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Mathematical models of n-player social dilemmas

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Abstract

The research of *social dilemmas* is a popular research field that combines social science and mathematics. In this field human behavior is studied in situations where people have to choose between doing what is the best for the group and making the best individual choice. These situations can be described in mathematical models and they can be analyzed to predict how humans behave. In game theoretic terms these models form a family of games sharing the same strategic structure. In this report different models of *social dilemmas* reported in the literature have been analyzed. Two new models are proposed and analyzed. First the well-known *common resource dilemma*, which previously has been studied with the assumption that the resource is linearly decreasing, is modified. In this report it has been changed with the assumption that the resource is decreasing nonlinearly. This models a situation where it is more difficult to exploit a resource the less there is left of it. Second, a novel type of social dilemma is described where the *public goods dilemma* and the *common resource dilemma* are combined into the model *the dormitory kitchen dilemma*. For both new models, the analysis includes computation and comparison of strategies and outcomes in the *social optimum* and the *Nash equilibrium*.

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Contents

Chapter 1

Introduction to game theory and social dilemmas

1.1 Introduction

A social dilemma is a situation in which the best individual choice is not what is the best for the group. This makes the situation a dilemma in the sense that the person has to choose between doing what is the best for the group and making the best individual choice. Situations like this can often be seen often in everyday life.

An example from current news is the countries fishing in the Baltic Sea. On one hand the countries could care only for themselves and catch as much fish as possible. On the other hand they could think about the group and catch a reasonable amount of fish. If everyone makes the best individual choice it will end with no fish in the Baltic and the countries will have nothing to fish. However, if one country chooses to fish less to prevent over-fishing it runs the risk of losing, both in the short run (less fish is caught) and in the long run (there is no fish left).

Similar problems occur in many real-life situations and affects individuals as well as organizations and countries. The study of social dilemmas has consequently become an important field of research. The research combines different disciplines such as social science, economics, political science, biology, environmental science and philosophy and it raises interesting mathematical questions, mainly in game theory. An important tool for studying social dilemmas is mathematical modeling, as described below.

The aim of this thesis is to review n-player social dilemmas that have been described in the literature, as well as to make new contributions by developing some new models and analyzing them. In this chapter I give a more detailed introduction to the theoretical framework and a summary of the work I have done.

1.2 A brief introduction to game theory

Game theory was originally developed by mathematicians and economists and it seems that the first formal theorem was formulated by E. Zermelo in the beginning of the 20th century. The theorem was about chess and it was a part of one of the first articles in game theory [15]. However, it is commonly acknowledged that John von Neumann can be seen as the founder of game theory because he and Oskar Morgenstern published the first book in this field [13]. They described the general notation of a game as a situation with a number of players, each of whom can choose between different strategies that yield different outcomes depending on what strategies the other players choose.

A common way to specify a game is to describe its so-called payoff table, which specifies the payoffs to the players of each possible combination of strategies.

The easiest game to illustrate this with is a two-player game. An example is the game *matching pennies* with two players, Kimmo and Pontus. The idea of the game is that Kimmo and Pontus each has a coin which they simultaneously show to each other. Kimmo wins if Pontus' coin shows the same side as him and Pontus wins if they show different sides. In other words they have two strategies to choose between: heads or tails. Pontus wins if they play different strategies and Kimmo wins if they play the same. The one who wins gets 1 point and the one who loses gets -1. The game is illustrated with a payoff table where Kimmo's strategies are the rows and his payoffs are shown in the lower part of the cells. Pontus' strategies are the columns and his payoffs are shown in the upper part of the cells.

	Heads	Tails
Heads	-1 +1	+1 -1
Tails	+1 -1	-1 +1

In general a game may have any number of players, usually denoted by n, and they can choose between different actions or strategies. A combination of one strategy per player is called a strategy profile. If Kimmo plays heads in *matching pennies* and Pontus plays tails, the strategy profile is the combination of heads and tails. To completely specify the game, the payoffs to every player must be known for every possible strategy profile. Every strategy profile is a possible outcome of the game.

However, if people are free to make their choices they will probably try to maximize their payoffs. The aim of game theory is to make mathematically based predictions of how players will behave in such a game.

The key concept is the *Nash equilibrium*, which can be described as follows. Consider the situation when the players in the game are asked to repeatedly play the same game. How will they change their strategies over time? Probably they will change it after every game to receive a higher payoff. What will happen with the strategy profile after a few repetitions? Will it converge to some profile? If so, to what profile?

This problem was attacked by the famous economist and mathematician John Nash. He defined what later became known as a *Nash equilibrium* as any strategy profile such that no player can increase her payoff by unilaterally changing her strategy. Nash formulated and proved the theorem that said that under generous assumptions, a game always has at least one *Nash equilibrium* [8]. Some games have multiple equilibria and in those cases the strategy profile in the repeated game converges to different profiles depending on what happens in the early part of the process.

1.3 The prisoner's dilemma

The *prisoner's dilemma* is probably the most famous of all game theoretic models. It has been studied in thousands of papers across many scientific disciplines. Albert W Tucker may have been the first to produce a mathematical description of the prisoner's dilemma [12]. The dilemma can be formulated in many different ways but an example is as follows:

Kimmo and Pontus are two gangsters, guilty of a major crime. The judge knows that they are guilty but he is unable to convict either of them unless one of them confesses. To make them confess he proposes a deal: If Kimmo confesses and Pontus fails to confess, Kimmo goes free and Pontus is convicted to 10 years in prison. If both confess, both get 9 years in prison and if both fail to confess they will be framed on a tax evasion charge so that each gets 1 year in prison [1].

