is that of a famous topological one.

Definition 1.3. [6] A *cell complex* (or *CW complex*) is constructed as follows:

(i) Let X^0, the 0-*cell*, be a discrete set.
(ii) We build the *n-skeleton* by setting $X^n = X^{n-1} \coprod_\alpha D^n_\alpha / \sim$, where $x \sim \phi(x)$ for a continuous function $\phi\colon S^{n-1} \to X^{n-1}$ and $x \in \partial D^n_\alpha$.
(iii) Finally we let $X = \cup_n X^n$ be given the weak topology

Since we have a homeomorphism $\Delta_p \simeq D^p$, we can identify the non-degenerate p-simplices of F with the p-cells on an induced CW structure on $|F|$.

2. Waldhausen categories

The definition of the K-groups will heavily be influenced by Waldhausens construction. Good references are ([1, 17]).

Definition 2.1. A category with *cofibrations* is a category $\mathcal{C}$ with a zero object 0 and a collection of morphisms of $\mathcal{C}$, $co(\mathcal{C})$, such that

C1: $\mathrm{Iso}(\mathcal{C}) \subseteq co(\mathcal{C})$;
C2: $0 \to X$ is a cofibration, for all $X \in \mathrm{Ob}(\mathcal{C})$;
C3: If $A \to B$ is a cofibration and $A \to C$ is any morphism in $\mathcal{C}$, then the pushout $B \cup_A C$ exists and the map $C \to B \cup_A C$ is a cofibration.

A cofibration is usually symbolized with the arrow $\rightarrowtail$ and we will stick to this notation. We also have morphisms called *weak equivalences*, $w(\mathcal{C})$, denoted by $\xrightarrow{\sim}$ and defined to fulfill the following conditions:

W1: $\mathrm{Iso}(\mathcal{C}) \subseteq w(\mathcal{C})$;
W2: The composition of weak equivalences is a weak equivalence;
W3: If we have a diagram of the form

$$\begin{tikzcd} C \arrow[d, "\sim"] & A \arrow[l] \arrow[r, tail] \arrow[d, "\sim"] & B \arrow[d, "\sim"] \\ C' & A' \arrow[l] \arrow[r, tail] & B', \end{tikzcd}$$

then the induced map $B \cup_A C \to B' \cup_{A'} C'$ is a weak equivalence.

Note that we again require that pushouts exist. Furthermore, in this chapter the map $\xrightarrow{\sim}$ always denotes a weak equivalence. In other chapters this is not the case, it might just denote an isomorphism of groups or even a homotopy equivalence. This will be clear from the context.

Definition 2.2. A *Waldhausen category* is a category $\mathcal{C}$ with cofibrations and weak equivalences.

The category $\mathcal{C}$ will be a Waldhausen category throughout this chapter.

Let $A \rightarrowtail B$ be a cofibration and $A \to 0$ the unique map, then we can consider the pushout $B/A := 0 \cup_A B$. The pushout is only unique up to isomorphism, so we have different choices for B/A. The canonical morphism $B \to B/A$ will be denoted by the arrow $\twoheadrightarrow$. This allows us the form the following construction.

Definition 2.3. The category $S_n\mathcal{C}$, $n \geq 0$, has as objects triangles

$$\begin{tikzcd}
 & & & & A_{n-1,n} \\
 & & & & \vdots \arrow[u, two heads] \\
 & & A_{2,3} \arrow[r, tail] & \cdots \arrow[r, tail] & A_{2,n} \arrow[u, two heads] \\
 & A_{1,2} \arrow[r, tail] & A_{1,3} \arrow[u, two heads] \arrow[r, tail] & \cdots \arrow[r, tail] & A_{1,n} \arrow[u, two heads] \\
A_1 \arrow[r, tail] & A_2 \arrow[u, two heads] \arrow[r, tail] & A_3 \arrow[u, two heads] \arrow[r, tail] & \cdots \arrow[r, tail] & A_n, \arrow[u, two heads]
\end{tikzcd}$$

where $A_1, \ldots, A_n \in \mathrm{Ob}(\mathcal{C})$ and $A_{ij} := A_j/A_i$ is a pushout with a specific choice in mind. We also require the diagram to commute. A morphism in $S_n\mathcal{C}$ is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r, tail] \arrow[d] & A_1 \arrow[r, tail] \arrow[d] & \cdots \arrow[r, tail] & A_n \arrow[d] \\
0 \arrow[r, tail] & B_1 \arrow[r, tail] & \cdots \arrow[r, tail] & B_n.
\end{tikzcd}$$

We can define such a morphisms to be weak equivalences if all the vertical arrows are weak equivalences. Dito holds for cofibrations. The reason why we only need to do this on the bottom row of the triangles, instead on all the elements of the triangle, follows from the use of **W3**.

We have that $S_\bullet\mathcal{C}$ is a simplicial category. Note that $S_1\mathcal{C} = \mathcal{C}$ and that $S_0\mathcal{C} = 0$, where the latter equality is by convention. The face maps are the following ones.

Definition 2.4. The functor $\partial_0 \colon S_n\mathcal{C} \to S_{n-1}\mathcal{C}$, $n \geq 0$, is defined by removing the bottom row of our triangle. The face map $\partial_i \colon S_n\mathcal{C} \to S_{n-1}\mathcal{C}$, for $i = 1, \ldots, n$, is defined by removing the i^{th} row (counted such that the bottom row is 0 in the sequence) and removing the column containing A_i.

This can be stretched further and $S_\bullet\mathcal{C}$ is actually a simplicial Waldhausen category.

We now consider the subcategory $wS_n\mathcal{C} \subseteq S_n\mathcal{C}$ which has the additional condition that the morphisms are $w(S_n\mathcal{C})$; weak equivalences. This allows us to consider the geometric realisation $|wS_\bullet\mathcal{C}|$ and make the following crucial definition.

Definition 2.5. Let $\mathcal{C}$ be a small Waldhausen category. Then we can define its K-groups as

$$K_i(\mathcal{C}) := \pi_{i+1}|wS_\bullet\mathcal{C}|.$$

Indeed, this is the correct definition that generalizes Quillen's constructions. We have the following example which shows a striking resemblance with how the Grothendieck group (cf. Def. 4.1) is defined for the category of algebraic varieties.

Proposition 2.1. *The Grothendieck group $K_0(\mathcal{C})$ is generated by elements $[A]$ over $\mathbb{Z}$, with $A \in Ob(\mathcal{C})$, modulo the following relations*

(i) $[A] = [B]$ *if there exists a weak equivalence* $A \xrightarrow{\sim} B$*;*

(ii) *For every* ***cofibration sequence*** $A \rightarrowtail B \twoheadrightarrow B/A$*, we have* $[B] = [B/A] + [A]$.

Proof. See ([17, Prop. 8.4]). $\square$

3. The group K_n of the category of algebraic varieties

Throughout the text $\mathbf{k}$ will be a field. By $\mathrm{Var}_{\mathbf{k}}$ we mean the category of algebraic $\mathbf{k}$-varieties, i.e., the category whose objects are algebraic $\mathbf{k}$-varieties and whose morphisms are morphisms of algebraic $\mathbf{k}$-varieties. We define $S_n\mathrm{Var}_{\mathbf{k}}$, for $n \geq 0$, to be the category with *objects* equal to sequences of closed subvarieties:

$$\emptyset \subseteq X_1 \subseteq \cdots \subseteq X_{n-1} \subseteq X_n, \tag{1}$$

$X_i \in \mathrm{Var}_{\mathbf{k}}$, and *morphisms* are commutative diagrams

$$\begin{tikzcd} \emptyset \arrow[d, equal] & \subseteq & X_1 \arrow[d, "f_1"] & \subseteq & \cdots & \subseteq & X_n \arrow[d, "f_n"] \\ \emptyset & \subseteq & Y_1 & \subseteq & \cdots & \subseteq & Y_n \end{tikzcd}, \tag{2}$$

where $f_1, \ldots, f_n \in \mathrm{Mor}(\mathrm{Var}_{\mathbf{k}})$. An *isomorphism* is a morphism where each f_i, $i = 1, \ldots, n$, is an isomorphism in $\mathrm{Var}_{\mathbf{k}}$. We will call the morphisms we built the sequence with (in our case we used closed inclusions $\subseteq$) for *cofibrations**. The fact that this indeed is a category is obvious.

Let the *weak equivalences* be the isomorphisms of $S_n\mathrm{Var}_{\mathbf{k}}$. Now consider $wS_n\mathrm{Var}_{\mathbf{k}}$, the subcategory of $S_n\mathrm{Var}_{\mathbf{k}}$ where

$$\mathrm{Ob}(wS_n\mathrm{Var}_{\mathbf{k}}) = \mathrm{Ob}(S_n\mathrm{Var}_{\mathbf{k}})$$

and

$$\mathrm{Mor}(wS_n\mathrm{Var}_{\mathbf{k}}) = \mathrm{Iso}(S_n\mathrm{Var}_{\mathbf{k}}).$$

This means that we consider the subcategory whose morphisms are solely the weak equivalences. The new construction $wS_\bullet\mathrm{Var}_{\mathbf{k}}$ is also a simplicial category where a face map ∂_i applied to (1) removes the i^{th} element in the sequence. The face map ∂_0 removes the empty set but also subtracts X_1 from all the algebraic varieties in the sequence. That is,

$$\partial_0(\emptyset \subseteq X_1 \subseteq \cdots \subseteq X_n) = \emptyset \subseteq X_2 \setminus X_1 \subseteq \cdots \subset X_n \setminus X_1.$$

A degeneracy map s_i simply adds an identity morphism in the obvious place:

$$s_i(\emptyset \subseteq X_1 \subseteq \cdots \subseteq X_n) = \emptyset \subseteq X_1 \subseteq \cdots \subseteq X_{i-1} \subseteq X_i \subseteq X_i \subseteq X_{i+1} \subseteq \cdots \subseteq X_n.$$

