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MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Lens spaces

by

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"All in the golden afternoon

Full leisurely we glide;

For both our oars, with little skill,

By little arms are plied,

While little hands make vain pretence

Our wanderings to guide."

(From the opening poem of *Alice's Adventures in Wonderland*, by Lewis Carroll)

Foreword

The intention of this paper was to be an unassuming monograph over classical lens spaces. In the process of writing it I came to the conclusion that it should include most of the general major results needed for the specific case at hand.

The paper is divided into four chapters. In the first chapter I discuss basics. The lens spaces are identifications spaces and therefore I discuss this subject in more generality. Then there come three different definitions. The definitions are actually recipes for constructing these spaces. It can be quite instructive to see how doing different things can lead to the same object. Indeed I show that the three definitions are equivalent. I then try to collect the apparently different spaces into homeomorphic classes.

Of capital importance when trying to characterize a topological space is the fundamental (homotopy) group. In order to define it in the second chapter I define homotopies and homotopy equivalence and then the thing itself. I show even how one could go about determining the fundamental group for certain classes of spaces.

The fundamental group lives essentially in two-dimensional space. If we want to investigate properties of spaces in higher dimensions we will need something more. Of course, there are the higher dimensional homotopy groups, a more advanced topic outside our horizon here. We shall instead look at homology and cohomology, the subject matter of the third and fourth chapters respectively. So in the third chapter I define homology groups and discuss some homological algebra. I then introduce the Mayer–Vietoris sequence and define the Euler characteristic. Using all that I do some "tricks" and compute the homology groups of the lens spaces. I repeat the procedure in chapter four where I define cohomology groups and compute.

Although two spaces need not be homeomorphic they can still be fairly alike in that homotopically equivalent loops can be "accommodated" in the same way in both spaces.

The lens spaces are topological objects defined in terms of two parameters, p and q. Both parameters are important for the characterization of these spaces but it turns out that the parameter q is essential both in determining when two lens spaces are homeomorphic and when they are homotopically equivalent in case they are not homeomorphic.

To settle this matter I define some (quite technical) products in cohomology and then prove a homotopy equivalence theorem in chapter five.

For all the fundamental theorems I rely heavily on Armstrong, Bredon and Hatcher in no particular order except the alphabetical one.

Contents

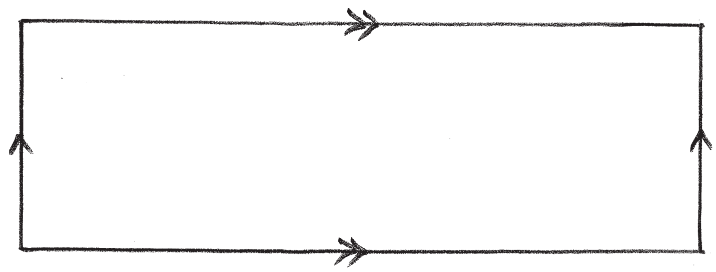
1 *Identification Spaces*

“ ’Curiouser and curiouser!’ cried Alice (she was so much surprised, that for the moment she quite forgot how to speak good English).”

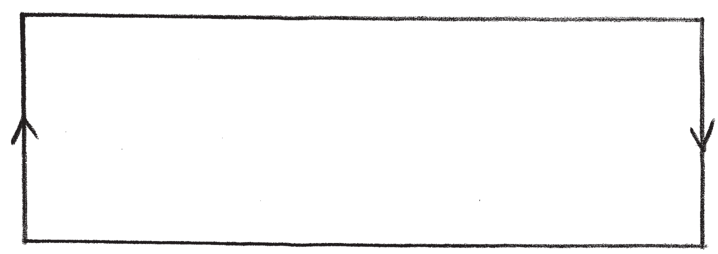
(*Alice′s Adventures in Wonderland*, ch. 2 *The Pool of Tears*)

1.1 Identification Topology

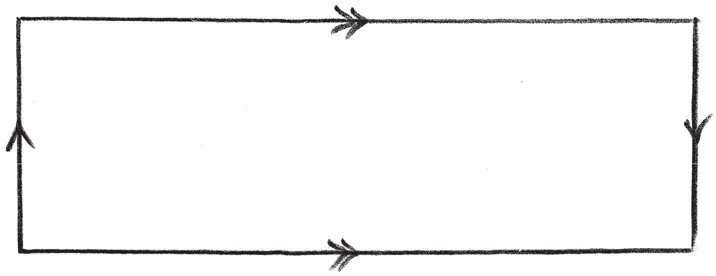
If we take a rectangular piece of paper and glue together/identify the two pairs of opposite edges we obtain a torus:



If we instead give the strip of paper half a twist first and then glue only the pair along the axis of the twist we get a Möbius strip:



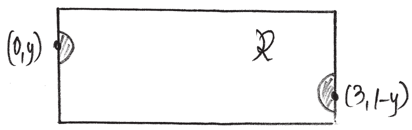
Even more ingenuously, if we glue the other pair too, we obtain a Klein bottle:

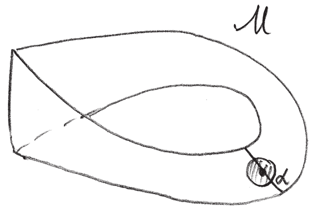


We start with something very simple, identify certain points and so we obtain more exotic geometric objects. This is the idea behind identification spaces. Let us make the idea mathematically more precise and construct a Möbius strip $\mathcal{M}$ in the process.

The rectangle can be identified with the subspace $\mathcal{R} \subseteq \mathbb{R}^2$ consisting of points (x, y) satisfying, say, $0 \leq x \leq 3$ and $0 \leq y \leq 1$. We shall be identifying pairs of points of the form $(0, y)$ and $(3, 1-y)$ whereas other points in $\mathcal{R}$ will be unaffected by our construction.

We partition $\mathcal{R}$ into subsets consisting either of a single point or two points, the latter being of the type $\{(0, y), (3, 1-y)\}$. These identifications are equivalent to our actually glueing the relevant edges.





The identification can be construed as a map π from $\mathcal{R}$ to $\mathcal{M}$ that sends each point of $\mathcal{R}$ to the corresponding subset of the partition in which it lies. If we want to consider the new geometric object as a topological space we must define the identification topology. It will be the largest topology for which π is continuous, i.e. a subset $\mathcal{O} \subseteq \mathcal{M}$ is defined to be open in the identification topology of $\mathcal{M}$ if and only if $\pi^{-1}(\mathcal{O})$ is open in the topology of $\mathcal{R}$.

If we write $\mathcal{R}_\star$ for $\mathcal{R}$ minus its vertical edges and denote the image under π of the vertical edges of $\mathcal{R}$ by $\mathcal{L}$, we realize that the restriction of π to $\mathcal{R}_\star$ is a homeomorphism between $\mathcal{R}_\star$ and $\mathcal{M} \setminus \mathcal{L}$. The neighbourhoods of points of $\mathcal{M} \setminus \mathcal{L}$ are pure and simple images under π of neighbourhoods of points of $\mathcal{R}_\star$. What about the neighbourhoods of points $\mathfrak{p} \in \mathcal{L}$? Then $\pi^{-1}(\{\mathfrak{p}\}) = \{(0, y), (3, 1-y) : 0 \leq y \leq 1\}$, two distinct points in $\mathcal{R}$. The union of half-disks (possibly quarter-disks) centred at $(0, y)$ and $(3, 1-y)$ respectively, will be mapped onto a disk (possibly half-disk) with centre $\mathfrak{p}$ on $\mathcal{L}$, and this is a neighbourhood of $\mathfrak{p}$. The half-disks (quarter–disks) are open in $\mathcal{R}$. The points of $\mathcal{L}$ have exactly the same type of neighbourhood as any other point in $\mathcal{M}$, and the identification topology coincides with the topology induced from $\mathbb{R}^3$ (the Möbius strip is embedded in $\mathbb{R}^3$).

This was quite pictorial but the construction is basically abstract and so shall we be in the sequel!

Let $\mathcal{X}$ be a topological space and let $\mathfrak{P}$ be a family of disjoint nonempty subsets of $\mathcal{X}$ such that $\mathcal{X} = \cup\mathfrak{P}$. ($\mathfrak{P}$, of course, is a partition of $\mathcal{X}$.) We define a new space $\mathcal{Y}$, the identification space, as follows:

- The points of $\mathcal{Y}$ are the elements of $\mathfrak{P}$ (the equivalence classes associated to the partition $\mathfrak{P}$).

- $\pi : \mathcal{X} \to \mathcal{Y}$ maps every point of $\mathcal{X}$ to its equivalence class in $\mathfrak{P}$.

- The topology of $\mathcal{Y}$ is the largest topology for which π is continuous.
- A subset $\mathcal{O} \subseteq \mathcal{Y}$ is open if and only if $\pi^{-1}(\mathcal{O})$ is open in $\mathcal{X}$.

This is the identification topology on $\mathcal{Y}$. Basically we obtain $\mathcal{Y}$ from $\mathcal{X}$ by identifying/"collapsing" each subset of $\mathfrak{P}$ to a single point.