In the payoff table below coop means that a player fails to confess and defect means that the player confesses.

If Pontus chooses to defect the best response for Kimmo is to also defect. Then he only gets 9 years in prison instead of 10. But if Pontus chooses to coop, the best reply for Kimmo is to defect because then he will go free instead of getting one year in prison on the tax evasion charge. No matter what strategy Pontus may choose, Kimmo will always get a higher payoff by defecting. By symmetry the same argument is valid for Pontus. The *Nash equilibrium* is therefore mutual defection. In the table below the best replies for an action is bolded. To defect is the action that gives the best payoff regardless of what the other player does.

	Defect	Coop
Defect	-9 -9	-10 **0**
Coop	**0** -10	**-1** **-1**

1.4 Why it is a dilemma: social optimum vs. Nash equilibrium

The interesting nature of the *prisoner's dilemma* is the fact that the *Nash equilibrium* is equivalent to mutual defection leading to confession from both of the prisoners (to the judge's delight) despite the fact that if both prisoners were cooperating they would get a better payoff. The latter outcome is known as the *social optimum* of the game.

The *social optimum* of a game is a strategy profile that yields the best possible outcome for the group as a whole. It is a strategy profile that maximizes the sum of the players' payoffs.

In the prisoner's dilemma the *social optimum* is the action of not confessing (coop). If both Kimmo and Pontus fail to confess, both of them will get a good payoff. The sum of their payoffs is maximized when both choose to coop. It is not the best for the judge but he is not a player and it is the best for the group, in this case the prisoners, and that is what *social optimum* is about.

The interesting about this is that the *social optimum* is different from the *Nash equilibrium* and that is why this situation becomes a dilemma. This makes the choice of strategy a choice between optimizing for oneself or attempting to optimize for the group.

For further reading on dilemmas, a good reference is a book by Michael Taylor [11].

1.5 Multi-player social dilemmas

Dilemmas similar to the *prisoner's dilemma* can occur in situations with more than two players. These situations go under the name *social dilemmas* or *collective action dilemmas.* There are several situations in reality that can be described as social dilemmas. One example is television financed by license fees. As individuals we are better off using public television without making any contribution, but if everyone acted like that financing would fail and there would be no public television [7].

Another example is a group of people going to a restaurant, already decided that they are going to split the bill. When each person orders a meal they can choose a meal with a reasonable price for the others to share, on the other hand they can order an expensive meal because they will get a favor from it [3].

In research on social dilemmas, much attention has recently been paid to the work of Elinor Ostrom on common resource dilemmas. These are situations like the fishing in the Baltic sea that was mentioned in the introduction, that is, situations where a common resource (like fish) will be over-exploited if all individuals follow their selfish inclinations, leading to often severely suboptimal outcomes. Elinor Ostrom won the 2009 Nobel Memorial Prize in Economic Sciences for her analysis of how humans sometimes manage to solve the problem of governing behavior in common resource problems to maintain a sustainable use of resources [10].

1.6 Fear and greed

To quantify the difficulty of a dilemma one can compute two measures known as *greed* and *fear*. The *greed* in a dilemma is the strength of the temptation to obtain the best outcome for oneself [7]. It can be seen as the winning for a player who plays the best individual strategy while the others plays *social optimum.*

The *greed* in the *prisoner's dilemma* described above is the units Kimmo raises his payoff with if he chooses to defect instead of cooping. Then he will increase his payoff with one unit.

The *fear* is, analogously the strength of the temptation to avoid being suckered [4]. The fear in this situation is the fear of throwing away one's effort on a lost cause [7]. It can be seen as the loss for a player who plays *social optimum* while one of the others is playing the best individual strategy.

The *fear* in the *prisoner's dilemma* is the amount of units that Kimmo loses if he coops and Pontus defects instead of cooping. In that situation he will go from getting -1 to -10 so the loss for him is 9 units. It can be seen that the *fear* is bigger than the *greed* in this dilemma.

1.7 Summary of contributions

Although all models of social dilemmas share the strategic structure that the *social optimum* is not the same as the *Nash equilibrium*, there may be important differences between different models, both conceptually and mathematically. The aim of this report is to explore such differences between different models of social dilemmas. My work on this is reported in the three remaining chapters.

Chapter 2 reviews the literature on models of social dilemmas with respect to which models are used. To anticipate, it seems that the literature is dominated by just two models: a standard formulation of the *common resource dilemma* (where players choose how much to exploit of a common resource) and a standard formulation of the *public goods dilemma* (where players choose how much to contribute to a public good). The rest of the report is about my own further developments of these models.

The focus in chapter 3 is on the *common resource dilemma* where a common resource is used by several players. In the standard model the value of the resource decreases linearly according to the effort that goes into exploiting the resource. I analyze the consequences of instead assuming that exploitation is less effective when less of the resource remains.

Chapter 4 develops a more radically novel model, *the dormitory kitchen dilemma*. This can be seen as a hybrid of the two dominating models in that the players can both contribute to a common resource and exploit the same resource. From the literature, it seems that this kind of dilemma has not been formulated mathematically before.

Chapter 2

A survey of mathematical models of multi-player social dilemmas

Here I report the results of my review of the literature on models of multi-player social dilemmas. Basically, I have used the search engine Google Scholar to identify any papers that deal with modeling the of social dilemmas. This search turned up no more than four different families of models: *public goods dilemmas*, *common resource dilemmas*, *threshold games* and *network games*. Below I describe these four types of models and the important differences between them. Of these four models, the *public goods dilemma* and the *common resource dilemma* are by far the most commonly used models of social dilemmas in the literature. For these two models I also present their mathematical formulation and derive the *Nash equilibrium* and the *social optimum*.

