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Kontsevich's Graph Complex and Operads of Graphs

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Abstract

Kontsevich graph complex is a universal version of the standard deformation complex of the Lie algebra of polyvector fields. It was proved very recently by Thomas Willwacher that the zeroth cohomology of this complex is precisely the Grothendieck-Teichmüller Lie algebra.

We develop an operadic approach to this complex based on the Kapranov-Manin theorem. This gives us relatively simple definitions of all the structures involved in the Kontsevich graph complex.

Acknowledgements

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Contents

0. Introduction

Kontsevich's graph complex is one of the most mysterious complexes in homological algebra and geometry. Kontsevich introduced this complex in [Ko1], together with his famous *formality conjecture* which he solved later in [Ko2].

Any smooth manifold M has an associated Lie algebra of vector fields, $T(M)$, which are derivations of the ring of smooth functions on M equipped with the standard Lie bracket,

$$[X,Y] := X \circ Y - Y \circ X.$$

It is well known that this Lie bracket can be extended to the skew-symmetric tensor algebra of $T(M)$,

$$\mathcal{T}_{poly}(M) := \bigoplus_{n=0}^{\dim M} \wedge^n T(M)$$

and this extension is called the *Schouten bracket*, and is denoted by the same symbol $[\ ,\ ]$. According to Chevalley-Eilenberg, the deformation complex of the Lie algebra $(\mathcal{T}_{poly}(M), [\ ,\ ])$, is equal to the vector space,

$$\bigoplus_{n=0}^{\infty} \mathrm{Hom}(\wedge^n \mathcal{T}_{poly}(M), \mathcal{T}_{poly}(M))$$

with the differential,

$$\begin{array}{rccc} d: & \mathrm{Hom}(\wedge^n \mathcal{T}_{poly}(M), \mathcal{T}_{poly}(M)) & \longrightarrow & \mathrm{Hom}(\wedge^{n+1} \mathcal{T}_{poly}(M), \mathcal{T}_{poly}(M)) \\ & f & \longrightarrow & df \end{array}$$

given (up to signs) by

$$\begin{aligned} df(v_0, v_1, \ldots, v_n) \quad := \quad & \sum_{i=0}^{n} (-1)^i [v_i, f(v_0, v_1, \ldots, \widehat{v_i}, \ldots, v_n)] \\ & \sum_{0 \le i < j \le n} (-1)^{i+j} f([v_i, v_j], v_0, \ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots, v_n). \end{aligned}$$

Note that, though this complex depends on the choice of a particular manifold M, but it makes sense for *any* manifold. Kontsevich made this *universal* nature of the standard Chevalley-Eilenberg deformation complex precise by inventing his famous graph complex GC in [Ko1]. This idea can be made even more precise when one uses the language of operads: there is an operad, $\mathcal{G}$, which admits a canonical representation,

$$\rho : \mathsf{GC} \longrightarrow \mathcal{E}nd_{\mathcal{T}_{poly}(R^d)},$$

into the vector space of polyvector fields on an affine space $\mathbb{R}^d$ for any dimension d. We show that the Kontsevich graph complex is equal to the dg Lie algebra associated to the operad $\mathcal{G}$ by the Kapranov-Manin theorem and controls, therefore, universal (i.e. independent of the dimension d) deformations of the Schouten algebra $(\mathcal{T}_{poly}(R^d),\ [\ ,\ ])$. This operadic approach to the Kontsevich graph complex is the main theme of our work.

There is a strong interest on the Kontsevich graph complex nowadays stemming from a deep result of Willwacher [Wi] which says that the 0-th cohomology of GC is equal to the Grothendieck-Teichmüller Lie algebra,

$$\mathrm{H}^0(\mathsf{GC}) = \mathfrak{grt}.$$

The Grothendieck-Teichmüller group GT is a pro-unipotent group introduced by Drinfel'd in [Dr]. There is much interest in this group in various areas of mathematics, especially in number theory and algebraic geometry, because it contains the absolute Galois group of $\mathbb{Q}$, that is, there exist an injection

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow GT\ .$$

Rather mysteriously the Grothendieck-Teichmüller group (or rather its graded version GRT) appears naturally in two mathematically rigorous quantization theories: the first is the Drinfel'd-Etingof-Kazhdan quantization theory of Lie bialgebras and the second one is the Kontsevich quantization theory of Poisson structures.

There are still many open problems left with the Kontsevich graph complex. It is quite desirable to compute the first cohomology group of that complex, the conjecture is that it is equal to zero, this would mean that the Schouten bracket is rigid, i.e that it can not be deformed in the category of L_∞-algebras. Computer simulations by Willwacher showed that the second cohomology group of this complex is non-zero. The full cohomology of the Kontsevich graph complex is a Lie algebra which contains $\mathfrak{grt}$, it is an open and urgent problem to compute it.

The main purpose of the thesis is to develop an operadic approach to the Kontsevich graph complex.

This thesis is divided up into three sections.

- In the first introductory section we explain those fact of category theory, the theory of graphs and homological algebra that will be needed in the thesis.
- In the second section we describe the basic notions of operads. We discuss the general definition, give the construction of the free operad and finally show some minimal models of operads. We also give a detailed discussion of the Kapranov-Manin Theorem, which associates to an operad in the category chain complexes a Lie algebra (in fact three).
- In the third and main section we will use all the structures discussed in the previous ones to introduce a certain operad of graphs and deduce from it, using the Kapranov-Manin theorem, the Kontsevich graph complex CG and finally give a detailed statement of Willwacher's theorem on the cohomology of the graph complex GC.

1. Preliminaries

1.1. **Basics of Category Theory.** The notions of category theory are pervasive to modern mathematics. In this subsection we will introduce the basics we need in this thesis.

Definition 1.1.1. *A category $\mathcal{D}$ is :*

- *a class of objects $ob(\mathcal{D})$,*
- *a class of objects (called morphisms) $\hom(\mathcal{D})$ and for every $f \in \hom(\mathcal{D})$ two objects $x, y \in ob(\mathcal{D})$, called the source object and target object of f, represented as $f : x \to y$. The class objects of $\hom(\mathcal{D})$ with common source object x and target object y is denoted $\hom(x, y)$*
- *a binary operation $\hom(x, y) \times \hom(y, z) \to \hom(x, z)$, subject to the rules:*
  - *(associativity) $(\hom(x, y) \times \hom(y, z)) \times \hom(z, w) \to \hom(x, w) = \hom(x, y) \times (\hom(y, z) \times \hom(z, w)) \to \hom(x, w)$*
  - *for every object x there exist an identity map id_x such that if $f : x \to y$, then $\mathrm{id}_y \circ f = f \circ \mathrm{id}_x$.*

Example 1.1.2. *Some common categories.*

1. *The prototypical example of a category is the category* Set, *where objects are sets and morphisms are functions between sets.*
2. *The finite sets with functions form the category* Set_{fin}.
3. *Topological spaces form a category with continuous functions as morphisms.*
4. *The modules over a ring R with R-linear homomorphisms form a category.*
5. *Chain complexes of R-modules with chain maps.*

The idea of duality is central in category theory, and we'll see it first in the construction of the *opposite category.*

Definition 1.1.3. *Given a category $\mathcal{C}$ there exist a dual category called the opposite of $\mathcal{C}$ and denoted $\mathcal{C}^{op}$ where $ob(\mathcal{C}) = ob(\mathcal{C}^{op})$ but where maps are reversed.*

Just as we have maps between object in categories we also maps between categories. These maps, called functors, are also expected to preserve the relative structures inside the categories.

Definition 1.1.4. *A covariant functor Γ is a map from a category $\mathcal{D}$ to a category $\mathcal{E}$ such that:*

- *each $D \in ob(\mathcal{D})$ there is an object $\Gamma(D) \in ob(\mathcal{E})$*
- *for each $f : D_1 \to D_2$ there is a map $\Gamma(f) : \Gamma(D_1) \to \Gamma(D_2)$ subject to the rules*
  - *$\Gamma(id_D) = id_{\Gamma(D)}$*
  - *$\Gamma(fg) = \Gamma(f)\Gamma(g)$, when $g : D_1 \to D_2$ and $f : D_2 \to D_3$.*

A contravariant functor $F : \mathcal{D} \to \mathcal{E}$ is a covariant functor $F : \mathcal{D}^{op} \to \mathcal{E}$. Denote the covariant functors from $\mathcal{D} \to \mathcal{E}$ with $Fun(\mathcal{D}, \mathcal{E})$.

Definition 1.1.5. *Let F and G be functors from $\mathcal{D}$ to $\mathcal{E}$. A natural transformation $\nu : F \to G$ is a map ν_x for every $x \in ob(\mathcal{D})$ such that given $f : x \to y$ the following diagram commutes*

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & G(x) \\ \downarrow{\scriptstyle \nu_x} & & \downarrow{\scriptstyle \nu_y} \\ F(y) & \xrightarrow{G(f)} & G(y) \end{array} .$$

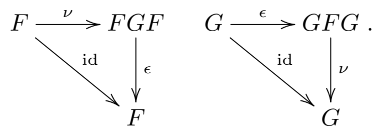
In this case we also say that ν is natural in x.

There is a weak notion of inverse for a functor called the adjoint.

Definition 1.1.6. *A left adjoint for a functor $F : \mathcal{D} \to \mathcal{E}$ is a functor $G : \mathcal{E} \to \mathcal{D}$ and two natural transformations*

i) $\nu : GF \to \mathrm{id}_{\mathcal{D}}$
ii) $\epsilon : \mathrm{id}_{\mathcal{E}} \to FG$

such that the following diagrams commute:

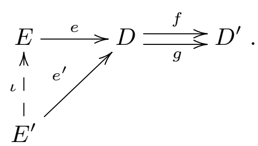


Alternatively we could say that we have a bijection of sets $\phi_{d,e} : \hom_{\mathcal{E}}(Fd, e) \cong \hom_{\mathcal{D}}(d, Ge)$ *for all* $(d, e) \in ob(\mathcal{D}) \times ob(\mathcal{E})$*, which is natural in* e *and* d*.*

Definition 1.1.7. *An equalizer of a diagram*

$$D \underset{g}{\overset{f}{\rightrightarrows}} D'$$

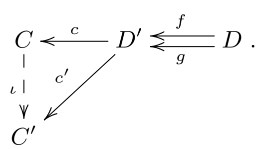
is an object E *and a map* $e : E \to D$ *such that* $fe = ge$ *and for any other object* E' *and map* $e' : E' \to D$ *such that* $fe' = fe'$ *there is a unique map* $\iota : E' \to E$ *which makes the following diagram commute*



The dual to the above construction is the coequalizer. A coequalizer of a digram

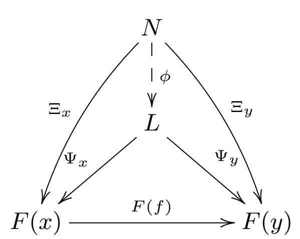
$$D' \underset{g}{\overset{f}{\leftleftarrows}} D$$

is an object C *and a map* $c : D' \to C$ *such that* $cf = cg$ *and for any other object* C' *and map* $c' : D' \to C'$ *such that* $c'f = c'g$ *there is a unique map* $\iota' : C' \to C$ *such that the following diagram commute*

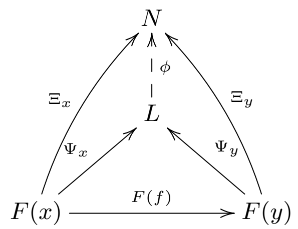


A more general construction which encompasses the equalizer is that of the limit of a functor.