Theorem 1. *Let $\mathcal{Y}$ be an identification space defined as above and let $\mathcal{Z}$ be an arbitrary topological space. A function $f : \mathcal{Y} \to \mathcal{Z}$ is continuous if and only if the composition $f\pi = f \circ \pi : \mathcal{X} \to \mathcal{Z}$ is continuous.*

Proof. Let $\mathcal{U}$ be an open subset of $\mathcal{Z}$. Then $f^{-1}(\mathcal{U})$ is open in $\mathcal{Y}$ if and only if $\pi^{-1}(f^{-1}(\mathcal{U}))$ is open in $\mathcal{X}$, i.e. if and only if $(f\pi)^{-1}(\mathcal{U})$ is open in $\mathcal{X}$. $\square$

Let $f : \mathcal{X} \to \mathcal{Y}$ be surjective and suppose that the topology on $\mathcal{Y}$ is the largest for which f is continuous. We call f an identification map. Any function $f : \mathcal{X} \to \mathcal{Y}$ determines a partition of $\mathcal{X}$ whose members are the subsets $\{f^{-1}(y) : y \in \mathcal{Y}\}$. Let $\mathcal{Y}_\star$ denote the identification space associated with this partition and $\pi : \mathcal{X} \to \mathcal{Y}_\star$ be the usual identification map.

Theorem 2. *If f is an identification map, then:*

(1) the spaces $\mathcal{Y}$ and $\mathcal{Y}_\star$ are homeomorphic

(2) a function $g : \mathcal{Y} \to \mathcal{Z}$ is continuous if and only if the composition $gf : \mathcal{X} \to \mathcal{Z}$ is continuous.

Proof. (2) is actually Theorem 1 above.

(1) The points of $\mathcal{Y}_\star$ are the sets $\{f^{-1}(y) : y \in \mathcal{Y}\}$.

Define $h : \mathcal{Y}_\star \to \mathcal{Y}$ by $h(\{f^{-1}(y)\}) = y$. Then h is obviously a bijection satisfying:

$$(h\pi)(x) = h(\pi(x)) = h([x]) = h(\{f^{-1}(y)\}) = y$$

$\because h\pi = f$, and

$$\left(h^{-1} f\right)(x) = h^{-1}(f(x)) = h^{-1}(y) = \left\{f^{-1}(y)\right\}$$

$\because h^{-1} f = \pi$

h is continuous by Theorem 1. h^{-1} is continuous by (2) above, since $h^{-1}f = \pi$ is continuous.

$\because$ h is a homeomorphism, so $\mathcal{Y}_\star \approx \mathcal{Y}$ $\square$

Theorem 3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be surjective. If f maps open sets of $\mathcal{X}$ to open sets of $\mathcal{Y}$, or closed sets of $\mathcal{X}$ to closed sets of $\mathcal{Y}$, then f is an identification map.*

Proof. We know that a topology is defined by specifying its open sets or, alternatively, its closed sets.

Suppose f maps open sets to open sets. Let $\mathcal{U}$ be a subset of $\mathcal{Y}$ such that $f^{-1}(\mathcal{U})$ is open in $\mathcal{X}$. Then $f(f^{-1}(\mathcal{U})) = \mathcal{U}$, since f is surjective, and then, because f maps open sets to open sets, $\mathcal{U}$ is open in the given topology on $\mathcal{Y}$. This topology is the largest for which f is continuous.

$\because f$ is an identification map

[Suppose f maps closed sets to closed sets. Let $\mathcal{V}$ be a subset of $\mathcal{Y}$ such that $f^{-1}(\mathcal{V})$ is open in $\mathcal{X}$. Then $f(f^{-1}(\mathcal{V})) = \mathcal{V}$ since f is surjective, and then, because f maps closed sets to closed sets, $\mathcal{V}$ is closed in the given topology on $\mathcal{Y}$. This topology is the largest for which f is continuous.

$\because f$ is an identification map] □

Corollary. *Let $f : \mathcal{X} \to \mathcal{Y}$ be surjective and continuous. If $\mathcal{X}$ is compact and $\mathcal{Y}$ is Hausdorff, then f is an identification map.*

Proof. A closed subset of $\mathcal{X}$ is compact and its image under the continuous map f is consequently a compact subset of $\mathcal{Y}$. A compact subset of $\mathcal{Y}$ must be closed because $\mathcal{Y}$ is Hausdorff. Thus f maps closed sets to closed sets and we conclude, by Theorem 3 above, that f is an identification map. □

Glueing

Let $\mathcal{X}$ and $\mathcal{Y}$ be subsets of a topological space. Give each of $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{X} \cup \mathcal{Y}$ the induced topology. If $f : \mathcal{X} \to \mathcal{Z}$ and $g : \mathcal{Y} \to \mathcal{Z}$ are functions agreeing on the intersection $\mathcal{X} \cap \mathcal{Y}$ we can define a new map $f \smile g : \mathcal{X} \cup \mathcal{Y} \to \mathcal{Z}$ by

$$(f \smile g)(x) = \begin{cases} f(x) & , x \in \mathcal{X} \\ g(x) & , x \in \mathcal{Y} \end{cases}$$

$f \smile g$ is the "glueing together" of f and g.

Glueing Lemma. *If $\mathcal{X}$ and $\mathcal{Y}$ are closed in $\mathcal{X} \cup \mathcal{Y}$, and if both $f : \mathcal{X} \to \mathcal{Z}$ and $g : \mathcal{Y} \to \mathcal{Z}$ are continuous, then $f \smile g : \mathcal{X} \cup \mathcal{Y} \to \mathcal{Z}$ is continuous.*

Proof. Let $\mathcal{C}$ be a closed subset of $\mathcal{Z}$. Then $f^{-1}(\mathcal{C})$ is closed in $\mathcal{X}$, by the continuity of f and therefore closed in $\mathcal{X} \cup \mathcal{Y}$, since $\mathcal{X}$ is closed in $\mathcal{X} \cup \mathcal{Y}$. Similarly, $g^{-1}(\mathcal{C})$ is closed in $\mathcal{X} \cup \mathcal{Y}$.

$(f \smile g)^{-1}(\mathcal{C}) = f^{-1}(\mathcal{C}) \cup g^{-1}(\mathcal{C})$ shows that $(f \smile g)^{-1}(\mathcal{C})$ is closed in $\mathcal{X} \cup \mathcal{Y}$.

$\because f \smile g$ is continuous □

Naturally, the glueing lemma can be stated in terms of open sets as well.

We define the disjoint union $\mathcal{X} \sqcup \mathcal{Y}$ of the topological spaces $\mathcal{X}$ and $\mathcal{Y}$, and the map $j : \mathcal{X} \sqcup \mathcal{Y} \to \mathcal{X} \cup \mathcal{Y}$ such that $j|_{\mathcal{X}} = \iota_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \cup \mathcal{Y}$, the inclusion of $\mathcal{X}$ in $\mathcal{X} \cup \mathcal{Y}$, and $j|_{\mathcal{Y}} = \iota_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{X} \cup \mathcal{Y}$, the inclusion of $\mathcal{Y}$ in $\mathcal{X} \cup \mathcal{Y}$. Obviously j is continuous and the composition $(f \smile g)j : \mathcal{X} \sqcup \mathcal{Y} \to \mathcal{Z}$ is continuous if and only if both f and g are continuous. These observations give us

Theorem 4. *If j is an identification map, and if both $f : \mathcal{X} \to \mathcal{Z}$ and $g : \mathcal{Y} \to \mathcal{Z}$ are continuous, then $f \smile g : \mathcal{X} \cup \mathcal{Y} \to \mathcal{Z}$ is continuous too.*

Proof. Use Theorem 2(2) □

If j is an identification map we can view $\mathcal{X} \cup \mathcal{Y}$ as an identification space obtained from $\mathcal{X} \sqcup \mathcal{Y}$ by identifying certain points of $\mathcal{X}$ with certain points of $\mathcal{Y}$, viz. if $x \in \mathcal{X} \cap \mathcal{Y}$, then x viewed as an element in $\mathcal{X}$ is identified with itself but viewed as an element in $\mathcal{Y}$. The open (closed) sets of $\mathcal{X} \cup \mathcal{Y}$ are those sets $\mathcal{U}$ such that $\mathcal{U} \cap \mathcal{X}$ and $\mathcal{U} \cap \mathcal{Y}$ are open (closed) in $\mathcal{X}$ and in $\mathcal{Y}$, respectively. Theorem 4 generalizes to arbitrary unions. Let $\{\mathcal{X}_\alpha : \alpha \in A\}$ be a family of subsets of a topological space. Each X_α and the union $\cup_{\alpha \in A}\mathcal{X}_\alpha$ are given the induced topology. Let $\mathcal{Z}$ be a space and suppose we have maps $f_\alpha : \mathcal{X}_\alpha \to \mathcal{Z}$, for each $\alpha \in A$, such that if $\alpha, \beta \in A$, $f_\alpha \mid \mathcal{X}_\alpha \cap \mathcal{X}_\beta = f_\beta \mid \mathcal{X}_\alpha \cap \mathcal{X}_\beta$. Define $F : \cup_\alpha \mathcal{X}_\alpha \to \mathcal{Z}$ by glueing together the f_α. This means, of course, that $F(x) = f_\alpha(x)$, if $x \in \mathcal{X}_\alpha$. Let $\sqcup_{\alpha \in A}\mathcal{X}_\alpha$ denote the disjoint union of the spaces $\mathcal{X}_\alpha$, and $j : \sqcup_\alpha \mathcal{X}_\alpha \to \cup_\alpha \mathcal{X}_\alpha$ be the function such that $j \mid \mathcal{X}_\alpha = \iota_{\mathcal{X}_\alpha} : \mathcal{X}_\alpha \to \cup_\alpha \mathcal{X}_\alpha$, the inclusion of $\mathcal{X}_\alpha$ in $\cup_\alpha \mathcal{X}_\alpha$.

