SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Hopf algebras in Lie theory and renormalization

av

Linda Winqvist

2010 - No 12

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

Hopf algebras in Lie theory and renormalization

Linda Winqvist

Självständigt arbete i matematik 30 högskolepoäng, grundnivå

Handledare: Sergei Merkulov

2010

Abstract. In this project we study bialgebras and Hopf algebras, bialgebras equipped with an antipode, in the context of the theory of Lie algebras and the theory of renormalisation of quantum field theories (QFT).

We study the classic *Poncaré-Birkhoff-Witt theorem* which states an isomorphism $s : S(L) \to U(L)$ between the symmetric algebra and the universal enveloping algebra for any Lie algebra L. We prove a slightly strengthened version of PBW theorem for vector spaces equipped with an arbitrary skew-symmetric binary operation not necessarily satisfying the Jacobi identity

We also study a Hopf algebra structure defined on vector spaces of graphs (following the ideas of Connes-Kreimer). Some specialised (tree) version of our Hopf algebra are proven by Connes-Kreimer to play an important role in the theory of renormalisation of QFT.

Contents

1. **Introduction**

We study the classic *Poncaré-Birkhoff-Witt theorem*, in the Lie theory, which can be formulated as follows: *Let L be a Lie algebra and let $U(L) = T(L)/J$ be the universal enveloping algebra of L. If $\{l_1, ..., l_p\}$ is a basis of L, then $U(L)$ is isomorphic, as a vector space, to $F[l_1, .., l_n]$, the polynomial ring generated by the formal variables $\{l_1, .., l_n\}$.* So there exists a vector space isomorphism $s : S(L) \to U(L)$ between the symmetric algebra and the universal enveloping algebra for any Lie algebra L. The universal enveloping algebra $(U(L), \star)$ is known to have a Hopf algebra structure, with the multiplication,$\star$, induced from the tensor product in $T(L)$ and the co-multiplication given by the formulas

$$\Delta(1) = 1 \otimes 1$$

$$\Delta(a) = 1 \otimes a + a \otimes 1, a \in L$$

and extended to all of $U(L)$ through $\Delta(a_1 \star ... \star a_n) = \Delta(a_1) \star ... \star \Delta(a_n)$. The antipode is defined by

$$S(1) = 1$$

$$S(a_1 \star ... \star a_n) = (-1)^n a_n \star a_{n-1} \star ... \star a_1$$

We note that $U(V) = T(V)/J$ make sense as a quotient associative algebra for arbitrary vector space V, equipped with *any* binary skew-symmetric operation $[,] : \Lambda^2 V \to V$, and prove a slightly strengthened version of PBW theorem which states that the natural symmetrization map $s : S(V) \to U(V)$ is an isomorphism if and only if the binary operation $[,]$ satisfies the Jacobi identity, i.e. if and only if V is a Lie algebra. In general $dimS_n(V) \geq dimU_n(V)$, that is, in general the universal enveloping algebra is smaller than the symmetric algebra over the same arbitrary vector space V.

We also study a Hopf algebra structure on a vector space of graphs (following the ideas of Connes-Kreimer)[Con, Kr 1][Con, Kr 2] . One of the most commonly used methods in OFT is pertubation calculations, in general resulting in ill-defined integral. The systematic treatment of these divergent integrals is known as renormalization. Connes and Kreimer develop a new technique of renormalization using Hopf algebras. They showed that the vector spaces generated by Feynman graphs, which govern the process of renormalization, have Hopf algebra structures. The combinatorics of renormalization can be described in terms of rooted trees and some other specialised families of graphs. In this paper we will study the vector space G spanned by linear combinations of admissible graphs of any genus. The multiplication is defined to be the disjoint union and co-multiplication is defined on connected admissible graphs by the formula

$$\Delta 1 = 1 \otimes 1$$

$$\Delta \mathcal{G} = 1 \otimes \mathcal{G} + \mathcal{G} \otimes 1 + \sum_{\mathcal{G}_A \subset \mathcal{G}} \mathcal{G}_A \otimes \mathcal{G}/\mathcal{G}_A$$

where summation goes over all admissible subgraphs $\mathcal{G}_A$ and extended to all of G by $\Delta(\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_n) = \Delta(\mathcal{G}_1)\Delta(\mathcal{G}_2), ..., \Delta(\mathcal{G}_n)$. We show in detail that this multiplication and co-multiplication make G into a Hopf algebra. That is, the Hopf algebra structure of rooted trees may be extended to all admissible graphs of any genus.

2. Multilinear maps and the tensor product

2.1. **Universality and Multilinear Maps.** Let A, S and X be sets and let f, g be functions with domain A such that $f : A \to S$ and $g : A \to X$. Further suppose that there exists a unique function $i : S \to X$ for which $g = i \circ f$, that is, that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g"'] & S \arrow[d, "i"] \\ & X \end{tikzcd}$$

commutes.

What does this say about the relationship between the functions f and g and their action on A? It says that any information in g is also in f, which is usually expressed by saying that g can be **factored through** f. Now let $\mathbf{S}$ be a family of sets including S and let $\mathbf{F}$ be a family of functions including f. Suppose that for all sets X in $\mathbf{S}$ and all functions $g : A \to X$ in $\mathbf{F}$ there exists a unique function $i : S \to X$ for which $g = i \circ f$ or equivalently, that the diagram above *commutes.* That is, every function g in $\mathbf{F}$ can be **factored through** f and since f itself is in $\mathbf{F}$ the information in f is precisely the same as the information in the entire family $\mathbf{F}$. So in this sense a single pair (S, $f : A \to S$) may capture the concept of a hole family of functions. This is the idea of *universality*, put in to a formal definition as follows:

2.1.1. **Definition.** Let $\mathbf{S}$ be a family of sets and let $\mathbf{F}$ be a family of functions from a set A to members of $\mathbf{S}$. Let $\mathbf{H}$ be a family of functions on members of $\mathbf{S}$. (Assume that $\mathbf{H}$ has the following properties: it contains the identity function for each member of $\mathbf{S}$, it is closed under composition of functions and composition of functions is associative. Also assume that for any $i \in \mathbf{H}$ and $f \in \mathbf{F}$, the composition is defined and a member of $\mathbf{F}$). A pair (S, $f : A \to S$), where $S \in \mathbf{S}$ and $f \in \mathbf{F}$ is a **universal pair for** $(\mathbf{F}, \mathbf{H})$, if for any $X \in \mathbf{S}$ and any $g : A \to X$ in $\mathbf{H}$ there exists a unique function $i : S \to X$ for which $g = i \circ f$. That is, every function g in $\mathbf{F}$ can be **factored through** f.

For this definition to make sense, we must show that if such universal pairs exist then they are essentially unique, that is, unique up to isomorphism. For otherwise functions in $\mathbf{F}$ may be factored through two essential different functions.

2.1.1. **Theorem (Universal pairs are essentially unique).** *Let (S, $f : A \to S$) and (T, $g : A \to T$) be universal pairs for (*$\mathbf{F}, \mathbf{H}$*). Then there is a bijective function $\mu \in \mathbf{H}$ for which $\mu(S) = T$.*

Proof. If (T, $g : A \to T$) and (S, $f : A \to S$) are both universal pairs for $(\mathbf{F}, \mathbf{H})$, then there exist unique functions $i : S \to T$ resp. $i' : T \to S$ for which $g = i \circ f$ resp $f = i' \circ g$. Hence combining this,

$$f = i' \circ i \circ f$$

So both $i' \circ i$ and id, the identity map, are members of $\mathbf{H}$ that make the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "f"'] & S \arrow[d, "i' \circ i = id"] \\ & S \end{tikzcd}$$

commute and the uniqueness requirement implies $i' \circ i = id$. The same argument gives $i \circ i' = id$, so i and i' are inverses and the bijective function $\mu \in \mathbf{H}$ for which $\mu(S) = T$. $\square$

Next let us restrict the family of functions, to include only multilinear ones. Let $V_1 \times ... \times V_n$ denote the Cartesian product of vector spaces $V_1, ..., V_n$, that is, the set of all n-tuples $(v_1, .., v_n)$ where $v_i \in V_i$.

2.1.2. **Definition.** Let $V_1, ..., V_n$ and W be vector spaces over the same base field F, a function $f : V_1 \times ... \times V_n \to W$ is called **multilinear** if it is *linear* in each variable separately, that is, if

$$\begin{aligned} f(v_1, ..., v_{m-1}, rv_m + sv'_m, v_{m+1}, ..., v_n) &= rf(v_1, ..., v_{m-1}, v_m, v_{m+1}, ..., v_n) + \\ & \quad sf(v_1, ..., v_{m-1}, v'_m, v_{m+1}, ..., v_n) \end{aligned}$$

for all $1 \leq m \leq n$ and scalars r, s belonging to the field F. For the special case $n = 2$, f is called **bilinear**.

The set of all such multilinear functions $f : V_1 \times ... \times V_n \to W$ for some fixed set $V_1, ..., V_n$, W of vector spaces will be denoted by $\mathbf{hom}(V_1, ..., V_n; W)$.

2.2. **Tensor Product.** Let $V_1, ..., V_n$ be vector spaces over the same base field F. We wish to define the **tensor product** $V_1 \otimes ... \otimes V_n$ as the vector space over F, which satisfies the *universal property*. That is, we wish to show that there exists a multilinear map $f : V_1 \times ... \times V_n \to V_1 \otimes ... \otimes V_n$ defined by $f(v_1, ..., v_n) \to v_1 \otimes ... \otimes v_n$ such that for any vector space W over F and for any multilinear map $g : V_1 \times ... \times V_n \to W$ there is a unique linear map $i : V_1 \otimes ... \otimes V_n \to W$ such that $g = i \circ f$. We shall show that such a product does in fact exists and is unique (up to isomorphism)

Let S be any set and F any field, then we can construct a vector space $span_F S$ as the set of all possible finite linear combinations of elements of S

$$span_F S = \{\lambda_1 s_1 + ... + \lambda_n s_n |\ n \geq 1, \lambda_i \in F, s_i \in S\}$$

Let $V_1, ..., V_n$ be vector spaces over the same field F and let $S = V_1 \times ... \times V_n$ be the Cartesian product of $V_1, ..., V_n$ as sets. Construct

$$M := span_F S = \{\lambda_1(v_1^1, ..., v_n^1) + ... + \lambda_n(v_1^n, ..., v_n^n) |\ \lambda_i \in F, v_j^i \in V_i\}$$

so M is clearly a vector space. Let M_0 be a subspace of M generated by all elements of the form

$$(v_1, ..., v_i + v'_i, ..., v_n) - (v_1, ..., v_i, ..., v_n) - (v_1, ..., v'_i, ..., v_n)$$
$$(v_1, ..., \lambda v_i, ..., v_n) - \lambda(v_1, ..., v_i, ..., v_n)$$

There is a natural map $f : V_1 \times ... \times V_n \to M/M_0$ given by

$$f(v_1, ..., v_n) = [(v_1, .., v_n)]$$

where $[(v_1, .., v_n)]$ denotes the equivalents class of $(v_1, .., v_n)$ in M/M_0.

2.2.1. **Theorem.** *The map f is multilinear*

Proof. We wish to verify that for all $1 \leq m \leq n$

$$\begin{aligned} f(v_1, ..., v_{m-1}, \lambda v_m + \lambda' v_m', v_{m+1}, ..., v_n) &= \lambda f(v_1, ..., v_{m-1}, v_m, v_{m+1}, ..., v_n) + \\ &\quad \lambda' f(v_1, ..., v_{m-1}, v_m', v_{m+1}, ..., v_n) \end{aligned}$$

for $\lambda, \lambda' \in F$. By definition of f we have

$$f(v_1, ..., v_{m-1}, \lambda v_m + \lambda' v_m', v_{m+1}, ..., v_n) = [(v_1, ..., v_{m-1}, \lambda v_m + \lambda' v_m', v_{m+1}, ..., v_n)]$$

Splitting addition and multiplication with scalar we wish to verify that,

$$(v_1, ..., v_i + v_i', ..., v_n) + M_0 = [(v_1, ..., v_i, ..., v_n) + M_0] + [(v_1, ..., v_i', ..., v_n) + M_0]$$
$$(v_1, ..., \lambda v_i, ..., v_n) + M_0 = \lambda(v_1, ..., v_i, ..., v_n) + M_0$$

but this follows immediately from the construction. Since M/M_0 is a quotient space, so $((v_1, ..., v_i, ..., v_n) + (v_1, ..., v_i', ..., v_n)) + M_0 = ((v_1, ..., v_i, ..., v_n) + M_0) + ((v_1, ..., v_i', ..., v_n) + M_0)$ and $(v_1, ..., v_i + v_i', ..., v_n)$ is congruent to $(v_1, ..., v_i, ..., v_n) + (v_1, ..., v_i', ..., v_n)$ mod M_0 so

$$\begin{aligned} (v_1, ..., v_i + v_i', ..., v_n) + M_0 &= [(v_1, ..., v_i, ..., v_n) + (v_1, ..., v_i', ..., v_n)] + M_0 = \\ &\quad [(v_1, ..., v_i, ..., v_n) + M_0] + [(v_1, ..., v_i', ..., v_n) + M_0] \end{aligned}$$

The second equality follows in the same way. Which completes the proof. □

Define the tensor product $V_1 \otimes ... \otimes V_n$ by

$$V_1 \otimes ... \otimes V_n = M/M_0$$
$$v_1 \otimes ... \otimes v_n = (v_1, ..., v_n) + M_0 \ \in V_1 \otimes ... \otimes V_n$$

Let us next show that the tensor product has the required universal property.