Definition 1.1.8. *Let* $F : \mathcal{D} \to \mathcal{E}$ *be a functor. A limit of* F *is an object* L *and a family of maps* $(\Psi_x : L \to F(x))_{ob(\mathcal{D})}$ *with the property that if* $f : x \to y$ *then* $F(f)\Psi_x = \Psi_y$ *such that given any other object* N *with maps* $(\Xi_x : N \to F(x))_{ob(\mathcal{D})}$ *with the property that if* $f : x \to y$ *then* $F(f)\Xi_x = \Xi_y$ *then it must also exist a unique map* $\phi : N \to L$ *such that the following diagram commutes*



The colimit L is characterized by the diagram produced if you in the above diagram turn all the arrows except $F(f)$ around;



,

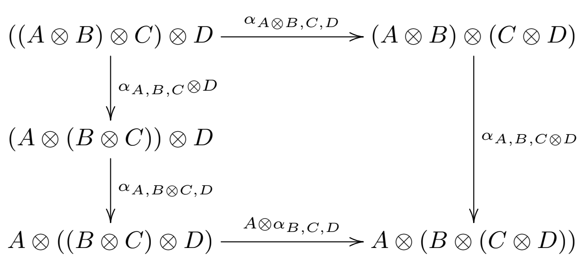
where it is obvious what should be changed in the description to give the diagram meaning.

Definition 1.1.9. *A monoidal category is a category $\mathcal{C}$ with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unit object I together with three natural isomorphisms,*

i) *the associator $\alpha_{A,B,C} : (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$*
ii) *the left unitor $\rho_A : I \otimes A \cong A$*
iii) *the right unitor $\nu_A : A \otimes I \cong A$,*

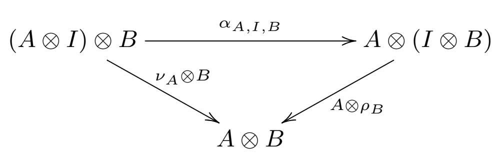
subject to the following coherence conditions:

(1) *For all $A, B, C, D \in \mathcal{C}$ the diagram*



commutes

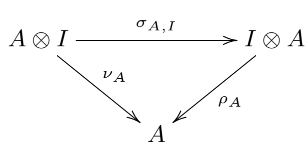
(2) *For all $A, B \in \mathcal{C}$ the diagram*



commutes.

Definition 1.1.10. *A symmetric monoidal category is a monoidal category $\mathcal{C}$ with an isomorphism $\sigma_{A,B} : A \otimes B \cong B \otimes A$ subject to the following coherence conditions:*

(1) *For all $A \in \mathcal{C}$ the digram*



commutes

(2) *for all $A, B, C \in \mathcal{C}$ the diagram*

$$\begin{array}{ccc}
(A \otimes B) \otimes C & \xrightarrow{\sigma_{A,B}\otimes C} & (B \otimes A) \otimes C \\
\downarrow{\scriptstyle \alpha_{A,B,C}} & & \downarrow{\scriptstyle \alpha_{B,A,C}} \\
A \otimes (B \otimes C) & & B \otimes (A \otimes C) \\
\downarrow{\scriptstyle \sigma_{A,B\otimes C}} & & \downarrow{\scriptstyle B\otimes\sigma_{A,C}} \\
(B \otimes C) \otimes A & \xrightarrow{\alpha_{A,B,C}} & B \otimes (C \otimes A)
\end{array}$$

commutes

(3) *for all $A, B \in \mathcal{C}$ the diagram*

$$\begin{array}{ccccc}
 & & B \otimes A & & \\
 & \nearrow^{\sigma_{A,B}} & & \searrow^{\sigma_{B,A}} & \\
A \otimes B & & \overset{A\otimes B}{=\!=\!=\!=} & & A \otimes B
\end{array}$$

commutes.

Example 1.1.11. *The following are symmetric monoidal categories*

- *Sets with the cartesian product,*
- *topological spaces with the cartesian product,*
- *Chain complexes of R-modules (for a commutative ring R) with the product*

$$(C_\bullet, d) \otimes (D_\bullet, d') = \left((C \otimes D)_n = \bigoplus_{i+j=n} C_i \otimes D_j, \partial \right)$$
$$\partial_n : t_i \otimes t_{n-i} \mapsto d_i(t_i) \otimes t_{n-i} + (-1)^i t_i \otimes d'_{n-i}(t_{n-i}).$$

1.2. **Theory of Graphs.** This subsection will contain the graph theoretical framework we need in the study of operads.

Definition 1.2.1. *A graph $G = (F, \Pi, \phi)$ is three pieces of data; a set of flags F, a partition Π of F and an involution $\phi : F \to F$. Where*

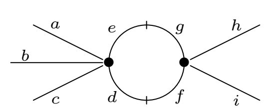
- *the vertices of G, $Vert(G)$, is the blocks of the partition Π,*
- *the edges of G, $Edge(G)$, is the 2-cycles of ϕ and*
- *the legs of G, $Leg(G)$, are the flags of F invariant under ϕ.*

A "classical" graph would be a graph without legs.

Every graph has a *geometric realization* given by

(1) associating to each flag a copy of $[0, 1/2]$,
(2) identifying the points $0 \in [0, 1/2]$ for all flags in the same block and
(3) identifying the points $1/2 \in [0, 1/2]$ for all flags in the same orbit of the involution.

Example 1.2.2. *Let Γ be the graph that has $\{a, b, c, \dots, i\}$ as the set of flags, with partition into blocks $\{a, b, c, d, e\}$, $\{f, g, h, i\}$ and involution $(df)(eg)$. The graphical representation of Γ is*



Definition 1.2.3. *A cycle in a graph G is a collection of vertices $v_1, \dots, v_n, \in Vert(G)$ and a collection of edges $e_1, \dots, e_n \in Edge(G)$ such that e_i is given as the 2-cycle (v_i, v_{i+1}), for $i = 1, \dots, n-1$ and e_n being the 2-cycle (v_n, v_1).*

From cycle follows the notion of *Tree*.

Definition 1.2.4. *A tree is a graph without a cycle.*

We will be using a more specific kind of tree, the *labeled tree*, when we later define the free operad.

Definition 1.2.5. *A rooted tree is a tree T where one leg $l \in Leg(T)$ have been singled out to be the root of the tree, the other legs are called leafs and make up the set $leaf(T)$. Let $vert(T)$ denote the set of internal vertices of T and let $in(v)$ denote the number of edges at an internal vertex minus the one going up, i.e. $in(v) =$ valence of $v - 1$.*

A labeled tree (T, l) is rooted tree T and a bijection $l : leaf(T) \to [n]$.

1.3. **Algebra.** This section will contain basic algebraic theory.

Definition 1.3.1. *An associative algebra A over a field k is a k-vector space with a multiplication map $\mu : A \otimes A \to A$ such that the following diagram commutes:*

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mathrm{id} \otimes \mu} & A \otimes A \\ \downarrow{\scriptstyle \mu \otimes \mathrm{id}} & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow{\mu} & A \end{array}$$

and a unit map $i : k \to A$.

Dual to this is the coalgebra

Definition 1.3.2. *A coassociative coalgebra C over a field k is a k-vector space with a comultiplication map $\Delta : C \to C \otimes C$ such that the following diagram commutes:*

$$\begin{array}{ccc} C \otimes C \otimes C & \xleftarrow{\mathrm{id} \otimes \Delta} & C \otimes C \\ \uparrow{\scriptstyle \Delta \otimes \mathrm{id}} & & \uparrow{\scriptstyle \Delta} \\ C \otimes C & \xleftarrow{\Delta} & C \end{array}$$

and a counit map $i : k \leftarrow C$.

Definition 1.3.3. *A Lie algebra $\mathfrak{g}$ is a vector space over a field k with a binary operation $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that*

- $[ax + by, z] = a[x, z] + b[y, z]$ *for all* $a, b \in k$ *and* $x, y, z \in \mathfrak{g}$
- $[x, x] = 0$ *for all* $x \in \mathfrak{g}$
- $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ *for all* $x, y, z \in \mathfrak{g}$.

Definition 1.3.4. *Let V be a vector space over a field k. We define the tensor algebra of V as the graded vector space $\bigoplus_{n \geq 0} V^{\otimes n}$ and with the multiplication given on monomials $(v_1 \otimes \ldots \otimes v_n) \cdot (u_1 \otimes \ldots \otimes u_n) := v_1 \otimes \ldots v_n \otimes u_1 \otimes \ldots \otimes u_n$.*

The symmetric algebra SV is defined as TV/I where I is the ideal of commutators,

$$I = \{v_i \otimes v_j - v_j \otimes v_i\}.$$

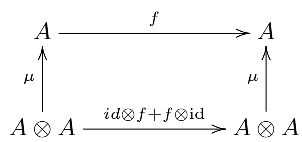
The tensor algebra TV is a coalgebra with comultiplication Δ defined on monomials as

$$\Delta TV \to TV \otimes TV$$

$$\Delta(v_1, \ldots, v_k) \mapsto \sum_{i=0}^{k} (v_1, \ldots, v_i) \otimes (v_{i+1}, \ldots, v_k).$$

The above also makes SV into a coalgebra.