By using the ideas we talked about previously we can apply the nerve pointwise on $wS_\bullet\mathrm{Var}_{\mathbf{k}}$ and then look at the diagonal of $\Delta^{op} \times \Delta^{op}$ in order to get a simplicial set. This allows us to talk about the simplicial set $\mathcal{N}_\Delta(wS_\bullet\mathrm{Var}_{\mathbf{k}})$ whose n-simplicies are diagrams of the form

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*We will use this terminology, which is inspired by Waldhausen, in order to ease things. We will later reuse the construction in this chapter with the slight modification of considering other categories, cofibrations and weak equivalences.

$$
\begin{array}{ccccccc}
\emptyset & \subseteq & X_1^0 & \subseteq & \cdots & \subseteq & X_n^0 \\
\| & & \downarrow\cong & & & & \downarrow\cong \\
\emptyset & \subseteq & X_1^1 & \subseteq & \cdots & \subseteq & X_n^1 \\
\| & & \downarrow\cong & & & & \downarrow\cong \\
\vdots & & \vdots & & \ddots & & \vdots \\
\| & & \downarrow\cong & & & & \downarrow\cong \\
\emptyset & \subseteq & X_1^n & \subseteq & \cdots & \subseteq & X_n^n.
\end{array}
$$

The face operator ∂_i, $1 \leq i \leq n$, simply just removes the i^{th} row and i^{th} column for $1 \leq i \leq n$, and ∂_0 maps the above diagram to

$$
\begin{array}{ccccccc}
\emptyset & \subseteq & X_2^1 \setminus X_1^1 & \subseteq & \cdots & \subseteq & X_n^1 \setminus X_1^1 \\
\| & & \downarrow\cong & & & & \downarrow\cong \\
\emptyset & \subseteq & X_2^2 \setminus X_1^2 & \subseteq & \cdots & \subseteq & X_n^2 \setminus X_1^2 \\
\| & & \downarrow\cong & & & & \downarrow\cong \\
\vdots & & \vdots & & \ddots & & \vdots \\
\| & & \downarrow\cong & & & & \downarrow\cong \\
\emptyset & \subseteq & X_2^n \setminus X_1^n & \subseteq & \cdots & \subseteq & X_n^n \setminus X_1^n
\end{array}
$$

The degeneracy map s_i, $0 \leq i \leq n$, simply adds a copy of the i^{th} row directly under the current i^{th} row, and in the same way adds a copy of the i^{th} column.

We are now ready to define our K-theory on the category of algebraic varieties. Take the geometric realisation of $\mathcal{N}_\Delta(wS_\bullet \mathrm{Var}_{\mathbf{k}})$ and call it $\mathcal{B}(wS_\bullet \mathrm{Var}_{\mathbf{k}})$ for short. Our K-groups are defined as

$$K_i(\mathrm{Var}_{\mathbf{k}}) = \pi_{i+1}(\mathcal{B}(wS_\bullet \mathrm{Var}_{\mathbf{k}}, \emptyset)),$$

for $i \geq 0$.

4. Justification for our K-theory

As it is now, we have simply written down a definition for our K-groups. We must also show that this definition, at degree 0, agrees with the *Grothendieck group* for algebraic varieties.

Recall the definition for the Grothendieck group.

Definition 4.1. The Grothendieck group on the category of algebraic varieties is the free abelian group on the objects of $\mathrm{Var}_{\mathbf{k}}$ modulo the following relations.

(i) If $A \cong B$, then $[A] = [B]$, and
(ii) if A is a closed subvariety of B, then $[B] = [B \setminus A] + [A]$.

To make this identification we must take a detour into the simplicial settings. We start with a general construction called the *simplicial fundamental group*, π_1^s, which is due to Gabriel-Zisman ([3])

Definition 4.2 (cf. [7])**.** A *precategory* consists of a set of *objects*, O, and a set of *arrows*, A. We also have have a pair of functions $A \rightrightarrows O$, ∂_0 and ∂_1. The *domain* of $f \in A$ is $\partial_0 f$ and its *codomain* is $\partial_1 f$.

A precategory is also sometimes called a *graph*; the objects are then called *vertices* and the arrows are called *edges*.

Definition 4.3. The *path category*, or the *free category*, of a precategory $A \rightrightarrows O$ is the category whose objects are O and morphisms $\mathrm{Hom}(a, \hat{a})$ are of the form

$$a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} a_n,$$

with $a_0, \dots, a_n \in O$, $f_1 \in A$ and $a = a_0, \hat{a} = a_n$.

Let F be a simplicial set and let $*$ be a 0-simplex.[†] We construct a precategory X_F by letting its objects equal to F_0, the arrows equal to F_1 and letting $\mathrm{A} \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} \mathrm{O}$ be the two face maps (cf. [4]). We then take the path category $\mathcal{P}(X_F)$ over our precategory X and consider the groupoid G_F (cf. page 33 in [3]). This means that we are dealing with sequences $(y_0, \dots, y_n)$, of arbitrary lengths, where y_i either equals an element x living in F_1, or equals a *formal* 1*-simplex* $\bar{x}$ such that

$$\partial_0 \bar{x} = \partial_1 x, \text{ and}$$
$$\partial_1 \bar{x} = \partial_0 x.$$

We also require that

$$\partial_0 y_0 = \partial_1 y_n = *, \text{ and} \tag{3}$$
$$\partial_0 y_i = \partial_1 y_{i-1}, \text{ for } 0 < i \leq n. \tag{4}$$

Composition of sequences is given by juxtaposition

$$(y_0, \dots, y_n) \times (y'_0, \dots, y'_m) = (y_0, \dots, y_n, y'_0, \dots, y'_m),$$

where we have the condition that $(y) \times (\bar{y}) = (\bar{y}) \times (y)$ is the identity.

[†] In our case the simplicial sets will only have a single 0-simplex.

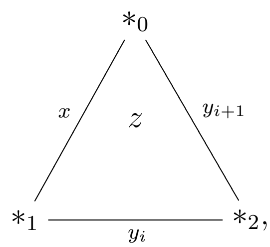
The conditions (3) and (4) will be trivial in our case because of the fact that we only have a single 0-simplex.

We also have the following relations. For all $* \in F_0$ and all $x \in F_2$:

(5) $$s_0 * = id_*, \text{ and}$$

(6) $$\partial_1 x = (\partial_0 x) \times (\partial_2 x).$$

That is, consider $(y_0, \ldots, y_i, y_{i+1}, \ldots, y_n)$ (with all $y_i \in F_1$) and the 2-simplex x



where $*_0, *_1, *_2$ are 0-simplices Then

$$(y_0, \ldots, y_i, y_{i+1}, \ldots, y_n) = (y_0, \ldots, y_{i-1}, x, y_{y+2}, \ldots, y_n),$$

where

$$\begin{aligned} \partial_0 z &= y_i; \\ \partial_1 z &= x; \\ \partial_2 z &= y_{i+1}. \end{aligned}$$

If $(\ldots, y_0, y_1, \ldots) = (\ldots, x, \ldots)$ we also set $(\ldots, \bar{y_0}, \bar{y_1}, \ldots) = (\ldots, \bar{x}, \ldots)$.

We can now construct $\pi_1^s(F, *)$, the simplicial fundamental group, to be the group which equals G_F but with the additional condition that we fix a 0-simplex $*$ and only consider sequences such that $\partial_0 y_0 = \partial_1 y_n = *$.

There is, for each 1-simplex y, a canonical path, $\hat{y}$, in $|F|$ given by

$$\begin{array}{rcl} \hat{y}\colon [0,1] = \Delta_1 & \to & |F| \\ t & \mapsto & (y, t) \end{array},$$

and for each $\bar{y}$ a path

$$\begin{array}{rcl} \hat{\bar{y}}\colon [0,1] = \Delta_1 & \to & |F| \\ t & \mapsto & (y, 1-t). \end{array}$$

Using the above construction one can show that the simplicial fundamental groupoid $\pi_1^s(F, *)$ is isomorphic to the "standard" fundamental group $\pi_1(|F|, *)$ where the isomorphism is given by

$$\begin{array}{rcl} \pi_1^s(F, *) & \to & \pi_1(|F|, *) \\ (y_0, \ldots, y_n) & \mapsto & \hat{y}_0 \boxdot \cdots \boxdot \hat{y}_n. \end{array}$$

The operator $\boxdot$ denotes the usual concatenation of paths in a topological space. See ([4]) for a proof of the isomorphism.

Consider the map

$$\begin{array}{rcl} \phi\colon \mathrm{Var}_{\mathbf{k}} & \to & \pi_1^s(wS_\bullet \mathrm{Var}_{\mathbf{k}}, \emptyset) \\ X & \mapsto & \langle X \rangle \end{array}$$

where $\langle X \rangle$ is the equivalence class of

$$\begin{tikzcd} \emptyset \arrow[d, equal] & \subseteq & X \arrow[d, equal] \\ \emptyset & \subseteq & X \end{tikzcd},$$

in $\pi_1^s(\mathrm{Var}_{\mathbf{k}}, \emptyset)$.

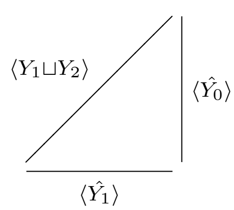
The results from (A.1) and (A.2) tell us that $\langle \emptyset \rangle$ acts like the identity element. By (A.3), (A.1), (A.2) and (A.4) we get that $X \cong Y \Rightarrow \langle X \rangle = \langle Y \rangle$. The second condition (4.1.ii) we needed to show was that $\langle X \rangle = \langle Y \rangle + \langle X \setminus Y \rangle$, when Y is a closed subvariety of X. We get this from (A.5). These properties also hold for the equivalence classes of the corresponding formal 1-simplices.

For ease of notation we will write $\pi_1^s(\mathrm{Var}_{\mathbf{k}})\colon = \pi_1^s(\mathrm{Var}_{\mathbf{k}}, *)$.