Theorem 5. *If j is an identification map, and each of the f_α is continuous, then F is continuous too.*

Proof. $F : \sqcup_\alpha \mathcal{X}_\alpha \to \mathcal{Z}$ is continuous if and only if each f_α is continuous, by Theorem 2(2). □

We say that $\cup_\alpha \mathcal{X}_\alpha$ has the identification topology if j is an identification map.

Caveat! If the family $\{\mathcal{X}_\alpha : \alpha \in A\}$ is finite then $\cup_\alpha \mathcal{X}_\alpha$ automatically has the identification topology but one can have surprises if the family is infinite.

Attaching maps

Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces, let $\mathcal{A}$ be a subspace of $\mathcal{Y}$, and let $f : \mathcal{A} \to \mathcal{X}$ be continuous. We form the disjoint union $\mathcal{X} \sqcup \mathcal{Y}$ and define a partition $\mathfrak{P}$ as follows: two points lie in the same class if and only if they are identified under f. Our classes will consist of:

(a) singletons = individual points of $\mathcal{Y} \setminus \mathcal{A}$

(b) singletons = individual points of $\mathcal{X} \setminus \operatorname{im} f$

(c) pairs of points of the form $\{a, f(a)\}, a \in \mathcal{A}$.

We denote the identification space associated with $\mathfrak{P}$ by $\mathcal{X} \cup_f \mathcal{Y}$. We call f the attaching map.

Caveat! If $\mathcal{Y}$ is an identification space formed from $\mathcal{X}$ then $\mathcal{Y}$ will inherit properties such as compactness, connectedness and path-connectedness from $\mathcal{X}$, if this is the case. Yet $\mathcal{Y}$ need not be Hausdorff even though $\mathcal{X}$ is. For example: let $\mathcal{X} = \mathbb{R}$ and identify $r \sim s$ if and only if $r - s \in \mathbb{Q}$. This will result in an *indiscrete* space.

1.2 Lens Spaces

Construction V

The result of glueing two 3-balls (n-balls) via a homeomorphism of their boundaries is known to be homeomorphic to S^3 (S^n).

Let us consider an analogous construction using two solid tori ($\mathsf{S}^1 \times \mathsf{B}^2$), $\mathcal{V}_1$ and $\mathcal{V}_2$, instead. If $h : \partial\mathcal{V}_2 \to \partial\mathcal{V}_1$ is a homeomorphism we may form the space $\mathcal{M}^3 = \mathcal{V}_1 \cup_h \mathcal{V}_2$ by identifying each $x \in \partial\mathcal{V}_2$ with $h(x) \in \partial\mathcal{V}_1$ in the disjoint union $\mathcal{V}_1 \sqcup \mathcal{V}_2$. This is just a glueing together of two solid tori along their boundaries. Since $\mathcal{M}^3$ is the result of attaching two closed, compact and path-connected spaces it must itself be closed, compact and path-connected. Since every point in $\mathcal{M}^3$, in an obvious way, can be enclosed in a 3–ball, the space must be locally homeomorphic to $\mathbb{R}^3$. It proves furthermore to be orientable. (See Construction S below.)

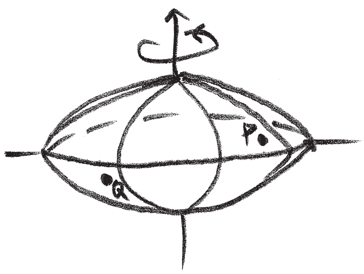
Given a homeomorphism $f : \mathsf{S}^1 \times \mathsf{B}^2 \to \mathcal{V}$ a solid torus $\mathcal{V}$ can be characterized by its meridian, m, a simple closed curve of the form $f(1 \times \partial\mathsf{B}^2)$, and its longitude, l, a simple closed curve of the form $f(\mathsf{S}^1 \times 1)$. The set of closed curves on the surface of a torus can be conceived as being the set of linear combinations of a fixed meridian m and a fixed longitude l. (We can think of $\{m, l\}$as the basis of the 2$-$dimensional $ml-$plane.) If we now let m_i and l_i, respectively, be meridian and longitude for $\partial\mathcal{V}_i$, $i = 1, 2$, we may define $h(m_2) = pl_1 + qm_1$, where p and q are coprime integers. Thus, $h(m_2) = pl_1 + qm_1$ will be a curve with slope $\frac{q}{p}$ winding around the torus $\partial\mathcal{V}_1$.

Surely h is a homeomorphism from $\partial\mathcal{V}_2$ to $\partial\mathcal{V}_1$. The resulting space $\mathcal{M}^3$ will depend, up to homeomorphisms, only upon $h(m_2)$ in $\partial\mathcal{V}_1$, or, more exactly, upon the homotopy class (getting ahead of ourselves!) of $h(m_2)$ in $\partial\mathcal{V}_1$. We shall call this space the lens space of type (p, q) and we shall denote it by $\mathsf{L}(\mathsf{p}, \mathsf{q})$.

Construction L

We shall give a more traditional definition.

Consider a lens–shaped solid ($\approx \mathsf{B}^3 \subseteq \mathbb{R}^3$) with edge/equator on the circle $x^2 + y^2 = 1$ and edge-angle $^{2\pi}/_p$ radians.

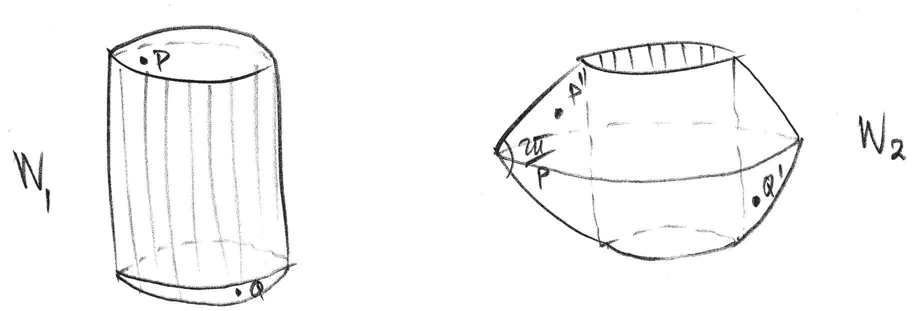


P and Q are points on the mantles (hemispheres) of the solid. Reflect the point P in the $x - y$ plane and then rotate the reflected image through an angle of $^{2\pi q}/_p$ radians, thus reaching Q. Identify P with Q. In this way each point on the upper hemisphere ($z > 0$) will be identified with exactly one point on the lower hemisphere ($z < 0$). A point on the equator will be identified with ($p-1$) other points on the equator. We shall call this homeomorphism of boundaries h and denote the resulting identification space B^3/h.

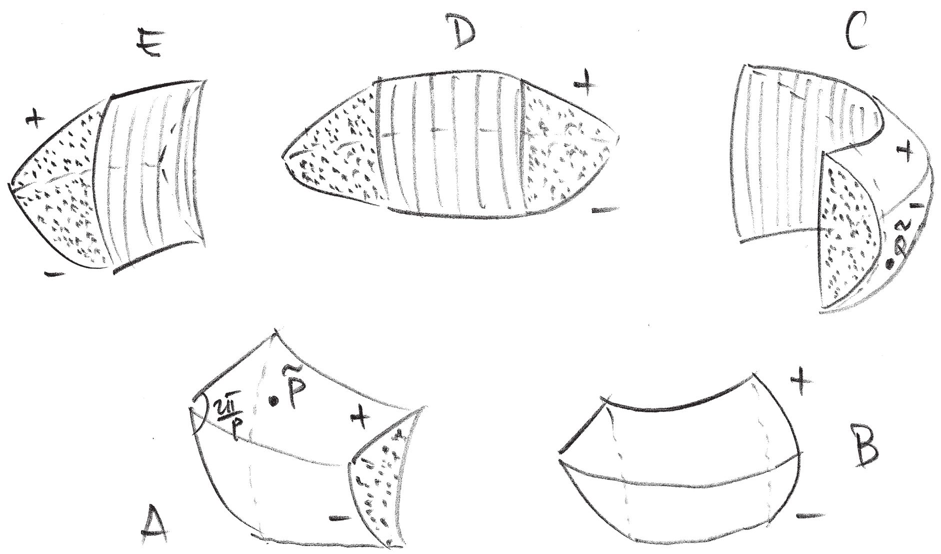
The equivalence V $\Longleftrightarrow$ L

Let $\mathcal{W}_1$ be the part of the lens-shaped space above lying inside the solid cylinder $x^2 + y^2 \leq {}^1/_2$, and let $\mathcal{W}_2$ be the closure of its complement. This closure is already homeomorphic to a solid torus which we shall call $\mathcal{V}_2$. The identifications described above will turn even $\mathcal{W}_1$ into a solid torus.