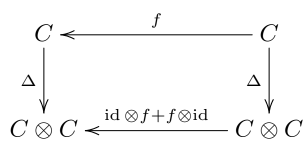
Definition 1.3.5. *A derivation of a k-algebra A is a k-linear map $f : A \to A$ such that the following diagram commutes*



The set of k-derivations of an algebra A form a k-vector space and is denoted $\mathrm{Der}_k(A)$, *or just* $\mathrm{Der}(A)$ *when it's clear from the context what field we are using.*

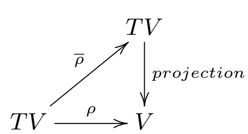
Dual to derivations of algebras are coderivations of coalgebras

Definition 1.3.6. *A coderivation of a k-coalgebra C is a k-linear map $f : C \to C$ such that the following diagram commutes*



The set of k-coderivations of a coalgebra C form a k-vector space and is denoted $\mathrm{CoDer}_k(C)$, *or just* $\mathrm{CoDer}(A)$ *when it's clear from the context what field we are using.*

Theorem 1.3.7. *(a) Given a map $\rho : V^{\otimes n} \to V$ of degree $|\rho|$, which can be viewed as a map $\rho : TV \to V$ by letting its only non-zero component being given by the original map on $V^{\otimes n}$. Then ρ lifts uniquely to a coderivation $\overline{\rho} : TV \to TV$ with*



by taking

$$\overline{\rho}(v_1, \ldots, v_k) := 0, \qquad k < n,$$

$$\overline{\rho}(v_1, \ldots, v_k) = \sum_{i=0}^{k-n} (-1)^{|\rho|(|v_1|+\ldots+|v_i|)}(v_1, \ldots, \rho(v_{i+1}, \ldots, v_{i+n}), \ldots, v_k), \qquad k \geq n.$$

(b) there is a one-to-one correspondence between coderivations $\sigma : TV \to TV$ and systems of maps $\{\rho_i : V^{\otimes i} \to V\}_{i \geq 0}$, given by $\sigma = \sum_{i \geq 0} \overline{\rho}_i$.

Proof. (a) Let $\overline{\rho}^j$ denote the component of $\overline{\rho}$ mapping $TV \to V^{\otimes j}$. Then $\overline{\rho}^1, \overline{\rho}^2, \ldots, \overline{\rho}^{m-1}$ will uniquely determine $\overline{\rho}^m$, by the coderivation property of $\overline{\rho}$. To make this clear consider the following equations

$$\begin{aligned}
\Delta(\overline{\rho}(v_1, \ldots, v_k)) &= (\overline{\rho} \otimes \mathrm{id} + \mathrm{id} \otimes \overline{\rho})(\Delta(v_1, \ldots, v_k)) \\
&= (\overline{\rho} \otimes \mathrm{id} + \mathrm{id} \otimes \overline{\rho})(\sum_{i=0}^{k} (v_1, \ldots, v_i) \otimes (v_{i+1}, \ldots, v_k)) \\
&= \sum_{i=0}^{k} \overline{\rho}(v_1, \ldots, v_i) \otimes (v_{i+1}, \ldots, v_k) + \\
&\qquad (-1)^{|\overline{\rho}|(|v_1|+\ldots+|v_i|)}(v_1, \ldots, v_i) \otimes \overline{\rho}(v_{i+1}, \ldots, v_k).
\end{aligned}$$

If we project both sides of the above to $\bigoplus_{i+j=m} V^{\otimes i} \otimes V^{\otimes j} \subset TV \otimes TV$ we get

$$\Delta(\overline{\rho}^m(v_1,\dots,v_k)) = \sum_{i=0}^{k} \overline{\rho}^{m+i-k}(v_1,\dots,v_i) \otimes (v_{i+1},\dots,v_k) + (-1)^{|\overline{\rho}|(|v_1|+\dots+|v_i|)}(v_1,\dots,v_i) \otimes \overline{\rho}^{m-i}(v_{i+1},\dots,v_k).$$

The right hand side will depend on $\overline{\rho}^i$ for $i < m$, except for the expressions $\overline{\rho}^m(v_1,\dots,v_k)\otimes 1$ and $1 \otimes \overline{\rho}^m(v_1,\dots,v_k)$, which are uninteresting right now. From this we can build an induction argument that proves that $\overline{\rho}^m$ is only non-zero on the component $V^{\otimes k}$ where $k = m+n-1$.

(b) The sum of coderivations is again a coderivation, so the map

$$\alpha : \{\{\rho_k : V^{\otimes k} \to V\}_{k\geq 0}\} \to \operatorname{CoDer} TV, \{\rho : V^{\otimes k} \to V\} \mapsto \sum \overline{\rho}_k$$

is well-defined. It's inverse β acts by giving the system of maps obtained by restricting and projecting; $\beta\sigma = \{pr_V \circ \sigma|_{V^{\otimes k}}\}_{k\geq 0}$. From the lifting property of (a) we see that $\beta \circ \alpha = \mathrm{id}$ and from the uniquness in the construction of α we see that $\alpha \circ \beta = \mathrm{id}$.

□

Corollary 1.3.8. *We have isomorphisms*

$$\operatorname{CoDer}(TV) \cong \prod_{k\geq 0} \operatorname{Hom}(V^{\otimes k}, V)$$

$$\operatorname{CoDer}(SV) \cong \prod_{k\geq 0} \operatorname{Hom}(V^{\otimes k}, V)^{\mathbb{S}_k}.$$

Definition 1.3.9. *Let R be a commutative ring. A chain complex C in the category of R-modules is a series $(C_i)_{i\in\mathbb{Z}}$ of R-modules and a series of homomorphisms $(d_i)_{i\in\mathbb{Z}}$ such that $d_i : C_i \to C_{i-1}$ and $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$. A chain complex is often written $(C_\bullet, d_\bullet)$ and the map $d_\bullet$ is called a differential.*

Given two chain complexes $(C_\bullet, d_\bullet)$ and $(D_\bullet, d'_\bullet)$, a series of homomorphisms $f_i : C_i \to D_i$ is called a chain map if the following infinite diagram is commutative

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & C_{i+1} & \xrightarrow{d_{i+1}} & C_i & \xrightarrow{d_i} & C_{i-1} & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle f_{i+1}} & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}} & & \\
\dots & \longrightarrow & D_{i+1} & \xrightarrow{d'_{i+1}} & D_i & \xrightarrow{d'_i} & D_{i-1} & \longrightarrow & \dots
\end{array}$$

Chain complexes form a category with chain maps.

Dual to chain complexes are cochain complexes. The only difference is that the differential on cochain complexes is increasing, $d_i : C_i \to C_{i+1}$.

An algebra in the category of (co)chain complexes is called a differential graded algebra.

Definition 1.3.10. *The homology of a chain complex $C = (C_\bullet, d_\bullet)$ is the chain complex $(\mathrm{H}_i(C), 0)_{i\in\mathbb{Z}}$ where $\mathrm{H}_i(C) = \operatorname{Ker} d_i / \operatorname{Im} d_{i+1}$. Dually, the cohomology of a cochain complex $D = (D_\bullet, d_\bullet)$ is the cochain complex $(\mathrm{H}^i(D), 0)$ where $\mathrm{H}^i(D) = \operatorname{Ker} d_i / \operatorname{Im} d_{i-1}$.*

2. Theory of Operads

An operad is a sophisticated combinatorial gadget which governs associativity of compositions on a countable collection of objects. Operads were invented by Peter May in the 70:s to classify loop spaces and since then they have seen uses in multiple areas of mathematics

The definition of an operad will be developed in a couple of steps. We do this to reduce the initial difficulty that one can be faced with in trying to learn about operads.

2.1. Non-unital Operads.

Before defining what a non-unital operad is we need to define group actions.

Definition 2.1.1. *Let G be a group and x an object in some category $\mathcal{C}$. A left action by G on x is a group homomorphism $G \to \mathrm{Aut}_{\mathcal{C}}(x,x)$, where $\mathrm{Aut}_{\mathcal{C}}(x,x)$ is the group of units in the monoid $\hom_{\mathcal{C}}(x,x)$. A right action by G on x is function $G \to \mathrm{Aut}_{\mathcal{C}}(x,x)$ such that it is a group homomorphism when composed with the inversion map $G \to G$*

Definition 2.1.2. *Let Σ be the category with objects the sets $[n] = \{1,\ldots,n\}$ and morphisms the elements of the symmetric groups. A Σ-module in a category $\mathcal{C}$ is an element in $Fun(\Sigma^{op},\mathcal{C})$. Alternatively we could say that a E is a Σ-module if there are objects $E(n)$ (where it is understood that $E([n]) = E(n)$) for all $n \geq 0$ with a right action of $\mathbb{S}_n$.*

While stripped of much of the useful structure, the first level of the definition will have the most important features of the operad, which is a generalized associative composition map and an action of the symmetric groups.

Definition 2.1.3. *A non-unital operad in a symmetric monoidal category $(\mathcal{C},\otimes,I)$ is a Σ-module $\{\mathcal{O}(n)\}_{n\geq 1}$ and a composition map*

$$\gamma : \mathcal{O}(k) \otimes \bigotimes_{r=1}^{k} \mathcal{O}(j_r) \to \mathcal{O}(\sum j_r)$$

such that the following diagrams commute:

(1) (associativity)

$$\begin{array}{ccc}
\mathcal{O}(k) \otimes (\bigotimes_{r=1}^{k} \mathcal{O}(j_r)) \otimes (\bigotimes_{t=1}^{\sum j_r} \mathcal{O}(i_t)) & \xrightarrow{\gamma\otimes \mathrm{id}} & \mathcal{O}(\sum_{r=1}^{k} j_r) \otimes (\bigotimes_{t=1}^{\sum j_r} \mathcal{O}(i_t)) \\
\downarrow{\scriptstyle shuffle} & & \\
\mathcal{O}(k) \otimes (\bigotimes_{r=1}^{k}(\mathcal{O}(j_r) \otimes (\bigotimes_{q=1+j_1+\ldots+j_{r-1}}^{j_1+\ldots+j_r} \mathcal{O}(i_q)) & & \downarrow{\scriptstyle\gamma} \\
\downarrow{\scriptstyle \mathrm{id}\otimes(\otimes_r\gamma)} & & \\
\mathcal{O}(k) \otimes (\bigotimes_{r=1}^{k} \mathcal{O}(\sum_{q=1}^{j_r} i_{j_1+\ldots+j_{r-1}+q})) & \xrightarrow{\gamma} & \mathcal{O}(\sum_{t=1}^{\sum j_r} i_t)
\end{array}$$