Theorem 4.1. *The set $\pi_1^s(Var_{\mathbf{k}})$ is an abelian group under the operation $\times$.*

Proof. Let $\langle Y_1 \rangle, \langle Y_2 \rangle \in \pi_1^s(\mathrm{Var}_{\mathbf{k}})$. We want to show that $\langle Y_1 \rangle \times \langle Y_2 \rangle = \langle Y_2 \rangle \times \langle Y_1 \rangle$. For this, consider the algebraic variety $Y_1 \sqcup Y_2$. It has two closed subvarieties $\hat{Y}_1, \hat{Y}_2 \subseteq Y_1 \sqcup Y_2$ such that $\hat{Y}_1 \cong Y_1$ and $\hat{Y}_1 \cong Y_1$. We also have that $Y_1 \sqcup Y_2 \setminus \hat{Y}_1 = \hat{Y}_2$ and $Y_1 \sqcup Y_2 \setminus \hat{Y}_2 = \hat{Y}_1$.

The theorem then readily follows from the following 2-simplex:



□

Note that the equivalence class of a formal 1-simplex, $\langle \bar{X} \rangle$, is the inverse of its associated 1-simplex $\langle X \rangle$.

Now we can consider the obvious homomorphism

$$\begin{array}{rcl} [\cdot]\colon \pi_1^s(\mathrm{Var}_{\mathbf{k}}) & \to & K_0(\mathrm{Var}_{\mathbf{k}}) \\ \langle X \rangle & \mapsto & [X] \end{array}$$

The equivalence class of a formal 1-simplex $\langle \bar{X} \rangle$ is mapped to $-[X]$.

We know that $\pi_1^s(wS_\bullet \mathrm{Var}_{\mathbf{k}},)$ is generated by the 1-simplices of the form $\langle X \rangle$. Furthermore we have shown that this is a well-defined map. Surjectivity follows easily as $K_0(\mathrm{Var}_{\mathbf{k}})$ is generated by elements of the form $[X]$. On the other hand we also have a map

$$\begin{array}{rcl} \langle \cdot \rangle \colon K_0(\mathrm{Var}_{\mathbf{k}}) & \to & \pi_1^s(\mathrm{Var}_{\mathbf{k}}) \\ [X] & \mapsto & \langle X \rangle \end{array}$$

that clearly is surjective. Composing the maps $[\cdot]$ and $\langle \cdot \rangle$ give us the identity maps which can be seen on the generators. This proves that our definition of the K-groups coincide with the Grothendieck group in degree 0.

5. Comma categories and some properties

Let $\mathcal{C}$ be a (small) category and X an object of $\mathcal{C}$.

Definition 5.1. The *comma category* $(\mathcal{C} \downarrow X)$ has as objects pairs (Y, f), or $f\colon Y \to X$, such that Y is an object of $\mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$. A morphism $g\colon (Y, f) \to (Y', f')$ is a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[dr, "f"'] & Y' \arrow[d, "f'"] \\ & X. \end{tikzcd}$$

Let X be an algebraic variety. We shall consider comma categories $\mathrm{Var}_X :\ = (\mathrm{Var}_{\mathbf{k}} \downarrow X)$ and their K-groups $K_i(\mathrm{Var}_X)$. These are defined in the obvious way: $\mathcal{N}_\Delta(wS_\bullet \mathrm{Var}_X)$ has as n-simplices

$$\begin{array}{ccccccccc} \emptyset & \subseteq & (X_1^0 \to X) & \subseteq & \cdots & \subseteq & (X_n^0 \to X) \\ \| & & \downarrow \cong & & & & \downarrow \cong \\ \vdots & & \vdots & & \ddots & & \vdots \\ \| & & \downarrow \cong & & & & \downarrow \cong \\ \emptyset & \subseteq & (X_1^n \to X) & \subseteq & \cdots & \subseteq & (X_n^n \to X), \end{array}$$

where $X_j^i \to X \in \mathrm{Var}_X$, for $0 \leq i \leq n, 1 \leq j \leq n$. We also want $X_j^i \subseteq X_{j+1}^i$, the cofibrations, and that $X_j^i \to X$ factors through $X_{j+1}^i \to X$ via the inclusion map. The weak equivalences are isomorphisms $X_j^i \to X_j^{i+1}$ such that we have a commutative diagram

$$\begin{tikzcd} X_j^i \arrow[rr] \arrow[dr] & & X_j^{i+1} \arrow[dl] \\ & X. & \end{tikzcd}$$

The rest is clear.

The category $\mathrm{Var}_{\mathbf{k}}$ has a final object, namely $\mathrm{Spec}(\mathbf{k})$, and therefore we have a natural isomorphism of the categories $\mathrm{Var}_{\mathbf{k}}$ and $\mathrm{Var}_{\mathrm{Spec}(\mathbf{k})}$. This is how this connects back to our theory.

Given a (covariant) functor $F\colon C \to C'$ of (small) categories, we have an induced function $BF\colon BC \to BC'$ (cf. [10]). To see this, note that we have an obvious map between the nerves of the categories, $F_N\colon NC \to NC'$, where the map is considered as a map of simplicial sets. The n-simplex

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n\ .$$

is mapped to

$$F_N(X_0) \xrightarrow{F_N f_1} F_N(X_1) \xrightarrow{F_N f_2} \cdots \xrightarrow{F_N f_n} F_N(X_n)\ .$$

This is the same for bisimplicial sets. Note that if we have functors

$$G \circ F \colon C \xrightarrow{F} C' \xrightarrow{G} C''$$

then $(G \circ F)_N = F_N \circ G_N$.

Next we assume that we are given a map of simplicial sets $F \colon X \to Y$. From it we get an induced map on the geometric realisations (cf. [13] chapter 2.1 and [8] §14)

$$\begin{array}{rccc} |F| \colon & |X| & \to & |Y| \\ & (x,t) & \mapsto & (F(x),t). \end{array}$$

Again note that if we have two maps of simplicial sets $F \colon X \to Y$ and $G \colon Y \to Z$, then $|G \circ F| = |G| \circ |F|$ since

$$|G \circ F|(x,t) = (G \circ F(x),t) = |G|(F(x),t) = |G| \circ |F|(x,t).$$

Furthermore, suppose that U, V, W are topological spaces and consider two continuous functions $F \colon U \to V$ and $G \colon V \to W$. Each map, say F, induces a homomorphism on the homotopy groups ($i > 0$)

$$\begin{array}{rccc} F_\pi \colon & \pi_i(U) & \to & \pi_i(V) \\ & \langle \alpha \rangle & \mapsto & \langle F \circ \alpha \rangle. \end{array}$$

Note once again that $(G \circ F)_\pi(\langle \alpha \rangle) = \langle G \circ F \circ \alpha \rangle = G_\pi(\langle F \circ \alpha \rangle) = G_\pi(F_\pi(\langle \alpha \rangle)) = (G_\pi \circ F_\pi)(\langle \alpha \rangle)$.

Rewinding back to our functor $F \colon C \to C'$, we see that we get an induced homomorphism $F_* \colon K_i(C) \to K_i(C')$ (assuming we can construct the K-groups in a relevant way). If we have another functor $G \colon C' \to C''$, then $(G \circ F)_* = G_* \circ F_* \colon K_i(C) \to K_i(C'')$.

Theorem 5.1. *Let $f \colon X \to X'$ be a morphism of algebraic varieties. Then we get an induced homomorphism $f_* \colon K_i(Var_X) \to K_i(Var_{X'})$. Moreover, if $g \colon X' \to X''$ is another morphism, then $(gf)_* = g_* \circ f_* \colon K_i(Var_X) \to K_i(Var_{X''})$.*

Proof. The morphism $f \colon X \to X'$ induces a functor $f_* \colon \mathrm{Var}_X \to \mathrm{Var}_{X'}$ on categories given by

$$\begin{tikzcd} Y \arrow[dd] & Y \arrow[d] \\ {} \arrow[r, mapsto] & X \arrow[d] \\ X & X'. \end{tikzcd}$$

A triangle

$$\begin{tikzcd} Y \arrow[r] \arrow[dr] & Y' \arrow[d] \\ & X \end{tikzcd}$$

gets mapped to

$$\begin{tikzcd} Y \arrow[r] \arrow[dr] & Y' \arrow[d] \\ & X \arrow[d] \\ & X'. \end{tikzcd}$$

Note that, given this definition of the induced map, the property that $(g \circ f)_* = g_* \circ f_* \colon \mathrm{Var}_X \to \mathrm{Var}_{X''}$ follows readily. The rest follows from the discussion above. $\square$

We also get a pullback.

Theorem 5.2. *Given the same maps as in the previous theorem, we get an induced homomorphism* $f^* \colon K_i(\mathit{Var}_{X'}) \to K_i(\mathit{Var}_X)$. *We also have that* $(g \circ f)^* = f^* \circ g^* \colon K_i(\mathit{Var}_{X''}) \to K_i(\mathit{Var}_X)$.

Proof. Define $f^* \colon \mathrm{Var}_{X'} \to \mathrm{Var}_X$ by

$$\begin{tikzcd} Y \arrow[d] \arrow[r, phantom, "\longmapsto"] & Y \times_{X'} X \arrow[d] \\ X' & X \end{tikzcd}$$

so that we have a Cartesian diagram

$$\begin{tikzcd} Y \times_{X'} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & X'. \end{tikzcd}$$

Calculations give us that, by applying $(g \circ f)^*$,

$$\begin{tikzcd} Y \arrow[d] \\ X'' \end{tikzcd}$$

is mapped to

$$\begin{tikzcd} Y \times_{X''} X \arrow[d] \\ X. \end{tikzcd}$$

On the other hand, if we first apply g^* we get

$$\begin{tikzcd} Y \times_{X''} X' \arrow[d] \\ X', \end{tikzcd}$$

and if we then apply f^* we get

$$\begin{tikzcd} (Y \times_{X''} X') \times_{X'} X \arrow[d] \\ X. \end{tikzcd}$$

But there is a natural isomorphism $(Y \times_{X''} X') \times_{X'} X \cong Y \times_{X''} X$, and so the theorem follows. $\square$

The proof is not as innocent as one might think. When applying a pullback we get choices of different Cartesian diagrams given by the universal definition of fibre products. Therefore $(f \circ g)^*$ and $f^* \circ g^*$ need not to be equal on a category theoretical level (as functors). The fix is easy: We first make a specific choice of Cartesian diagram when pulling back by g and then a specific choice when pulling back by f. We then select a specific Cartesian diagram when pulling back by $(f \circ g)$ such that we tautologically have an equality $(f \circ g)^* = f^* \circ g^*$. When we work over the K-groups the choices do not matter and we will always have an equivalence of homomorphisms. The reason for this is that natural isomorphisms between functors on the category of groups reduces to an equivalence of homomorphisms between groups.