2.2.2. **Theorem (The universal property for the Tensor product).** *Let $V_1, ..., V_n$ be vector spaces over the field F. The pair $(V_1 \otimes ... \otimes V_n, f : V_1 \times ... \times V_n \to V_1 \otimes ... \otimes V_n)$, where f is defined by $f(v_1, ..., v_n) \to v_1 \otimes ... \otimes v_n$ has the following property. If $g : V_1 \times ... \times V_n \to W$ is any multilinear function from $V_1 \times ... \times V_n$ to W over F then there exists a unique linear transformation $i : V_1 \otimes ... \otimes V_n \to W$ such that $g = i \circ f$ or equal, that the diagram*

$$\begin{tikzcd} V_1 \times ... \times V_n \arrow[r, "f"] \arrow[dr, "g"'] & V_1 \otimes ... \otimes V_n \arrow[d, "i"] \\ & W \end{tikzcd}$$

commutes.

Proof. Assume first that i exists, then the condition $g = i \circ f$ uniquely determines the value of i on $v_1 \otimes ... \otimes v_n$.

$$i(v_1 \otimes ... \otimes v_n) = i \circ f(v_1, .., v_n) = g(v_1, .., v_n)$$

i.e. if i exists, it is unique.

To prove the existence define for all $v_1 \otimes ... \otimes v_n \in V_1 \otimes ... \otimes V_n$ the function i to be

$$i(v_1 \otimes ... \otimes v_n) = g(v_1, ..., v_n) \quad (*)$$

We have to show that this definition makes sense. Take any $v_1 \otimes ... \otimes v_n$ and let V be any element in $V_1 \times ... \times V_n$ such that

$$f(V) = v_1 \otimes ... \otimes v_n$$

Such an element obviously exists, take for example $V = (v_1, .., v_n)$. If V' is another element such that $f(V') = v_1 \otimes ... \otimes v_n$, then $f(V - V') = 0$ so $V - V' \in kerf$. That is, $V - V' = V''$ where

$$V'' \in M_0$$

moreover, by multilinearity of g, g vanish on M_0. Finally i does not depend on the choice of V since if V' is another element such that $f(V') = v_1 \otimes ... \otimes v_n$, then

$$g(V) = g(V' + V'') = g(V') + g(V'') = g(V')$$

So the map i, given by $(*)$, is well-defined and makes the diagram commute. $\square$

From the construction of the tensor product it thus follows that for all $0 \leq i \leq n$ and $a \in F$ we have the following two formulas,

$$(v_1 \otimes ... \otimes (v_i + v_i') \otimes ... \otimes v_n) = (v_1 \otimes ... \otimes v_i \otimes ... \otimes v_n) + (v_1 \otimes ... \otimes v_i' \otimes ... \otimes v_n)$$

$$(v_1 \otimes ... \otimes \lambda v_i \otimes ... \otimes v_n) = \lambda(v_1 \otimes ... \otimes v_i \otimes ... \otimes v_n)$$

2.2.1. **Remark.** The tensor product may be called *associative* in the sense that there exists a canonical isomorphism

$$\tau : (V_1 \otimes ... \otimes V_n) \otimes (W_1 \otimes ... \otimes W_m) \to V_1 \otimes ... \otimes V_n \otimes W_1 \otimes ... \otimes W_m$$

for which

$$\tau((v_1 \otimes ... \otimes v_n) \otimes (w_1 \otimes ... \otimes w_m)) = v_1 \otimes ... \otimes v_n \otimes w_1 \otimes ... \otimes w_m$$

The tensor product $(v_1 \otimes v_2) \otimes v_3 \in (V_1 \otimes V_2) \otimes V_3$ can therefore be canonically identified with $v_1 \otimes (v_2 \otimes v_3) \in V_1 \otimes (V_2 \otimes V_3)$ and hence can be viewed as one and the same element $v_1 \otimes v_2 \otimes v_3$ in $V_1 \otimes V_2 \otimes V_3$.

3. Associative algebras

3.1. Unital associative algebra.

3.1.1. **Definition.** An **associative algebra** is a vector space A over a base field F together with bilinear map $\psi : A \otimes A \to A$ which is associative. The associativity is expressed by the commutativity of the following diagram.

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\psi \otimes id"] \arrow[d, "id \otimes \psi"'] & A \otimes A \arrow[d, "\psi"] \\ A \otimes A \arrow[r, "\psi"'] & A \end{tikzcd}$$

The diagram implies that for all $a, b, c \in A$ we have $(ab)c = a(bc)$.

3.1.2. **Definition.** The algebra A is **unital** if moreover there is a unit $\mathbf{1}$ in it. This is expressed by the commutativity of the following diagram

$$\begin{tikzcd} F \otimes A \arrow[r, "\mu\otimes id"] \arrow[dr, "\sim"'] & A \otimes A \arrow[d, "\psi"] & A \otimes F \arrow[l, "id\otimes\mu"'] \arrow[dl, "\sim"] \\ & A & \end{tikzcd}$$

where $\mu : F \to A$ defined by $\mu(\lambda) = \lambda\mathbf{1}$.

3.1.3. **Definition.** An unital associative algebra is said to be **commutative** if further the following diagram

$$\begin{tikzcd} A \otimes A \arrow[rr, "T"] \arrow[dr, "\psi"'] & & A \otimes A \arrow[dl, "\psi"] \\ & A & \end{tikzcd}$$

where $T : A \otimes A \to A \otimes A$ is the twist map defined by $T(a \otimes b) = (b \otimes a)$ commutes.

3.1.4. **Definition.** Let A and A' be two unital associative algebras, a linear map $f : A \to A'$ is called a **linear map of algebras** if the following two diagrams

$$\begin{tikzcd} A \otimes A \arrow[r, "f"] \arrow[d, "\psi"'] & A' \otimes A' \arrow[d, "\psi'"] \\ A \arrow[r, "f"'] & A' \end{tikzcd}$$

$$\begin{tikzcd} & F & \\ A \arrow[ur, "\mu"] \arrow[rr, "f"'] & & A' \arrow[ul, "\mu'"'] \end{tikzcd}$$

commute.

3.2. **Tensor algebra.** Our first example of an associative algebra is the tensor algebra. For any vector space V over F, and any nonnegative integer p the p^{th} tensor power of V is the tensor product of V with itself p times, $V \otimes ... \otimes V$, denoted $T^p(V)$ or $V^{\otimes p}$. So $T^p(V)$ consist of all tensors on V of rank p.

3.2.1. **Definition.** To any vector space V one can associate its **tensor algebra** $T(V)$, defined by:

$$T(V) = \bigoplus_{0 \le k} V^{\otimes k}$$

with $V^{\otimes 0} = F$.

The space $T(V)$ has a natural algebraic structure $\psi : T(V) \otimes T(V) \to T(V)$ defined by

$$\psi(v_1 \otimes ... \otimes v_n, u_1 \otimes ... \otimes u_m) = v_1 \otimes ... \otimes v_n \otimes u_1 \otimes ... \otimes u_m$$

the embedding of the ground field into $T(V)$ gives the unit map μ, that is, the unit in $T(V)$ is the unit in F. Which completes the definition of the algebraic structure. The tensor algebra is obviously associative since the tensor product is.

3.3. **Endomorphism algebra.** A second example of an associative algebra is the endomorphism algebra. For any vector space, the set $\mathbf{End}(P) = \mathbf{hom}(P, P)$. Indeed, $\mathbf{End}(P)$ is a vector space. The product in $\mathbf{End}(P)$ is defined by the composition of maps,

$$\psi : \mathbf{End}(P) \times \mathbf{End}(P) \to \mathbf{End}(P)$$

$$f \times g \to f \circ g$$

where $f \circ g(a) = f(g(a))$ is the composition map in the usual sense.

3.4. **The symmetric algebra.** A third example of an associative algebra is the symmetric tensor algebra.

3.4.1. **Definition.** Let A be an associative algebra. A subspace $I \subset A$ is called a **(two-sided) ideal** if for any $r \in I$, $a \in A$

$$ra \in I, ar \in I$$

3.4.2. **Definition.** Let A be an associative algebra. An ideal of the form

$$< a_1, ..., a_n >= \{\sum b_1 a_1 c_1 + ... + b_n a_n c_n | b_i, c_i \in A\}$$

is called the **ideal generated by** $a_1, .., a_n \in A$.

As the next lemma shows, for any algebra A, the quotient vector space of A by any (two-sided) ideal is itself an algebra with an induced algebraic structure given below.

3.4.1. **Lemma.** *For any (two-sided) ideal I in A, the quotient vector space*

$$B = A/I$$

has an induced algebraic structure defined by

$$[b_1][b_2] = [b_1 b_2]$$

Proof. Since $[b_1] = b_1 + I$ and $[b_2] = b_2 + I$ the product $[b_1][b_2]$ might be written as

$$[b_1][b_2] = (b_1 + I)(b_2 + I) = b_1 b_2 + b_1 I + I b_2 + II$$

Since I is an two-sided ideal $b_1 I + I b_2 + II$ is in the ideal and hence

$$[b_1][b_2] = b_1 b_2 + I = [b_1 b_2]$$

□

The symmetric tensor algebra may now be defined in the following way

3.4.3. **Definition.** Consider a vector space V and let J be an ideal in the tensor algebra $T(V)$ generated by all elements of the form

$$< v_i \otimes v_j - v_j \otimes v_i >$$

where $v_i, v_j \in V$. The quotient of $T(V)$ by the ideal J is called the **symmetric tensor algebra**.

$$S(V) = T(V)/J$$

There is a natural projection $\pi : T(V) \to S(V)$ defined by,

$$\pi(v_1 \otimes ... \otimes v_n) = v_1 \odot ... \odot v_n$$

and from lemma it follows that $S(V)$ is an associative algebra, with an induced natural structure of a commutative associative algebra $\psi : S(V) \otimes S(V) \to S(V)$ given by

$$\psi(v_1 \odot ... \odot v_n, u_1 \odot ... \odot u_n) \to v_1 \odot ... \odot v_n \odot u_1 \odot ... \odot u_n$$

3.4.4. **Remark.** For $charF = 0$ the symmetric tensor algebra may alternatively be regarded as a subspace of the tensor algebra rather than a quotient space. Let σ be any permutation in S_p, the multilinear map $f_\sigma : V^{\times p} \to T^p(V)$ defined by $f_\sigma(v_1, ..., v_p) = (v_{\sigma(1)} \otimes ... \otimes v_{\sigma(p)})$ determines, by universality, a unique linear operator λ_σ on $T^p(V)$ for which $\lambda_\sigma(v_1 \otimes ... \otimes v_p) = v_{\sigma(1)} \otimes ... \otimes v_{\sigma(p)}$. Let $B = \{e_1, ..., e_n\}$ be a basis for V, then the set $B = \{e_{i_1} \otimes ... \otimes e_{i_p} | e_{i_j} \in B\}$ is a basis for $T^p(V)$. λ_σ is a bijection of B so λ_σ is an isomorphism of $T^p(V)$. A tensor $t \in T^p(V)$ is called **symmetric** if $\lambda^\sigma(t) = t$ for all permutations $\sigma \in S_p$. If $charF = 0$ we can identify $S(V)$ with a subspace of $T(V)$ by the following mapping

$$v_1 \odot ... \odot v_n \to \sum_{\sigma \in S_n} \frac{1}{p!} v_{\sigma(1)} \otimes ... \otimes v_{\sigma(p)}$$

If we choose a base $B = \{e_1, ..., e_n\}$ in V, then $S(V)$ can be identified with the polynomial ring, $F[e_1, .., e_n]$, generated by the formal variables $e_1, ..., e_n$.

3.5. **Lie algebra.** We shall be interested below in associative algebras which are associated to Lie algebras, let us first give a definition of the latter concept.