(2) (equivariance)

$$\begin{array}{ccc}
\mathcal{O}(k) \otimes (\bigotimes_{r=1}^{k} \mathcal{O}(j_r)) & \xrightarrow{\sigma\otimes\sigma^{-1}} & \mathcal{O}(k) \otimes \bigotimes_{r=1}^{k} \mathcal{O}(j_{\sigma(r)}) \\
\downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} \\
\mathcal{O}(\sum_{r=1}^{k} j_r) & \xrightarrow{\sigma(j_1,\ldots,j_k)} & \mathcal{O}(\sum_{r=1}^{k} j_r)
\end{array}$$

$$\begin{CD}
\mathcal{O}(k) \otimes (\bigotimes_{r=1}^{k} \mathcal{O}(j_r)) @>{\mathrm{id} \otimes (\tau_1 \otimes \ldots \otimes \tau_k)}>> \mathcal{O}(k) \otimes \bigotimes_{r=1}^{k} \mathcal{O}(j_r) \\
@V{\gamma}VV @VV{\gamma}V \\
\mathcal{O}(\sum_{r=1}^{k} j_r) @>{\tau_1 \oplus \ldots \oplus \tau_k}>> \mathcal{O}(\sum_{r=1}^{k} j_r)
\end{CD}$$

for $\sigma \in \mathbb{S}_k$ *and* $\tau_i \in \mathbb{S}_{j_i}$*, where* $\sigma(j_1, \ldots, j_k) \in \mathbb{S}_{\sum j_r}$ *is the induced permuation action on the* k *blocks* r_j *and where* $\tau_1 \oplus \ldots \oplus \tau_k \in \mathbb{S}_{\sum j_r}$ *is the block sum permutation.*

Definition 2.1.4. *A pseudo operad in a symmetric monoidal category* $(\mathcal{C}, \otimes, I)$ *is a* Σ*-module* $\{\mathcal{O}(n)\}_{n \geq 1}$ *and with composition maps*

$$\circ_j : \mathcal{O}(n) \otimes \mathcal{O}(m) \to \mathcal{O}(n+m-1) \quad 1 \leq j \leq n$$

such that the following conditions are fulfilled

- *(associativity) For iterated compositions of* $\mathcal{O}(n) \otimes \mathcal{O}(m) \otimes \mathcal{O}(p)$ *the following apply*

$$\circ_i(\circ_j \otimes \mathrm{id}) = \begin{cases} \circ_{j+p-1}(\circ_i \otimes \mathrm{id})(\mathrm{id} \otimes \tau) & \text{for } 1 \leq i \leq j-1, \\ \circ_j(\mathrm{id} \otimes \circ_{i-j+1}) & \text{for } j \leq i \leq j+n-1 \text{ and} \\ \circ_j(\circ_{i-n+1} \otimes \mathrm{id})(\mathrm{id} \otimes \tau) & \text{for } j+n \leq i, \end{cases}$$

where τ *is the transposition* $\mathcal{O}(n) \otimes \mathcal{O}(m) \to \mathcal{O}(m) \otimes \mathcal{O}(n)$.

We can also express these relations in commutative diagrams. For $1 \leq i \leq j-1$*:*

$$\begin{CD}
\mathcal{O}(n) \otimes \mathcal{O}(m) \otimes \mathcal{O}(p) @>{\mathrm{id} \otimes \tau}>> \mathcal{O}(n) \otimes \mathcal{O}(p) \otimes \mathcal{O}(m) @>{\circ_i \otimes \mathrm{id}}>> \mathcal{O}(n+p-1) \otimes \mathcal{O}(m) \\
@V{\circ_j \otimes \mathrm{id}}VV @. @VV{\circ_{j+p-1}}V \\
\mathcal{O}(n+m-1) \otimes \mathcal{O}(p) @>{\circ_i}>> @>>> \mathcal{O}(n+m+p-2),
\end{CD}$$

for $j \leq i \leq j+n-1$ *:*

$$\begin{CD}
\mathcal{O}(n) \otimes \mathcal{O}(m) \otimes \mathcal{O}(p) @>{\mathrm{id} \otimes \circ_{i-j+1}}>> \mathcal{O}(n) \otimes \mathcal{O}(m+p-1) \\
@V{\circ_j \otimes \mathrm{id}}VV @VV{\circ_j}V \\
\mathcal{O}(n+m-1) \otimes \mathcal{O}(p) @>{\circ_i}>> \mathcal{O}(n+m+p-2),
\end{CD}$$

for $j+n \leq i$ *:*

$$\begin{CD}
\mathcal{O}(n) \otimes \mathcal{O}(m) \otimes \mathcal{O}(p) @>{\mathrm{id} \otimes \tau}>> \mathcal{O}(n) \otimes \mathcal{O}(p) \otimes \mathcal{O}(m) @>{\circ_{i-n+1} \otimes \mathrm{id}}>> \mathcal{O}(n+p-1) \otimes \mathcal{O}(m) \\
@V{\circ_j \otimes \mathrm{id}}VV @. @VV{\circ_j}V \\
\mathcal{O}(n+m-1) \otimes \mathcal{O}(p) @>{\circ_i}>> @>>> \mathcal{O}(n+m+p-2).
\end{CD}$$

- *(equivariance) For compositions* $\mathcal{O}(n) \otimes \mathcal{O}(m)$ *the following apply:*

$$\circ_i(\sigma \otimes \rho) = (\sigma \circ_i \rho)\circ_{\sigma(i)}$$

where $\sigma \in \mathbb{S}_n$*,* $\rho \in \mathbb{S}_m$ *such that* $\sigma \circ_i \rho \in \mathbb{S}_{m+n-1}$ *with* $\sigma \circ_i \rho = \sigma_{1,\ldots,1,m,1,\ldots,1} \circ (1 \times \cdots \times 1 \times \rho \times 1 \times \cdots \times 1)$*, and where* $\sigma_{1,\ldots,1,m,1,\ldots,1}$ *is the block permutation on the* n *blocks* $1, \ldots, 1, m, 1, \ldots, 1$*. Or, expressed in a diagram*

$$\begin{CD}
\mathcal{O}(n) \otimes \mathcal{O}(m) @>{\sigma \otimes \rho}>> \mathcal{O}(n) \otimes \mathcal{O}(m) \\
@V{\circ_{\sigma(i)}}VV @VV{\circ_i}V \\
\mathcal{O}(n+m-1) @>{\sigma \circ_i \rho}>> \mathcal{O}(n+m-1).
\end{CD}$$

2.2. **Operads.**

Definition 2.2.1. *An operad in a symmetric monoidal category $(\mathcal{C}, \otimes, I)$ is a Σ-module $\{\mathcal{O}(n)\}_{n\geq 1}$, a unit map $\nu : I \to \mathcal{O}(1)$ and a composition map*

$$\gamma : \mathcal{O}(k) \otimes \bigotimes_{r=1}^{k} \mathcal{O}(j_r) \to \mathcal{O}\left(\sum j_r\right)$$

such that the following diagrams commute:

(1) (associativity)

$$\begin{array}{ccc}
\mathcal{O}(k) \otimes (\bigotimes_{r=1}^{k} \mathcal{O}(j_r)) \otimes (\bigotimes_{t=1}^{\sum j_r} \mathcal{O}(i_t)) & \xrightarrow{\gamma\otimes\mathrm{id}} & \mathcal{O}(\sum_{r=1}^{k} j_r) \otimes (\bigotimes_{t=1}^{\sum j_r} \mathcal{O}(i_t)) \\
\downarrow{\scriptstyle shuffle} & & \\
\mathcal{O}(k) \otimes (\bigotimes_{r=1}^{k}(\mathcal{O}(j_r) \otimes (\bigotimes_{q=1+j_1+\ldots+j_{r-1}}^{j_1+\ldots+j_r} \mathcal{O}(i_q)) & & \downarrow{\scriptstyle\gamma} \\
\downarrow{\scriptstyle \mathrm{id}\otimes(\otimes_r \gamma)} & & \\
\mathcal{O}(k) \otimes (\bigotimes_{r=1}^{k} \mathcal{O}(\sum_{q=1}^{j_r} i_{j_1+\ldots+j_{r-1}+q})) & \xrightarrow{\gamma} & \mathcal{O}(\sum_{t=1}^{\sum j_r} i_t)
\end{array}$$

(2) (unitality)

$$\begin{array}{ccc}
\mathcal{O}(k) \otimes (I)^{\otimes k} & & \\
{\scriptstyle \mathrm{id}\otimes(\nu^{\otimes k})}\downarrow & \searrow^{\cong} & \\
\mathcal{O}(k) \otimes (\mathcal{O}(1)^{\otimes k}) & \xrightarrow{\gamma} & \mathcal{O}(k)
\end{array}
\qquad
\begin{array}{ccc}
I \otimes \mathcal{O}(k) & & \\
{\scriptstyle \nu\otimes\mathrm{id}}\downarrow & \searrow^{\cong} & \\
\mathcal{O}(1) \otimes \mathcal{O}(k) & \xrightarrow{\gamma} & \mathcal{O}(k)
\end{array}$$

(3) (equivariance)

$$\begin{array}{ccc}
\mathcal{O}(k) \otimes (\bigotimes_{r=1}^{k} \mathcal{O}(j_r)) & \xrightarrow{\sigma\otimes\sigma^{-1}} & \mathcal{O}(k) \otimes \bigotimes_{r=1}^{k} \mathcal{O}(j_{\sigma(r)}) \\
\downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} \\
\mathcal{O}(\sum_{r=1}^{k} j_r) & \xrightarrow{\sigma(j_1,\ldots,j_k)} & \mathcal{O}(\sum_{r=1}^{k} j_r)
\end{array}$$

$$\begin{array}{ccc}
\mathcal{O}(k) \otimes (\bigotimes_{r=1}^{k} \mathcal{O}(j_r)) & \xrightarrow{\mathrm{id}\otimes(\tau_1\otimes\ldots\otimes\tau_k)} & \mathcal{O}(k) \otimes \bigotimes_{r=1}^{k} \mathcal{O}(j_r) \\
\downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} \\
\mathcal{O}(\sum_{r=1}^{k} j_r) & \xrightarrow{\tau_1\oplus\ldots\oplus\tau_k} & \mathcal{O}(\sum_{r=1}^{k} j_r)
\end{array}$$

for $\sigma \in \mathbb{S}_k$ and $\tau_i \in \mathbb{S}_{j_i}$, where $\sigma(j_1, \ldots, j_k) \in \mathbb{S}_{\sum j_r}$ is the induced permutation action on the k blocks r_j and where $\tau_1 \oplus \ldots \oplus \tau_k \in \mathbb{S}_{\sum j_r}$ is the block sum permutation.

We can also give a partial definition of the operadic composition map.