Suppose we are given a morphism of varieties $f \colon U \to X$. We can use it to construct a functor $F_f \colon \mathrm{Var}_Y \to \mathrm{Var}_{X\times Y}$ given by

$$\begin{tikzcd} W \arrow[d] & U \times W \arrow[d] \\ Y \arrow[r, phantom, "\longmapsto"] & X \times Y. \end{tikzcd}$$

We know from the previous discussions that it induces a homomorphism

$$F_{f*} \colon K_i(\mathrm{Var}_Y) \to K_i(\mathrm{Var}_{X\times Y}).$$

The Grothendieck group $K_0(\mathrm{Var}_X)$ is generated by morphisms $U \to X$, with $U \in \mathrm{Var}_{\mathbf{k}}$, and so we get a homomorphism $K_0(\mathrm{Var}_X) \times K_i(\mathrm{Var}_Y) \to K_i(\mathrm{Var}_{X\times Y})$ that takes

$$\left(\begin{tikzcd} U \arrow[d, "f"'] \\ X \end{tikzcd}, x \right) \mapsto F_{f*}(x)$$

and is expanded linearly.

If $U \to X, V \to X$ are equal under the equivalence relations of the grothendieck group, i.e. if $[U \to X] = [V \to X] \in K_0(\mathrm{Var}_X)$, then clearly we must have that $[U \to X]$ and $[V \to X]$ induce the same homomorphism $K_i(\mathrm{Var}_Y) \to K_i(\mathrm{Var}_{X\times Y})$. The same thing goes for the second equivalence relation, i.e. Definition 4.1.ii. Therefore me must also make sure $K_0(\mathrm{Var}_X) \times K_i(\mathrm{Var}_Y) \to K_i(\mathrm{Var}_{X\times Y})$ obeys the equivalence relations of the Grothendieck group $K_0(\mathrm{Var}_X)$. We start off with the first equivalence relation, Definition 4.1.i. Consider the commutative diagram

$$\begin{tikzcd} U \arrow[r, "\cong"] \arrow[dr, "f"'] & U' \arrow[d, "f'"] \\ & X. \end{tikzcd}$$

We need to show that $F_{f'*} = F_{f*}$. Now, F_{f*} is the function induced by the map

$$\begin{tikzcd} W \arrow[dd, "g"'] \arrow[r, phantom, ""] & U \times W \arrow[dd, "f\times g"] \\ {} \arrow[r, mapsto] & {} \\ Y & X \times Y \end{tikzcd}$$

and $F_{f'*}$ is the function induced by the map

$$\begin{tikzcd} W \arrow[dd, "g"'] & U' \times W \arrow[dd, "f'\times g"] \\ {} \arrow[r, mapsto] & {} \\ Y & X \times Y. \end{tikzcd}$$

Note that since $U \cong U'$ we have that $U \times W \cong U' \times W$. We do have a strict equivalence if we pick a fix fibre product so that they all agree no matter what representative we choose of a class in $K_0(\mathrm{Var}_X)$. This gives us an equality of homomorphisms $F_{f*} = F_{f'*}$.

Let $U \subseteq V \in \mathrm{Var}_{\mathbf{k}}$ be a closed subvariety with $f\colon U \to X, g\colon V \setminus U \to X \in \mathrm{Var}_X$ such that there exists $h : V \to X \in \mathrm{Var}_X$ and we have the equivalences $f = h|_U, g = h|_{V\setminus U}$. We need to show that $F_{h*} = F_{f*} + F_{g*}$.

The homomorphism $F_{f*} + F_{g*}$ is induced by the composition of the following functors

$$\mathrm{Var}_Y \to \mathrm{Var}_{X\times Y} \times \mathrm{Var}_{X\times Y} \xrightarrow{\sqcup} \mathrm{Var}_{X\times Y}$$

where the first functor is given by

$$\begin{matrix} W \\ \downarrow \\ Y \end{matrix} \mapsto \left(\begin{matrix} U \times W & (V \setminus U) \times W \\ \downarrow & \downarrow \\ X \times Y, & X \times Y \end{matrix} \right).$$

In order to show that the induced homomorphism equals F_{h*}, we need to consider a new construction. Define $\mathrm{Var}_X^{\hookrightarrow}$ to be the category whose objects are commutative triangles

$$\begin{tikzcd} & X & \\ A \arrow[ur] \arrow[rr, hook] & & B, \arrow[ul] \end{tikzcd}$$

with $A \to X, B \to X \in \mathrm{Var}_X$, and morphisms are Cartesian diagrams

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & B \arrow[d] \\ C \arrow[r, hook] & D. \end{tikzcd}$$

We have two functors $\mathrm{Var}_X^{\hookrightarrow} \rightrightarrows \mathrm{Var}_X$ given by

$$\begin{tikzcd} A \arrow[rr, hook] \arrow[dr] & & B \arrow[dl] & & A \arrow[d] \\ & X & & \mapsto & X \end{tikzcd}$$

and

$$\begin{tikzcd} A \arrow[rr, hook] \arrow[dr] & & B \arrow[dl] & & (B \setminus A) \arrow[d] \\ & X & & \mapsto & X. \end{tikzcd}$$

We can construct the K-groups of $\mathrm{Var}_X^{\hookrightarrow}$ be letting cofibrations be Cartesian diagrams of the form

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, hook] & B \arrow[d, hook] \\ C \arrow[r, hook] & D \end{tikzcd}$$

and weak equivalences are Cartesian diagrams of the form

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "\simeq"] & B \arrow[d, "\simeq"] \\ C \arrow[r, hook] & D. \end{tikzcd}$$

We have the following additivity theorem

Theorem 5.3 ([2])**.** *The product of the two functors from above induces a homotopy equivalence*

$$\mathcal{B}(wS_\bullet \mathit{Var}_X^{\hookrightarrow}) \xrightarrow{\sim} B(wS_\bullet \mathit{Var}_X) \times B(wS_\bullet \mathit{Var}_X).$$

Let $U \to X, V \to X$ be the morphisms previously mentioned and consider the two functors $\mathrm{Var}_Y \rightrightarrows \mathrm{Var}_{X\times Y}^{\hookrightarrow}$ given by

$$\begin{tikzcd} h_1 \colon W \mapsto & U \times W \arrow[rr, hook] \arrow[dr] & & V \times W \arrow[dl] \\ & & X \times Y & \end{tikzcd}$$

and

$$\begin{tikzcd} h_2 \colon W \mapsto & U \times W \arrow[rr, hook] \arrow[dr] & & U \times W \sqcup (V \setminus U) \times W. \arrow[dl] \\ & & X \times Y & \end{tikzcd}$$

Consider the composition

$$\mathrm{Var}_Y \to \mathrm{Var}^{\hookrightarrow}_{X\times Y} \xrightarrow{\sim} \mathrm{Var}_{X\times Y} \times \mathrm{Var}_{X\times Y},$$

where the last map is the homotopy equivalence in the additivity theorem. This composition gives the same result no matter if the first functor is h_1 or h_2. By the additivity theorem (Theorem 5.3), the canonical maps $\mathcal{B}(h_1)$ and $\mathcal{B}(h_2)$ are homotopic and therefore induce the same maps on the K-groups.

If we now look at the functor $\mathrm{Var}^{\hookrightarrow}_{X\times Y} \to \mathrm{Var}_{X\times Y}$ given by

$$\begin{tikzcd} A \arrow[rr, hook] \arrow[dr] & & B \arrow[dl] \\ & X \times Y & \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} B \arrow[d] \\ X \times Y, \end{tikzcd}$$

we see that it induces the homomorphisms $F_{f*} + F_{g*}$ and F_{h*} when composed with h_1 and h_2, respectively. Since we already have seen that h_1 and h_2 are homotopic, we can conclude that the compositions induce the same homomorphisms, i.e., $F_{h*} = F_{f*} + F_{g*}$.

We now turn to a theorem on the interaction of the pullbacks and pushouts.

Theorem 5.4 (Base-change formula)**.** *Suppose we are given a Cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y. \end{tikzcd}$$

We then have $(f')_*(g')^* = g^* f_* \colon K_i(\mathit{Var}_X) \to K_i(\mathit{Var}_{X'})$.