3.5.1. **Definition.** A vector space L over a field F is called a **Lie algebra** if there is a bilinear map, called the **Lie bracket**

$$[,] : L \otimes L \to L$$
$$(l_i, l_j) \to [l_i, l_j]$$

that satisfies the conditions,

$$[l_i, l_j] = -[l_j, l_i]$$
$$[l_i, [l_j, l_k]] + [l_j, [l_k, l_i]] + [l_k, [l_i, l_j]] = 0$$

for all $l_i, l_j, l_k \in L$. The second condition is called **the Jacobi identity**.

3.5.2. **Remark.** Let A be an associative algebra. Define the commutator of l_i and l_j to be $[l_i, l_j] = l_i l_j - l_j l_i$, where then product is the associative product of the algebra A. This commutator obvious satisfies the two conditions above making A a Lie algebra, usually denoted $Lie(A)$. As a vector space $Lie(A)$ is isomorphic to A.

3.6. **Graded and filtered algebras.** Let us end this section on associative algebras by defining the concept of graded and filtered algebras.

3.6.1. **Definition.** Let A be an associative algebra over F. A is said to be **graded** if for each integer $n \geq 0$, there is a subspace A_n of A such that:

(G.1) A is the direct sum of all the A_n, and $1 \in A_0$

(G.2) $A_m A_n \subseteq A_{m+n} \quad \forall m, n \geq 0$

3.6.2. **Definition.** Let A be an associative algebra over F. A is said to be **filtered** if for $n \geq 0$, there is a subspace $A^{(n)}$ of A such that:

(F.1) $1 \in A^0$ and $A^{(0)} \subseteq A^{(1)} \subseteq A^{(2)} \subseteq ...$ and $\cup A^{(n)} = A$

(F.2) $A^{(m)} A^{(n)} \subseteq A^{(m+n)} \forall m, n \geq 0$

So what is the connection between graded and filtered algebras? How can one construct a filtered algebra starting from a graded one and vice versa, i.e. how to construct a graded algebra starting from a filtered?

Let A be a graded algebra and let A_n a subspace of A satisfying condition (G.1) and (G.2) above. Put $A^{(n)} = \sum_{p=0}^{n} A_p$, then it is easily verified that A becomes a filtered algebra, called **the filtered algebra associated with the graded algebra** A.

Now let A be a filtered algebra and let $A^{(n)}$ a subspace of A for filling condition (F.1) and (F.2) in the definition. Put $B_n = A^{(n)}/A^{(n-1)}$ and let π_n be the natural map of $A^{(n)}$ onto B_n, and denote the direct sum of B_n by B. Construct the product in B in the following way: given $b'_1 \in B_n$ and $b'_2 \in B_m$ choose $b_1 \in A^{(n)}$ and $b_2 \in A^{(m)}$ such that $\pi_n(b_1) = b'_1$ and $\pi_m(b_2) = b'_2$. Define $b'_1 b'_2 = \pi_{n+m}(b_1 b_2)$. It is easy to verify that the product is well-defined and independent of the choices of b_1, b_2. The map from $B_n \times B_m$ into B_{n+m} defined by $(b'_1, b'_2) \to b'_1 b'_2$ is bilinear and extends to a bilinear map $B \times B \to B$ given by $(b'_1, b'_2) \to b'_1 b'_2$. We call it **the graded algebra associated with the filtered algebra** A, and denote it from now on by A^{gr}.

4. Coalgebras

4.1. **Co-unital co-associative coalgebras.** Coalgebras are objects that are dual to algebras. Axioms for coalgebras can be produced from axioms of algebras by just inverting arrows in all diagram.

4.1.1. **Definition.** A **co-associative coalgebra** is a vector space C over a base field F together with a bilinear map $\Delta : C \to C \otimes C$ which is co-associative. The co-associativity is the same as the commutativity of the following diagram,

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta \otimes id"] \\ C \otimes C \arrow[r, "id \otimes \Delta"'] & C \otimes C \otimes C \end{tikzcd}$$

4.1.2. **Definition.** A coalgebra C is called **co-unital** if there is a co-unit. That is, if there exists a linear function $\varepsilon : C \to F$ such that the following diagram

$$\begin{tikzcd} & C \arrow[dl, "\sim"'] \arrow[d, "\Delta"] \arrow[dr, "\sim"] & \\ F \otimes C & C \otimes C \arrow[l, "\varepsilon\otimes id"] \arrow[r, "id\otimes\varepsilon"'] & C \otimes F \end{tikzcd}$$

commutes.

4.1.3. **Definition.** A coalgebra is called **co-commutaive** if the following diagram

$$\begin{tikzcd} & C \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ C \otimes C \arrow[rr, "T"'] & & C \otimes C \end{tikzcd}$$

where $T : A \otimes A \to A \otimes A$ is the twist map defined by $T(a \otimes b) = (b \otimes a)$ commutes.

4.1.4. **Definition.** Let C and C' be two co-unital co-associative coalgebras, a linear map $g : C \to C'$ is called a **linear map of coalgebras** if the following two diagrams

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "\Delta"'] & C' \arrow[d, "\Delta'"] \\ C \otimes C \arrow[r, "g\otimes g"'] & C' \otimes C' \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[rr, "g"] \arrow[dr, "\varepsilon"'] & & C' \arrow[dl, "\varepsilon'"] \\ & F & \end{tikzcd}$$

commute.

4.2. **Tensor coalgebra.** To any vector space V one can associate its tensor coalgebra $T^C(V)$ which, as a vector space, can be identified with $T(V)$.

4.2.1. **Definition.** Let V be any vector space and let $T^C(V)$ denote its **tensor coalgebra** with the co-product $\Delta : T(V) \to T(V) \bar{\otimes} T(V)$ defined by

$$\Delta(v_1 \otimes ... \otimes v_n) = 1 \bar{\otimes} v_1 \otimes ... \otimes v_n + v_1 \otimes ... \otimes v_n \bar{\otimes} 1 + \sum_{p=1}^{n-1} (v_1 \otimes ... \otimes v_p) \bar{\otimes} (v_{p+1} \otimes ... \otimes v_n)$$

and extended by linearity to all of $T(V)$.

It is easy to verify that Δ is in fact co-associative. For example, for $v_1 \otimes v_2 \in T^C(V)$ we have

$$\begin{aligned} (\Delta \bar{\otimes} id)\Delta(v_1 \otimes v_2) \quad = \quad & (\Delta \bar{\otimes} id)(1 \bar{\otimes} v_1 \otimes v_2 + v_1 \otimes v_2 \bar{\otimes} 1 + v_1 \bar{\otimes} v_2) = \\ & \Delta(1) \bar{\otimes} v_1 \otimes v_2 + \Delta(v_1 \otimes v_2) \bar{\otimes} 1 + \Delta(v_1) \bar{\otimes} v_2 = \\ & 1 \bar{\otimes} 1 \bar{\otimes} v_1 \otimes v_2 + 1 \bar{\otimes} v_1 \otimes v_2 \bar{\otimes} 1 + v_1 \otimes v_2 \bar{\otimes} 1 \bar{\otimes} 1 + \\ & v_1 \bar{\otimes} v_2 \bar{\otimes} 1 + 1 \bar{\otimes} v_1 \bar{\otimes} v_2 + v_1 \bar{\otimes} 1 \bar{\otimes} v_2 \end{aligned}$$

and

$$
\begin{aligned}
(id \bar{\otimes} \Delta)\Delta(v_1 \otimes v_2) \quad &= \quad (id \bar{\otimes} \Delta)(1 \bar{\otimes} v_1 \otimes v_2 + v_1 \otimes v_2 \bar{\otimes} 1 + v_1 \bar{\otimes} v_2) = \\
& \quad 1 \bar{\otimes} \Delta(v_1 \otimes v_2) + v_1 \otimes v_2 \bar{\otimes} \Delta(1) + v_1 \bar{\otimes} \Delta(v_2) = \\
& \quad 1 \bar{\otimes} 1 \bar{\otimes} v_1 \otimes v_2 + 1 \bar{\otimes} v_1 \otimes v_2 \bar{\otimes} 1 + 1 \bar{\otimes} v_1 \bar{\otimes} v_2 + \\
& \quad v_1 \otimes v_2 \bar{\otimes} 1 \bar{\otimes} 1 + v_1 \bar{\otimes} 1 \bar{\otimes} v_2 + v_1 \bar{\otimes} v_2 \bar{\otimes} 1
\end{aligned}
$$

so that $(\Delta \bar{\otimes} id)\Delta(v_1 \otimes v_2) = (id \bar{\otimes} \Delta)\Delta(v_1 \otimes v_2)$. The co-unit is given by the natural projection of $T^C(V)$ onto F.

5. Bialgebras and Hopf algebras

5.1. Bialgebra.

5.1.1. **Definition.** A **bialgebra** B is a vector space, endowed with an algebra structure (defined by ψ, μ) and a coalgebra structure (defined by Δ, ε) such that the following three diagrams

$$
\begin{tikzcd}
B \otimes B \otimes B \otimes B \arrow[rr, "id\otimes t\otimes id"] & & B \otimes B \otimes B \otimes B \arrow[d, "\psi\otimes\psi"] \\
B \otimes B \arrow[u, "\Delta\otimes\Delta"] \arrow[r, "\psi"'] & B \arrow[r, "\Delta"'] & B \otimes B
\end{tikzcd}
$$

(where $t : B \otimes B \to B \otimes B$ is the transposition map $b_i \otimes b_j \to b_j \otimes b_i$.)

$$
\begin{tikzcd}
B \otimes B \arrow[r, "\varepsilon\otimes\varepsilon"] \arrow[d, "\psi"'] & F \otimes F \arrow[d, "\sim"] \\
B \arrow[r, "\varepsilon"'] & F
\end{tikzcd}
$$

$$
\begin{tikzcd}
B \otimes B & F \otimes F \arrow[l, "\mu\otimes\mu"'] \\
B \arrow[u, "\Delta"] & F \arrow[l, "\mu"] \arrow[u, "\sim"']
\end{tikzcd}
$$

commute.

5.1.2. **Definition.** An element v in a bialgebra B is called **primitive** if

$$\Delta(v) = 1 \otimes v + v \otimes 1$$

5.1.3. **Definition.** Let B and B' be two bialgebras, a linear map $f : B \to B'$ is called a **linear map of bialgebras** if it is a linear map of algebras and also a linear map of coalgebras.

5.2. **Hopf algebra.**

5.2.1. **Definition.** Let A be an algebra and C a coalgebra over the same field F. Define an algebraic structure on $\mathbf{hom}(C, A)$, the set of all linear maps from C to A, by

$$* : \mathbf{hom}(C, A) \otimes \mathbf{hom}(C, A) \to \mathbf{hom}(C, A)$$
$$f \otimes g \to f * g$$

where $f * g : C \to A$ is given by the composition

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f\otimes g} A \otimes A \xrightarrow{\psi} A$$

This product is called the **convolution product**.

The convolution product is associative since A is associative and C co-associative.

5.2.2. **Definition.** Let H be a bialgebra, a linear map $S : H \to H$ is called an **antipode** of the bialgebra H if S is the inverse of the identity map $id : H \to H$ with respect to the convolution product in $\mathbf{hom}(H^C, H^A)$, that is, if the following diagram

$$\begin{tikzcd}
& H \otimes H \arrow[rr, "S\otimes id"] & & H \otimes H \arrow[dr, "\psi"] & \\
H \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rr, "\varepsilon"] & & F \arrow[rr, "\mu"] & & H \\
& H \otimes H \arrow[rr, "id\otimes S"'] & & H \otimes H \arrow[ur, "\psi"'] &
\end{tikzcd}$$

commutes. A bialgebra H having an antipode is called a **Hopf algebra**.

5.2.3. **Example. (The group algebra)** Let G be a group, and let $G(F)$ be the associative group algebra over the field F. This is an F-vector space with basis $\{g_i | g_i \in G\}$ so its elements are of the form $\sum \lambda_i g_i$.

The associative product $\psi : G(F) \otimes G(F) \to G(F)$ is defined by the product of G extended to a bilinear map from $G(F) \times G(F)$ to G(F):

$$(\lambda_1 g_1)(\lambda_2 g_2) = (\lambda_1 \lambda_2)(g_1 g_2)$$

for any $\lambda_1, \lambda_2 \in F$ and $g_1, g_2 \in G$. The unit μ is given by the neutral element e of G i.e. $\mu : F \to G(F)$ is given by $\mu(\lambda) = \lambda e$.

The space $G(F)$ has a coalgebra structure $\Delta : G(F) \to G(F) \otimes G(F))$ given by:

$$\Delta(\sum \lambda_i g_i) = \sum \lambda_i g_i \otimes g_i$$

with the co-unit $\varepsilon : G(F) \to F$ given by $\varepsilon(\sum \lambda_i g_i) = \sum \lambda_i$.