Definition 2.2.2. *An operad in a symmetric monoidal category $(\mathcal{C}, \otimes, I)$ is a Σ-module $\{\mathcal{O}(n)\}_{n\geq 1}$, a unit map $\nu : I \to \mathcal{O}(1)$ and n composition maps*

$$\circ_j : \mathcal{O}(n) \otimes \mathcal{O}(m) \to \mathcal{O}(n + m - 1)$$

such that $\circ_j = \gamma\sigma\pi$ where

$$\pi : \mathcal{O}(n) \otimes \mathcal{O}(m) \cong \mathcal{O}(n) \otimes I^{j-1} \otimes \mathcal{O}(m) \otimes I^{n-j}$$

$$\sigma : \mathcal{O}(n) \otimes I^{j-1} \otimes \mathcal{O}(m) \otimes I^{n-j} \to \mathcal{O}(n) \otimes \mathcal{O}(1)^{j-1} \otimes \mathcal{O}(m) \otimes \mathcal{O}(1)^{n-j}$$

$$\gamma : \mathcal{O}(n) \otimes \mathcal{O}(1)^{j-1} \otimes \mathcal{O}(m) \otimes \mathcal{O}(1)^{n-j} \to \mathcal{O}(n + m - 1).$$

Example 2.2.3. *Let $\mathcal{C}$ be a symmetric monoidal category with internal* hom*-functor* Hom *and let X be an object in $\mathcal{C}$. The endomorphism operad of X, $\mathcal{E}nd_X$, is the objects* $\mathrm{Hom}(X^{\otimes k}, X)$ *with the composition map γ is given as the composition of the following maps*

$$\mathrm{Hom}(X^{\otimes n}, X) \otimes \mathrm{Hom}(X^{\otimes k_1}, X) \otimes \ldots \otimes \mathrm{Hom}(X^{\otimes k_n}) \to \mathrm{Hom}(X^{\otimes(\sum k_i)}, X)$$

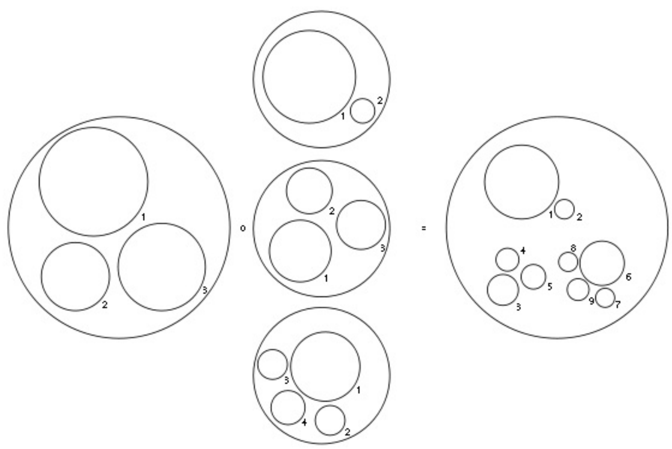
acting on $f \in \mathrm{Hom}(X^{\otimes n}, X)$ and $g_i \in \mathrm{Hom}(X^{\otimes k_i}, X)$ such that

$$\gamma(f, g_1, \ldots, g_n) = f(g_1(-), \ldots, g_n(-)) \in \mathrm{Hom}(X^{\otimes(\sum k_i)}, X).$$

Example 2.2.4. ***The little k-disk operad.*** *Let D denote the standard k-disk in $\mathbb{R}^k$. Consider the set of m ordered non-intersecting k-discs contained in D, let these be denoted $\mathcal{L}(m)$. Let $d = \{d_1, \ldots, d_m\}$ be an element from $\mathcal{L}(m)$ and $a_i = \{a_{i1}, \ldots, a_{ik_i}\}$ be an element from $L(k_i)$ for $i = 1, \ldots, m$ then the composition of $d, a_1, \ldots, a_m$ is the set*

$$\{a'_{11}, \ldots, a'_{1k_1}, a'_{21}, \ldots, a'_{2k_2}, \ldots, a'_{m1}, \ldots, a'_{mk_m}\}$$

each a'_{ij} is a k-disc and where the position of a'_{ij} is to d_i as a_{ij}'s position was to D. A picture illustrates



The collection $\mathcal{L}(m)_{m\geq 0}$ is an operad of topological spaces together with the above described map.

Definition 2.2.5. *Let $\mathcal{O} = \{\mathcal{O}(n)\}_{n\geq 1}$ and $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 1}$ be operads. A morphism $\phi : \mathcal{O} \to \mathcal{P}$ is a sequence of maps $\phi(n) : \mathcal{O}(n) \to \mathcal{P}(n)$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{O}(n) \otimes (\bigotimes_i \mathcal{O}(k_i)) & \xrightarrow{\mathrm{id} \otimes (\bigotimes_i \phi(k_i))} & \mathcal{P}(n) \otimes (\bigotimes_i \mathcal{P}(k_i)) \\ \downarrow \gamma_{\mathcal{O}} & & \downarrow \gamma_{\mathcal{P}} \\ \mathcal{O}(\sum k_i) & \xrightarrow{\phi(\sum k_i)} & \mathcal{P}(\sum k_i) \end{array}$$

where $\gamma_{\mathcal{O}}$ is the composition map in $\mathcal{O}$ and $\gamma_{\mathcal{P}}$ is the composition map in $\mathcal{P}$.

Definition 2.2.6. *Let $\mathcal{O}$ be an operad in a symmetric monoidal category $\mathcal{C}$. An algebra A over $\mathcal{O}$ is an object from $\mathcal{C}$ and a morphism of operads $\theta : \mathcal{O} \to \mathcal{E}nd_X$.*

Definition 2.2.7. *An ideal $\mathcal{I}$ in an operad $\mathcal{O}$ is a collection of subobjects $\mathcal{I}(n) \subset \mathcal{O}(n)$ such that whenever $i \in \mathcal{I}$ then $\gamma(\ldots, i, \ldots) \in \mathcal{I}$.*

Given a family of elements, $(x_i)_{i\in I}$, from an operad $\mathcal{O}$. The smallest ideal in $\mathcal{O}$ that contains all the x_i is *the ideal generated by the family* $(x_i)_{i\in I}$.

Definition 2.2.8. *The quotient of an operad $\mathcal{O}$ by an ideal $\mathcal{I}$ is the operad $(\mathcal{O}/\mathcal{I})(n) := \mathcal{O}(n)/\mathcal{I}(n)$ and with the induced composition map from $\mathcal{O}$.*

2.3. **The Free Operad.**

Definition 2.3.1. *Let* Op *denote the category of operads (in some fixed but notationally supressed symmetric monoidal category $\mathcal{C}$) and let* $\Psi\mathrm{Op}$ *denote the category of non-unitary operads.*

Given a non-unitary operad $\mathcal{S}$ there is a forgetful functor F which takes $\mathcal{S}$ to it's underlying Σ-module. From this we can define the free non-unitary functor on the category Σ-mod as the left-adjoint to F, i.e. the functor taking the object A to $\Psi(A)$ where $\hom_{\Sigma-mod}(A, F(\mathcal{S})) \cong \hom_{\Psi\mathrm{Op}}(\Psi(A), E)$.

Definition 2.3.2. *Given a set Y of cardinality n and an assignment of objects A_y in $\mathcal{C}$ for each $y \in Y$. Let $Ord(Y)$ denote the set of bijections $\{Y \to [n]\}$. Let $g \in Ord(Y)$, then for each $\sigma \in \mathbb{S}_n$ there exist an induced map $\sigma^* : \bigotimes_{i=1}^n A_{g^{-1}(i)} \to \bigotimes_{i=1}^n A_{(\sigma\circ g)^{-1}(i)}$. We define the unorded product over Y as*

$$\bigotimes_Y A_y = \mathrm{coequalizer}_{\sigma\in\mathbb{S}_n} \left\{ \sigma^* : \coprod_{y\in Ord(Y)} \bigotimes_{i=1}^n A_{y^{-1}(i)} \to \coprod_{y\in Ord(Y)} \bigotimes_{i=1}^n A_{y^{-1}(i)} \right\}.$$

Given a non-unitary Σ-module A and a labeled tree (T, l) we form the unordered product

$$A(T,l)(n) = \bigotimes_{v\in vert(T)} A(in(v)),$$

this product is a functor from Tree_n to $\Psi\mathrm{Op}$, where Tree_n is the category of labeled trees with morphisms the label-preserving isomorphisms.

Definition 2.3.3. *The free non-unitary operad on the Σ-module A is defined as*

$$\Psi(A)(n) = \mathrm{colim}_{(T,l)\in\mathrm{Tree}_n} A(T,l)$$

and the composition maps are given as grafting of trees.

There exist a functor which takes a non-unitary operad A to an operad by formally adjoining the unit

$$U : A \to I \coprod A.$$

The composition $U\Psi$ is the free operad functor on Σ-modules.

2.4. **Minimal Models of Operads.**

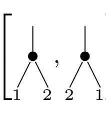
Definition 2.4.1. *A differential graded Σ-module A is a Σ-module of differential graded vector spaces $(A(n), d_n)$ such that the map $d : A(n)^i \to A(n)^{i+1}$ is k-linear and $\mathbb{S}_n$-equivariant.*

A differential graded operad (dg-operad) is a differential graded Σ-module with the structure of an operad and where the composition maps are morphism of differential graded vector spaces.

In this subsection we will assume that the operads are dg-operads over the field k. Theorems are stated without proof in this thesis but the book by Markl, Shnider and Stasheff ([MaShSt]) contains the omitted matter.

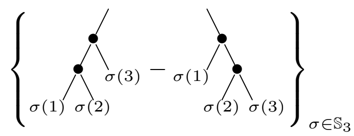
Let k be a field and define the Σ-module E as

$$E(n) = \begin{cases} k[\mathbb{S}_2] = k\left[\;\cdot\;,\;\cdot\; \right] & \text{if } n = 2 \\ 0 & \text{if } n \neq 2. \end{cases}$$

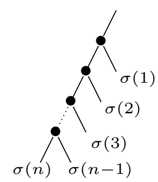


Consider the free operad on E, $\Psi(E)$. It consist of binary trees decorated with elements of $k[\mathbb{S}_2]$. We define the associative operad Ass as the operad $\Psi(E)$, modulo an ideal I.

$$Ass = \Psi(E)/I, \quad \text{where } I =$$



This associative condition makes it possible to rewrite all trees on the following form

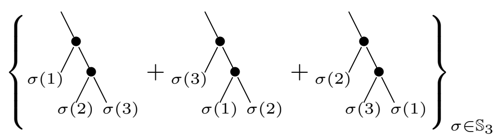


for some $\sigma \in \mathbb{S}_n$

so that it becomes clear that $\Psi(E)(n) \cong k[\mathbb{S}_n]$. Note that algebras over the operad Ass is the same thing as associative algebras.

Consider the operadic ideal

$$J =$$



Algebras over the quotient $Lie = \Psi(E)/J$ is the same thing as Lie algebras.