Proof. As before, it is important to remember that we do not always have an equality of functors, but we do have an equality on the K-groups. Pick $U \to X \in \mathrm{Var}_X$. We have

$$g^* \circ f_*(U \to X) = g^*(U \to X \to Y) = U \times_Y Y' \to Y'.$$

We also have

$$(f')_* \circ (g')^*(U \to X) = (f')_*(U \times_X X' \to X') = U \times_X X' \to X' \to Y'.$$

But if we use the fact that the diagram above is a Cartesian diagram the result follows since

$$U \times_X X' = U \times_X (X \times_Y Y') \cong U \times_Y Y'.$$

This means that we have $U \times_X X' \cong U \times_Y Y'$ and we do get a strict equality if we make the correct choises of Cartesian diagrams, which we can. This concludes the proof. □

We will now slightly manipulate the product formula

$$\times \colon K_0(\mathrm{Var}_X) \times K_i(\mathrm{Var}_Y) \to K_i(\mathrm{Var}_{X\times Y}).$$

Let $Y = X$ and consider the diagonal morphism $\Delta \colon X \to X \times X$. This induces a homomorphisms $\Delta \colon K_i(\mathrm{Var}_X) \to K_i(\mathrm{Var}_{X\times X})$. There exists a canonical homomorphism

$\cdot\colon K_0(\mathrm{Var}_X) \times K_i(\mathrm{Var}_X) \to K_i(\mathrm{Var}_X)$ such that we have a commutative diagram

$$\begin{tikzcd}
K_0(\mathrm{Var}_X) \times K_i(\mathrm{Var}_X) \arrow[r] \arrow[dr] & K_i(\mathrm{Var}_X) \arrow[d, "\Delta"] \\
& K_i(\mathrm{Var}_{X\times X}),
\end{tikzcd}$$

where $K_0(\mathrm{Var}_X) \times K_i(\mathrm{Var}_X) \to K_i(\mathrm{Var}_{X\times X})$ is our previous product formula and is called the *exterior product*. The latter product $K_0(\mathrm{Var}_X) \times K_i(\mathrm{Var}_X) \to K_i(\mathrm{Var}_X)$ is called the *interior product*. The interior product is defined as follows. If $[U \to X] \in K_0(\mathrm{Var}_X)$, then we have a functor

$$\mathrm{Var}_X \to \mathrm{Var}_X$$

given by

$$\begin{tikzcd}
V \arrow[d] & U \times_X V \arrow[d] \\
X \arrow[r, phantom, "\longmapsto"] & X.
\end{tikzcd}$$

This functor induces a homomorphism $K_i(\mathrm{Var}_X) \to K_i(\mathrm{Var}_X)$ that is then, as before, expanded linearly. This definition gives us the following theorem.

Theorem 5.5 (Projection formula)**.** *Let $u \in K_0(Var_X)$, $v \in K_i(Var_Y)$ and $f\colon Y \to X$ be a morphism of varieties. Then $f_*(f^*u\cdot v) = u\cdot f_*v$, where we use the interior products.*

Proof. It is enough to show this on the level of categories because the result will then also be true for the induced homomorphisms on the K-groups. Let $u = g\colon U \to X \in K_0(\mathrm{Var}_X)$ and recall the definition $U\cdot = F_{g*}\colon K_i(\mathrm{Var}_X) \to K_i(\mathrm{Var}_X)$,

$$\begin{tikzcd}
W \arrow[d] & U \times_X W \arrow[d] \\
X \arrow[r, phantom, "\longmapsto"] & X.
\end{tikzcd}$$

We have the maps

$$f_*\colon \mathrm{Var}_Y \to \mathrm{Var}_X,$$

$$f^*\colon \mathrm{Var}_X \to \mathrm{Var}_Y$$

and

$$f^*U\cdot\colon \mathrm{Var}_Y \to \mathrm{Var}_Y,$$

where the latter is defined by

$$\begin{tikzcd} W \arrow[d] & U \times_X Y \times_Y W \arrow[d] \\ Y \arrow[r, phantom, "\longmapsto"] & Y. \end{tikzcd}$$

If we for each $V \to Y \in \mathrm{Var}_Y$ fix a fibre product we can simply write $f^*U \cdot (V \to Y)$ as $U \times_X W$ in an unambiguous way since $U \times_X Y \times_Y W \cong U \times_X W$.

We need to show that the diagram

$$\begin{tikzcd} \mathrm{Var}_Y \arrow[r, "f_*"] \arrow[d, "f^*U\cdot"'] & \mathrm{Var}_X \arrow[d, "U\cdot"] \\ \mathrm{Var}_Y \arrow[r, "f_*"] & \mathrm{Var}_X \end{tikzcd}$$

is commutative. Let $W \to Y \in \mathrm{Var}_Y$. Then

$$U \cdot (f_*(V \to Y)) = U \cdot (V \to X) = V \times_X U \to X.$$

On the other hand, going in the other direction gives

$$f_*(f^*U \cdot (V \to Y)) = f_*(U \times_X V \to Y) = U \times_X V \to X$$

and so these are equal, completing the proof. $\square$

6. A homotopy equivalence

Consider the two functors

$$\begin{array}{rcl} F\colon \mathrm{Var}_{\mathbf{k}} & \to & \mathrm{Sch}_{\mathbf{k}} \\ X & \mapsto & X \end{array}$$

and

$$\begin{array}{rcl} G\colon \mathrm{Sch}_{\mathbf{k}} & \to & \mathrm{Var}_{\mathbf{k}} \\ X & \mapsto & X_{\mathrm{red}} \end{array}.$$

Suppose that the composition $G \circ F$ is the identity on $\mathrm{Var}_{\mathbf{k}}$. Suppose also that we have a canonical morphism $F \circ G \hookrightarrow \mathrm{id}_{\mathrm{Sch}_{\mathbf{k}}}$ and we would like to show that this somehow gives us a homotopy equivalence between $K_i(F \circ G)$ and $K_i(\mathrm{id}_{\mathrm{Sch}_{\mathbf{k}}})$. This would mean that $K_i(\mathrm{Var}_{\mathbf{k}}) = K_i(\mathrm{Sch}_{\mathbf{k}})$ for all $i \geq 0$. Note that in order to construct $K_i(\mathrm{Sch}_{\mathbf{k}})$ we follow the usual construction method and let the cofibrations be inclusion of closed subschemes and weak equivalences are isomorphism of schemes.

Recall the following famous theorem (cf. [10, Prop. 2] and [11, Prop. 2.1]).

Theorem 6.1. *Let C, C' be categories, $S, T\colon C \to C'$ functors and $\varphi\colon S \to T$ a natural transformation. Then $\mathcal{B}(S) \sim \mathcal{B}(T)$ are homotopic.*

The proof is due to Segal.

Proof. Let Δ_1 be the category consisting of two objects, 0 and 1. Let the morphisms of Δ_1 be the following three: $0 \to 0$, $0 \to 1$ and $1 \to 1$. Regard φ as a functor $\varphi\colon C \times \Delta_1 \to C'$. This induces a morphism $\mathcal{B}(\varphi)\colon \mathcal{B}(C \times \Delta_1) \to C'$. But we do have a canonical homeomorphism $\mathcal{B}(C \times \Delta_1) \to B(C) \times \mathcal{B}(\Delta_1)$ ([10]) and $\mathcal{B}(\Delta_1) = I$, the unit interval. This shows that $\mathcal{B}(\varphi)$ is a homotopy between $\mathcal{B}(S)$ and $\mathcal{B}(T)$. □

Lets elaborate this proof. Define the functor $F \rightleftharpoons G\colon C \times \Delta_1 \to C'$ to take values $F \rightleftharpoons G(c, 0) = F(c)$ and $F \rightleftharpoons G(c, 1) = G(c)$. Note that we have canonical morphisms $F \rightleftharpoons G((c, 0) \to (c', 0)) = F(c) \to F(c')$ and $F \rightleftharpoons G((c, 1) \to (c', 1)) = G(c) \to G(c')$. Recall that for a morphism $f : c \to c'$ the natural transformation has the property that we get a commutative diagram

$$\begin{tikzcd} F(c) \arrow[r, "F(f)"] \arrow[d, "\varphi_C"'] & F(c') \arrow[d, "\varphi_{c'}"] \\ G(c) \arrow[r, "G(f)"'] & G(c'). \end{tikzcd}$$

Therefore we have that $F \rightleftharpoons G((c, 0) \to (c, 1)) = F(c) \xrightarrow{\varphi_c} G(c)$, $F \rightleftharpoons G((c', 0) \to (c', 1)) = F(c') \xrightarrow{\varphi_{c'}} G(c')$ and $F \rightleftharpoons G((c, 0) \to (c', 1)) = F(c) \to G(c')$, where the last arrow is the compositions $G(f) \circ \varphi_c = \varphi_{c'} \circ F(f)$.

Next we take the step towards the nerves. Note that the n-simplices of $\mathcal{N}(\Delta_1)_\bullet$ are of the form

$$\mathcal{N}(\Delta_1)_n : 0 \to 0 \to \cdots \to 0 \to 1 \to \cdots \to 1.$$

We will from now on assume the last 0 in the sequences is in the i^{th} place. The functor $F \rightleftharpoons G$ induces a map of simplicial sets given by

$$\begin{aligned}\mathcal{N}(F \rightleftharpoons G)(c_0 \to \cdots \to c_n, 0 \to \cdots \to 0 \to 1 \to \cdots \to 1) \\ = F(c_0) \to \cdots \to F(c_i) \to G(c_{i+1}) \to \cdots \to G(c_n).\end{aligned}$$

This, in turn, induces a homotopy on the geometric realisations

$$\mathcal{B}(F \rightleftharpoons G) \colon \mathcal{B}(C) \times I \to \mathcal{B}(C'),$$

giving a homotopy $\mathcal{B}(F) \sim \mathcal{B}(G)$.

Going back to our construction of $K_i(\mathrm{Sch}_{\mathbf{k}})$, we can see that the nerves are of the form

$$Y_0 \xrightarrow{\sim} \cdots \xrightarrow{\sim} Y_n,$$

where each $Y_i \in wS_j\mathrm{Sch}_{\mathbf{k}}$, for some j. The maps "$\xrightarrow{\sim}$" above are induced by isomorphisms of schemes. So if we would try to copy the previous construction of a homotopy equivalence we would get

$$\begin{aligned}\mathcal{N}(FG)(Y_0 \xrightarrow{\sim} \cdots \xrightarrow{\sim} Y_n, 0 \to \cdots \to 0 \to 1 \to \cdots \to 1) \\ = FG(Y_0) \xrightarrow{\sim} \cdots \xrightarrow{\sim} FG(Y_i) \xrightarrow{\varphi} Y_{i+1} \xrightarrow{\sim} \cdots \xrightarrow{\sim} Y_n \\ = (Y_0)_{\mathrm{red}} \xrightarrow{\sim} (Y_i)_{\mathrm{red}} \xrightarrow{\varphi} Y_{i+1} \xrightarrow{\sim} \cdots \xrightarrow{\sim} Y_n.\end{aligned}$$

The problem is that we want φ to be a weak equivalence when it in fact is a map induced by the cofibration

$$(Y_i)_{\mathrm{red}} \hookrightarrow Y_{i+1},$$

i.e., a map induced by inclusion of closed subschemes. This can be fixed but we need some more theory in order to do that.