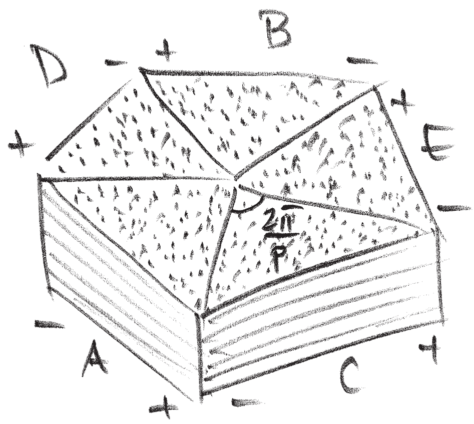
W_1 is a finite solid cylinder with lens-shaped caps. The top cap has been twisted $^{2\pi q}/_{p}$ radians and then glued to the bottom cap. Surely the result is a solid torus which we shall call $\mathcal{V}_1$.



Let us chop $\mathcal{W}_2$ into pieces and, for the sake of illustration, let us choose $p = 5$ and $q = 2$. This means that we divide the lens-shaped solid in 5 congruent sections and rotate the equivalent of 2 sectors.



The upper mantle (+) will be attached to the lower (−) in the order $A - C - E - B - D$. Reassembling we get a " cake", something like this:



We see now why the edge–angle had to be ${}^{2\pi}/_{p}$ radians. The p "slices of cake" must fit nicely together.

Furthermore, in order to respect the identifications described, the dotted sides must stay attached. So even the sliced $\mathcal{W}_2$ resurrects into a solid torus which we can call $\mathcal{V}_2$. Let us follow a meridinal curve (call it m_2) on $\partial\mathcal{V}_2$. This is the striped region of the reassembled cake. The striped sides of $\mathcal{W}_1$ and of the sliced $\mathcal{W}_2$ must stick together and so we have a space consisting of two solid tori glued along their boundaries. Our curve viewed on the striped region of $\mathcal{W}_2$ will consist of p vertical segments equally spaced on the surface of $\mathcal{W}_1$. If we follow one such segment we realize that its upper end will be twisted ${}^{2\pi q}/_{p}$ radians and then will be reflected onto the lower end of another such segment. So these segments, besides being thus twisted, will actually build a closed curve on what is $\partial\mathcal{V}_1$. An initially meridinal curve around $\mathcal{V}_2$ will wind itself in the end, with slope $\frac{q}{p}$, around $\mathcal{V}_1$. We can call this transformation h and so we have $h(m_2) = pl_1 + qm_1$. (I thank Rolfsen, 1990, for the "cake".)

The argument above shows that the two constructions lead to the same identification space, in other words $\mathsf{B}^3/h \approx \mathsf{L(p,q)}$.

$\because V \iff L$

Construction S

We shall give a third definition which will present the lens spaces as orbit spaces of a certain group action.

Generally, if $\mathcal{X}$ is a topological space and $\mathcal{G}$ is a group we call the map $\mathcal{G} \times \mathcal{X} \to \mathcal{X}$ an action of $\mathcal{G}$ on $\mathcal{X}$ and we call $\mathcal{O}(x) = \{(g,x) = gx : g \in \mathcal{G}\}$ the orbit of x. The orbits are either disjoint or identical so they partition the space. The set of these orbits with the quotient (identification) topology given by the map $\mathcal{X} \to \mathcal{X}/\mathcal{G}$ defined by $x \mapsto \mathcal{O}(x)$ is called the orbit space. The action of $\mathcal{G}$ on $\mathcal{X}$ is called properly discontinuous if each point $x \in \mathcal{X}$ has a neighbourhood $\mathcal{U}$ such that $g(\mathcal{U}) \cap \mathcal{U} \neq \emptyset \Longrightarrow g = e$, the identity element in $\mathcal{G}$. The lens-shaped solid above is embedded in $\mathbb{R}^3$. Each of its solid hemispheres is homeomorphic to a 3–ball. In the equatorial plane the boundaries of these two 3–balls are glued together. The first step described in Definition L leads to the total glueing of the two 3–balls along their boundaries resulting thus in S^3 .

Consider the homeomorphism $\tau_{(p,q)} : \mathsf{S}^3 \to \mathsf{S}^3$, given by $\tau_{(p,q)}(u, v) = (\varepsilon u, \varepsilon^q v)$, where $\varepsilon = e^{2\pi i/p}$ is the principal primitive pth root of unity. Then, surely, $\tau_{(p,q)}$ is periodic with period p and, therefore, generates a $\mathbb{Z}_p$–action on S^3.

(*Nota bene*: $\mathbb{Z}_p \cong <x> = \{id, \tau, \tau^2, \ldots, \tau^{p-1}\}$). We shall define the lens space to be the orbit space associated with the action of $\tau_{(p,q)}$ on S^3, where $x \sim y$ if and only if $y = \tau^k(x)$ for some $k \in \mathbb{Z}$. (Here we have written τ instead of $\tau_{(p,q)}$). Thus we get $\mathsf{S}^3/\mathbb{Z}_p$ with points $\mathcal{O}(x) = [x]$. It should be clear that the action of $\mathbb{Z}_p$ on S^3 is properly discontinuous.

Quite generally we can say that if $\mathcal{X}$ (where $\mathcal{X}$ is locally homeomorphic to $\mathbb{R}^n$) is an orientable manifold and the group G acts by orientation-preserving homeomorphisms on $\mathcal{X}$ then the orbit space $\mathcal{X}/\mathsf{G}$ is orientable too. Given two charts $(\varphi,\mathcal{U})$ and $(\psi,\mathcal{V})$ on $\mathcal{X}/\mathsf{G}$ we consider the coordinate transformation $\theta = \varphi\psi^{-1}$. This can be lifted to $\mathcal{X}$ and there we may consider $\hat{\theta} = \varphi\tau\psi^{-1}$, where τ is an orientation-preserving homeomorphism in G.

Since $\mathcal{X}$ is orientable the determinant of the Jacobian for $\widehat{\theta}$ must be positive and so it must hold that det $\mathsf{J}\hat{\theta} =$ det $\mathsf{J}(\varphi\tau\psi^{-1}) =$ det $\mathsf{J}(\varphi\psi^{-1}) \cdot$ det $\mathsf{J}\tau > 0$. We know that det $\tau > 0$ because τ is orientation-preserving, so det $\mathsf{J}\theta=$ det $\mathsf{J}(\varphi\psi^{-1}) > 0$. The conclusion is that $\mathcal{X}/\mathsf{G}$ is orientable. In particular, $\mathsf{S}^3/\mathbb{Z}_p$ is orientable. Furthermore, defining the lens spaces as $\mathsf{S}^3/\mathbb{Z}_p$, we can instantly recognize them as so called *spherical* $3-$*manifolds*.

The equivalence $L \iff S$

S^3 lies in $\mathbb{C} \times \mathbb{C}$. Let us divide the unit circle in the first factor $\mathbb{C}$ into p sectors by picking as vertices the points $e^{2\pi k/p} \in \mathsf{S}^1$, $k = 1, 2, \ldots, p$. Join the kth vertex to the unit circle in the second factor $\mathbb{C}$ by arcs of great circles on S^3. This way you will get a 2–dimensional ball, call it B^2_k, bounded by the latter unit circle. Take now the edge of the kth sector and join it to this unit circle thus obtaining a 3–dimensional ball, call it B^3_k, which will be bounded by B^2_k and B^2_{k+1}.

Let us follow B^3_k through S^3 under $\tau_{(p,q)}$. Going in the counter-clockwise direction along S^1 in the first factor $\mathbb{C}$ the points on the upper hemisphere of B^3_k, which is B^2_k, will be carried over to the lower hemisphere, another copy of B^2_k, after a twist of ${}^{2\pi}/_{p}$ radians in the first factor and ${}^{2\pi q}/_{p}$ radians in the second. Since the action of $\mathbb{Z}_p$ on S^3 is properly discontinuous we can view S^3 as the covering space and B^3_k as a fundamental region. The fate of B^3_k under $\tau_{(p,q)}$ is that it gets its hemispheres glued by the homeomorphism h of Construction L. This shows that $\mathsf{S}^3/\mathbb{Z}_p \approx \mathsf{B}^3/h$.

$\because L \iff S$

Now we can decide to call these identification spaces lens spaces and denote them $\mathsf{L(p,q)}$.

We can generalize Construction S.