To show that $G(F)$ is an Hopf algebra we must show that the product and coproduct are compatible, that is turns $G(F)$ into a bialgebra, and that it has an antipode.

For any to elements g_1, g_2 in G

$$\Delta(g_1 g_2) = (g_1 g_2) \otimes (g_1 g_2) = (g_1 \otimes g_1)(g_2 \otimes g_2) = \Delta(g_1)\Delta(g_2)$$

that is, Δ is an algebra morphism.

It remains to show the existence of an antipode. Let $S : G(F) \to G(F)$ be defined by

$$S(g) = g^{-1}$$

for any $g \in G$ and extended linearly. This is an antipode of the bialgebra $G(F)$ since

$$\psi(S \otimes id)\Delta(g) = \psi(S \otimes id)g \otimes g = \psi(S(g) \otimes g) = S(g)g = g^{-1}g = e$$

and $e = 1 \circ \varepsilon(g)$ for any $g \in G$, and similarly for $\psi(id \otimes S)\Delta(g)$

5.3. **Graded connected bialgebras.** Finally let us show that for any graded connected bialgebra the antipode comes for free making it a Hopf algebra.

5.3.1. **Definition.** Let B be a bialgebra $(B, \psi, \Delta, \varepsilon, \mu)$. B is said to be a **graded connected** bialgebra if it permits a decomposition into a direct sum

$$B = \bigoplus_{n \geq 0} B_n$$

such that

(1) the multiplication and co-multiplication preserves the grading, i.e. for all $b \in B_n$ and $b' \in B_m$

$$\psi(b, b') \in B_{m+n}$$

$$\Delta(b) \in \bigoplus_{n=k+l} B_k \otimes B_l$$

(2) the unit and co-unit maps ε, μ are graded i.e.

$$\varepsilon : B_0 \to F$$

$$\mu : F \to B_0$$

so the image of ε is zero on all $B_{n \geq 1}$ and the image of μ is in B_0.

the connectedness is expressed by the following condition

(3) B_0 is identified with the base field F

$$B_0 = F$$

Any graded bialgebra is obviously filtered by the canonical filtration associated with the grading

$$B^{(n)} = \bigoplus_{m=0}^{n} B_m$$

Let, as usual, $\mathbf{hom}(B, B)$ denotes the set of all linear maps from B onto itself. Because of linearity this may be written as

$$\mathbf{hom}(B, B) = \mathbf{hom}(\bigoplus_{n \geq 0} B_n, \bigoplus_{m \geq 0} B_m) = \bigoplus_{n,m} \mathbf{hom}(B_n, B_m)$$

5.3.2. **Definition.** For any natural number k

$$\mathbf{hom}_k(B, B) := \bigoplus_{n \geq 0} \mathbf{hom}(B_n, B_{n+k})$$

Now let $k = n - m$ and we can write

$$\mathbf{hom}(B, B) = \bigoplus_k \mathbf{hom}_k(B_n, B_{n+k})$$

so any function $f \in \mathbf{hom}(B, B)$ may be written as a direct sum

$$f = \bigoplus_k f_k$$

for some $f_k \in \mathbf{hom}_k(B, B)$. Such functions are called homogeneous of degree k.

5.3.1. **Lemma.** *Let $f_k \in \mathbf{hom}_k(B, B)$ and $g_l \in \mathbf{hom}_l(B, B)$, then*

$$f_k * g_l \in \mathbf{hom}_{k+l}(B, B)$$

Proof. Since the co-product preserves the grading, for any $b \in B_n$ we have

$$\Delta(b) = \sum_{i+j=n} b'_i \otimes b''_j$$

where $i, j \geq 1$. So the convolution product becomes

$$f_k * g_l(b) = \sum_{i+j=n} \psi(f_k(b'_i)g_l(b''_j))$$

but $f_k(b'_i) \in B_{k+i}$ and $g_l(b''_j) \in B_{l+j}$ plus remembering that the product also respects the grading we may conclude

$$f_k * g_j(B_n) \in B_{n+k+l}$$

□

From lemma we may conclude that for any $f, g \in \mathbf{hom}_0(B, B)$, the set of all linear maps $f : B \to B$ such that $f(B_n) \subset B_n$, their convolution product $f * g$ also lies in $\mathbf{hom}_0(B, B)$

5.3.2. **Theorem.** *Any connected graded bialgebra B has canonically a unique antipode S, making it a Hopf algebra. The antipode can be given explicitly, using the following notation, for any $b \in B_n, n \geq 1$ let*

$$\sum_{n=i+j \;\; i,j \neq n} b'_i \otimes b''_j = \Delta(b) - (b \otimes 1 + 1 \otimes b)$$

By induction on the grading we define the antipode by

$$S(1) = 1$$

$$S(b) = -b - \sum_{n=i+j \;\; i,j \neq n} S(b'_i)b''_j = -b - \sum_{n=i+j \;\; i,j \neq n} b'_i S(b''_j)$$

Proof. Let $e := \mu \circ \varepsilon$. Since the image of ε is zero on $B_{n\geq 1}$ and just the idenity map on B_0 we have that,

$$e(B_0) = B_0$$

$$e(B_{n\geq 1}) = 0$$

For any $f_0 \in \mathbf{hom}_0(B, B) = \bigoplus_{n\geq 0} \mathbf{hom}(B_n, B_n)$ we may write

$$f_0 = f_0^0 \oplus f_0^1 \oplus f_0^2 \oplus ...$$

we can extend f_0^n to an element in $\mathbf{hom}_0(B,B)$ by assuming that $f_0^n(B_k)=0$ for $k\geq n$. Using this notation we can write $e=e^0\oplus e^1\oplus ...$ where $e^0=id$ and $e^{n\geq 1}=0$ and $id=id^0\oplus id^1\oplus ...$

The map $S:B\to B$ is an antipode if the diagram

$$\begin{tikzcd} B\otimes B \arrow[rr, "id\otimes S"] & & B\otimes B \arrow[d, "\psi"] \\ B \arrow[u, "\Delta"] \arrow[r, "\varepsilon"] & B \arrow[r, "\mu"] & B \end{tikzcd}$$

commutes, that is, if

$$id*S=e$$

We define S by induction over n,

$$S=S_0\oplus S_1\oplus ...\oplus S_n\oplus ...$$

$S_0=id$ solves $id*S(B_0)=e(B_0)$. Assume we constructed $S=S_0\oplus S_1\oplus ...\oplus S_n$ such that $id*S(b)=e(b)$ for any $b\in B^{(n)}$. Let us find S_{n+1} such that $id*S(b)=e(b)$ holds for any $b\in B^{(n+1)}$. We may assume that $b\in B_{n+1}$ so the equation becomes $id*S(b)=0$, i.e.

$$1S(b)+bS(1)+\sum_{n+1=i+j,\ i,j\neq n+1} b_i'b_j''=0$$

For $b\in B_{n+1}$ we have $S(b)=S_{n+1}(b)$, thus

$$S_{n+1}(b)=-b-\sum_{n+1=i+j,\ i,j\neq n+1} b_i'S_j(b_j'')$$

but we already constructed $S_{j<n+1}$, so we see that there is a unique formula for S_{n+1}. This completes the induction. A similar calculation shows that this also satisfies $S*id=e$ which completes the proof. $\square$

6. The universal enveloping algebra

6.1. **The universal enveloping algebra.** We have already seen that every associative algebra A can be turned into a Lie algebra $Lie(A)$ by replacing its multiplication by the commutator $[l_i,l_j]=l_il_j-l_jl_i$. Now consider the reverse situation, starting from a Lie algebra L we wish to find an associative algebra A such that the Lie algebra $Lie(A)$ contains L. This algebra will be called the universal enveloping algebra, and will be denoted by $U(L)$.

6.1.1. **Definition.** Let L be a Lie algebra with Lie bracket $[,]$. Let J be the ideal of $T(L)$ generated by all elements of the form

$$<l_i\otimes l_j-l_j\otimes l_i-[l_i,l_j]>$$

then the **universal enveloping algebra** of L is define as the quoitent algebra

$$U(L)=T(L)/J$$

The canonical mapping $\phi : L \to T(L)$ induces a mapping $\sigma : L \to T(L) \to U(L)$ called the canonical mapping of L onto the quotient algebra $U(L) = T(L)/J$ such that for all $l_i, l_j \in L$

$$\sigma(l_i)\sigma(l_j) - \sigma(l_j)\sigma(l_i) = \sigma([l_i, l_j])$$

6.1.1. **Theorem (The universal property for the universal enveloping algebra).** *Let L be a Lie algebra over a field F. The pair $(U(L), \sigma : L \to U(L))$, where σ is the canonical mapping has the following property. If A is any algebra, and $g : L \to Lie(A)$ is any Lie algebra homomorphism, then there exists a unique algebra homomorphism $i_\sigma : U(L) \to A$ such that $g = i_\sigma \circ \sigma$.*

$$\begin{tikzcd} L \arrow[r, "\sigma"] \arrow[dr, "g"'] & U(L) \arrow[d, "i_\sigma"] \\ & A \end{tikzcd}$$

Proof. Since the algebra $U(L)$ is generated by 1 and $\sigma(L)$ the algebra homomorphism i_σ,if it exists, is clearly unique.

Let $\phi : L \to T(L)$ be the canonical map inducing σ

$$\begin{tikzcd} L \arrow[r, "\phi"] \arrow[dr, "g"'] & T(L) \arrow[r] \arrow[d, "i_\phi"] & U(L) \arrow[dl, "i_\sigma"] \\ & A & \end{tikzcd}$$

Let i_ϕ be the unique homomorphism of T into A such that $g = i_\phi \circ \phi$. For all $l_i, l_j \in L$

$$g(l_i)g(l_j) - g(l_j)g(l_i) = g([l_i, l_j])$$

so

$$i_\phi(l_i \otimes l_j - l_j \otimes l_i - [l_i, l_j]) = g(l_i)g(l_j) - g(l_j)g(l_i) - g([l_i, l_j]) = 0$$

hence $i_\phi(J) = 0$ and, by passage to quotient, i_ϕ defines a homomorphism i_σ of $U(L)$ such that $g = i_\sigma \circ \sigma$. □

We have now shown that the universal enveloping algebra, as defined above, exists, is universal in the usual sense of the word and unique up to isomorphism. Based on this result, we may consider $U(L)$ as the unique universal enveloping algebra.

6.2. **The Poincaré-Birhoff-Witt theorem.** Let $\{l_1, .., l_n\}$ be a basis in L, and let $I = (i_1, .., i_p)$ be any finite sequence of integers in the set $\{1, 2, ..., n\}$.

6.2.1. **Definition.** By a **monomial** we mean any tensor which is either 1 or of the form

$$l_{i_1} \otimes ... \otimes l_{i_p}$$

for $p \geq 1$ and $i_1, .., i_p \in I$. Let I be linearly ordered, a **standard monomial** is a tensor which is either 1 or of the form $l_{i_1} \otimes ... \otimes l_{i_p}$ for $p \geq 1$ and $i_1 \leq ... \leq i_p \in I$.

It is clear that for any basis $\{l_1, .., l_n\}$ in L the image under the canonical map in $U(L)$ of tensor monomials $l_{i_1} \otimes ... \otimes l_{i_p}$ span the universal enveloping algebra over F, since they span the tensor algebra. In fact we will show, in this section, that even the canonical image of the standard monomials span $U(L)$, and does in fact form a basis for $U(L)$. By using the Lie brackets we will show that the image of any monomial in $U(L)$ may be rewritten as a sum of standard ones.

6.2.2. **Definition.** Let $U_p(L)$ denote the canonical image in $U(L)$ of

$$\bigoplus_{0 \le k \le p} T^k(L)$$

that is,

$$U_0(L) = F$$
$$U_1(L) = F \oplus \sigma(L)$$

If I is a linearly ordered set one can define the *defect* as the number of terms in a standard monomial that is 'out of place' relative to there order in the index set. More precisely,

6.2.3. **Definition.** Let $d =$**defect**$(l_{i_1} \otimes ... \otimes l_{i_p})$ denote the number of pairs (r, s) such that $1 \le r < s \le p$ but $i_r > i_s$.

We write $U_p^d(L)$ for the linear span formed by the image of all monomials of degree p and defect d. That the defect is zero if and only if the monomial is standard, moreover $U_p(L)$ is obviously the sum over all possible defects $U_p^d(L)$.

6.2.4. **Definition.** Denote the canonical image of l_i in $U(L)$ by u_i and set $u_I = u_{i_1} ... u_{i_p}$. For any integer i we write $i \le I$ if $i \le i_1, ..., i \le i_p$.