When $X_{\bullet,\bullet}$ is a bisimplicial set we can talk about its nerve $|X_{\bullet,\bullet}|$, which by definition is a simplicial set and not a topological space. We do not want to give the exact definition but will instead use the following proposition which essentially says that it equals to $X_{\bullet,\bullet} \circ \Delta \colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ where $\Delta \colon \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ is the diagonal functor.

Lemma 6.2 ([4])**.** *Let $X_{\bullet,\bullet}$ be a bisimplicial set. Then $|X_{\bullet,\bullet}|$ and $X_{\bullet,\bullet} \circ \Delta$ are isomorphic.*

Theorem 6.3 ([16, Lem. 1.7] and [4, Prop. 1.7])**.** *Let $X_{\bullet,\bullet}$ and $Y_{\bullet,\bullet}$ be bisimplicial sets. If $f_{\bullet,\bullet} \colon X_{\bullet,\bullet} \to Y_{\bullet,\bullet}$ is a map such that $f_{k,\bullet} \colon X_{k,\bullet} \to Y_{k,\bullet}$ is a homotopy equivalence for all $k \geq 0$, then $|f_{\bullet,\bullet}| : |X_{\bullet,\bullet}| \to |Y_{\bullet,\bullet}|$ is a homotopy equivalence.*

Consider the bisimplicial set $\mathcal{N}_\bullet(wS_\bullet \mathrm{Sch}_{\mathbf{k}})$ and a *bisimplex of bidegree* (n,m):

$$\begin{array}{ccccccc}
\emptyset & \subseteq & X_1^0 & \subseteq & \cdots & \subseteq & X_n^0 \\
\Vert & & \downarrow\cong & & & & \downarrow\cong \\
\emptyset & \subseteq & X_1^1 & \subseteq & \cdots & \subseteq & X_n^1 \\
\Vert & & \downarrow\cong & & & & \downarrow\cong \\
\cdot & & \cdot & & & & \cdot \\
\cdot & & \cdot & & \ddots & & \cdot \\
\cdot & & \cdot & & & & \cdot \\
\Vert & & \downarrow\cong & & & & \downarrow\cong \\
\emptyset & \subseteq & X_1^m & \subseteq & \cdots & \subseteq & X_n^m.
\end{array}$$

In order to apply Theorem 6.3, we fix m and look at the simplicial set $\mathcal{N}_m(wS_\bullet \mathrm{Sch}_{\mathbf{k}})$. We are going to construct a homotopy equivalence $\varphi_\bullet\colon \mathcal{N}_m(wS_\bullet \mathrm{Sch}_{\mathbf{k}}) \times \mathcal{N}_\bullet(\Delta_1) \to \mathcal{N}_m(wS_\bullet \mathrm{Sch}_{\mathbf{k}})$ such that

$$\varphi_n|_{\mathcal{N}_m(wS_n \mathrm{Sch}_{\mathbf{k}})\times\{0\to\cdots\to 0\}} = \mathcal{N}(F\circ G)$$

and

$$\varphi_n|_{\mathcal{N}_m(wS_n \mathrm{Sch}_{\mathbf{k}})\times\{1\to\cdots\to 1\}} = \mathcal{N}(\mathrm{id}_{\mathrm{Sch}_{\mathbf{k}}}),$$

for all $n \geq 0$.

Consider the n-simplex S given by

$$\begin{array}{ccccccc}
\emptyset & \subseteq & X_1^0 & \subseteq & \cdots & \subseteq & X_n^0 \\
\Vert & & \downarrow\cong & & & & \downarrow\cong \\
\emptyset & \subseteq & X_1^1 & \subseteq & \cdots & \subseteq & X_n^1 \\
\Vert & & \downarrow\cong & & & & \downarrow\cong \\
\cdot & & \cdot & & & & \cdot \\
\cdot & & \cdot & & \ddots & & \cdot \\
\cdot & & \cdot & & & & \cdot \\
\Vert & & \downarrow\cong & & & & \downarrow\cong \\
\emptyset & \subseteq & X_1^m & \subseteq & \cdots & \subseteq & X_n^m.
\end{array}$$

Define φ_n by letting $\varphi_n(S, 0\to\cdots\to 0\to 1\to\cdots\to 1)$ equal to the n-simplex where we substitute all X_p^ℓ for $(X_p^\ell)_{\mathrm{red}}$ in S, $\ell = 0,\ldots,m$, whenever $p \leq i$. Note that, as usual, the last 0 in the sequence $0\to\cdots\to 1$ is on the i^{th} position.

We need to show that $\varphi_\bullet$ indeed is a simplicial map in order for us to have the desired homotopy. What we need to do is to show that $\varphi_\bullet$ commutes with the face and degeneracy maps (cf. Definition 1.2). When dealing with the degeneracy maps this is straightforward. Consider the degeneracy map σ_k with $k \leq i$. If we apply σ_k to $\varphi_n(S, 0 \to \cdots \to 1)$, the result is just obtained by adding a copy of the k^{th} row and column. This extra column will contain reduced schemes since $k \leq i$. On the other hand, if we first take $\sigma_k \times \sigma_k(S, 0 \to \cdots \to 1)$ we expand S with a copy of the k^{th} row and column. Since $k \leq i$ we also add an extra 0 to $0 \to \cdots \to 1$ and so again we take the reduced schemes of the first $(i+1)$ columns and get the same result as above. The case $k > i$ is just as easy to check. The difference is that we add an 1 to $0 \to \cdots \to 1$ and, on the other hand, also expand the column which we have not applied the reduced scheme operator on. This shows that $\varphi_\bullet$ commutes with the degeneracy maps.

The face map ∂_i, $i > 0$, are just as easy. We essentially just remove rows, columns and digits instead of adding new ones. The verification is straightforward and it all turn out well. The issue is with the face map ∂_0 since it does not only remove the columns, it also take complements. To see the issue, let S equal to

$$\begin{array}{ccccc}
\emptyset & \subseteq & X_1^0 & \subseteq & X_2^0 \\
\| & & \downarrow\cong & & \downarrow\cong \\
\emptyset & \subseteq & X_1^1 & \subseteq & X_2^1
\end{array}$$

and consider the sequence $0 \to 0 \to 1 \in \mathcal{N}_2(\Delta_1)$. If we first apply φ_2 we get

$$\begin{array}{ccccc}
\emptyset & \subseteq & (X_1^0)_{\text{red}} & \subseteq & X_2^0 \\
\| & & \downarrow\cong & & \downarrow\cong \\
\emptyset & \subseteq & (X_1^1)_{\text{red}} & \subseteq & X_2^1
\end{array}$$

and then, by applying ∂_0, get

$$\begin{array}{ccc}
\emptyset & \subseteq & X_2^0 \setminus (X_1^0)_{\text{red}} \\
\| & & \downarrow\cong \\
\emptyset & \subseteq & X_2^1 \setminus (X_1^1)_{\text{red}}.
\end{array}$$

On the other hand, if we first apply $\partial_0 \times \partial_0$ we get

$$\begin{array}{ccc}
\emptyset & \subseteq & X_2^0 \setminus X_1^0 \\
\| & & \downarrow\cong \\
\emptyset & \subseteq & X_2^1 \setminus X_1^1.
\end{array}$$

Then, by applying φ_1, we get

$$\begin{array}{ccc}
\emptyset & \subseteq & X_2^0 \setminus X_1^0 \\
\| & & \downarrow\cong \\
\emptyset & \subseteq & X_2^1 \setminus X_1^1.
\end{array}$$

The question is whether or not these are equal. Let X be a closed subscheme of Y. Since X and X_{red} are homeomorphic as topological spaces, the underlying spaces of $Y \setminus X$ and $Y \setminus (X)_{\mathrm{red}}$ will be equal and their structure sheaves will both be the restriction of $\mathcal{O}_Y$ to this space. Thus, the simplices above are equal.

If we instead look at $0 \to 0 \to 0$ we will face the simplices

$$\begin{tikzcd} \emptyset \arrow[d, equal] & \subseteq & (X_2^0 \setminus X_1^0)_{\mathrm{red}} \arrow[d, "\cong"] \\ \emptyset & \subseteq & (X_2^1 \setminus X_1^1)_{\mathrm{red}} \end{tikzcd}$$

and

$$\begin{tikzcd} \emptyset \arrow[d, equal] & \subseteq & (X_2^0)_{\mathrm{red}} \setminus (X_1^0)_{\mathrm{red}} \arrow[d, "\cong"] \\ \emptyset & \subseteq & (X_2^1)_{\mathrm{red}} \setminus (X_1^1)_{\mathrm{red}}. \end{tikzcd}$$

However, $(Y \setminus X)_{\mathrm{red}} = Y_{\mathrm{red}} \setminus X_{\mathrm{red}}$ and so these are also equal. These are only two special cases arising when considering the face map ∂_0. The rest will follow easily, making $\varphi_\bullet$ commute with the face map. Thus $\varphi_\bullet$ is a simplicial map. We can then, by using Theorem 6.3, conclude that $K_i(\mathrm{Var}_{\mathbf{k}}) = K_i(\mathrm{Sch}_{\mathbf{k}})$.

7. Finite sets, monoids and non-triviality

Let Set^f be the category whose objects are finite sets and morphisms are functions between (finite) sets. We can define the K-groups in the exact same way that we defined them for $\mathrm{Var}_{\mathbf{k}}$. The cofibrations are inclusions and the weak equivalences are bijections. Thus, the category $wS_n\mathrm{Set}^f$ will have as objects sequences of inclusions

$$\emptyset \subseteq S_1 \subseteq \cdots \subseteq S_n,$$

$S_i \in \mathrm{Set}^f$, for $i = 1, \ldots, n$, and morphisms will be commutative diagrams

$$\begin{array}{ccccccccc} \emptyset & \subset & S_1 & \subset & \cdots & \subset & S_n \\ \| & & \downarrow & & & & \downarrow \\ \emptyset & \subset & S_1' & \subset & \cdots & \subset & S_n', \end{array}$$

where the vertical maps are bijections.