Definition 2.4.2. *Let $\mathcal{O}$ be a dg-operad $\{\mathcal{O}(n)\}_{n\geq 1}$, with $\mathcal{O}(n) = \{\mathcal{O}(n)^i\}_{i\in\mathbb{Z}}$. The homology of $\mathcal{O}$ is the operad of cohomology complexes, $[\mathrm{H}(\mathcal{O})(n)]^i = \mathrm{H}^i(\mathcal{O}(n))$.*

Definition 2.4.3. *A quasi-isomorphism $\pi : \mathcal{O} \to \mathcal{P}$ of dg-operads is a morphism of operads such that the induced map on homology is an isomorphism,* $\mathrm{H}(\pi) : \mathrm{H}(\mathcal{O}) \cong \mathrm{H}(\mathcal{P})$.

Quasi-isomorphisms induce an equivalence relation on operads. Two operads $\mathcal{Q}$ and $\mathcal{S}$ are *weakly equivalent* if they are connected by a chain of quasi-isomorphisms in the following way

$$Q \leftarrow \mathcal{P}_1 \rightarrow \mathcal{P}_2 \leftarrow \cdots \rightarrow \mathcal{P}_{s-1} \leftarrow \mathcal{P}_s \rightarrow S.$$

Definition 2.4.4. *Let $\mathcal{O}$ be an operad with $\mathcal{O}(1) = k$. Then the decomposables $\mathcal{DO} = (\mathcal{DO}(n))_{n\geq 1}$ is the elements*

$$\gamma(o, o_1, \ldots, o_n) \quad o \in \mathcal{O}(n), o_i \in \mathcal{O}(k_i)$$

where at least two of $n, k_1, \ldots, k_n$ are greater than 1.

The decomposables of an operad is an ideal.

Definition 2.4.5. *A minimal operad $\mathfrak{M} = (\Psi(E), \partial)$ is a free dg-operad on a Σ-module E with $E(1) = 0$ and a differential ∂ such that $\partial(E) \subset \mathcal{D}\mathfrak{M}$.*

Theorem 2.4.6. *Minimal operads are isomorphic if and only if they are weakly equivalent.*

Definition 2.4.7. *Let $\mathcal{O}$ be a dg-operad. A minimal model of $\mathcal{O}$ is a minimal operad $\mathfrak{M}$ and a quasi-isomorphism $q : \mathfrak{M} \to \mathcal{O}$.*

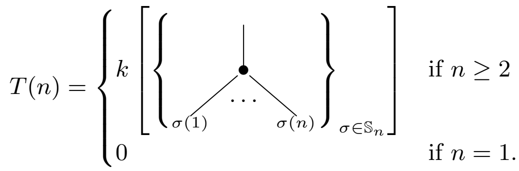
Theorem 2.4.8. *Every dg-operad $\mathcal{S} = (\mathcal{S}, \partial)$ such that*

$$\mathrm{H}(\mathcal{S})(1) = k$$

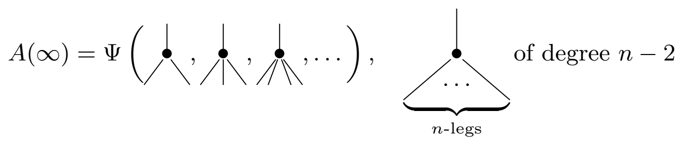
admits a minimal model $q : \mathfrak{M} \to \mathcal{S}$.

We can consider *Ass* as a dg-operad with trivial differential, in which case $\mathrm{H}(Ass) = Ass$.

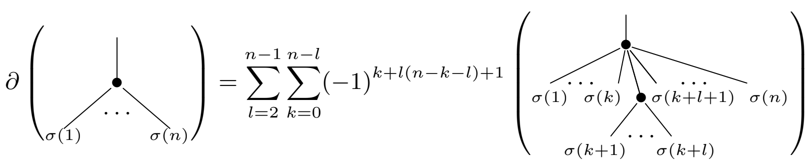
The minimal model of *Ass* is the operad $A(\infty)$. It's the free operad on the Σ-module T, with



We write that

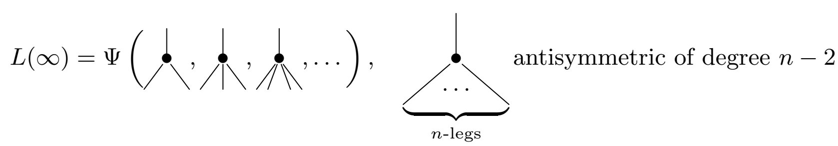


Where A is any corolla with n legs. The differential ∂ on $A(\infty)$ is defined as

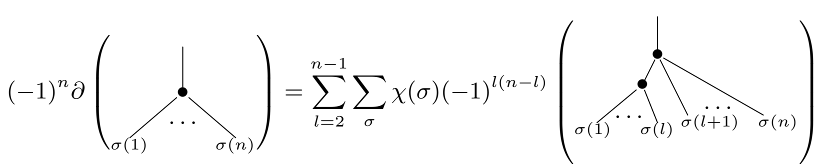


The operad *Lie* can in the same way be considered as a dg-operad with trivial differential.

The minimal model of *Lie* is the operad $L(\infty)$. It's free on the Σ-module T, as above with $A(\infty)$. We can write this



And the differential is given as follows



where the second summation is taken over all $(l, n-l)$-unshuffles σ, i.e. such that

$$\sigma(1) < \sigma(2) < \ldots < \sigma(l) \text{ and } \sigma(l+1) < \sigma(l+2) < \ldots < \sigma(n).$$

The term $\chi(\sigma)$ is a sign defined as follows. The *Koszul sign convention* states that whenever we have elements x and y with degrees $\deg x = p$ and $\deg y = q$ we will add a sign $(-1)^{pq}$ when we commute x with y in a formula. If $\pi \in \mathbb{S}_n$ and

$$x_1 \wedge x_2 \wedge \ldots \wedge x_n \in \bigwedge^n (x_1, \ldots, x_n)$$

then we will let $\epsilon(\pi, x_1, \ldots, x_n) := \epsilon(\pi)$ be the sign implied by the Koszul sign rule which makes the following equality correct

$$x_1 \wedge x_2 \wedge \ldots \wedge x_n = \epsilon(\pi) x_{\pi(1)} \wedge x_{\pi(2)} \wedge \ldots \wedge x_{\pi(n)}.$$

We now define $\chi(\sigma) = \mathrm{sgn}(\pi)\epsilon(\pi)$.

Algebras over the operad $A(\infty)$ are called *strongly homotopy Ass-algebras* or A_∞-algebras and algebras over the operad $L(\infty)$ are called *strongly homotopy Lie-algebras* or L_∞-algebras. Let us describe them explicitly.

Definition 2.4.9. *An A_∞-algebra A is a differential graded vector space $(V, d) = (\bigoplus_{i\in\mathbb{Z}} V^i, d)$ with a set of multi-linear maps $\{m_n\}_{n\geq 2}$, $m_n : V^{\otimes n} \to V$ where $\deg m_n = n-2$. The maps act as follows:*

$$\begin{aligned} 0 &= [m_2, d](a,b) \\ m_2(m_2(a,b),c) - m_2(a, m_2(b,c)) &= [m_3, d](a,b,c) \\ m_3(m_2(a,b),c,d) - m_3(a,m_2(b,c),d) + m_3(a,b,m_2(c,d)) & \\ -m_2(m_3(a,b,c),d)) - (-1)^{|a|} m_2(a, m_3(b,c,d)) &= [m_4, d](a,b,c,d) \\ &\vdots \end{aligned}$$

$$\sum_{\substack{i+j=n+1\\ i,j\geq 2}} \sum_{s=0}^{n-j} u\, m_i(a_1,\dots,a_s, m_j(a_{s+1},\dots,a_{s+j}), a_{s+j+1},\dots,a_n) = [m_n, d](a_1,\dots,a_n)$$

where u is the sign

$$(-1)^{j+s(j+1)+j(|a_1|+\dots+|a_{s-1}|)}$$

and where $[m_n, d]$ is the induced differential in the complex $\mathrm{Hom}(V^{\otimes n}, V)$

$$[m_n, d](a_1,\dots,a_n) := \sum_{s=1}^{n} (-1)^{|a_1|+\dots+|a_{s-1}|} m_n(a_1,\dots,da_s,\dots,a_n) - (-1)^n d m_n(a_1,\dots,a_n)$$

for $a_1,\dots,a_n \in V$.

Definition 2.4.10. *An $L(\infty)$-algebra L is a differential graded vector space $(V, d) = (\bigoplus_{i\in\mathbb{Z}} V^i, d)$ with a system of maps $\{l_n\}_{n\geq 2}$, $l_n : V^{\otimes n} \to V$ with $\deg l_n = n-2$ and subject to the rules*

$$l_n(a_1,\dots,a_n) = \chi(\pi) l_n(a_{\pi(n)},\dots,a_{\pi(n)})$$

$$\sum_{\substack{i+j=n+1\\ i,j\geq 2}} \sum_{\sigma} \chi(\sigma)(-1)^{i(j-1)} l_j(l_i(a_{\sigma(1)},\dots,a_{\sigma(i)})), a_{\sigma(i+1)},\dots,a_{\sigma(n)})) = (-1)^n [d, l_n](a_1,\dots,a_n)$$

for all $\pi \in \mathbb{S}_n$, and where the sum is taken over all $(i, n-i)$-unshuffles σ, χ is the same as above, in the discussion on the $L(\infty)$-operad.

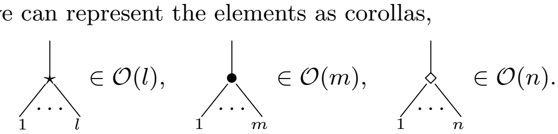
2.5. **A Theorem by Kapranov-Manin.**

Theorem 2.5.1. *Let $\mathcal{O}(n)_{n\geq 1}$ be a dg-operad over some field k. Then $L = \bigoplus_{n\geq 1} \mathcal{O}(n)$ is a Lie algebra with bracket*

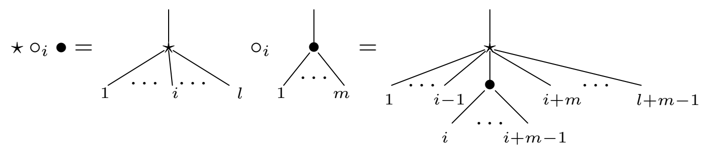
$$[a,b] = \sum_{i=1}^{m} a \circ_i b - (-1)^{|a||b|} \sum_{j=1}^{n} b \circ_j a \qquad a \in \mathcal{O}(m), b \in \mathcal{O}(n),$$

where $\circ_k$ is the partial composition. Furthermore the subspaces $L_{\mathbb{S}} = \bigoplus_{n\geq 1} \mathcal{O}(n)_{\mathbb{S}_n}$ and $L^{\mathbb{S}} = \bigoplus_{n\geq 1} \mathcal{O}(n)^{\mathbb{S}_n}$ are also Lie algebras with the induced bracket.