Given integer p and integers $q_1, q_2, \ldots, q_n$, relatively prime to p, we define the $(2n-1)$–dimensional lens space $\mathsf{L}(p; q_1, q_2, \ldots, q_n)$ to be the orbit space $\mathsf{S}^{2n-1}/\mathbb{Z}_p$ of the unit sphere $\mathsf{S}^{2n-1} \subseteq \mathbb{C}^n$ with the action generated by $\tau(z_1, z_2, \ldots, z_n) = (\varepsilon^{q_1} z_1, \varepsilon^{q_2} z_2, \ldots, \varepsilon^{q_n} z_n)$, where $\varepsilon = e^{2\pi i/p}$ is the principal primitive pth root of unity. $\mathsf{L}(p,q)$ becomes the special case $\mathsf{L}(p; 1, q)$.

1.3 Homeomorphisms

Let us remind ourselves of what a heomeomorphism is.

A function $h\colon \mathcal{X} \longrightarrow \mathcal{Y}$ is called a *homeomorphism* if it is

- continuous
- invertible
- and its inverse h^{-1} is continuous

If such a function exists we say that $\mathcal{X}$ and $\mathcal{Y}$ are *homeomorphic* or *topologically equivalent*. We shall write $\mathcal{X} \approx \mathcal{Y}$.

Of course, two such spaces are essentially the same.

We shall try to classify the lens spaces with respect to homeomorphisms and it seems only natural to try to understand the role played by the parameters.

In Construction L we rotate the upper hemisphere of B^3 ${}^{2\pi q}/_{p}$ radians and then reflect it in the xy–plane, or vice versa. Obviously if p and q are not coprime we can cancel the factor $\gcd(p,q)$ so we rotate in fact through another angle and, therefore, we shall request that $\gcd(p,q) = \mathtt{1}$. Could $p = \mathtt{0}$ be allowed? In that case it is necessary that $q = \mathtt{1}$ because of coprimality.

$\star$ $\mathsf{L}(0,1)$

We realize that we cannot use the constructions L or S. In Construction V, though, we get $h(m_2) = m_1$ which means that the meridian m_2 of $\mathcal{V}_2$ is glued to the meridian m_1 of $\mathcal{V}_1$ and, since a meridian univocally defines a solid torus, this entails that $\mathsf{L}(0,1)$ is simply $\mathcal{V}_1 \cup_h \mathcal{V}_2$, two solid tori glued along their boundaries without any twisting.

We can make this attachment more explicit, as

$$\mathcal{V}_1 \cup_h \mathcal{V}_2 \approx (\mathsf{S}^1 \times \mathsf{B}^2) \underset{<id,h'>}{\cup} (\mathsf{S}^1 \times \mathsf{B}^2) \approx (\mathsf{S}^1 \underset{id}{\cup} \mathsf{S}^1) \times (\mathsf{B}^2 \underset{h'}{\cup} \mathsf{B}^2) \approx \mathsf{S}^1 \times \mathsf{S}^2$$

$\because \mathsf{L}(0,1) \approx \mathsf{S}^1 \times \mathsf{S}^2$

Some writers do not count this as a lens space, probably because of the absence of twisting. Others consider it a degenerate case.

$\star$ $\mathsf{L}(1,\mathsf{q})$

We rotate the upper hemisphere of B^3 through an angle of $2\pi q$ radians thus coming back to our starting point and then we glue the upper hemisphere to the lower hemisphere. The second step means that we glue two 3–dimensional balls along their boundaries and the result of that is known to be S^3.

$\because \mathsf{L}(1,\mathsf{q}) \approx \mathsf{S}^3$

Some writers do not count this either as a lens space and others consider it degenerate but we can actually go through with the "cake construction" we described earlier or let $\mathbb{Z}_1 = <\text{id}>$ act on S^3!

Let us see what happens if the parameters are negative integers. We start with $p, q > 0$ and consider

$\star$ $\mathsf{L}(-\mathsf{p},-\mathsf{q})$, $\mathsf{L}(-\mathsf{p},\mathsf{q})$, $\mathsf{L}(\mathsf{p},-\mathsf{q})$

In the first case we rotate through the angle $^{2\pi(-q)}/_{-p} = {}^{2\pi q}/_{p}$ and then identify, so we get $\mathsf{L}(\mathsf{p},\mathsf{q})$. In the other two cases we obviously rotate in the negative direction but the result is bound to be the same, viz. $\mathsf{L}(\mathsf{p},\mathsf{q})$, because, after the initial rotation and identification, we can rotate back twice in the positive direction (a homeomorphism) . Or, just reflect the ball in a mirror and rotate then!

$\because \mathsf{L}(-\mathsf{p},-\mathsf{q}) \approx \mathsf{L}(-\mathsf{p},\mathsf{q}) \approx \mathsf{L}(\mathsf{p},-\mathsf{q}) \approx \mathsf{L}(\mathsf{p},\mathsf{q})$

Thus we shall ignore negative integers.

$\star\ \mathsf{L(p, q + np)}$

Here we rotate n full times and then rotate through $2\pi q/p$ radians more.
$\because\ \mathsf{L(p, q + np)} \approx \mathsf{L(p, q)}$

The above considerations lead to the conclusion that

$$(\mathsf{p}' = \pm\mathsf{p}) \wedge (\mathsf{q}' \overset{p}{\equiv} \pm\mathsf{q}) \Longrightarrow \mathsf{L(p', q')} \approx \mathsf{L(p, q)}$$

From now on we shall restrict our attention to $1 < q < p$, $\gcd(p, q) = 1$.

$\star\ \mathsf{L(2, 1)}$

This is a much more interesting case. By construction S we let $\tau_{(2,1)}$ act on S^3 so $\tau_{(2,1)}(u, v) = (e^{\pi i}u, e^{\pi i}v) = -(u, v)$, and $\tau^2_{(2,1)} = \mathrm{id}$. In other words $\tau_{(2,1)}$ defines a $\mathbb{Z}_2$–action on S^3 and the essence of this action is the identification of antipodal points. We recognize the result as the real 3–dimensional projective space.
$\because\ \mathsf{L(2, 1)} \approx \mathbb{R}\mathsf{P}^3$

If you recall the generalization to higher dimensions then

$$\mathsf{L(2; q_1, q_2, \ldots, q_n)} \approx \mathbb{R}\mathsf{P}^{2n-1}.$$

The last case we shall consider is

$\star\ \mathsf{L(p, q^{-1})}$

Let S^3 be embedded in $\mathbb{C} \times \mathbb{C}$. Interchange axis of rotation and axis of reflection. In the end you will get $\mathsf{L(p, q)}$.

But, let us define $\hat{\tau}_{(p,q)}(u, v) = (\varepsilon^q u, \varepsilon v)$ and look at $\tau^q_{(p,q')}(u, v) = (\varepsilon^q u, \varepsilon v)$, where $q' = q^{-1}$. Obviously $\hat{\tau}_{(p,q)} = \tau^q_{(p,q')}$, so $\hat{\tau}_{(p,q)}$ defines the same $\mathbb{Z}_p$–action on S^3 as does $\tau_{(p,q)}$ itself.
$\because\ \mathsf{L(p, q^{-1})} \approx \mathsf{L(p, q)}$

We have in fact shown that

$$\mathsf{q}' \equiv \pm\mathsf{q}^{\pm 1} \implies \mathsf{L(p, q')} \approx \mathsf{L(p, q)}.$$

The converse is also true although much harder to prove, totally outside the scope of this paper and much beyond my abilities. It has been proved by Brody (1960), using knot theory, and by Cohen (1973), using advanced algebraic methods. I shall state it here as

The Homeomorphism Theorem for Lens Spaces :

$$\mathsf{L}(\mathsf{p}', \mathsf{q}') \approx \mathsf{L}(\mathsf{p}, \mathsf{q}) \iff (\mathsf{p}' = \pm\mathsf{p}) \wedge (\mathsf{q}' \equiv \pm\mathsf{q}^{\pm 1})$$

(Reidermeister, 1935).

2 Homotopy

“ ’Who are *you*?’ said the Caterpillar. This was not an encouraging opening for a conversation. Alice replied, rather shyly, ’I – I hardly know, sir, just at present – at least I knew who I *was* when I got up this morning, but I think I must have changed several times since then.’ ”

(*Alice's Adventures in Wonderland*, ch. 5 *Advice from a Caterpillar*)

2.1 Homotopic Maps

The main purpose of algebraic topology is to encode geometric properties of spaces in algebraic statements and then to study the algebraic structures instead of the geometric objects themselves. One could try to associate certain groups to spaces and then hope that maps between the spaces would induce homomorphisms between the groups. Studying these homomorphisms one could then draw conclusions about the spaces. It is the idea of functors from the category of topological spaces and maps to the category of groups and homomorphisms.