6.2.1. **The Poincaré-Birhoff-Witt theorem.** *Let L be a Lie algebra and let $U(L) = T(L)/J$ be the universal enveloping algebra of L. Suppose that $\{l_1, ..., l_p\}$ is an ordered basis of L. Then $u_I = u_{i_1} ... u_{i_p}$ form a basis of $U(L)$ as a vector space over F.*

Before attending to prove the theorem in general we will first do a study of the cases where t is a monomial of defect at most 2 and degree at most 4, and show that all such monomials can be uniquely rewritten in terms of standard ones, to highlight some of the important steps of the proof. In fact the proof will take the form of an induction over the defect and the degree.

6.2.5. **Example.** For $p = 0, 1$ the situation is clear.

If $p = 2$ every monomial $x \otimes y$ with $x > y$, may be rewritten according to

$$x \otimes y = y \otimes x + [x, y]$$

so every none standard monomial can obviously be rewritten in the standard form since the commutator is in $U_1(L)$.

If $p = 3$, there are two possibilities for $x \otimes y \otimes z$ not to be standard it can either have defect 1 or 2. First consider the case $p = 3, d = 1$. Let us assume $x > y$ but $x, y < z$. Using the previous result

$$x \otimes y \otimes z = (y \otimes x + [x, y]) \otimes z = y \otimes x \otimes z + [x, y] \otimes z$$

the first term on the right-hand side being standard and the second of lower degree.

The other possibilities follow analogously. There is also the second case where $p = 3, d = 2$. That is $x > y > z$, there is now two ways to rearranges them either first interchanging x and y (then x and z and finally y and z) or first interchanging y and z. So it remains to show not only that the monomial can be rewritten in this way but also that rewriting in terms of standard monomials are unique.

$$\begin{aligned} x \otimes y \otimes z &= (y \otimes x + [x, y]) \otimes z = \\ &\quad y \otimes x \otimes z + [x, y] \otimes z = \\ &\quad y \otimes z \otimes x + y \otimes [x, z] + [x, y] \otimes z = \\ &\quad z \otimes y \otimes x + [y, z] \otimes x + y \otimes [x, z] + [x, y] \otimes z \end{aligned}$$

The last three terms may be rewritten again using the same formula

$$\begin{aligned} x \otimes y \otimes z &= z \otimes y \otimes x + [y, z] \otimes x + y \otimes [x, z] + [x, y] \otimes z = \\ &\quad z \otimes y \otimes x + x \otimes [y, z] + [[y, z], x] + [x, z] \otimes y + [y, [x, z]] + z \otimes [x, y] + [[x, y], z] \end{aligned}$$

Using the second order for rearrangement

$$\begin{aligned} x \otimes y \otimes z &= x \otimes z \otimes y + x \otimes [y, z] = \\ &\quad z \otimes x \otimes y + [x, z] \otimes y + x \otimes [y, z] = \\ &\quad z \otimes y \otimes x + z \otimes [x, y] + [x, z] \otimes y + x \otimes [y, z] \end{aligned}$$

so the two rearrangements differ by a factor

$$[[y, z], x] + [y, [x, z]] + [[x, y], z]$$

which is, exactly the Jacobi identity i.e. it vanishes.

Last consider the case $p = 4$ and $d = 2$ but the two defects do not interact. Again there are two possible ways to do this

$$\begin{aligned} x \otimes y \otimes z \otimes q &= y \otimes x \otimes z \otimes q + [x, y] \otimes z \otimes q = \\ &\quad y \otimes x \otimes q \otimes z + y \otimes x \otimes [z, q] + [x, y] \otimes z \otimes q = \\ &\quad y \otimes x \otimes q \otimes z + y \otimes x \otimes [z, q] + [x, y] \otimes q \otimes z + [x, y] \otimes [z, q] \end{aligned}$$

and

$$\begin{aligned} x \otimes y \otimes z \otimes q &= x \otimes y \otimes q \otimes z + x \otimes y \otimes [z, q] = \\ &\quad y \otimes x \otimes q \otimes z + [x, y] \otimes q \otimes z + x \otimes y \otimes [z, q] = \\ &\quad y \otimes x \otimes q \otimes z + x \otimes y \otimes [z, q] + [x, y] \otimes q \otimes z + [x, y] \otimes [z, q] \end{aligned}$$

so they are in fact identical.

Proof. The proof progresses through a series of lemmas. To show that $u_I = u_{i_1} ... u_{i_p}$ form a basis for $U(L)$ one must show that the elements are linearly independent over $U(L)$ and that they span $U(L)$. We start with proving the latter. For a monomial not to be standard it must have some indices which are not correctly ordered, that is, there exist a least one index j such that $i_j > i_{j+1}$. As noted previously the canonical image of the monomials span $U(L)$ and from the following lemma, we shall see, that so do the standard monomials.

6.2.2. **Lemma.** *Let $l_1, .., l_p \in L$, σ be the canonical mapping of L into $U(L)$ and let π be a permutation of the numbers 1,...,p, then*

$$\sigma(l_1)...\sigma(l_p) - \sigma(l_{\pi(1)})...\sigma(l_{\pi(p)})$$

is in $U_{p-1}(L)$.

Proof. By induction over the degree of tensors and, for every fixed degree, by induction over the defect, it suffices to consider the case when $\pi = (j, j+1)$, that is the transposition of j and $j+1$. Which follows directly from the equality

$$\sigma(l_j)\sigma(l_{j+1}) - \sigma(l_{j+1})\sigma(l_j) = \sigma([l_j, l_{j+1}])$$

□

Proving the linearly independence of standard monomials take some more work. Let P be the algebra $F[r_1, .., r_n]$ of polynomials in n indeterminates $r_1, .., r_n$. For any non-negative integer i let P_i be the set of elements of P of degree less or equal to i. If $I = (i_1, .., i_p)$ is a sequence of integers between 1 and n, let $r_I = r_{i_1} r_{i_2} ... r_{i_p}$.

6.2.6. **Definition.** A **representation** of a Lie algebra in a vector space V is a bilinear map

$$L \times V \to V$$

$$(l, v) \to l \circ v$$

such that

$$l_1 \circ (l_2 \circ v) - l_2 \circ (l_1 \circ v) = [l_1, l_2] \circ v$$

for any $l_1, l_2 \in L$ and $v \in V$

By the next lemma we will show that there exists a bilinear map ψ of $L \times P$ into P such that

$$\psi(l_i, r_I) = r_i r_I, \quad \forall i \leq I$$

$$\psi(l_i, \psi(l_j, r_J)) = \psi(l_j, \psi(l_i, r_J)) + \psi([l_i, l_j], r_j) \quad \forall i, j, J$$

that is, there exists a representation g of L in P such that

$$g(l_i) r_I = r_i r_I \quad \forall i \leq I$$

Since the existence of a map $g : L \times P \to P$ is equivalent to the existence of a map $g : L \to \mathbf{End}(P)$ and since, by the universal property of $U(L)$, there exists a unique algebra homomorphism i of $U(L)$ into $\mathbf{End}(P)$ such that $g = i \circ \sigma$ we conclude that

$$r_i r_I = g(l_i) r_I = (i \circ \sigma(l_i)) r_I = i(u_i) r_I \quad \forall i \leq I$$

If $i_1 \leq i_2 \leq ... \leq i_p$ we may deduce step by step that since the elements r_I, for any increasing I, are independent in P so are the elements u_I, and we have

$$i(u_{i_1} ... u_{i_p}) \cdot 1 = r_{i_1} ... r_{i_p}$$

where 1 is the identity in P.

The following lemma shows that such a representation exists.

6.2.3. **Lemma.** *For any non-negative integer p, there exists a unique homomorphism ψ from $L \otimes P_p$ to P such that*

(1) $\psi_p(l_i \otimes r_I) = r_i r_I$ *for* $i \leq I, r_I \in P_p$

(2) $\psi_p(l_i \otimes r_I) - r_i r_I \in P_q$ *for* $r_I \in P_q, q \leq p$

(3) $\psi_p(l_i \otimes \psi_p(l_j \otimes r_J)) = \psi_p(l_j \otimes \psi_p(l_i \otimes r_J)) + \psi_p([l_i, l_j] \otimes r_J)$ *for* $r_J \in P_{p-1}$

The restriction of ψ_p to $L \otimes P_{p-1}$ is ψ_{p-1}. (Note that the terms in condition (3) are meaningful by virtue of (2).)

Proof. The lemma is shown by induction on p.

Let $p = 0$, from (1) we have $\psi_p(l_i \otimes 1) = r_i$ and the remaining conditions follows immediately.

Assume the existence and uniqueness of ψ_{p-1} for some p greater then zero. First note that if ψ_p exists, then the restriction of ψ_p to $L \otimes P_{p-1}$ satisfies (1)-(3) so it is equal to ψ_{p-1}.

Thus it remains to show ψ_{p-1} has a unique extension that satisfies conditions (1)-(3). That is we must define $\psi_p(l_i \otimes r_I)$ for any increasing sequence I of length p. Suppose $i \leq I$, from (1) $\psi_p(l_i \otimes r_I)$ must be defined as $r_i r_I$, on the other hand if $i > I$ let j be the first element of I, delete j from I and call the new sequence J, that is, $I = (j, J)$. Then $i > j \leq J$ and

$$r_I = r_j r_J = \psi_{p-1}(l_j \otimes r_J)$$

so from (1)

$$\psi_p(l_i \otimes \psi_{p-1}(l_j \otimes r_J)) = \psi_p(l_i \otimes r_I)$$

and from (3)

$$\psi_p(l_i \otimes r_I) = \psi_p(l_j \otimes \psi_{p-1}(l_i \otimes r_J)) + \psi_{p-1}([l_i, l_j] \otimes r_J)$$

By (2) $\psi_{p-1}(l_i \otimes r_J) = r_i r_J + w$ for $w \in P_{p-1}$ so

$$\psi_p(l_i \otimes r_I) = r_j r_i r_I + \psi_p(l_j \otimes w) + \psi_{p-1}([l_i, l_j] \otimes r_J)$$

This is a unique extension of ψ_{p-1} that satisfies (1)-(2) plus (3) for $i > j \leq J$. Because of anti-symmetry $[l_i, l_j] = -[l_j, l_i]$ condition (3) is also satisfied for $i < j \leq J$. Since (3) is trivially true for $i = j$, we conclude that (3) is true if $i \leq J$ or $j \leq J$. Suppose neither $i \leq J$ nor $j \leq J$ are satisfied. Clearly the length of J is grater then zero, so let k be the first element of J, delete k from J, and call the new sequence K, that is $J = (k, K)$, then $k \leq K$ and $k < i, k < j$. By the induction assumption

$$\begin{aligned}\psi_p(l_i \otimes l_j) &= \psi_p(l_j \otimes \psi_p(l_k \otimes r_K)) = \\ &\quad \psi_p(l_k \otimes \psi_p(l_j \otimes r_K)) + \psi_p([l_j, l_k] \otimes r_K)) = \\ &\quad \psi_p(l_k \otimes r_j r_K) + \psi_p(l_k \otimes w) + \psi_p([l_j, l_k] \otimes r_K))\end{aligned}$$

where $w = \psi_p(l_j \otimes r_K) - r_j r_K \in P_{p-2}$. Therefore,

$$\begin{aligned}\psi_p(l_i \otimes \psi_p(l_j \otimes l_J)) &= \psi_p(l_i \otimes \psi_p(l_k \otimes r_j r_K)) + \\ &\quad \psi_p(l_i \otimes \psi_p(l_k \otimes w)) + \psi_p(l_i \otimes \psi_p([l_j, l_k] \otimes r_K))\end{aligned}$$

since $k \leq K$ and $k < j$ (3) can be applied to the first term on the right-hand side. By the induction assumption (3) may also be applied to the other two terms. This yields

$$\begin{aligned}\psi_p(l_i \otimes \psi_p(l_j \otimes l_J)) &= \psi_p(l_k \otimes \psi_p(l_i \otimes \psi_p(l_k \otimes r_K))) + \psi_p([l_i, l_k] \otimes (l_j \otimes r_K)) + \\ &\quad \psi_p([l_j, l_k] \otimes (l_i \otimes r_K)) + \psi_p([l_i [l_j, l_k]] \otimes r_K)\end{aligned}$$

Interchanging i and j and cancelling term by term

$$\begin{aligned}
\psi_p(l_i \otimes \psi_p(l_j \otimes l_J)) - \psi_p(l_j \otimes \psi_p(l_i \otimes l_J)) &= \psi_p(l_k \otimes (\psi_p(l_i \otimes \psi_p(l_j \otimes r_K)) - \psi_p(l_j \otimes \psi_p(l_i \otimes r_K))) \\
&\quad + \psi_p([l_i[l_j, l_k]] \otimes r_K) - \psi_p([l_j[l_i, l_k]] \otimes r_K) = \\
&\quad \psi_p(l_k \otimes \psi_p([l_i, l_j] \otimes r_K)) + \psi_p([l_i[l_j, l_k]] \otimes r_K) - \\
&\quad \psi_p([l_j[l_i, l_k]] \otimes r_K) = \\
&\quad \psi_p([l_i, l_j] \otimes \psi_p(l_k \otimes r_K)) + \psi_p([l_k[l_i, l_j]] \otimes r_K) + \\
&\quad \psi_p([l_i[l_j, l_k]] \otimes r_K) - \psi_p([l_j[l_i, l_k]] \otimes r_K)
\end{aligned}$$

using the Jacobi identity

$$\psi_p(l_i \otimes \psi_p(l_j \otimes l_J)) - \psi_p(l_j \otimes \psi_p(l_i \otimes l_J)) = \psi_p([l_i, l_j] \otimes r_K)$$

Thus (3) for all possibilities. □

So the elements $u_I = u_{i_1} ... u_{i_p}$ are linearly independent over $U(L)$ and that they span $U(L)$ which completes the proof. □

6.3. **The universal enveloping algebra as a filtered algebra.** Let L be a Lie algebra over F, and let $U(L)$ be the universal enveloping algebra over L. As before, let $U_p(L)$ denote the canonical image in $U(L)$ of $\bigoplus_{k=0}^{p} T^k(L)$, then the $U_p(L)$ are subspace of $U(L)$ and defines an increasing filtration of $U(L)$

$$U_0(L) \subset U_1(L) \subset ... \subset U_p(L) \subset ...$$

However $U(L)$ is not a graded algebra.