2.1 The public goods dilemma

In a *public goods dilemma* individuals are faced with an immediate cost that generates a benefit that is shared by all. The best for the individual is to avoid the cost, but the total benefit to the group of the public good is larger than the total cost [7].

An example of a *public goods dilemma* is taxation. As individuals we are better off by not paying the tax but if everyone would think like that there would be no money to finance for example public school and health care.

In this model every player has an endowment of e units to contribute or keep. x_i denotes the contribution from player i and λ is the marginal payoff a player can get. For a social dilemma to occur the marginal payoff λ should lie between $1/n$ and 1. Otherwise there is no benefit for the group to share the public good. The payoff for player i is:

$$\pi_i = (e - x_i) + \lambda \sum_{i=1}^{n} x_i$$

It is easy to see that the *Nash equilibrium* is equivalent to making no contribution but this is how it is derived:
Given what the other players have done the payoff function for player i is:

$$\pi_i = (e - x_i) + \lambda(t^* + x_i)$$
$$t^* = \sum_{j \neq i} x_j$$

The only maximum of this function is on the boundaries and assuming that λ is less than 1 the maximum is when $x_i = 0$. The *Nash equilibrium* strategy, denoted by x_{ne}, is therefore to make no contribution, $x_{ne} = 0$.

To derive the *social optimum* the payoff function where every one makes the same contribution,x, has to be maximized:

$$\pi = (e - x) + nx\lambda$$

Assuming that $\lambda \geq \frac{1}{n}$, so that $n\lambda \geq 1$, the best action for the group is to contribute e. The *social optimum* strategy, denoted by x_{so}, is therefore $x_{so} = e$.

2.2 The common resource dilemma

The common resource dilemma, also called *the tragedy of the commons*, has been known at least since the 19th century but Hardin made it famous in 1968 [6] when he analyzed it in an article published in *Science*. The basic idea of the dilemma is that a group of individuals share a common resource. The resource is limited and the individuals can act in self-interest and choose to exploit just as much as they want from the resource. The result is less of the resource for the other individuals to use and if everyone acts in self-interest the outcome is a disaster where nothing is left of the resource. Therein lies the dilemma.

The standard mathematical formulation of *the common resource dilemma* is as follows. There is a group of n individuals who can choose to spend their time to exploit from a private pool or a common pool. The exploitation of a pool is often called an extraction. They have e units to extract either from the private pool or the common pool. From the private pool they get w per units they extract and from the common pool they get A minus the sum of all players' extractions times B. The payoff for player i is:

$$\pi_i = w(e - x_i) + (A - B\sum_{j=1}^{n} x_j)x_i \tag{2.1}$$

Here x_i denotes how much player i extracts from the common pool. Player i gets a higher payoff the less the other players extract from the common pool. If all players try to extract a lot from the common pool they will receive a low payoff.

The *Nash equilibrium* for the dilemma can be calculated. Ellinor Ostrom gives an example on how it is done for a *common resource dilemma* in one of her articles [9]. Holding the other players' strategies fixed, the payoff to player i is differentiated with respect to x_i:

$$-w + A - B\sum_{j\neq i} x_j - 2x_i$$

The optimal strategy of player i is obtained when this derivative is zero. The *Nash equilibrium* is obtained when all players optimize their strategies. Because the problem is symmetric, in the *Nash equilibrium* all players will play the same strategy x_{ne}, which gives the following equation:

$$A - w - B(n - 1 + 2)x_{ne} = 0 \Longleftrightarrow x_{ne} = \frac{A - w}{B(n+1)}$$

This equation determines the *Nash equilibrium* strategy:

$$x_{ne} = \frac{A - w}{B(n+1)} \tag{2.2}$$

The payoff in *Nash equilibrium* is:

$$\pi_{ne} = \frac{-n(-1+w)^2 + B(1+n)(-1+w)^2 + B^2e(1+n)^2w}{B^2(1+n)^2} \tag{2.3}$$

The *social optimum* is calculated by assuming that all players play the same strategy and then maximizing the payoff.

$$\pi = w(e - x) + (A - Bnx)x$$

Differentiation with respect to x yields:

$$-w + A - 2Bnx$$
$$-w + A - 2Bnx = 0 \Longleftrightarrow x = \frac{A - w}{2Bn}$$

The *social optimum* strategy x_{so} is:

$$x_{so} = \frac{A - w}{2Bn} \tag{2.4}$$

The payoff in *social optimum* is:

$$\pi_{so} = \frac{-(-1+w)^2 + 2B(-1+w)^2 + 4B^2enw}{4B^2n} \tag{2.5}$$

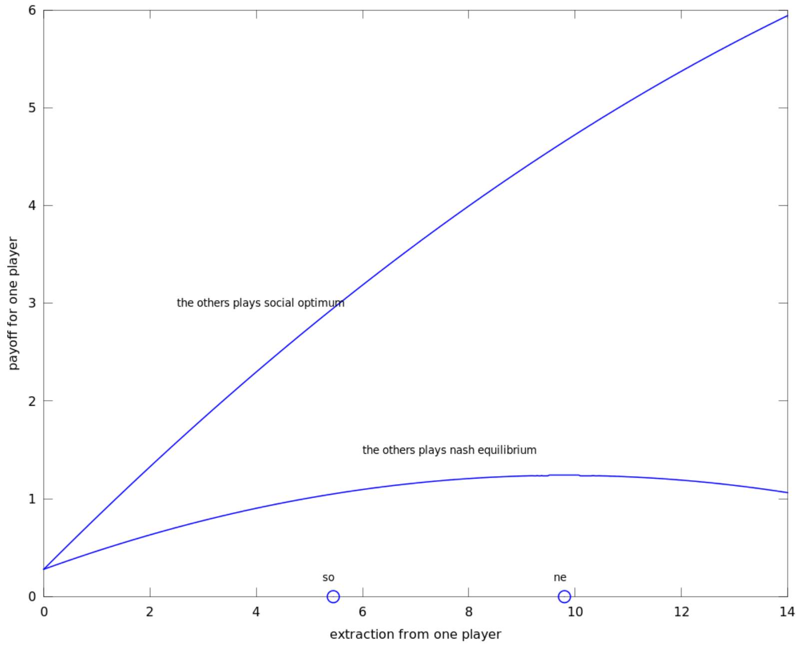

Figure 2.1: Payoff for a player while the others play *Nash equilibrium* or *social optimum*

Figure 2.1 illustrates the superiority of the *social optimum* and the incentive of the player to defect. Assuming that all other players choose either the *social optimum* strategy or the *Nash equilibrium* strategy, the figure shows how the payoff of one player varies with the strategy of that player. Here the parameters are set to: $w = 0.02$, $B = 0.01$, $n = 9$, and $e = 14$. The *social optimum* strategy and the *Nash equilibrium* strategy is marked on the x-axis.

2.3 Threshold games

A threshold game is a variant of *the public goods dilemma*. The difference is that in this case, to get a part from the common goods the contributions has to meet a certain threshold level [5]. An example of a dilemma of this kind is the spread of a rumor. People have to hear a rumor to pass it forward to a new person, but people vary in their credulity so to believe it they may have to hear it from a couple of others before they can spread it [2].

This threshold level affects the *Nash equilibrium* and the *social optimum*. In this game there are two kinds of *Nash equilibria*, one is the same as in the *public goods dilemma*, to make no contribution and the other one is to contribute the amount that is required to reach the level. There is therefore an infinite amount of *Nash equilibria* since there is an infinite amount of combinations of the contributions from every player that sum up to the level. The only symmetric *Nash equilibrium* is for each player to contribute an identical amount [2].

The *social optimum* is the same as in the *public goods dilemma*, to make the largest possible contribution. However, versions of *threshold games* have been proposed where there is no extra benefit of exceeding the threshold. For instance, if a group of people is going to build a house together and are contributing by working. The level that has to be reached is when the house is ready. After that there is no need to contribute with more than that level so the *social optimum* in such cases is the same as a *Nash equilibrium* (but the presence of the other *Nash equilibrium*, at zero contributions, still makes it a kind of social dilemma).

2.4 Network games

A fourth family of models of social dilemmas is formed by the *network games* that were introduced by Yamagishi and Cook [14]. In these games, people can choose to give resources to a person to which they are connected in a network. In the example given by Yamagishi and Cook, the network is a directed n-cycle, where every player is given an endowment and can choose to give some of it to the successor in the cycle, in which case the donated sum is multiplied by some factor less than one (as in the *public goods dilemma*). This is a social dilemma, because the *Nash equilibrium* is obviously that everyone makes zero donations whereas the *social optimum* is obviously that everyone donates the entire endowment. It is different to the *public goods dilemma* in that each player can affect only one other player's decision and is affected by only one other player's decision, which will have impact on *fear* (but not on *greed*).

Chapter 3

Analysis of a common resource dilemma with diminishing efficiency of resource extraction

In the *common resource dilemma* treated in chapter 2, it is difficult to make sense of the terms of the equation. The value of spending an effort on exploiting the common pool resources decreases with exploitation. This seems to suggest that the exploitation effort becomes increasingly less effective as the common resources decreases. How much a player can exploit from the resource should be affected by how much the players before have been exploiting. But if this is so, then it is not realistic that the decrease is linear with the effort because if exploitation becomes increasingly less effective, then the rate with which the resource decreases ought to decrease in a corresponding way. In other words the resource should decrease nonlinearly.

An example is a group of fishermen who has a common pool where they can catch fish. The amount of fish they can get when they choose to fish from the common resource depends on how much time the others have spent there before. If the others spend much time there it is harder to get any fish. An additional unit of effort will be less effective in decreasing the amount of fish.

In this chapter the payoff function is adapted according to the assumption that the resource is decreasing nonlinearly. The new function is analyzed and compared with the one in section 2.2 to see how the new assumption affects the dilemma. At first the new payoff function is derived and then the *Nash equilibrium* and the *social optimum* is derived.

3.1 Payoff function

The payoff function is here derived in terms of the fishing example. The amount of fish in the sea is normalized to 1. One unit of fishing will yield a catch that is a fixed proportion v of the current amount of fish in the sea. It follows that the amount of fish that is still left after j units of fishing is:

$$(1-v)^j$$

In a total of t units of fishing, the average catch per unit is:

$$\frac{1}{t}\sum_{j=1}^{t}(1-v)^{j-1}v$$

Summation of the geometric series yields an average catch per unit:

$$\frac{1-(1-v)^t}{t} \tag{3.1}$$

To obtain the payoff function corresponding to (2.1), set $u = 1 - v$ and let t denote the total exploitation effort of all players. Then the payoff function is obtained:

$$\pi_i = w(e - x_i) + \frac{1-u^t}{t}x_i \tag{3.2}$$

To obtain a dilemma w has to be smaller than v because otherwise no player will want to spend even a single unit of effort on the common pool.