We shall try to construct a group out of loops within the space. We shall define a *loop* to be a continuous map α from the interval $\mathsf{I} = [0,1]$ to the space $\mathcal{X}$ such that $\alpha(1) = \alpha(0)$ and say that the loop is *based* at $x_0 = \alpha(0)$. Since we want to construct a group out of these loops we need to define a binary operation on the set of loops which we shall call multiplication. We shall, in fact, consider first *paths* instead of loops where the condition $\alpha(1) = \alpha(0)$ is not required. Tentatively we might say that if α and β are paths in $\mathcal{X}$ and if $\beta(0) = \alpha(1)$ then we define the product $\alpha{\cdot}\beta$ by the formulas

$$(\alpha\star\beta)(s) \begin{cases} \alpha(2s) & , 0 \le s \le \frac{1}{2} \\ \beta(2s-1) & , \frac{1}{2} \le s \le 1 \end{cases}$$

With this definition $\alpha \star \beta$ is continuous and maps the interval $\left[0, \frac{1}{2}\right]$ onto the image of α in $\mathcal{X}$ and the interval $\left[\frac{1}{2}, 1\right]$ onto the image of β in $\mathcal{X}$. But we have a problem: this product is not even associative, in general. In order that the involved products be defined we should have to define

$$((\alpha\star\beta)\star\gamma)(s) \begin{cases} \alpha(4s) & , 0 \le s \le \frac{1}{4} \\ \beta(4s-1) & , \frac{1}{4} \le s \le \frac{1}{2} \\ \gamma(2s-1) & , \frac{1}{2} \le s \le 1 \end{cases}$$

and

$$(\alpha\star(\beta\star\gamma))(s) \begin{cases} \alpha(2s) & , 0 \le s \le \frac{1}{2} \\ \beta(4s-2) & , \frac{1}{2} \le s \le \frac{3}{4} \\ \gamma(4s-3) & , \frac{3}{4} \le s \le 1 \end{cases}$$

Obviously, $(\alpha\star\beta)\star\gamma \neq \alpha\star(\beta\star\gamma)$. But there are too many paths anyway so we might try to collect them in equivalence classes and start again.

We define first a *homotopy* of paths to be a family $f_t : \mathsf{I} \longrightarrow \mathcal{X}$, $0 \le t \le 1$ such that

- the endpoints $f_t(0) = x_0$ and $f_t(1) = x_1$ are independent of t. If $x_1 = x_0$ we have a loop.
- the associated map $\mathsf{F}\colon \mathsf{I}\times\mathsf{I} \longrightarrow \mathcal{X}$ defined by $\mathsf{F}(s,t) = f_t(s)$ is continuous.

Two paths $f = f_0$ and $g = f_1$ related like this by a homotopy f_t are called *homotopic* with notation $f \simeq g$ rel $\partial\mathsf{I}$. ($\mathsf{I} = [0,1]$, $\partial\mathsf{I} = \{0,1\}$).

Theorem 6. *The relation of homotopy on paths with fixed endpoints in any space is an equivalence relation*

Proof. Reflexivity. This is self-evident: taking as homotopy the constant homotopy$f_t = f$ we have immediately $f \simeq f$.

Symmetry. This is quite easy too. If $f \simeq g$ via the homotopy f_t then the inverse homotopy f_{1-t} gives us $g \simeq f$.

Transitivity. Suppose that $f \simeq g$ via f_t , $g \simeq h$ via g_t, and $f_1 = g_0$. Then we can define the homotopy

$$h_t = \begin{cases} f_{2t} & ,0 \leq t \leq \frac{1}{2} \\ g_{2t-1} & ,\frac{1}{2} \leq t \leq 1 \end{cases}$$

The two "branches" agree at $t = \frac{1}{2}$ since we assumed that $f_1 = g_0$. *Nota bene*: the associated map $\mathsf{H}(s,t) = h_t(s)$ is continuous since it is defined on the union of two closed sets and its restrictions to the two sets are continuous. In our case we have

$$\mathsf{H}(s,t) = \begin{cases} \mathsf{F}(s,2t) & ,0 \leq t \leq \frac{1}{2} \\ \mathsf{G}(s,2t-1) & ,\frac{1}{2} \leq t \leq 1 \end{cases}$$

F and G are the maps associated with the homotopies f_t and g_t, respectively. So H is continuous on $\mathsf{I} \times \mathsf{I}$ being continuous on both $\mathsf{I} \times \left[0,\frac{1}{2}\right]$ and $\mathsf{I} \times \left[\frac{1}{2},1\right]$.

□

The equivalence class of a path f under the equivalence relation of homotopy will be denoted $[[f]]$ although, for ease of notation, we shall occasionally write f where there is no danger of confusion. We shall furthermore restrict our attention to loops or, actually, equivalence classes of loops based at the *base point* x_0. The set of all such homotopy classes $[[f]]$ of loops $f : \mathsf{I} \longrightarrow \mathcal{X}$ at the base point x_0 will be denoted $\pi_1(\mathcal{X}, x_0)$.

Now we can define a product on homotopy classes via the previous product defined on paths/loops.

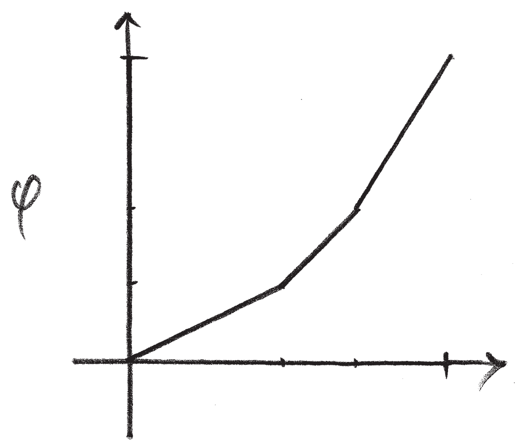
2.2 The Fundamental Group

Theorem 7. *$\pi_1(\mathcal{X},\ x_0)$ is a group with respect to the product* $[[f]][[g]] = [[f \star g]]$.

Proof. Since we consider only loops based at x_0 the product $f \star g$ of any two such loops is defined. This product respects homotopy classes because if $f_0 \simeq f_1$ via f_t and $g_0 \simeq g_1$ via g_t and $f_0(1) = g_0(0)$, in which case $f_0 \star g_0$ is defined, then $f_t \star g_t$ is defined and, in fact, provides the homotopy $f_0 \star g_0 \simeq f_1 \star g_1$. Consequently, the product $[[f]][[g]] = [[f \star g]]$ is well–defined.

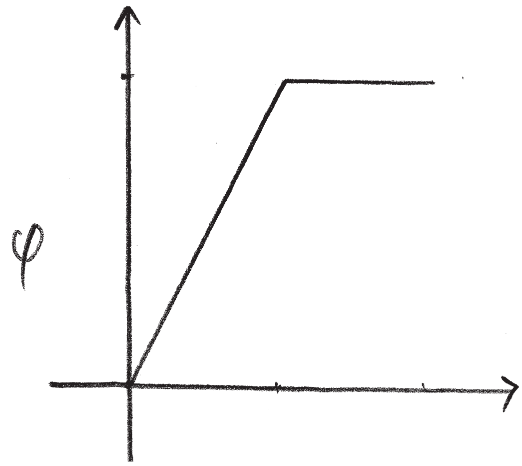
The composition $f\varphi$, where f is a path and φ: $\mathsf{I} \longrightarrow \mathsf{I}$ is a continuous map, with $\varphi(0) = 0$ and $\varphi(1) = 1$, is called a *reparametrization* of f. Reparametrizations preserve homotopy classes because $f\varphi \simeq f$ via the homotopy $f\varphi_t$ with $\varphi_t(s) = (1-t)\varphi(s) + ts$ which gives $\varphi_0 = \varphi$ and $\varphi_1(s) = s$. By convexity $(1-t)\varphi(s) + ts$ lies between $\varphi(s)$ and s, and therefore is in I. This guarantees that $f\varphi_t$ is defined.

Consider now paths f, g and h with $f(1) = g(0)$ and $g(1) = h(0)$. The products $(f \star g)\star h$ and $f\star(g \star h)$ are defined and the latter is a reparametrization of the former by the function φ of the type

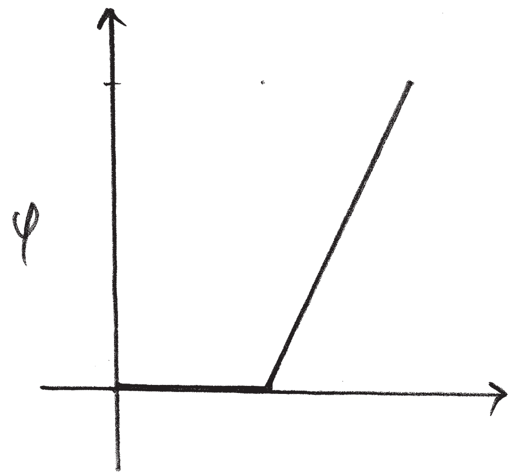


We conclude, by the above, that $(f \star g)\star h \simeq f\star(g \star h)$, and this means that the product in $\pi_1(\mathcal{X},\ x_0)$ is associative.

Given an arbitrary path $f : \mathsf{I} \longrightarrow \mathcal{X}$ we can define the constant path c at $f(1)$ by $c(s) = f(1)$ for all $s \in \mathsf{I}$. $f \star c$ becomes, of course, a reparametrization of f via a function φ with graph



so $f \star c \simeq f$.
Taking c such that $c(s) = f(0)$ and using a function φ with graph



we get $c \star f \simeq f$. But since $f(0) = f(1)$ we have found the two-sided identity in $\pi_1(\mathcal{X}, x_0)$, the constant loop at x_0.