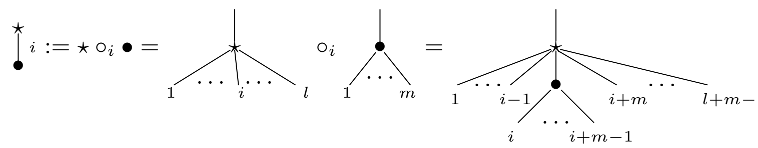
Proof. We will prove this in the ungraded case. Suppose that $\star \in \mathcal{O}(l)$, $\bullet \in \mathcal{O}(m)$ and $\diamond \in \mathcal{O}(n)$. Then we can represent the elements as corollas,



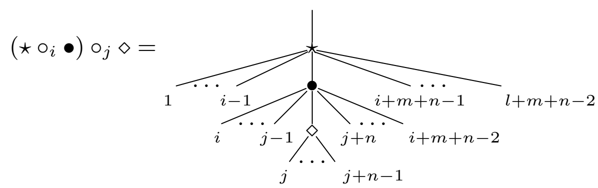
The partial composition is given as the grafting of trees in the following manner:



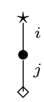
The last tree is rather large and we would like to have an abbreviated notation for it. We define



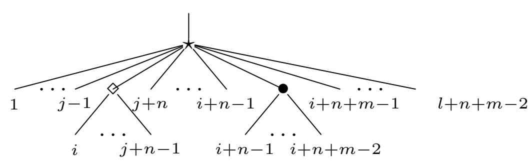
where the i signifies that the grafting took place at the i:th leg. The expression $(\star \circ_i \bullet) \circ_j \diamond$ can be of essentially two types. If $i \leq j \leq i+m$ then



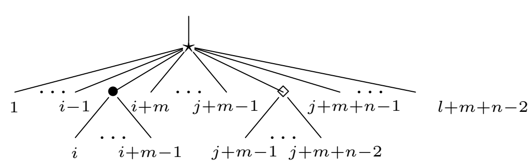
for which we define the following abbreviative notation



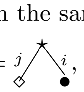
In the case $j < i$ we get



and in the case $j > m+i-1$ we get



we will abbreviate them both with the same tree

$(\star \circ_i \bullet) \circ_j \diamond =$  $,\quad$ if $j < i$ or $j > m+1-1$

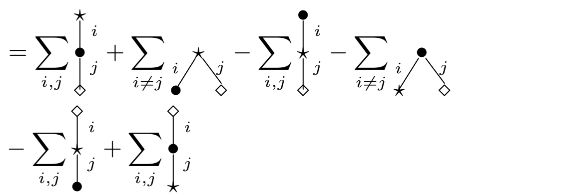
where the order in which the legs appear have no significance on what number of i and j is larger.

If we let $\sigma = (\star, \bullet, \diamond)$ the three-cycle permuting the elements then the Jacobi identity states that

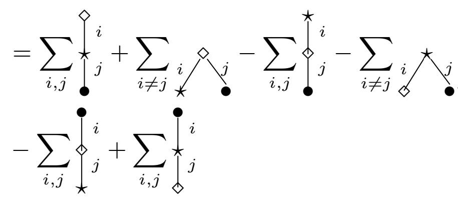
$$(1+\sigma+\sigma^2)[[\star,\bullet],\diamond] = 0.$$

This identity is the only difficult part of the proof and hence the only part we will give. The result will be clear when the brackets are expanded and rewritten with the rules given above.

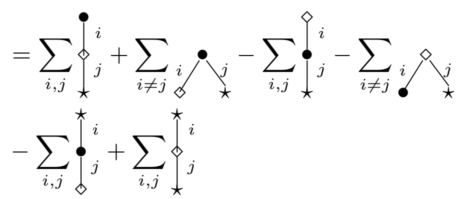
$$
\begin{aligned}
[[\star,\bullet],\diamond] &= \sum_i [\star,\bullet]\circ_i \diamond - \sum_i \diamond \circ_i [\star,\bullet] \\
&= \sum_{i,j} (\star \circ_i \bullet) \circ_j \diamond - \sum_{i,j} (\star \circ_i \bullet) \circ_j \diamond - \sum_{i,j} \diamond \circ_i (\star \circ_j \bullet) \\
&+ \sum_{i,j} \diamond \circ (\bullet \circ \star)
\end{aligned}
$$



$$\sigma[[\star,\bullet],\diamond] = [[\diamond,\star],\bullet]$$



$$\sigma^2[[\star,\bullet],\diamond] = [[\bullet,\diamond],\star]$$



Where all sums are taken in an exhaustive manner. It's now easy to check that the terms cancel. $\square$

3. The Operad of Graphs

In this section we will define the operad of graphs and study some of its properties. The section concludes with a definition of the Kontsevich graph complex and a statement of Willwacher's theorem.

3.1. **Graph complexes.** Let $G_{n,l}$ be the set of graphs G with n vertices, $Vert(G)$, ordered with $[n] = \{1, 2, \ldots, n\}$ and l edges , $Edge(G)$, totally ordered up to an even permutation. The group $\mathbb{Z}_2$ acts on a graph by reversing the direction of the total order, the orbit of a graph Γ is the set $\{\Gamma, \Gamma_{opp}\}$. Let $\mathcal{G}_{n,l}$ be the vector space spanned by the isomorphism classes $[\Gamma]$, for graphs $\Gamma \in G_{n,l}$ and with the relation $[\Gamma] = -[\Gamma_{opp}]$

$$\mathcal{G}_{n,l} = \frac{\operatorname{span}_k \langle \{[\Gamma] | \Gamma \in G_{n,l}\} \rangle}{[\Gamma] = -[\Gamma_{opp}]}.$$

From this we can form the graded vector space

$$\mathcal{G}(n) = \bigoplus_{l \geq 0} \mathcal{G}_{n,l}[2n - l - 2].$$

There is a natural action of $\mathbb{S}_n$ on $\mathcal{G}(n)$ where we permute the vertices. It is clear that $\mathcal{G}(n)_{n \geq 1}$ is a Σ-module.

Consider a collection of graphs, $\Gamma_0 \in \mathcal{G}(n)$ and $\Gamma_1 \in \mathcal{G}(k_1), \ldots, \Gamma_n \in \mathcal{G}(k_n)$. In relation to these graphs we can define the functions

$$f_i \in \hom(Edge(i), Vert(\Gamma_i)) \text{ for } 1 \leq i \leq n.$$

Each of the functions describe a way to connect the edges of the $i:th$ vertex of Γ_0 to the vertices of Γ_i. We use this to construct a composition map on the Σ-module $\{\mathcal{G}(n)\}_{n \geq 1}$. Let $\Gamma_{f_1, \ldots, f_n}$ be the graph where the vertices $1, \ldots, n$ have been replaced with the graphs $\Gamma_1, \ldots, \Gamma_n$ and edges previously connected to i are reconnected to $Vert(\Gamma_i)$ according to how f_i acts.

The composition map is then given as

$$\gamma : \mathcal{G}(n) \otimes \mathcal{G}(k_1) \otimes \ldots \otimes \mathcal{G}(k_n) \to \mathcal{G}(k_1 + \ldots + k_n)$$

$$\Gamma_0 \otimes \Gamma_1 \otimes \ldots \otimes \Gamma_n \mapsto \sum_{f = (f_1, \ldots, f_n) \in \prod_{i=1}^{n} \hom(Edge(i), Vert(\Gamma_i))} (-1)^{\sigma_f} \Gamma_f$$

This makes $\{\mathcal{G}(n)\}_{n \geq 1}$ into an operad. Furthermore the vertices of the graph Γ_f are labeled and ordered in a lexicographically way such that the vertices that come from Γ_i will have labels $\{i, 1\}, \ldots, \{i, k_i\}$ and that $\{i, j\} < \{i', j'\}$ if and only if $i < i'$ or if $i = i'$ and $j < j'$. The vertices are then relabeled with the minimal string of numbers

$$1, 2, \ldots, k_1 + \ldots + k_n$$

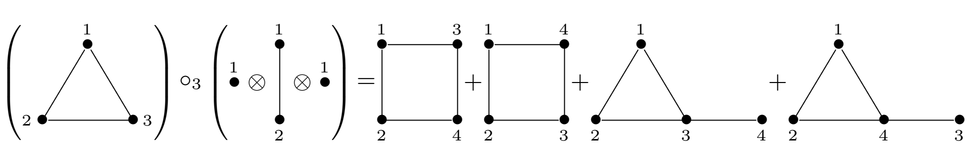
such that the previous ordering is preserved. The sign $(-1)^{\sigma_f}$ is determined so that

$$\bigwedge_{e \in Edge(\Gamma_f)} e = (-1)^{\sigma_f} \bigwedge_{e_0 \in Edge(\Gamma_0)} e_0 \wedge \bigwedge_{e_1 \in Edge(\Gamma_1)} e_1 \wedge \ldots \wedge \bigwedge_{e_n \in Edge(\Gamma_n)} e_n.$$

For graphs $\Gamma_0 \in \mathcal{G}(n)$ and $\Gamma_1 \in \mathcal{G}(m)$ we also have partial composition

$$\circ_i = \gamma|_{\mathcal{G}(1)^{i-1} \otimes \mathcal{G}(m) \otimes \mathcal{G}(1)^{n-i-1}}$$

for all vertices $i \in Vert(\Gamma_0)$. This is an example of how a composition can look:



3.2. **Representation of the graphs operad.** In the introduction to this thesis we mentioned that there is representations of the operad $\mathcal{G}$ which are Lie algebras of polyvector fields, this is explained in more detail now.