3.2 Social optimum

The *social optimum* is calculated by using the same method as in section 2.1 and 2.2. If all players choose the same strategy the payoff function is:

$$w(e-x) + \frac{1-u^{nx}}{nx}x$$

The *social optimum* occurs when the derivative with respect to x is zero:

$$\begin{aligned} &-w - u^{nx}\log u \\ &-w - u^{nx}\log u = 0 \iff \\ x_{so} &= \frac{\log(-\frac{w}{\log u})}{n\log u} \end{aligned} \tag{3.3}$$

$$\tag{3.4}$$

The total exploitation effort in the social optimum is $t = nx_{so}$. Plugging these values of t and x_{so} into the payoff function, and simplifying, yields the following payoff in the social optimum:

$$\pi_{so} = \frac{w + \log(u) + enw\log(u) - w\log(-\frac{w}{\log(u)})}{n\log(u)} \tag{3.5}$$

3.3 Nash equilibrium

To obtain the Nash equilibrium, fix the strategies for the other players and let $t^* = \sum_{j \neq i}^{n} x_j$ be their total exploitation effort. Differentiation with respect to x_i yields the following equation for the optimal strategy of player i:

$$-w - \frac{(1-u^{t^*+x_i})x_i}{(t^*+x_i)^2} + \frac{1-u^{t^*+x_i}}{t+x_i} - \frac{u^{t^*+x_i}x_i\log u}{t^*+x_i}$$

To find the symmetric *Nash equilibrium* all players are treated the same. Then the equation takes the form:

$$-w - \frac{(1-u^{nx})x}{(nx)^2} + \frac{1-u^{nx}}{nx} - \frac{u^{nx}x\log u}{nx} = 0$$

Since the equation contains x both in a linear form and in the exponential it is difficult to solve the equation. A Taylor expansion of the term u^{nx} is therefore done, keeping just a few terms, to obtain an approximation of the solution.

$$u^{nx} = 1 + n\log ux + \frac{1}{2}n^2\log u^2x^2 + \frac{1}{6}n^3\log u^3x^3 + O(x)^4$$

Plugging in this approximation into the equation yields a quadratic equation:

$$(-w - \log u) + (-\frac{1}{2}\log u^2 - \frac{1}{2}n\log u^2)x + (-\frac{1}{3}n\log u^3 - \frac{1}{6}n^2\log u^3)x^2 = 0$$

Solving this equation yields the following approximation of the *Nash equilibrium*:

$$x = \frac{(-3\log u^2 - 3n\log u^2 + \sqrt{3}\sqrt{-16nw\log u^3 - 8n^2w\log u^3 + 3\log u^4 - 10n\log u^4 - 5n^2\log u^4})}{(2(2n\log u^3 + n^2\log u^3))}$$

For any given set of parameter values, it is of course possible to solve the original equation numerically, in order to check the accuracy of the approximation.

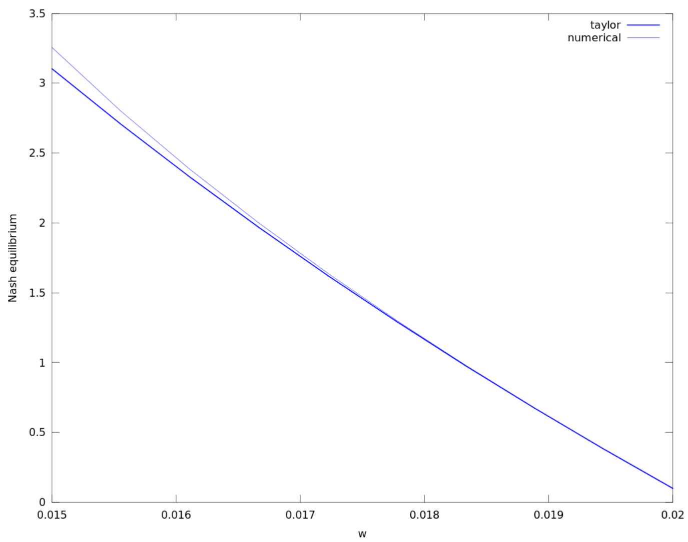

Figure 3.1: The Taylor approximation and the numerical solution

Figure 3.1 shows the approximation together with the numerical solution for different values of w when $n = 9$,$v = 0.02$ and $e = 14$.

3.4 Comparison with the standard model

To get a measurement of the dilemma the ratio between the extraction in the *social optimum* and the *Nash equilibrium*, the *SO:NE* ratio, can be calculated. This ratio shows how much players will need to adapt their equilibrium behavior in order to achieve the social optimum. If the ratio is close to 1 it indicates that the dilemma is not so difficult. In this section the *SO:NE* ratios for the standard model and the new model is compared and a general result for large values of n is concluded: The *SO:NE* ratio is closer to one in the standard model than in the new model, i.e., the dilemma is worse in the new model. In the standard model (from chapter 2) the *SO:NE* ratio is:

$$Q_1 = \frac{x_{so}}{x_{ne}} = \frac{\frac{A-w}{2Bn}}{\frac{A-w}{B(n+1)}} = \frac{n+1}{2n}$$

One can see that for large groups (large values of n) the ratio converges to $\frac{1}{2}$, which may be interpreted as that the players in *Nash equilibrium* must cut their extraction down to half to achieve the *social optimum*.