6.4. **The graded algebra associated with the universal enveloping algebra.** Let $U^{gr}(L)$ denote *the graded algebra associated with* $U(L)$, in accordance with section 3.6, that is

$$U^{gr}(L) = \sum U_n^{gr}(L)$$

where

$$U_n^{gr}(L) = U_n(L)/U_{n-1}(L)$$

and the map: $U_m(L) \times U_n(L) \to U_{m+n}(L)$ given by $(a, b) \to ab$ defines (by passage to quotient) a bilinear map: $U_m^{gr}(L) \times U_n^{gr}(L) \to U_{m+n}^{gr}(L)$, so $U^{gr}(L)$ is a graded algebra.

By the PBW theorem we have a canonical isomorphism as vector spaces between the universal enveloping algebra $U(L)$ and the symmetric algebra $S(L)$ and an isomorphism of algebras between the graded algebra associated with the universal enveloping algebra $U^{gr}(L)$ and the symmetric algebra $S(L)$. So we can state that the symmetric algebra is the graded algebra associated with $U(L)$.

6.5. **The universal enveloping algebra as a Hopf algebra.** Let us now show that the universal enveloping algebra $U(L)$ has a Hopf algebra structure.

Define the multiplication $\psi : U(L) \otimes U(L) \to U(L)$ by

$$\psi(a_1 \otimes a_2) = a_1 \star a_2$$

for all $a_1, a_2 \in U(L)$. This clearly defines an associative algebra since (U(L),$\star$) is a qoutient of the associative tensor algebra $T(L)$ by an ideal.

Define the co-multiplication $\Delta : U(L) \to U(L) \otimes U(L)$ by setting its action on 1 and L to be

$$\Delta(1) = 1 \otimes 1$$

$$\Delta(a) = 1 \otimes a + a \otimes 1$$

for all $a \in L \subset U(L)$ and extend to all of U(L) through

$$\Delta(a_1 \star ... \star a_n) = \Delta(a_1) \star ... \star \Delta(a_n)$$

where each $\Delta(a_i) \subset U(L) \otimes U(L)$. One may check that this is well defined, take for example $a_1 * a_2$ then

$$\begin{aligned}
(\Delta \otimes id)\Delta(a_1 * a_2) \quad = \quad & (\Delta \otimes id)(1 \otimes a_1 * a_2 + a_2 \otimes a_1 + a_1 \otimes a_2 + a_1 * a_2 \otimes 1) = \\
& \Delta(1) \otimes a_1 * a_2 + \Delta(a_2) \otimes a_1 + \Delta(a_1) \otimes a_2 + \Delta(a_1 * a_2) \otimes 1 = \\
& 1 \otimes 1 \otimes a_1 * a_2 + 1 \otimes a_2 \otimes a_1 + a_2 \otimes 1 \otimes a_1 + \\
& 1 \otimes a_1 \otimes a_2 + a_1 \otimes 1 \otimes a_2 + 1 \otimes a_1 * a_2 \otimes 1 + \\
& a_2 \otimes a_1 \otimes 1 + a_1 \otimes a_2 \otimes 1 + a_1 * a_2 \otimes 1 \otimes 1
\end{aligned}$$

$$\begin{aligned}
(id \otimes \Delta)\Delta(a_1 * a_2) \quad = \quad & (id \otimes \Delta)(1 \otimes a_1 * a_2 + a_2 \otimes a_1 + a_1 \otimes a_2 + a_1 * a_2 \otimes 1) = \\
& 1 \otimes \Delta(a_1 * a_2) + a_2 \otimes \Delta(a_1) + a_1 \otimes \Delta(a_2) + a_1 * a_2 \otimes \Delta(1) = \\
& 1 \otimes 1 \otimes a_1 * a_2 + 1 \otimes a_2 \otimes a_1 + 1 \otimes a_1 \otimes a_2 + \\
& 1 \otimes a_1 * a_2 \otimes 1 + a_2 \otimes 1 \otimes a_1 + a_2 \otimes a_1 \otimes 1 + \\
& a_1 \otimes 1 \otimes a_2 + a_1 \otimes a_2 \otimes 1 + a_1 * a_2 \otimes 1 \otimes 1
\end{aligned}$$

It is well known that the antipode S on $U(L)$ is given by

$$S(1) = 1$$

$$S(a_1 \star ... \star a_n) = (-1)^n a_n \star a_{n-1} \star ... \star a_1$$

se for example [Sh,St] for a detailed proof.

6.6. **A generalisation of the Poncaré-Birkhoff-Witt theorem.** Let us now look at the problem from a new perspective. We know that for any Lie algebra L there exists an isomorphism as vector spaces between the symmetric algebra $S(L)$ and the universal enveloping algebra $U(L)$. Now note that the definition of the universal enveloping algebra makes sense for arbitrary vector space equipped with any skew-symmetric map $[,] : \Lambda^2 V \to V$. We define the universal enveloping algebra of $(V, [,])$ as before,

$$U(V) = T(V) / < v_i \otimes v_j - v_j \otimes v_i - [v_i, v_j] >$$

There is a natural projection:

$$\pi : T(V) \to U(V)$$

define by

$$v_1 \otimes ... \otimes v_n \to v_1 \star ... \star v_n$$

where $v_1 \star ... \star v_n$ denotes the equivalence class of $[v_1 \otimes ... \otimes v_n]$ mod $< v_i \otimes v_j - v_j \otimes v_i - [v_i, v_j] >$ so there exists a canonical map

$$s : S(V) \to U(V)$$

defined by

$$v_1 \odot ... \odot v_n \to \sum_{\sigma \in S_n} \frac{1}{p!} v_{\sigma(1)} \star ... \star v_{\sigma(p)}$$

We know that if V is a Lie algebra then the map s is an isomorphism by the PBW theorem. Now we can invert this statement.

6.6.1. **The Poincaré-Birkhoff-Witt theorem (*a generalisation*).** *Let V be a vector space equipped with an arbitrary skew-symmetric map $[,] : \Lambda^2 V \to V$, then the map*

$$s : S(V) \to U(V)$$

is an isomorphism if and only if $(V, [,])$ is a Lie algebra.

Proof. By the PBW theorem it remains to show that if s is an isomorphism then $(V, [,])$ is a Lie algebra. Assume that s is an isomorphism. We have already shown that the symmetric algebra and the universal enveloping algebra have natural filtrations such that the mapping s preserves them. If s is an isomorphism so is $s_1 : S_1(V) \to U_1(V)$

$$F \oplus V \to F \oplus \pi(V)$$

Since the map $[,]$ is anti-symmetric it suffices to show that it fulfils the Jacobi identity. Since the Jacobi identity holds trivially for $dimV < 3$, assume $dimV \geq 3$ and let e_1, e_2, e_3 be distinct base vectors in V.

For any pair of vectors (v_i, v_j) in $U(L)$ we have the identity

$$v_i \star v_j = v_j \star v_i + [v_i, v_j]$$

So in particular for the base vectors e_1, e_2, e_3 we have,

$$\begin{aligned}(e_1 \star e_2) \star e_3 &= e_2 \star e_1 \star e_3 + [e_1, e_2] \star e_3 = \\ &\quad e_2 \star e_3 \star e_1 + e_2 \star [e_1, e_3] + [e_1, e_2] \star e_3 = \\ &\quad e_3 \star e_2 \star e_1 + [e_2, e_3] \star e_1 + e_2 \star [e_1, e_3] + [e_1, e_2] \star e_3\end{aligned}$$

and

$$e_1 \star (e_2 \star e_3) = e_3 \star e_2 \star e_1 + e_3 \star [e_1, e_2] + [e_1, e_3] \star e_2 + e_1 \star [e_2, e_3]$$

and their difference may be written as

$$\begin{aligned}(e_1 \star e_2) \star e_3 - e_1 \star (e_2 \star e_3) &= [e_2, e_3] \star e_1 + e_2 \star [e_1, e_3] + [e_1, e_2] \star e_3 - \\ &\quad e_3 \star [e_1, e_2] + [e_1, e_3] \star e_2 + e_1 \star [e_2, e_3] = \\ &\quad ([e_2, e_3] \star e_1 - e_1 \star [e_2, e_3]) + (e_2 \star [e_1, e_3] - \\ &\quad [e_1, e_3] \star e_2) + ([e_1, e_2] \star e_3 - e_3 \star [e_1, e_2]) = \\ &\quad [[e_2, e_3], e_1] + [e_2, [e_1, e_3]] + [[e_1, e_2], e_3] = \\ &\quad [[e_1, e_2], e_3] + [[e_2, e_3], e_1] + [[e_3, e_1], e_2]\end{aligned}$$

Let $\{e_i\}_{i\in I}$ be a basis in V, then

$$[e_i, e_j] = \sum_{k\in I} c_{i,j}^k e_k$$

for some $c_{i_i}^k \in F$. Now $[[e_1, e_2], e_3] + [[e_2, e_3], e_1] + [[e_3, e_1], e_2]$ may be written as

$$\sum_{m,l} (c_{1,2}^m c_{m,3}^l + c_{2,3}^m c_{m,1}^l + c_{3,1}^m c_{m,2}^l) e_l = \sum \lambda^l(c) e_l$$

which is in fact in $U_1(V)$. But U(V) is associative so $(v_1 \star v_2) \star v_3 - v_1 \star (v_2 \star v_3) = 0$ that is

$$\sum \lambda^l(c) e_l = 0$$

Now $\{e_i\}$ form a basis for $U_1(V)$ if and only if $\lambda^l(c) = 0$ for all l. But $U_1(V)$ is isomorphic to V, as a vector spaces, since $S_1(V) = V$ and s_1 is an isomorphism. So since $\{e_i\}_{i\in I}$ are linearly independent in V they are also linearly independent in $U_1(V)$, and the Jacobi identity holds for all elements in V making $(V, [,])$ a Lie algebra. □

6.6.2. **Corollary.** *Assume that V is finite dimensional, then (in the assumption of Theorem 6.5.1) we have*

$$dim U_n^{gr}(V) \leq dim S_n(V)$$

Proof. Let us construct

$$\sum \lambda^l(c) e_l = 0$$

as in the last paragraph of the proof above. The $\{e_i\}$ forms a basis for $U_1(V)$ if and only if $\lambda^l(c) = 0$ for all l. But $\lambda^l(c) = 0$ for all l exactly when V satisfies the Jacobi identity so $dim U_1(V) \leq dim V = dim S_1(V)$. □

6.6.1. **Remark.** Equality holds if and only if $(V, [,])$ is an Lie algebra, so the universal enveloping algebra gets smaller if the Jacobi identity does not hold for the underlying algebra.

7. Combinatorial graphs as a Hopf algebra

7.1. **Combinatorial graphs.** We will start by introducing the basic concepts of combinatorial graphs.