Surely the inverse of a path f must be $f^{-1}(s) = f(1-s)$. Let us write $f(1-s) = \bar{f}$ for the moment. Define the homotopy $h_t = f_t \star g_t$, where f_t equals f on $[0, 1-t]$ and is stationary on $[1-t, 1]$ while $g_t\ (s) = f_t(1-s)$, i. e. is the inverse path of f_t.

h_t is a homotopy from $f \star \bar{f}$ to $c \star \bar{c} = c$ so $f \star \bar{f} \simeq c$. Analogously we get $\bar{f} \star f \simeq c$. Since f is a loop at the base point x_0 we deduce that $[[\bar{f}]]$ is a two-sided inverse for $[[f]]$ in $\pi_1(\mathcal{X}, x_0)$.

$\because$ $\pi_1(\mathcal{X}, x_0)$ is a group as claimed. (It need not be abelian though!)

$\square$

$\pi_1(\mathcal{X}, x_0)$ is called the *fundamental group* or the *Poincaré* group of the space $\mathcal{X}$.

I shall also mention that one can define in an analogous way homotopy groups π_n where we change the interval I involved in the argument by the $n-$dimensional cube I^n. Thus the fundamental group is the first in a large family of homotopy groups. If it proves many times difficult to compute the fundamental group of a space it turns out to be even more difficult to compute higher-dimension homotopy groups.

2.3 Homotopy type

The fundamental group is a space invariant. Two homeomorphic spaces will have the same fundamental group. But there are other maps besides the homeomorphic maps which leave the fundamental group invariant.

Two spaces $\mathcal{X}$ and $\mathcal{Y}$ are said to have the same *homotopy type* or are called *homotopically equivalent* if there exist maps $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ and $\mathcal{Y} \xrightarrow{g} \mathcal{X}$ such that $g \circ f \simeq \mathrm{id}_{\mathcal{X}}$ and $f \circ g \simeq \mathrm{id}_{\mathcal{Y}}$. We shall write $\mathcal{X} \simeq \mathcal{Y}$.

g is called a *homotopy inverse* for f .

Lemma 8. *The relation $\simeq$ is an equivalence relation on the set of topological spaces.*

Proof. Reflexivity. Obvious. *Symmetry*. Obvious.

Transitivity. Suppose $\mathcal{X} \simeq \mathcal{Y}$ and $\mathcal{Y} \simeq \mathcal{Z}$. Then there exist maps $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ and $\mathcal{Y} \xrightarrow{g} \mathcal{X}$ such that $g \circ f \simeq \mathrm{id}_{\mathcal{X}}$ and $f \circ g \simeq \mathrm{id}_{\mathcal{Y}}$, and there exist maps $\mathcal{Y} \xrightarrow{h} \mathcal{Z}$ and $\mathcal{Z} \xrightarrow{k} \mathcal{Y}$ such that $k \circ h \simeq \mathrm{id}_{\mathcal{Y}}$ and $h \circ k \simeq \mathrm{id}_{\mathcal{Z}}$.

We note first that if

$$\mathcal{X} \xrightarrow{u} \mathcal{Y} \xrightarrow{w} \mathcal{Z} \text{ and } \mathcal{X} \xrightarrow{v} \mathcal{Y} \xrightarrow{w} \mathcal{Z}$$

and $u \simeq v$ rel $\partial\mathsf{I}$ via a homotopy F then $w \circ u \simeq w \circ v$ rel $\partial\mathsf{I}$ via the homotopy $w \circ F$.

We get a similar result for the situation

$$\mathcal{X} \xrightarrow{u} \mathcal{Y} \xrightarrow{v} \mathcal{Z} \text{ and } \mathcal{X} \xrightarrow{u} \mathcal{Y} \xrightarrow{w} \mathcal{Z}\ .$$

This shows that homotopy is well behaved under composition of maps. Then, coming back to our initial situation, we have $\mathcal{X} \xrightarrow{h \circ f} \mathcal{Z}$, $\mathcal{Z} \xrightarrow{g \circ k} \mathcal{X}$,

$$(g \circ k) \circ (h \circ f) \simeq g \circ \mathrm{id}_{\mathcal{Y}} \circ f = g \circ f \simeq \mathrm{id}_{\mathcal{X}}$$

and

$$(h \circ f) \circ (g \circ k) \simeq h \circ \mathrm{id}_{\mathcal{Y}} \circ k = h \circ k \simeq \mathrm{id}_{\mathcal{Z}}.$$

$\because \mathcal{X} \simeq \mathcal{Z}$.

$\because \simeq$ is an equivalence relation on the set of topological spaces as claimed.

□

Let $\mathcal{A}$ be a subspace of $\mathcal{X}$ and define a homotopy F: $\mathcal{X} \times \mathsf{I} \longrightarrow \mathcal{X}$ relative to $\mathcal{A}$ such that for all $x \in \mathcal{X}$

$$\begin{cases} F(x, \mathtt{0}) = & x \\ F(x, \mathtt{1}) \epsilon & \mathcal{A} \end{cases}$$

This is called a *deformation retraction*. We shall simply state the obvious: if there exists a deformation retraction of $\mathcal{X}$ onto $\mathcal{A}$ then $\mathcal{X} \simeq \mathcal{A}$. The inclusion $\mathcal{A} \longrightarrow \mathcal{X}$ and the map $\mathcal{X} \longrightarrow \mathcal{A}$ given by $x \longmapsto F(x,\mathtt{1})$ will be homotopy inverses to each other.

A space $\mathcal{X}$ is called *contractible* if the identity map $\mathrm{id}_{\mathcal{X}}$ is homotopically equivalent to the constant map at some point in $\mathcal{X}$. A space is contractible if and only if it is homotopically equivalent to a point.

We shall finally state without proof

Theorem 9. *Two path-connected spaces of the same homotopy type have isomorphic fundamental groups.*

Proof. For a proof see Armstrong, 1983, or Bredon, 1993, or Hatcher, 2002.

□

2.4 Computations

Simple spaces

Let $\mathcal{X}$ be a simply connected space with base point x_0. We can shrink any loop at x_0 to the constant/trivial loop at x_0, hence $\pi_1(\mathcal{X}, x_0) = \mathtt{0}$, the trivial group. Incidentally, this shows that $\pi_1(\{x_0\}, x_0) = \mathtt{0}$.

Let $\mathcal{X} = \mathsf{S}^1$, the unit circle in $\mathbb{R}^2 \approx \mathbb{C}$. Define π: $\mathbb{R} \longrightarrow \mathsf{S}^1$ as the exponential map $x \longmapsto e^{2\pi i x}$. All integers are identified to $\mathtt{1}$ so we let the base point be $x_0 = \mathtt{1}$. Given any integer $n \epsilon\, \mathbb{Z}$ define the path $\gamma_n(s) = ns$, $s \in \mathsf{I}$. Under π γ_n is projected to a loop based at $\mathtt{1} \in \mathsf{S}^1$ winding round the circle n times, in a positive or negative direction, depending on n.

The following theorem should not come as a surprise.

Theorem 10. *The map φ: $\mathbb{Z} \longrightarrow \pi_1(\mathsf{S}^1, 1)$ defined by $\varphi(n) = [[\pi \circ \gamma_n]]$ is an isomorphism.*

Yet, proving that $\pi_1(\mathsf{S}^1, x_0) \cong \mathbb{Z}$ requires much more machinery. The theorem could be proved after our discussion on covering spaces.

Let $\mathcal{X} = \mathsf{S}^2$, the unit sphere in $\mathbb{R}^3$. Every loop at the base point x_0 can be shrunk to the trivial loop.

$\because \pi_1(\mathsf{S}^2, x_0) = 0$

Analogously one might reason for the general case, $n > 2$. So $\pi_1(\mathsf{S}^n, x_0) = 0$, $n \geq 2$.

Product spaces

Theorem 11. *If $\mathcal{X}$ and $\mathcal{Y}$ are path-connected spaces $\pi_1(\mathcal{X}\times\mathcal{Y})$ is isomorphic to $\pi_1(\mathcal{X})\times\pi_1(\mathcal{Y})$.*

Proof. Choose base points $x_0 \in \mathcal{X}$, $y_0 \in \mathcal{Y}$ and $(x_0, y_0) \in \mathcal{X} \times \mathcal{Y}$, respectively. The projections p_1: $\mathcal{X} \times \mathcal{Y} \longrightarrow \mathcal{X}$ and p_2: $\mathcal{X} \times \mathcal{Y} \longrightarrow \mathcal{Y}$ induce homomorphisms $p_{1\star}$: $\pi_1(\mathcal{X}\times\mathcal{Y}) \longrightarrow \pi_1(\mathcal{X})$ and $p_{2\star}$: $\pi_1(\mathcal{X}\times\mathcal{Y}) \longrightarrow \pi_1(\mathcal{Y})$ thus providing the homomorphism

$$\pi_1(\mathcal{X}\times\mathcal{Y}) \xrightarrow{\psi} \pi_1(\mathcal{X})\times\pi_1(\mathcal{Y})$$

given by

$$[[\alpha]] \longmapsto ([[p \circ \alpha]], [[p \circ \alpha]])$$

Let α be a loop at (x_0, y_0) in $\mathcal{X}\times\mathcal{Y}$. If $p_1 \circ \alpha \simeq \beta$ at x_0 with respect to $\mathsf{F}(s, t)$ and $p_2 \circ \alpha \simeq \gamma$ at y_0 with respect to $\mathsf{G}(s, t)$, then it must hold that $\alpha \simeq (\beta, \gamma)$ at (x_0, y_0) with respect to $\mathsf{H}(s, t) = (\mathsf{F}(s, t), \mathsf{G}(s, t))$. This shows that ψ above is one-to-one.