The *SO:NE* ratio from the new model (using the approximation) is:

$$Q_2 = \frac{x_{so}}{x_{ne}} = 2(2n\log u^3 + n^2 \log u^3)\log(-\frac{w}{\log u})/(n \log u(-3\log u^2 - 3n\log u^2 + \\ + \sqrt{3}\sqrt{-16nw\log u^3 - 8n^2 w \log u^3 + 3\log u^4 - 10n\log u^4 - 5n^2 \log u^4}))$$

The ratio between Q_2 and Q_1 is calculated to see which one of them is the largest.

$$\frac{Q_2}{Q_1} = (4(2n\log u^3 + n^2\log u^3)\log -(\frac{w}{\log u}))/((1+n)\log u(-3\log u^2 - 3n\log u^2 + \\ + \sqrt{3}\sqrt{-16nw\log u^3 - 8n^2 w\log u^3 + 3\log u^4 - 10n\log[u]^4 - 5n^2\log u^4}))$$

Assuming that n is large the equation can be simplified to:

$$\frac{4\log u^2 \log(-\frac{w}{\log u})}{-3\log u^2 + \sqrt{3}\sqrt{-8w\log u^3 - 5\log u^4}}$$

Setting $z = \frac{w}{\log(1-v)}$ the expression can be simplified further:

$$\frac{4\log(-z)}{-3+\sqrt{3}\sqrt{-8z-5}}$$

This expression takes real values only for $z < -5/8$ and by doing a Taylor expansion of $\log(1-v)$:

$$\log(1-v) = -v - \frac{v^2}{2} - \frac{v^3}{3} - \frac{v^4}{4} - \frac{v^5}{5} - \frac{v^6}{6} - \frac{v^7}{7}\dots$$
$$z = -w/(v + \frac{v^2}{2} + \frac{v^3}{3} + \frac{v^4}{4} + \frac{v^5}{5} + \frac{v^6}{6} + \frac{v^7}{7}\dots)$$

From the conditions $w < v$ and $0 < v < 1$ and $z < -5/8$ it follows that z must lie between -1 and -5/8.

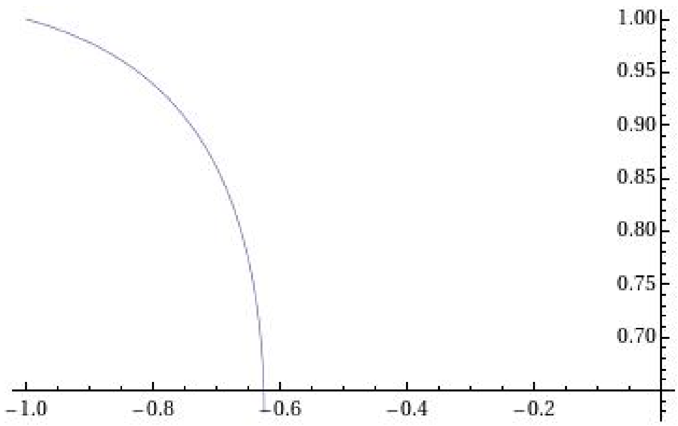

Figure 3.2: Values for different z

Figure 3.2 shows the values of the ratio $\frac{Q_2}{Q_1}$ as z varies between -1 and $-5/8$, demonstrating that the ratio takes values between 0 and 1. This means that Q_2 is less than Q_1. This leads to the conclusion that the distance between the extraction that yields *social optimum* and *Nash equilibrium* is larger in the new model than in the standard model.

3.5 Fear and greed

Other measures of the severity of a dilemma are the *fear* and *greed* measures that were presented in the introduction. To compare the two models with respect to *fear* and *greed*, they must first be normalized in some way that makes them comparable. Here the *Nash equilibrium* is set to be equal in both models. The function (2.1) is normalized by putting $A = 1$. The value of the parameter v has to be chosen, which is arbitrarily set to 0.02. The conditions on the parameters for the game to be a dilemma are: $0 < B < 1 - w$, $0 < w < v$, $e > 0$, and $n \geq 3$.

To begin with the *worst* action is calculated. It is the action that yields the best payoff when the others play *social optimum*. The *fear* is then calculated by taking the difference between the payoff for a player who plays *social optimum* when the others play *social optimum* and the payoff for one who plays *social optimum* when one of the other is playing the *worst*.

The *greed* is calculated by taking the difference between the payoff for a player who plays *social optimum* when the other plays the same and the payoff for a player who plays the *worst* when the other plays *social optimum*.

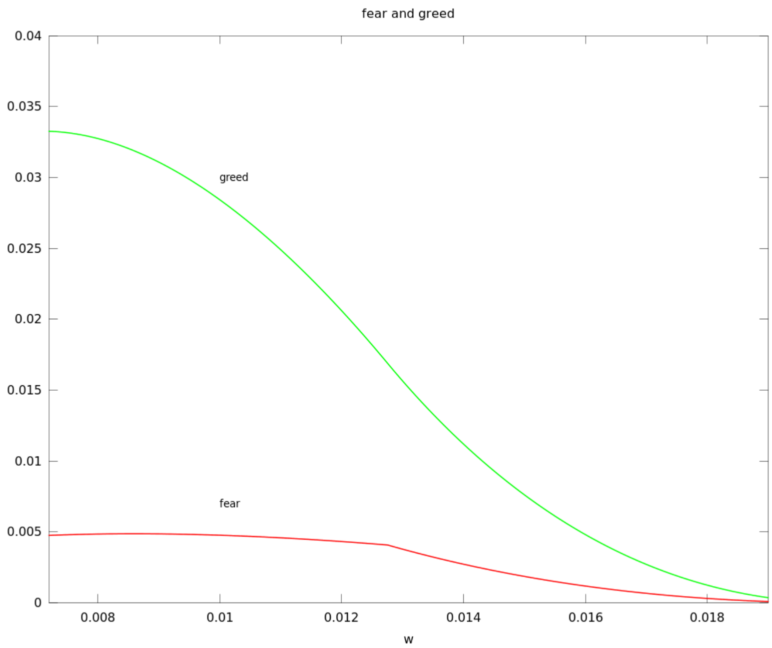

Figure 3.3: Fear and greed for (3.2)