We can think of the polyvector field on $\mathbb{R}^d$, $\mathcal{T}_{poly}(\mathbb{R}^d)$, as the graded commutative algebra $C^\infty(\mathbb{R}^d)[\Psi_1, \ldots, \Psi_d]$, with $|\Psi_i| = 1$, i.e. subject to the rule $\Psi_i\Psi_j = -\Psi_j\Psi_i$. Fix some system of local coordinates $x^1, \ldots, x^d$ for $\mathbb{R}^d$. An element of $\mathcal{T}_{poly}(\mathbb{R}^d)$ will be of the form

$$\sum C^{\alpha_1,\ldots,\alpha_p}(x^1, \ldots, x^d)\Psi_{\alpha_1} \wedge \ldots \wedge \Psi_{\alpha_p}$$

where $C^{\alpha_1,\ldots,\alpha_p}(x_1, \ldots, x_d)$ are smooth functions on the variables $x_1, \ldots, x_d$. Consider the operator

$$\Delta = \sum_\alpha \frac{\partial^2}{\partial x^\alpha \partial \Psi_\alpha}.$$

We have the properties that $\Delta^2 = 0$ and that Δ is $\mathsf{Aff}(\mathbb{R}^d)$-invariant. The Schouten bracket on $\mathcal{T}_{poly}(\mathbb{R}^d)$ is then defined on homogeneous elements as

$$[\gamma_1, \gamma_2]_s := (-1)^{|\gamma_1|}\Delta(\gamma_1\gamma_2) - (-1)^{|\gamma_1|}\Delta(\gamma_1)\gamma_2 - \gamma_1\Delta(\gamma_2).$$

We have a representation of the operad $\mathcal{G}$,

$$\rho : \mathcal{G} \to \mathcal{E}nd_{\mathcal{T}_{poly}(\mathbb{R}^d)};$$

a sequence of maps

$$\rho_n : \mathcal{G}(n) \to \mathcal{E}nd_{\mathcal{T}_{poly}(\mathbb{R}^d)}(n) = \mathrm{Hom}(\mathcal{T}_{poly}(\mathbb{R}^d)^{\otimes n}, \mathcal{T}_{poly}(\mathbb{R}^d)),$$

defined on graphs $\Gamma \in \mathcal{G}(n)$ in the following manner $\Gamma \mapsto \Phi_\Gamma = (\mathcal{T}_{poly}(\mathbb{R}^d)^{\otimes n} \to \mathcal{T}_{poly}(\mathbb{R}^d))$. The map Φ_Γ is given as the composition of two maps, $\mu \circ \phi$, where μ is just the regular multiplication map $a \otimes b \mapsto ab$ and where $\phi = \prod_{e \in Edge(G)} \Delta_e$, the product is taken over the edges in their associated ordering. The map Δ_e is defined on an edge $e = \overset{i}{\bullet}\!\!-\!\!\!-\!\!\overset{j}{\bullet}$ as follows

$$\Delta_e = \sum_\alpha \mathrm{id}^{\otimes i-1} \otimes \frac{\partial}{\partial x^\alpha} \otimes \mathrm{id}^{\otimes j-i-1} \otimes \frac{\partial}{\partial \Psi_\alpha} \otimes \mathrm{id}^{\otimes n-j} + \mathrm{id}^{\otimes i-1} \otimes \frac{\partial}{\partial \Psi_\alpha} \otimes \mathrm{id}^{\otimes j-i-1} \otimes \frac{\partial}{\partial x^\alpha} \otimes \mathrm{id}^{\otimes n-j}.$$

3.3. **The Grothendieck-Teichmüller Lie Algebra.** Let $\mathbb{F}_2 = k\langle\langle x, y\rangle\rangle$ be the free completed algebra on two generators. This algebra has a comultiplication defined on primitive elements x and y as

$$\Delta x = x \otimes 1 + 1 \otimes x$$
$$\Delta y = y \otimes 1 + 1 \otimes y$$

Definition 3.3.1. *An element Φ is called group like if $\Delta\Phi = \Phi \widehat{\otimes} \Phi$.*

Equivalently we can say that $\Phi = \exp\phi$ is group like if $\phi \in \widehat{\mathbb{F}}_{Lie}(x, y) \subset \mathbb{F}_2$.

Definition 3.3.2. *The Drinfel'd Kohno Lie algebra,* $\mathfrak{t}(n)$, *is generated by the indeterminants $t_{ij} = t_{ji}$ with $1 \leq i, j \leq n$ and $i \neq j$ and subject to*

$$[t_{ij}, t_{ik} + t_{kj}] = 0 \quad \text{for distinct } i, j, k$$
$$[t_{ij}, t_{kl}] = 0 \quad \text{for distinct } i, j, k, l.$$

Definition 3.3.3. *Let $\mu \in \overline{k}$ and $t_{ij} \in \mathfrak{t}(n)$ for $\{i, j\} \subset \{1, 2, 3, 4\}$. The group-like solutions $\Phi \in \mathbb{F}_2$ to the system*

(1)

$$\Phi(t_{12}, t_{23} + t_{24})\Phi(t_{13} + t_{23}, t_{34}) = \Phi(t_{23}, t_{34})\Phi(t_{12} + t_{13}, t_{23} + t_{34})\Phi(t_{12}, t_{23})$$

(2) $$\exp(\mu(t_{13} + t_{23})/2) = \Phi(t_{13}, t_{12}) \exp(\mu t_{13}/2)\Phi(t_{13}, t_{23})^{-1} \exp(\mu t_{23}/2)\Phi(t_{12}, t_{23})$$

(3) $$\Phi(x, y) = \Phi(y, x)^{-1}$$

are called Drinfel'd associators when $\mu \neq 0$ and elements of the Grothendieck-Teichmüller group (GRT) when $\mu = 0$.

The following theorem was proved by Furusho [Fu].

Theorem 3.3.4. *Any group-like solution $\Phi \in \mathbb{F}_2$ to (1) will automatically be a solution to (2) and (3) if $\mu = \sqrt{24 c_2(\Phi)}$, where $c_2(\Phi)$ is the coefficient of xy in $\Phi(x, y)$.*

There is a group structure on GRT, the multiplication is given as

$$\Phi \circ \Phi'(x, y) = \Phi(x, y)\Phi'(\Phi(x, y)^{-1} x \Phi(x, y), y).$$

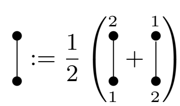
Associated to the group GRT is the *Grothendieck-Teichmüller Lie algebra,* $\mathfrak{grt}$. It is given as the Lie series $\phi \in \widehat{\mathbb{F}}_{Lie}(x, y)$ such that

$$\begin{aligned} \phi(t_{12}, t_{23} + t_{24}) + \phi(t_{13} + t_{23}, t_{34}) &= \phi(t_{23}, t_{34}) + \phi(t_{12} + t_{13}, t_{24} + t_{34}) + \phi(t_{12}, t_{23}) \\ 0 &= \phi(x, y) + \phi(y, -x - y) + \phi(-x - y, x) \\ 0 &= \phi(x, y) + \phi(y, x). \end{aligned}$$

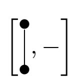
3.4. **The Kontsevich Graph Complex.** From the operad of graphs, $\{\mathcal{G}(n)\}_{n \geq 1}$, we use the Kapranov-Manin theorem to form the Lie algebra $(\bigoplus_{n \geq 1} \mathcal{G}(n)^{\mathbb{S}_n}, [\ ,\ ])$ where

$$[\Gamma, \Gamma'] = \sum_{i=1}^{n} \Gamma \circ_i \Gamma' - \sum_{j=1}^{m} \Gamma \circ_j \Gamma' \text{ for } \Gamma \in \mathcal{G}(n), \Gamma' \in \mathcal{G}(m).$$

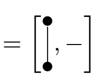
The map $[\ ,\ ] : \mathcal{G}(n) \otimes \mathcal{G}(m) \to \mathcal{G}(n + m - 1)$ has degree 0. To see this take two graphs Γ_1 and Γ_2, with l_1 and l_2 edges, respectively. The left hand side, $\Gamma_1 \otimes \Gamma_2$, will be of degree $(2m - l_1 - 2) + (2n - l_2 - 2) = 2(n + m - 1) - (l_1 + l_2) - 2$, which is the degree of the right hand side, $[\Gamma_1, \Gamma_2]$, since the number of edges is not changed when the bracket is applied. The element



has degree 1 so that the commutator



gives us a differential graded Lie algebra

$$\left(\mathsf{fGC} = \bigoplus_{n \geq 1} \mathcal{G}(n)^{\mathbb{S}_n}, [-, -], d = \right.$$  $$\left.\right).$$

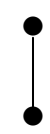
Consider the representation of operads $\rho : \mathcal{G} \to \mathcal{E}nd_{\mathcal{T}_{poly}(\mathbb{R}^d)}$. Using the Kapranov-Manin Theorem on the left side we get the Lie algebra fGC and using it on the right hand side we get the *Chevalley-Eilenberg complex* of $\mathcal{T}_{poly}(\mathbb{R}^d)$. This complex is denoted as $\mathrm{CE}(\mathcal{T}_{poly}(\mathbb{R}^d), \mathcal{T}_{poly}(\mathbb{R}^d))$, from the construction we have that

$$\mathrm{CE}(\mathcal{T}_{poly}(\mathbb{R}^d), \mathcal{T}_{poly}(\mathbb{R}^d)) := \bigoplus_{n \geq 1} \mathrm{Hom}(\mathcal{T}_{poly}(\mathbb{R}^d)^{\otimes n}, \mathcal{T}_{poly}(\mathbb{R}^d))^{\mathbb{S}_n}.$$

It follows that there exists an induced representation

$$\rho^{ind} : \mathsf{fGC} \to \mathrm{CE}(\mathcal{T}_{poly}(\mathbb{R}^d), \mathcal{T}_{poly}(\mathbb{R}^d))$$

and that the graph



is mapped to the Schouten bracket $[\ ,\ ]_s \in \mathrm{Hom}(\mathcal{T}_{poly}(\mathbb{R}^d)^{\otimes 2}, \mathcal{T}_{poly}(\mathbb{R}^d))^{\mathbb{S}_2}$ under ρ^{ind}. In fact more can be said; elements ω in fGC satisfying the equation

$$d\omega + \frac{1}{2}[\omega, \omega] = 0,$$

so called Maurer-Cartan elements, correspond to L_∞ structures on $\mathcal{T}_{poly}(\mathbb{R}^d)$.

Let GC be the subalgebra spanned by connected graphs where each vertex has at least 3 edges. The differential and bracket from fGC is inherited to GC and the resulting differential graded Lie algebra is known in the literature as the *(odd) Kontsevich graph complex*. Very recently this remarkable theorem was proved by Willwacher [Wi]:

Theorem 3.4.1. *The non-positive cohomology groups of* GC *are given as*

$$\mathrm{H}^i(\mathsf{GC}, d) = \begin{cases} \mathfrak{grt} & \text{if } i = 0 \\ 0 & \text{if } i \leq 0. \end{cases}$$

It's an important open problem to compute the full cohomology of GC. It's been conjectured that the first cohomology of the Kontsevich graph complex is trivial, which would mean that the Schouten bracket is undeformable in the category of L_∞-algebras.

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