7.1.1. **Definition.** A **graph** $\mathcal{G}$ is defined by two set

(1) a finite set $V_{\mathcal{G}}$ called **vertices**.
(2) a finite set $F_{\mathcal{G}}$ called **flags**.

such that

(a) each flag $f \in F_{\mathcal{G}}$ is incident to exactly one vertex $v \in V_{\mathcal{G}}$, denoted $v = \delta_{\mathcal{G}}(f)$, defining a map $\delta_{\mathcal{G}} : F_{\mathcal{G}} \to V_{\mathcal{G}}$.
(b) some pairs of flags may be connected, that is, there is a function $j_{\mathcal{G}} : F_{\mathcal{G}} \to F_{\mathcal{G}}$ such that $(j_{\mathcal{G}})^2 = id$.

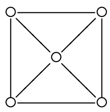
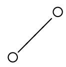
7.1.2. **Definition.** Pairs of flags $f \neq f'$ with $j_{\mathcal{G}}(f) = f'$ form a set called **edges**, denoted $E_{\mathcal{G}}$.

7.1.3. **Definition.** We call $\mathcal{G}$ **connected** if its geometric realisation is such.

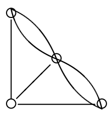
7.1.4. **Definition.** Let $\mathcal{G}$ and $\mathcal{G}'$ be two graphs. The graph $\mathcal{G}'$ is called a **subgraph** of $\mathcal{G}$ if, $\mathcal{G}'$ is non-empty, $V_{\mathcal{G}'} \subseteq V_{\mathcal{G}}$ and $E_{\mathcal{G}'} \subseteq E_{\mathcal{G}}$, where each edge in $E_{\mathcal{G}'}$ is incident with vertices in $V_{\mathcal{G}'}$.

7.1.5. **Definition.** Let $\mathcal{G}'$ be a subgraph of $\mathcal{G}$ different from the empty graph and from the whole of $\mathcal{G}$, the **contracted graph**, denoted $\mathcal{G}/\mathcal{G}'$, is the graph obtained by shrinking all edges of $\mathcal{G}'$ to one point, inside $\mathcal{G}$.

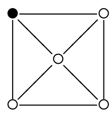
7.1.6. **Example.** Let $\mathcal{G}$ and $\mathcal{G}'$ be the two graphs,

$\mathcal{G} =$  *and* $\mathcal{G}' =$ 

such that $\mathcal{G}'$ is the subgraph of $\mathcal{G}$ consisting of the centre vertex and vertex in the up right corner of together with the edge connecting them. The contracted graph $\mathcal{G}/\mathcal{G}'$ is then,

$\mathcal{G}/\mathcal{G}' =$ 

7.2. **Hopf algebra of admissible graphs.** Let $\mathcal{G}$ be an arbitrary graph and select among its vertices one witch will be "special" and market by $\bullet$.

$\mathcal{G} =$ 

7.2.1. **Definition.** Call such a graph with one "special" vertex **admissible**. A subgraph $\mathcal{G}' \subset \mathcal{G}$ will be called admissible if it contains that special vertex.

Let G_C denote the set of all connected graphs under the condition that every vertex has at least one edge connected to it. Define a function $\Delta : G_C \to G_C \otimes G_C$ by

$$\Delta 1 = 1 \otimes 1$$

$$\Delta \mathcal{G} = 1 \otimes \mathcal{G} + \mathcal{G} \otimes 1 + \sum_{\mathcal{G}_A \subset \mathcal{G}} \mathcal{G}_A \otimes \mathcal{G}/\mathcal{G}_A$$

Here summation is over all possible proper admissible subgraphs $\mathcal{G}'$, that is, all except the empty graph and the whole of $\mathcal{G}$. Using the convention that whenever the "special" vertex and some other vertex is contracted the new vertex will inherit the property of being "special".

7.2.1. **Theorem.** *The co-product Δ of an admissible graph is co-associative*

Proof. We wish to show that this Δ is a coassociative, that is,

$$(\Delta \otimes Id)\Delta\mathcal{G} = (Id \otimes \Delta)\Delta\mathcal{G}$$

using he definition of Δ we get,

$$\begin{aligned}
(\Delta \otimes Id)\Delta\mathcal{G} &= (\Delta \otimes Id)(1 \otimes \mathcal{G} + \mathcal{G} \otimes 1 + \sum_{\mathcal{G}_A \subset \mathcal{G}} \mathcal{G}_A \otimes \mathcal{G}/\mathcal{G}_A) = \\
& 1 \otimes 1 \otimes \mathcal{G} + \Delta(\mathcal{G}) \otimes 1 + \sum_{\mathcal{G}_A \subset \mathcal{G}} \Delta(\mathcal{G}_A) \otimes \mathcal{G}/\mathcal{G}_A = \\
& 1 \otimes 1 \otimes \mathcal{G} + 1 \otimes \mathcal{G} \otimes 1 + \mathcal{G} \otimes 1 \otimes 1 + \sum_{\mathcal{G}_A \subset \mathcal{G}} \mathcal{G}_A \otimes \mathcal{G}/\mathcal{G}_A \otimes 1 \\
& + \sum_{\mathcal{G}_A \subset \mathcal{G}} 1 \otimes \mathcal{G}_A \otimes \mathcal{G}/\mathcal{G}_A + \sum_{\mathcal{G}_A \subset \mathcal{G}} \mathcal{G}_A \otimes 1 \otimes \mathcal{G}/\mathcal{G}_A + \\
& \sum_{\mathcal{G}_B \subset \mathcal{G}_A \subset \mathcal{G}} \mathcal{G}_B \otimes \mathcal{G}_A/\mathcal{G}_B \otimes \mathcal{G}/\mathcal{G}_A
\end{aligned}$$

as well as

$$\begin{aligned}
(Id \otimes \Delta)\Delta\mathcal{G} &= (Id \otimes \Delta)(1 \otimes \mathcal{G} + \mathcal{G} \otimes 1 + \sum_{\mathcal{G}_A \subset \mathcal{G}} \mathcal{G}_A \otimes \mathcal{G}/\mathcal{G}_A) = \\
& 1 \otimes \Delta(\mathcal{G}) + \mathcal{G} \otimes 1 \otimes 1 + \sum_{\mathcal{G}_A \subset \mathcal{G}} \mathcal{G}_A \otimes \Delta(\mathcal{G}/\mathcal{G}_A) = \\
& 1 \otimes 1 \otimes \mathcal{G} + 1 \otimes \mathcal{G} \otimes 1 + \mathcal{G} \otimes 1 \otimes 1 + \sum_{\mathcal{G}_A \subset \mathcal{G}} 1 \otimes \mathcal{G}_A \otimes \mathcal{G}/\mathcal{G}_A + \\
& \sum_{\mathcal{G}_A \subset \mathcal{G}} \mathcal{G}_A \otimes 1 \otimes \mathcal{G}/\mathcal{G}_A + \sum_{\mathcal{G}_A \subset \mathcal{G}} \mathcal{G}_A \otimes \mathcal{G}/\mathcal{G}_A \otimes 1 + \\
& \sum_{\mathcal{G}_{B'} \subset \mathcal{G}/\mathcal{G}_A} \mathcal{G}_A \otimes \mathcal{G}_{B'} \otimes (\mathcal{G}/\mathcal{G}_A)/\mathcal{G}_{B'}
\end{aligned}$$

so it remains to show that

$$\sum_{\mathcal{G}_B \subset \mathcal{G}_A \subset \mathcal{G}} \mathcal{G}_B \otimes \mathcal{G}_A/\mathcal{G}_B \otimes \mathcal{G}/\mathcal{G}_A = \sum_{\mathcal{G}_A \subset \mathcal{G}, \mathcal{G}_{B'} \subset \mathcal{G}/\mathcal{G}_A} \mathcal{G}_A \otimes \mathcal{G}_{B'} \otimes (\mathcal{G}/\mathcal{G}_A)/\mathcal{G}_{B'}$$

That is, that there is an one-to-one correspondence between the set $\{\mathcal{G}_B \subset \mathcal{G}_A \subset \mathcal{G}\}$ of all pairs of admissible subgraphs $\mathcal{G}_A, \mathcal{G}_B$, such that $\mathcal{G}_B \subset \mathcal{G}_A$, and the set $\{\mathcal{G}_A \subset \mathcal{G}, \mathcal{G}_{B'} \subset (\mathcal{G}/\mathcal{G}_A)\}$ of all pairs of admissible subgraphs $\mathcal{G}_A, \mathcal{G}_{B'}$, such that $\mathcal{G}_A \subset \mathcal{G}$ and $\mathcal{G}_{B'} \subset (\mathcal{G}/\mathcal{G}_A)$

Let $\mathcal{G}_B$ denote the pre-image in $\mathcal{G}$ of the subgraph $\mathcal{G}_{B'}$ in $\mathcal{G}/\mathcal{G}_A$, that is, the subgraph in $\mathcal{G}$ such that its image in $\mathcal{G}/\mathcal{G}_A$ is $\mathcal{G}_{B'}$.

Construct two functions $\psi : \{\mathcal{G}_B \subset \mathcal{G}_A \subset \mathcal{G}\} \to \{\mathcal{G}_A \subset \mathcal{G}, \mathcal{G}_{B'} \subset (\mathcal{G}/\mathcal{G}_A)\}, \phi : \{\mathcal{G}_A \subset \mathcal{G}, \mathcal{G}_{B'} \subset (\mathcal{G}/\mathcal{G}_A)\} \to \{\mathcal{G}_B \subset \mathcal{G}_A \subset \mathcal{G}\}$ such that

$$\psi(\mathcal{G}_B \subset \mathcal{G}_A \subset \mathcal{G}) = \{\mathcal{G}_B \subset \mathcal{G}, (\mathcal{G}_A/\mathcal{G}_B) \subset (\mathcal{G}/\mathcal{G}_B)\}$$

$$\phi(\mathcal{G}_A \subset \mathcal{G}, \mathcal{G}_{B'} \subset \mathcal{G}/\mathcal{G}_A) = \{\mathcal{G}_A \subset \mathcal{G}_B \subset \mathcal{G}\}$$

Let us show that the two composition maps are in-fact the identity

$$\phi \circ \psi(\mathcal{G}_B \subset \mathcal{G}_A \subset \mathcal{G}) = \phi(\{\mathcal{G}_B \subset \mathcal{G}, (\mathcal{G}_A/\mathcal{G}_B) \subset (\mathcal{G}/\mathcal{G}_B)\})$$

Using the notation $\mathcal{G}_{A'}$ for $\mathcal{G}_A/\mathcal{G}_B$ to mirror the notation above, so that the pre-image of $\mathcal{G}_{A'}$ is $\mathcal{G}_A$ we get

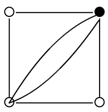
$$\phi(\{\mathcal{G}_B \subset \mathcal{G}, (\mathcal{G}_A/\mathcal{G}_B) \subset (\mathcal{G}/\mathcal{G}_B)\}) = \{\mathcal{G}_B \subset \mathcal{G}_A \subset \mathcal{G}\}$$

that is $\phi \circ \psi = id$. Reversing the order

$$\psi \circ \phi(\mathcal{G}_A \subset \mathcal{G}, \mathcal{G}_{B'} \subset \mathcal{G}/\mathcal{G}_A) = \psi(\{\mathcal{G}_A \subset \mathcal{G}_B \subset \mathcal{G}\}) = \{\mathcal{G}_A \subset \mathcal{G}, (\mathcal{G}_B/\mathcal{G}_A) \subset (\mathcal{G}/\mathcal{G}_A)\}$$

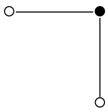
once again making us of the fact that $\mathcal{G}_B$ denotes the pre-image of $\mathcal{G}_{B'}$ in $\mathcal{G}/\mathcal{G}_A$, so also $\psi \circ \phi = id$ which completes the proof. $\square$

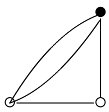
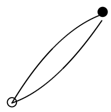
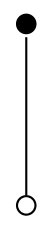
7.2.2. **Example.** As an example, let us study the following admissible graph

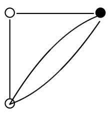
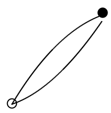
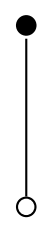
$\mathcal{G} =$ 

and all possible sets of admissible subgraphs $\{\mathcal{G}_A \subset \mathcal{G}, \mathcal{G}_{B'} \subset \mathcal{G}/\mathcal{G}_A\}$ and $\{\mathcal{G}_B \subset \mathcal{G}_A \subset \mathcal{G}\}$.

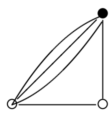
Let us begin with the letter, there are 3 possible admissible subgraphs $\mathcal{G}_A$ each of which has 2 admissible subgraph $\mathcal{G}_B$ such that $\mathcal{G}_B \subset \mathcal{G}_A$.