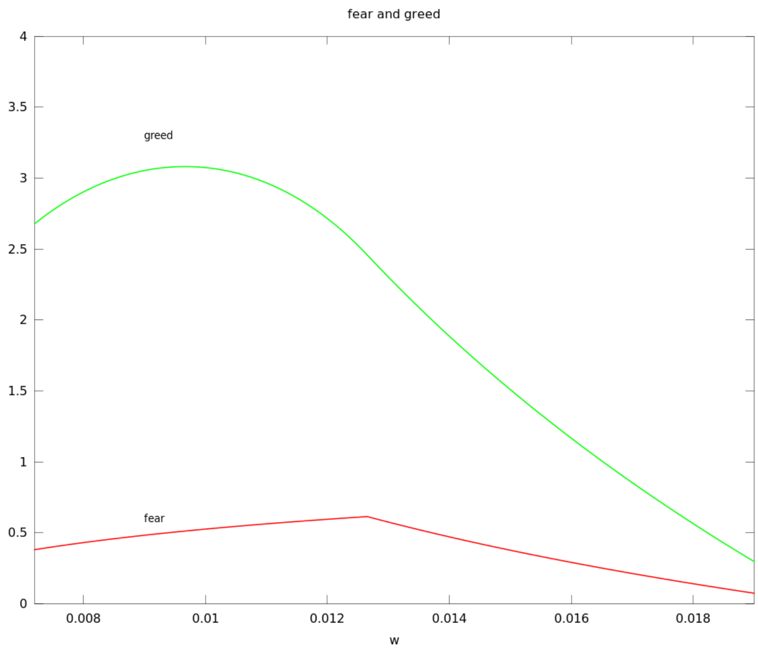

Figure 3.4: Fear and greed for (2.1)

Figure 3.3 shows the *fear* and *greed* for the new model when varying the parameter w. The other parameters are set to: $n = 9$ and $e = 14$. From the figure it seems like the *greed* is greater than the *fear* in this model.

Figure 3.4 shows the same thing for the model from section 2.2. The parameters n and e are the same as above and B is calculated by putting the extraction that yields *Nash equilibrium* to be the same as in the model above. It also seems that the *greed* is greater than the *fear* in this model.

The *fear* and the *greed* have also been calculated while varying n and e and they show the same thing: the *greed* is greater than the *fear* in both models. Interestingly, it seems that both measures behave qualitatively in the same way when parameter values change. However, it remains to be seen if this holds for more general cases.

3.6 Further analysis

The things that can be concluded from the analysis in this chapter is that the distance between the *social optimum* and the *Nash equilibrium* is bigger in the changed model. This can indicate that the dilemma introduced in this chapter is even more serious than in the standard version. In the model in this chapter it requires a larger effort to go from the extraction that yields the *Nash equilibrium* to the extraction that yields the *social optimum.* The fact that the resource is decreasing nonlinearly has consequences for the behavioral change that players must undergo to achieve the *social optimum.*

Here are some ideas that I have not been able to explore in this work, but which may be worthy of further analysis.(1) It is interesting that the ratio for the model in section 2.2 only depends on the group size. No matter what the other parameters are the ratio will always be the same. (2) It would be interesting to find a general relationship between the models in terms of *greed* and *fear.* (3) It would be interesting to analyze the differences between the payoffs for *Nash equilibrium* and *social optimum.* How do they differ between the both models? (4) In these analyses the game is only played once but often the situations that have been described here are repeated. It would be interesting to see what happens if the game is iterated and consider the evolution of the game. (5) In the models in chapter two there are psychological aspects that affects the behavior of the players in the game. There are also some psychological aspects that can be thought of in the model in this chapter. (6) Introducing an underlying nonlinearity in the payoff function may be relevant not only in the *common resource dilemma* but also in the *public goods dilemma.*

Chapter 4

The dormitory kitchen dilemma

The *public goods dilemma* and the *common resource dilemma* are the two most common models of social dilemmas. One of them describes the build-up of a resource and the other one describes the exploitation of the resource. But what if they occur at the simultaneously?

Situations as this occur in real life. An example is a common kitchen in a student dormitory. The people sharing the kitchen are both responsible for the messing up and the cleaning of the kitchen. The dilemma occurs when people mess up but are free riding on the others by not cleaning. This is a common problem in many situations in everyday life and this simple problem irritates people and it often ends up in an argument. It can be seen in the common kitchen at a workplace, at the common patio of an apartment building or in the property of a voluntary association.

Yamagishi and Cook describe these situations as a *general exchange dilemma* and they approach it as a type of *public goods dilemma* [14]. Everyone gives a contribution by working towards a nice and clean kitchen which is then shared by all. The difference from a usual *public goods dilemma* is that it is difficult to share the common good. In other words, how do you divide a clean kitchen? The benefits one can get from a contribution are not directly contingent and Yamagishi and Cook refer to that as *general exchange* and analyze it in their article.