The functor $wS_\bullet\mathrm{Set}^f$ is a simplicial category. We can take its nerve, the diagonal functor, the geometric realisation and finally the homotopy groups, just as in our construction of $K_i(\mathrm{Var}_{\mathbf{k}})$. This gives us the K-groups

$$K_i(\mathrm{Set}^f) := \pi_{i+1}(\mathcal{B}(wS_\bullet\mathrm{Set}^f)).$$

Assume that $\mathbf{k}$ is a finite field. We then have functors

$$\begin{array}{rcl} F\colon \mathrm{Set}^f & \to & \mathrm{Var}_{\mathbf{k}} \\ S & \mapsto & \bigsqcup_S \mathrm{Spec}\ \mathbf{k}, \end{array}$$

and

$$\begin{array}{rcl} G\colon \mathrm{Var}_{\mathbf{k}} & \to & \mathrm{Set}^f \\ X & \mapsto & X(\mathbf{k}). \end{array}$$

Define $S^* := \bigsqcup_S \mathrm{Spec}\ \mathbf{k}$. A sequence of subsets $S \subset T$ will be mapped to a sequence of closed subvarieties $S^* \subset T^*$. It is also obvious that a sequence of closed subvarieties will be mapped to a sequence of subsets via the latter functor. Note that the composition $\mathrm{Set}^f \to \mathrm{Var}_{\mathbf{k}} \to \mathrm{Set}^f$ is the identity. This means that $K_i(\mathrm{Set}^f)$ is a direct factor of $K_i(\mathrm{Var}_{\mathbf{k}})$, so in particular if $K_i(\mathrm{Set}^f) \neq 0$, then we must have $K_i(\mathrm{Var}_{\mathbf{k}}) \neq 0$.

Construct $\mathfrak{S}_n\mathrm{Set}^f$ to be the category whose objects are

$$\begin{array}{llllll} S_\bullet: & \emptyset \hookrightarrow & S_1 \hookrightarrow & S_2 \hookrightarrow & \cdots \hookrightarrow & S_n \\ & & \emptyset \hookrightarrow & S_2^1 \hookrightarrow & \cdots \hookrightarrow & S_n^1 \\ & & & & & \vdots \\ & & & & \emptyset \hookrightarrow & S_n^{n-1} \end{array}$$

with bijections $\phi_{mk}\colon S_k^m \to S_k \setminus S_m$. Note that the rows are sequencesof subsets (in spite of our use of arrows) and not just sequences of injections. Furthermore, we also want the following diagrams to commute

$$\begin{tikzcd} S^i_j \arrow[r, hook] \arrow[d, "\phi_{ij}"'] & S^i_{j+1} \arrow[d, "\phi_{i(j+1)}"] \\ S_j \setminus S_i \arrow[r, hook] & S_{j+1} \setminus S_i, \end{tikzcd}$$

for all $0 \leq i < j \leq n$. A morphism $F_\bullet \colon S_\bullet \to T_\bullet$ is a collection of morphisms $\{f^i_j\}_{0\leq i<j\leq n}$ with $f^i_j \colon S^i_j \to T^i_j$ a bijection of (small) sets. We furthermore want the following diagrams to commute

$$\begin{tikzcd} S^i_j \arrow[r, hook] \arrow[d, "f^i_j"] & S^i_{j+1} \arrow[d, "f^i_{j+1}"] \\ T^i_j \arrow[r, hook] & T^i_{j+1}. \end{tikzcd}$$

This makes $\mathfrak{S}_n\mathrm{Set}^f$ into a category.

We can define functors

$$\partial_i \colon \mathfrak{S}_n\mathrm{Set}^f \to \mathfrak{S}_{n-1}\mathrm{Set}^f$$

where ∂_0 removes the top row. The functor ∂_i, $1 \leq i \leq n$, removes the i^{th} column and $(i+1)^{\text{th}}$ row (counted downwards). Thus

$$\partial(S_\bullet)^k_l = \begin{cases} S^k_l, & l, k < i \\ S^{k+1}_{l+1}, & k \geq i \\ S^k_{l+1}, & k < i, l \geq i. \end{cases}$$

We also have functors

$$s_i \colon \mathfrak{S}_n\mathrm{Set}^f \to \mathfrak{S}_{n+1}\mathrm{Set}^f$$

where we add a copy of the i^{th} row so that this new copy is on the $(i+1)^{\text{th}}$ row and every row above it is given a new i^{th} column that is a copy of the previous i^{th} column.

Example 7.1. *If*

$$\begin{tikzcd} S_4\colon & \emptyset \arrow[r, hook] & S_1 \arrow[r, hook] & S_2 \arrow[r, hook] & S_3 \arrow[r, hook] & S_4 \\ & & \emptyset \arrow[r, hook] & S^1_2 \arrow[r, hook] & S^1_3 \arrow[r, hook] & S^1_4 \\ & & & \emptyset \arrow[r, hook] & S^2_3 \arrow[r, hook] & S^2_4 \\ & & & & \emptyset \arrow[r, hook] & S^3_4, \end{tikzcd}$$

then

$$\begin{tikzcd}[column sep=small]
s_2(S_4)\colon & \emptyset \arrow[r, hook] & S_1 \arrow[r, hook] & S_2 \arrow[r, hook] & S_2 \arrow[r, hook] & S_3 \arrow[r, hook] & S_4 \\
& & \emptyset \arrow[r, hook] & S_2^1 \arrow[r, hook] & S_2^1 \arrow[r, hook] & S_3^1 \arrow[r, hook] & S_4^1 \\
& & & \emptyset \arrow[r, hook] & \emptyset \arrow[r, hook] & S_3^2 \arrow[r, hook] & S_4^2 \\
& & & & \emptyset \arrow[r, hook] & S_3^2 \arrow[r, hook] & S_4^2 \\
& & & & & \emptyset \arrow[r, hook] & S_4^3.
\end{tikzcd}$$

It is easy to verify that this makes $\mathfrak{S}_\bullet\mathrm{Set}^f$ into a simplicial category.

Notice that we have a canonical simplicial map

$$F_\bullet\colon S_\bullet\mathrm{Set}^f \to \mathfrak{S}_\bullet\mathrm{Set}^f.$$

The map F_n just takes a sequence

$$\emptyset \hookrightarrow S_1 \hookrightarrow \ldots \hookrightarrow S_n$$

to the triangle with $S_j^i := S_j \setminus S_i$. On each level, $n \geq 0$, we also have a map

$$G_n\colon \mathfrak{S}_n\mathrm{Set}^f \to S_n\mathrm{Set}^f$$

which simply forgets everything in the triangle but the first row. This gives us the property that $G_n \circ F_n = \mathrm{id}_{S_n\mathrm{Set}^f}$. It is important to note that the canonical function $G_\bullet\colon \mathfrak{S}_\bullet\mathrm{Set}^f \to S_\bullet\mathrm{Set}^f$ is not a simplicial map, that is, it does not satisfy Definition 1.2. The map $F_\bullet$ is, however, a simplicial map. The problem with the former function has to do with the fact that $\partial_0 \circ G_n \neq G_{n-1} \circ \partial_0$.

Construct the K-groups of $\mathfrak{S}_\bullet\mathrm{Set}^f$ in the way we did it for $S_\bullet\mathrm{Set}^f$ and call them $K_i(\mathfrak{Set}^f)$.

Theorem 7.2. *Given the constructions above,*

$$K_i(\mathfrak{Set}^f) \cong K_i(Set^f).$$

Proof. Given the claim that on each level G_n and F_n are adjoint functors, by Theorem 6.3 the result follow. To prove the claim we need to show that

$$\mathrm{Hom}_{S_n\mathrm{Set}^f}(A, G_nB) = \mathrm{Hom}_{\mathfrak{S}_n\mathrm{Set}^f}(F_nA, B),$$

for all $A \in S_n\mathrm{Set}^f$ and $B \in \mathfrak{S}_n\mathrm{Set}^f$. Suppose we are given $f\colon F_nA \to B$, then we have a canonical map $\hat{f}\colon A \to G_nB$ given by only looking at the map f restricted to the first row.

Going in the other direction. Suppose we are given $f\colon A \to G_nB$, where

$$A : \emptyset \hookrightarrow T_1 \hookrightarrow \ldots \hookrightarrow T_n$$

and

$$B:\ \emptyset \hookrightarrow S_1 \hookrightarrow \cdots \hookrightarrow S_n$$

$$\vdots$$

$$\emptyset \hookrightarrow S_n^{n-1}.$$

We want to define a map $\hat{f}\colon F_nA \to B$. On the first row there is the obvious choice; $\hat{f}_j^0 := f_j$. For $\hat{f}_j^i\colon T_j \setminus T_i \to S_j^i$ we proceed as follows. Remember that we have a commutative diagram

$$\begin{tikzcd} S_j^i \arrow[r, hook] \arrow[d, "\phi_{ij}"'] & S_{j+1}^i \arrow[d, "\phi_{i(j+1)}"] \\ S_j \setminus S_i \arrow[r, hook] & S_{j+1} \setminus S_i. \end{tikzcd}$$

The obvious definition is then $\hat{f}_j^i := \phi_{ij}^{-1} \circ f_j|_{T_j \setminus T_i}$. This shows that the functors are adjoint. $\square$

Consider the category Σ of finite ordinals. We have a monoid structure on Σ given by

$$\begin{array}{ccc} \Sigma \times \Sigma & \to & \Sigma \\ (m,n) & \mapsto & m+n. \end{array}$$

We can take its classifying space and construct its K-groups $K_i(\Sigma)$. We can also use the method of constructing triangles

$$0 \hookrightarrow n_1 \hookrightarrow n_1+n_2 \hookrightarrow n_1+n_2+n_3 \hookrightarrow \cdots \hookrightarrow n_1+\ldots n_m$$

$$0 \hookrightarrow n_2^1 \hookrightarrow n_2^1+n_3^1 \hookrightarrow \cdots \hookrightarrow n_2^1+\cdots+n_m^1$$

$$\vdots$$

$$0 \hookrightarrow n_m^{m-1},$$

with bijections $n_l^k \to (n_1+\ldots n_l) \setminus (n_1+\cdots+n_k)$ that induce commutative diagrams

$$\begin{tikzcd} n_{i+1}^i+\cdots+n_j^i \arrow[r, hook] \arrow[d] & n_{i+1}^i+\cdots+n_{j+1}^i \arrow[d] \\ (n_1+\cdots+n_j) \setminus (n_1+\cdots+n_i) \arrow[r, hook] & (n_1+\cdots+n_{j+1}) \setminus (n_1+\cdots+n_i). \end{tikzcd}$$

It is important to note that $(n_1+n_2) \setminus n_1 \neq n_2$, but we do have a natural bijection between the two sets.