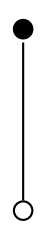
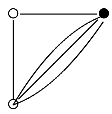
$\mathcal{G}_A =$  *with subgraphs* $\mathcal{G}_B =$ *or* $\mathcal{G}_B =$ 

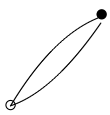
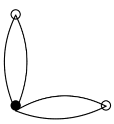
$\mathcal{G}_A =$  *with subgraphs* $\mathcal{G}_B =$  *or* $\mathcal{G}_B =$ 

$\mathcal{G}_A =$  *with subgraphs* $\mathcal{G}_B =$  *or* $\mathcal{G}_B =$ 

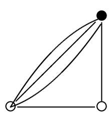
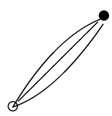
If we instead study the first set there are ones again 3 admissible subgraphs $\mathcal{G}_A$ such that there is a subgraph $\mathcal{G}_{B'} \subset \mathcal{G}/\mathcal{G}_A$.

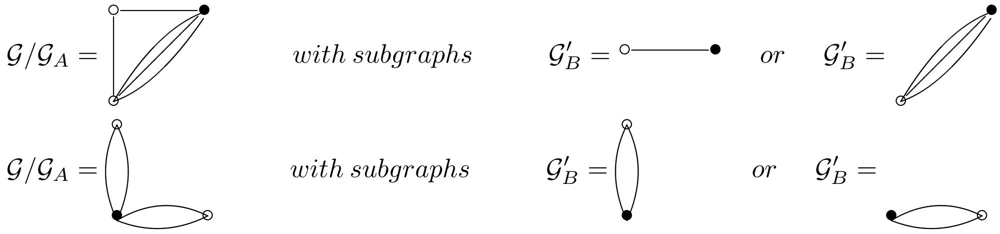
$\mathcal{G}_A =$ *which gives* $\mathcal{G}/\mathcal{G}_A =$ 

$\mathcal{G}_A =$  *which gives* $\mathcal{G}/\mathcal{G}_A =$ 

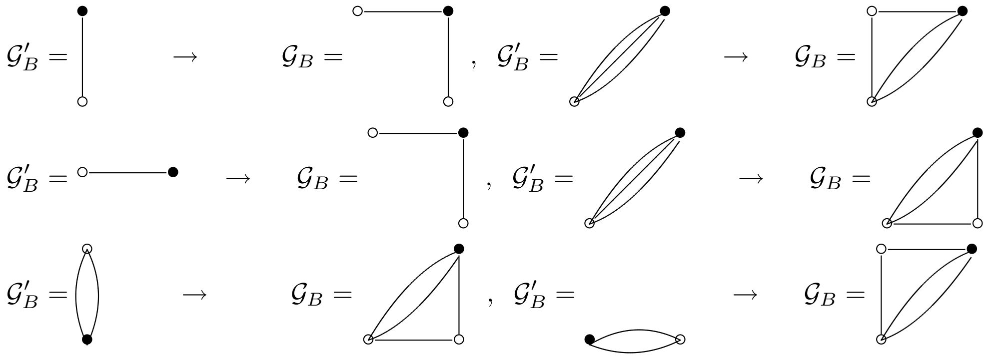
$\mathcal{G}_A =$  *which gives* $\mathcal{G}/\mathcal{G}_A =$ 

each $\mathcal{G}/\mathcal{G}_A$ has 2 admissible subgraphs $\mathcal{G}'_B$

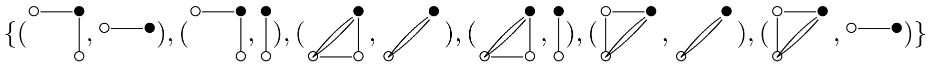
$\mathcal{G}/\mathcal{G}_A =$  *with subgraphs* $\mathcal{G}'_B =$  *or* $\mathcal{G}'_B =$ 



so both sets consists of six elements. To see the one-to-one correspondents consider the pre-image of all the graphs $\mathcal{G}_{B'}$ respectively.



so they both equal to the set



7.2.3. **Remark.** It is a necessary condition for the graph $\mathcal{G}$ to be admissible for Δ to be co-associative. If not the one-to-one correspondence will fail since the set $\{\mathcal{G}_A \subset \mathcal{G}, \mathcal{G}_{B'} \subset (\mathcal{G}/\mathcal{G}_A)\}$ of all pairs of subgraphs $\mathcal{G}_A, \mathcal{G}_{B'}$, such that $\mathcal{G}_A \subset \mathcal{G}$ and $\mathcal{G}_{B'} \subset (\mathcal{G}/\mathcal{G}_A)$ is getting bigger including subgraphs $\mathcal{G}_{B'}$ such that the pre-image $\mathcal{G}_B$ does not necessarily obey the relation $\mathcal{G}_A \subset \mathcal{G}_B$

Let G be the vector space generated by all linear combinations of connected graphs. We now wish to extend Δ to include the whole of G. Let $\mathcal{G}$ be some non-connected graph such that $\mathcal{G} = (\mathcal{G}_i, \mathcal{G}_j)$ (the disjoint union of the two graphs $\mathcal{G}_i$ and $\mathcal{G}_j$) and set

$$\Delta(\mathcal{G}) = \Delta(\mathcal{G}_i)\Delta(\mathcal{G}_j)$$

Both $\mathcal{G}_i$ and $\mathcal{G}_j$ may be non-connected but repeating this procedure eventually jails

$$\Delta(\mathcal{G}) = \Delta(\mathcal{G}_1)\Delta(\mathcal{G}_2), ..., \Delta(\mathcal{G}_n)$$

for $(\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_n)$ connected. Since $\Delta(\mathcal{G}_i) \in G \otimes G$ for connected $\mathcal{G}_i$, so is $\Delta(\mathcal{G})$.

7.2.2. **Corollary.** *The co-product Δ on linear combinations of admissible graph is co-associative*

Proof. We wish to show $\Delta \otimes Id(\Delta(\mathcal{G})) = Id \otimes \Delta(\Delta(\mathcal{G}))$ for some arbitrary $\mathcal{G} \in G$. Since $\mathcal{G}$ is in G its a linear combination of some connected graphs $\mathcal{G}_1, \mathcal{G}_2 ... \mathcal{G}_n$, using the definition we have

$$\Delta(\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_n) = \Delta(\mathcal{G}_1)\Delta(\mathcal{G}_2), ..., \Delta(\mathcal{G}_n)$$

so

$$\Delta \otimes Id(\Delta(\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_n)) = \Delta \otimes Id(\Delta(\mathcal{G}_1))\Delta \otimes Id(\Delta(\mathcal{G}_2))...\Delta \otimes Id(\Delta(\mathcal{G}_n))$$

But $\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_n$, are in G_C so $\Delta \otimes Id(\Delta(\mathcal{G}_i)) = Id \otimes \Delta(\Delta(\mathcal{G}_i))$ for all $1 \leq i \leq n$ which gives,

$$\begin{aligned}\Delta \otimes Id(\Delta(\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_n)) &= \Delta \otimes Id(\Delta(\mathcal{G}_1))\Delta \otimes Id(\Delta(\mathcal{G}_2))...\Delta \otimes Id(\Delta(\mathcal{G}_n)) = \\ & Id \otimes \Delta(\Delta(\mathcal{G}_1))Id \otimes \Delta(\Delta(\mathcal{G}_2))...Id \otimes \Delta(\Delta(\mathcal{G}_n)). = \\ & Id \otimes \Delta(\Delta(\mathcal{G}_1, \mathcal{G}_2, .., \mathcal{G}_n))\end{aligned}$$

Which completes the proof. □

We wish to show that vector space of all linear combinations of connected admissible graphs, with at least one edge, may be given a Hopf algebra structure. Define the co-product $\Delta : G \to G \otimes G$ as above and the product $\psi : G \otimes G \to G$ as the disjoint union of graphs (this product is clearly associative). Let the unit element $\mathbf{1}$ of G br the empty graph (the graph with no vertices), define the unit map $\mu : F \to G$ by $\mu(\lambda) = \lambda\mathbf{1}$ and the co-unit map $\varepsilon : G \to F$ by being zero on all of $G_{n\geq 1}$ and on G_0 $\varepsilon(\lambda\mathbf{1}) = \lambda$.

By construction we have $\Delta(\mathcal{G}, \mathcal{G}') = \Delta(\mathcal{G})\Delta(\mathcal{G}')$ for any two elements $\mathcal{G}, \mathcal{G}' \in G$, so $(G, \psi, \Delta, \varepsilon, \mu)$ has a bialgebra structure. Infact $(G, \psi, \Delta, \varepsilon, \mu)$ is a *connected graded bialgebra* since it permits a decomposition into subspaces

$$G = \bigoplus_{l \geq 0} G_l$$

where G_l is the set of all graph with l edges, such that both the product and co-product respects this grading, since for any $\mathcal{G} \in G_n$ and $\mathcal{G}' \in G_m$, clearly

$$\psi(\mathcal{G}, \mathcal{G}') \in G_{n+m}$$

and for any $\mathcal{G} \in G_n$

$$\Delta\mathcal{G} = \bigoplus_{n=l+k} G_k \otimes G_l$$

Finally G is connected since the set of graphs with no edge, under the condition that every graph has at least one edge, is the empty set, i.e. $G_0 = F$.

7.2.3. **Theorem.** *Admissible graphs has a Hopf algebra structure*

Proof. Follows immediately from Theorem 5.3.1. Since $(G, \psi, \Delta, \varepsilon, \mu)$ is a connected graded bialgebra its also a Hopf algebra. □

7.3. **Trees.**

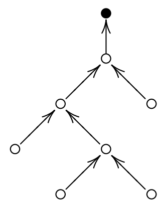
7.3.1. **Definition.** A **tree** $\mathcal{T}$ is defined by the following data;

(1) a finite set $V_\mathcal{T}$ called **vertices**.
(2) a distinguished element $r_\mathcal{T} \in V_\mathcal{T}$ called **root vertex**.
(3) two sets $V^i_\mathcal{T}$ called **internal vertices** and $V^t_\mathcal{T}$ called **tail vertices**.

such that

(a) $V^i_\mathcal{T}$ and $V^t_\mathcal{T}$ are both subsets of $V_\mathcal{T} \setminus \{r_\mathcal{T}\}$.
(b) $V_\mathcal{T} = \{r_\mathcal{T}\} \cup V^i_\mathcal{T} \cup V^t_\mathcal{T}$
(c) there is a map a $N_\mathcal{T} : V_\mathcal{T} \to V_\mathcal{T}$ for which $N_\mathcal{T}(r_\mathcal{T}) = r_\mathcal{T}$, for all $v \in V_\mathcal{T}$, $N^k_\mathcal{T}(v) = r_\mathcal{T}$ for $k >> 1$, and there exists a unique vertex $v \in V_\mathcal{T}$, $v \neq r_\mathcal{T}$ such that $N_\mathcal{T} = r_\mathcal{T}$.

7.3.2. **Definition.** Pairs $(v, N_{\mathcal{T}}(v))$, where $v \neq r_{\mathcal{T}}$ form a set called **edges**, denoted $E_{\mathcal{T}}$.



Trees are a subset of our G. Moreover such trees have naturally a distinguished vertex i.e. the root vertex. It is easy to see that our Δ preserves this subspace and hence makes it into a Hopf algebra. This Hopf algebra has been studied by Connes-Kreimer who used it to give a new renormalization procedure in a class of QFT.

References

[Con, Kr 1] A. Connes, D. Kreimer *Hopf algebras, renormalization and noncommutative geometry*, arXiv:hep-th/9808042v1(1998).

[Con, Kr 2] A. Connes, D. Kreimer *Renormalization in quantum field theory and the Riemann-Hilbert problem I. The Hopf algebra structure of graphs and the main theorem*, Commun. Math. Phys 210, no. 1, 249-273 (2000).

[Di] J. Dixmier, *Enveloping algebras*, American Mathematical Society (1996).

[Manchon] D. Manchon. *Hopf algebras from basics to applications to renormalization*, preprint arXiv:0408405v2

[Ma] Yu.I. Manin. *Renornalization and Computation I: Motivation and background*, preprint arXiv:09?4.4921v2

[Ko, So] M. Kontsevich and Y. Soibelman, *Deformations of algebras over operads and the Deligne conjecture*, arXiv:math/0001151v1

[Ko, Ma] A. Kostrikin and Y. Manin, *Linear Algebra and Geometry*, Gordon and Breach Science Publisher (1981).

[Ro] S. Roman, *Advanced Linear Algebra*, Springer (2005).

[Se] J-P. Serre, *Lie algebras and Lie groups*, W. A. Benjamin, Inc. (1965).

[Sh,St] S. Shnider and S. Sternberg, *Quantum Groups from coalgebra to Drinfeld algebra*, International Press (1993).

[Va] V.S. Varadarajan, *Lie groups, Lie algebras, and their representations*, Prentice-Hall, Inc. (1974).