In this chapter a different approach is used. Here the situation is described as a hybrid of the *public goods dilemma* and the *common resource dilemma*. The new model is first described mathematically with a payoff function and it is then analyzed in terms of the *Nash equilibrium* and the *social optimum*.

4.1 Model of the dormitory kitchen dilemma

In this dilemma n people are sharing a kitchen. Player i decides to clean x_i and mess up y_i units. The payoff for cleaning is A times x_i, B times y_i for the messing up. The payoff for having a messy kitchen is C times the mess that remains. The constants A, B and C are marginal payoffs and can be assumed to lie between 0 and 1. To introduce some boundaries on how much a player can mess up y_i can be any number between 1 and 0. That means that x_i can lie between 0 and y_i. The payoff for player i is then:

$$\pi_i = -Ax_i + By_i - C\sum_{i=1}^{n}(y_i - x_i) \tag{4.1}$$

Cleaning the kitchen as well as having a messy kitchen gives a negative payoff but to mess up gives a positive payoff. That should be realistic since it is an advantage for the individual to be able to use the kitchen and mess it up, while not having to clean so much. On the other hand it is not nice to have a dirty kitchen.

4.2 Analysis

The analysis of the model has different outcomes depending on the relation between A,B and C. There are six combinations but not all of them are realistic. This can for example be concluded by considering the case when a single person uses the kitchen and see if it matches the description of the dilemma above.

In the general case the *Nash equilibrium* is calculated by finding the value of x_i and y_i that maximizes the payoff for player i:

$$\pi_i = (C - A)x_i + (B - C)y_i - C\sum_{j \neq i}(y_j - x_j)$$

If $C < A$ and $B < C$ the action that gives the best payoff for player i is $x_i = 0$ and $y_i = 0$. By considering the case of a single person, it can be concluded that this is not realistic. If one person uses the kitchen, it gives a negative payoff both for cleaning and messing up.

It is also not realistic when $C > A$ and $B < C$ because then it gives a positive payoff for cleaning but not for messing up. And people should want to use the kitchen. The *Nash equilibrium* in this case is when $x_i = 0$, $y_i = 0$, because x_i can not be larger than y_i.

If $C > A$ and $B > C$ the *Nash equilibrium* is given by $x_i = 1$ and $y_i = 1$ which means that people would want to clean and mess up as much as possible. This is not completely unrealistic but does not satisfy the description above.

The interesting case is when $C < A$ and $B > C$. This is realistic as the relation gives the *Nash equilibrium* $x_i = 0$ and $y_i = 1$. This means that people want to mess up as much as possible while not cleaning at all. It is also compatible with the single person case.

The *social optimum* is found by maximizing this function where every player has the same strategy, x:

$$\pi_{so} = -Ax + By - C(ny - nx) = Cn(x - y) + By - Ax$$

There are three possible outcomes:

$$x = 1, y = 1 \iff \pi = B - A$$
$$x = 0, y = 1 \iff \pi = B - Cn$$
$$x = 0, y = 0 \iff \pi = 0$$

The *social optimum* for the two cases where $B < C$ is the same and depends only on the relation between A and B. If $B < A$ the *social optimum* is given by $x = 1$ and $y = 1$. If $B > A$ it is given by $x = 0$ and $y = 0$. But the case $B < A < C$ can then be excluded since that case will have the same *Nash equilibrium* as *social optimum* and that does not cause a dilemma.

In the case of $B > C$ it can be assumed that $B < Cn$ because n is probably large and B and C are assumed to be small. The *social optimum* is then determined by the relation between A and B and is then the same as above.

An interesting case is when $C < A < B$ because the *Nash equilibrium* is $x = 0$ and $y = 1$ and the *social optimum* is $x = 1$ and $y = 1$. This means that it is best for the individual not to clean and to mess up as much as possible but for the group it is best to clean and to mess up as much as possible. In the other case where $C < B < A$ it is also best for the individual not to clean and to mess up as much as possible but the best for the group is not to clean and not to mess up.

Intuitively, these last two cases seem to correspond to reality. It should be the best for the individual not to clean and it should be the best for the group if everyone did not mess up or if they do, that they will clean as much as they have messed up.

4.3 Further analysis

This was a simple model of the dilemma and as in chapter 3 there are several aspects that can be further analyzed.

(1) The assumption that the payoff function is linear is probably not realistic and can be changed. It is probably more difficult for a person to start to clean than to continue cleaning once started. (2) The assumption that C is the same for everybody can be changed, since people probably have different opinions about how a clean kitchen should look. (3) There are also psychological aspects like peer pressure. The fear of making the others in the group angry by not cleaning could also be included. If the people who share the kitchen know each other they probably care for each other and will therefore clean more. Often people organize days when everyone helps out to clean the kitchen. This may have a positive affect as people realize that the work could be avoided if everyone cleaned after themselves. (4) It is also common to have rules in a common kitchen saying that one has do a certain amount of work. If there is a penalty for not following the rules it would make it harder for a person to free-ride. (5) In many common kitchens there are some things for everyone's use and some things that are individual property. It can be easier for people to clean their individual property than the common property. (6) Another aspect is that things in a kitchen might break so there are less things to use. There could also be things that do not have to be cleaned on a regularly basis.

As for the models described in chapter 2 and 3 there are many aspects that can be discussed and it is difficult to find a model that completely reflects reality. It is a challenge to model social behavior but it is of great importance for solving problems in society.

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