This triangle is a simplicial category for exactly the same reason as the triangle constructed out of Set^f was. Call the K-groups for $K_i(\Sigma^T)$ [‡]. Once again we have maps

$$F\colon S_\bullet\Sigma \to \mathfrak{S}_\bullet\Sigma$$

and on each level we have a functor

$$G_n\colon \mathfrak{S}_n\Sigma \to S_n\Sigma,$$

where G_n forgets everything but the first row. The map F_m takes

$$n_1 \hookrightarrow n_1 + n_2 \hookrightarrow \ldots \hookrightarrow n_1 + \cdots + n_m$$

to

$$\begin{array}{ccccccccccc}
0 & \hookrightarrow & n_1 & \hookrightarrow & n_1 + n_2 & \hookrightarrow & n_1 + n_2 + n_3 & \hookrightarrow & \cdots & \hookrightarrow & n_1 + \cdots + n_m \\
 & & 0 & \hookrightarrow & n_2 & \hookrightarrow & n_2 + n_3 & \hookrightarrow & \cdots & \hookrightarrow & n_2 + \ldots n_m \\
 & & & & 0 & \hookrightarrow & n_3 & \hookrightarrow & \cdots & \hookrightarrow & n_3 + \cdots + n_m \\
 & & & & & & & & & & \vdots \\
 & & & & & & & & 0 & \hookrightarrow & r_m.
\end{array}$$

Again, $F_\bullet$ is a simplicial map but there is no way of making the functors G_i to a simplicial map. For the same reason as before, this does not matter and we get that

$$K_i(\Sigma) \cong K_i(\Sigma^T).$$

The category Σ is actually a subcategory of Set^f as we can view ordinals as sets.

$$\begin{aligned} 0 &= \emptyset \\ 1 &= \{0\} \\ n+1 &= n \cup \{n\}. \end{aligned}$$

This gives a simplicial map

$$f_\bullet\colon \mathfrak{S}_\bullet\Sigma \to \mathfrak{S}_\bullet\mathrm{Set}^f.$$

On each level we have a functor

$$g_n\colon \mathfrak{S}_n\mathrm{Set}^f \to \mathfrak{S}_n\Sigma,$$

induced by the functor that takes a set to its cardinality. By composing these functors level-wise we will always have that for every $n \geq 0$, $g_n \circ f_n = id_{\mathfrak{S}_n\Sigma}$ and so $f_\bullet$ is an equivalence.

It is a well known fact that $\mathcal{B}(\Sigma)$ equals to the sphere-spectrum (cf. [12]). This means that $K_i(\Sigma) = \pi_i^s(S^0)$, where the s means that we consider the stable homotopy groups. These groups are known to be non-trivial for arbitrarily large i. We conclude this chapter with the following theorem that sums the chapter up.

[‡]T as in Triangle.

Theorem 7.3. *For $\boldsymbol{k}$ a finite field, the algebraic K-groups of $Var_{\boldsymbol{k}}$ are non-trivial for an infinite number of indices.*

Appendix A. Some relations in π_1^s

1. If X and Y are isomorphic:

$$\left(\begin{array}{ccccccc} \emptyset & \subseteq & \emptyset & \emptyset & \subseteq & Y \\ \| & & \| & \| & & \downarrow\cong \\ \emptyset & \subseteq & \emptyset \,, & \emptyset & \subseteq & X \end{array}\right) \sim \left(\begin{array}{ccc} \emptyset & \subseteq & Y \\ \| & & \downarrow\cong \\ \emptyset & \subseteq & X \end{array}\right)$$

via the 2-simplex

$$\begin{array}{ccccc} \emptyset & \subseteq & Y & \subseteq & Y \\ \| & & \| & & \| \\ \emptyset & \subseteq & Y & \subseteq & Y \\ \| & & \| & & \| \\ \emptyset & \subseteq & X & \subseteq & X. \end{array}$$

2. If X and Y are isomorphic:

$$\left(\begin{array}{cccccc} \emptyset & \subseteq & Y & \emptyset & \subseteq & \emptyset \\ \| & & \downarrow\cong & \| & & \| \\ \emptyset & \subseteq & X \,, & \emptyset & \subseteq & \emptyset \end{array}\right) \sim \left(\begin{array}{ccc} \emptyset & \subseteq & Y \\ \| & & \downarrow\cong \\ \emptyset & \subseteq & X \end{array}\right)$$

via the 2-simplex

$$\begin{array}{ccccc} \emptyset & \subseteq & \emptyset & \subseteq & Y \\ \| & & \| & & \| \\ \emptyset & \subseteq & \emptyset & \subseteq & Y \\ \| & & \| & & \| \\ \emptyset & \subseteq & \emptyset & \subseteq & X. \end{array}$$

3. If X and Y are isomorphic:

$$\left(\begin{array}{cccccc} \emptyset & \subseteq & \emptyset & \emptyset & \subseteq & Y \\ \| & & \| & \| & & \downarrow\cong \\ \emptyset & \subseteq & \emptyset \,, & \emptyset & \subseteq & X \end{array}\right) \sim \left(\begin{array}{ccc} \emptyset & \subseteq & Y \\ \| & & \downarrow\cong \\ \emptyset & \subseteq & Y \end{array}\right)$$

via the 2-simplex

$$
\begin{tikzcd}
\emptyset \arrow[r, phantom, "\subseteq"] \arrow[d, equal] & Y \arrow[r, phantom, "\subseteq"] \arrow[d, equal] & Y \arrow[d, equal] \\
\emptyset \arrow[r, phantom, "\subseteq"] \arrow[d, equal] & X \arrow[r, phantom, "\subseteq"] \arrow[d, equal] & X \arrow[d, equal] \\
\emptyset \arrow[r, phantom, "\subseteq"] & Y \arrow[r, phantom, "\subseteq"] & Y.
\end{tikzcd}
$$

4. If X and Y are isomorphic:

$$
\left(
\begin{tikzcd}
\emptyset \arrow[r, phantom, "\subseteq"] \arrow[d, equal] & Y \arrow[d, "\cong"] \\
\emptyset \arrow[r, phantom, "\subseteq"] & X
\end{tikzcd}
,
\begin{tikzcd}
\emptyset \arrow[r, phantom, "\subseteq"] \arrow[d, equal] & \emptyset \arrow[d, equal] \\
\emptyset \arrow[r, phantom, "\subseteq"] & \emptyset
\end{tikzcd}
\right)
\sim
\left(
\begin{tikzcd}
\emptyset \arrow[r, phantom, "\subseteq"] \arrow[d, equal] & X \arrow[d, "\cong"] \\
\emptyset \arrow[r, phantom, "\subseteq"] & X
\end{tikzcd}
\right)
$$

via the 2-simplex

$$
\begin{tikzcd}
\emptyset \arrow[r, phantom, "\subseteq"] \arrow[d, equal] & \emptyset \arrow[r, phantom, "\subseteq"] \arrow[d, equal] & X \arrow[d, equal] \\
\emptyset \arrow[r, phantom, "\subseteq"] \arrow[d, equal] & \emptyset \arrow[r, phantom, "\subseteq"] \arrow[d, equal] & Y \arrow[d, equal] \\
\emptyset \arrow[r, phantom, "\subseteq"] & \emptyset \arrow[r, phantom, "\subseteq"] & X.
\end{tikzcd}
$$

5. If Y is a closed subvariety of X:

$$
\left(
\begin{tikzcd}
\emptyset \arrow[r, phantom, "\subseteq"] \arrow[d, equal] & X \setminus Y \arrow[d, "\cong"] \\
\emptyset \arrow[r, phantom, "\subseteq"] & X \setminus Y
\end{tikzcd}
,
\begin{tikzcd}
\emptyset \arrow[r, phantom, "\subseteq"] \arrow[d, equal] & Y \arrow[d, "\cong"] \\
\emptyset \arrow[r, phantom, "\subseteq"] & Y
\end{tikzcd}
\right)
\sim
\left(
\begin{tikzcd}
\emptyset \arrow[r, phantom, "\subseteq"] \arrow[d, equal] & X \arrow[d, "\cong"] \\
\emptyset \arrow[r, phantom, "\subseteq"] & X
\end{tikzcd}
\right)
$$

via the 2-simplex

$$
\begin{tikzcd}
\emptyset \arrow[r, phantom, "\subseteq"] \arrow[d, equal] & Y \arrow[r, phantom, "\subseteq"] \arrow[d, equal] & X \arrow[d, equal] \\
\emptyset \arrow[r, phantom, "\subseteq"] \arrow[d, equal] & Y \arrow[r, phantom, "\subseteq"] \arrow[d, equal] & X \arrow[d, equal] \\
\emptyset \arrow[r, phantom, "\subseteq"] & Y \arrow[r, phantom, "\subseteq"] & X.
\end{tikzcd}
$$

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