Consider loops β at x_0 in $\mathcal{X}$ and γ at y_0 in $\mathcal{Y}$ and form the loop $\alpha(s) = (\beta(s), \gamma(s))$ in $\mathcal{X}\times\mathcal{Y}$. $p_1 \circ \alpha = \beta$ and $p_2 \circ \alpha = \gamma$ by construction. Hence $\psi([[\alpha]]) = ([[\beta]], [[\gamma]])$ showing that ψ is onto.

We do have an isomorphism.

□

We can immediately apply this theorem to the torus and conclude that $\pi_1(\mathsf{T}^2) \cong \pi_1(\mathsf{S}^1 \times \mathsf{S}^1) \cong \pi_1(\mathsf{S}^1) \times \pi_1(\mathsf{S}^1) \cong \mathbb{Z} \oplus \mathbb{Z}$. (Here I used the so far unproved result that $\pi_1(\mathsf{S}^1) \cong \mathbb{Z}$.) We can do the same for $\pi_1(\mathsf{S}^m \times \mathsf{S}^n)$ getting $\pi_1(\mathsf{S}^m \times \mathsf{S}^n) \cong \pi_1(\mathsf{S}^m) \times \pi_1(\mathsf{S}^n) = 0$, the trivial group, for $m, n \geq 2$, whereas, if $m = 1$ and $n \geq 2$, for instance, we get $\pi_1(\mathsf{S}^1 \times \mathsf{S}^n) \cong \pi_1(\mathsf{S}^1) \times \pi_1(\mathsf{S}^n) \cong \mathbb{Z} \oplus 0 = \mathbb{Z}$.

Orbit spaces and covering spaces

A map $\varphi : \mathcal{X} \to \mathcal{Y}$ is called a covering map, and $\mathcal{X}$ is called a covering space of $\mathcal{Y}$, if $\mathcal{X}$ and $\mathcal{Y}$ are Hausdorff, path-connected, and if each point $y \in \mathcal{Y}$ has a path-connected neighbourhood $\mathcal{U}$ such that $\varphi^{-1}(\mathcal{U})$ is a nonempty union of sets $\mathcal{U}_\alpha$ (the path-connected components of $\varphi^{-1}(\mathcal{U})$) on which $\varphi \mid \mathcal{U}_\alpha$ is a homeomophism $\mathcal{U}_\alpha \approx \mathcal{U}$. $\mathcal{U}$ is called an *elementary set*. (We have already seen that $\pi : \mathsf{S}^3 \to \mathsf{S}^3/\mathbb{Z}_p \approx \mathsf{L}(\mathsf{p}, \mathsf{q})$ is a covering map.) In the sequel $\mathsf{I} = [0, 1]$ and $\partial\mathsf{I} = \{0, 1\}$.

The path lifting theorem. *Let $p : \mathcal{X} \to \mathcal{Y}$ be a covering map and let $f : \mathsf{I} \to \mathcal{Y}$ be a path. Let $x_0 \in \mathcal{X}$ be such that $p(x_0) = f(0)$. Then there exists a unique path $g : \mathsf{I} \to \mathcal{X}$ such that $pg = f$ and $g(0) = x_0$. In other words the following commutative diagram can be completed uniquely:*

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\
I \arrow[r, "f"'] \arrow[ur, dashed, "g"] & Y
\end{tikzcd}$$

Proof. By Lebesgue Lemma there is a an $n \in \mathbb{N}$ such that $f\left[\frac{i}{n}, \frac{i+1}{n}\right]$ lies in an elementary set $\mathcal{U}$ of $p(x_0)$. Since $p^{-1}(\mathcal{U}) = \bigcup\limits_\alpha \mathcal{U}_\alpha$ and $\mathcal{U}_\alpha \approx \mathcal{U}$ we can lift f by induction on i. At each step of the induction, the lift is already defined at the left-hand end point, leading to the uniqueness of the lift since it singles out exactly one component $\mathcal{U}_\alpha$ above the elementary set $\mathcal{U}$ which must be used. And the induction begins with $f\left[0, \frac{1}{n}\right]$ thus giving $g(0) = x_0$. In the reverse direction we get that $p \circ g = f$ as desired. Thus g is the unique lift of f. $\square$

The covering homotopy lemma. *Let $\mathcal{W}$ be an arbitrary space and let $\{\mathcal{U}_\alpha\}$ be an open covering of $\mathcal{W} \times \mathsf{I}$. Then for any point $w \in \mathcal{W}$ there is a neighbourhood $\mathcal{N}$ of w in $\mathcal{W}$ and a positive integer n such that $\mathcal{N} \times [i/n, i+1/n] \subset \mathcal{U}_\alpha$, for some α, for each $0 \leq i < n$.*

Proof. We can cover $\{w\} \times \mathsf{I}$ by a refinement of $\mathcal{U}_\alpha$ of the form $\mathcal{N}_1 \times \mathcal{V}_1$, $\mathcal{N}_2 \times \mathcal{V}_2, \ldots, \mathcal{N}_k \times \mathcal{V}_k$, by the compactness of I and the definition of the product topology. By Lebesgue Lemma there is an $n > 0$ such that $\left[\frac{i}{n}, \frac{i+1}{n}\right]$ is contained in one of the $\mathcal{V}_j$.

Take this n and set $\mathcal{N} = \bigcap_i \mathcal{N}_i$. $\square$

The covering homotopy theorem. *Let $\mathcal{W}$ be a locally connected space and let $p : \mathcal{X} \to \mathcal{Y}$ be a covering map.*

Let $\mathsf{F} : \mathcal{W} \times \mathsf{I} \to \mathcal{Y}$ *be a homotopy and let* $f : \mathcal{W} \times \{\mathtt{0}\} \to \mathcal{X}$ *be a lifting of the restriction of* F *to* $\mathcal{W} \times \{\mathtt{0}\}$. *Then there is a unique homotopy* $\mathsf{G} : \mathcal{W} \times \mathsf{I} \to \mathcal{X}$ *making the following diagram commute:*

$$\begin{tikzcd} W \times \{0\} \arrow[r, "f"] \arrow[d, hook] & X \arrow[d, "p"] \\ W \times I \arrow[r, "F"'] \arrow[ur, dashed, "G"] & Y \end{tikzcd}$$

Proof. Let $w \in \mathcal{W}$. By the above we can define a homotopy G on each $\{w\} \times \mathsf{I}$ and this in a unique way.

Then we can find a connected neighbourhood $\mathcal{N}$ of w in $\mathcal{W}$ and an integer $n > 0$ such that each $\mathsf{F}\left[\frac{i}{n}, \frac{i+1}{n}\right]$ lies in some elementary set $\mathcal{U}_i$. Assuming that G is continuous on $\mathcal{N} \times \left\{\frac{i}{n}\right\}$ we see that $\mathsf{G}(\mathcal{N} \times \left\{\frac{i}{n}\right\})$, being connected, must be contained in a single component of $p^{-1}(\mathcal{U}_i)$, say $\mathcal{V}$.

But then, on $\mathcal{N} \times \left[\frac{i}{n}, \frac{i+1}{n}\right]$, the lift G must be F composed with the inverse of the homeomorphism $p \mid \mathcal{V} : \mathcal{V} \to \mathcal{U}_i$, by connectedness.

This entails that G is continuous on all of $\mathcal{N} \times \left[\frac{i}{n}, \frac{i+1}{n}\right]$ and so, by induction, G is continuous on each $\mathcal{N} \times \mathsf{I}$ and, hence, everywhere. As an extra consequence we get that if F is a homotopy rel $\mathcal{W}'$ for some $\mathcal{W}' \subseteq \mathcal{W}$, then so is G by its construction. □

The covering homotopy corollary. *Let* $\psi : \mathcal{X} \to \mathcal{Y}$ *be a covering map. Let* f_0 *and* f_1 *be paths in* $\mathcal{Y}$ *with* $f_0 \simeq f_1$ *rel* $\partial\mathsf{I}$.

Let $\widetilde{f_0}$ *and* $\widetilde{f_1}$ *be liftings of* f_0 *and* f_1 *such that* $\widetilde{f_0}(\mathtt{0}) = \widetilde{f_1}(\mathtt{0})$. *Then* $\widetilde{f_0}(\mathtt{1}) = \widetilde{f_1}(\mathtt{1})$ *and* $\widetilde{f_0} \simeq \widetilde{f_1}$ *rel* $\partial\mathsf{I}$.

Proof. $\widetilde{f}_0(\mathtt{1}) = \widetilde{f}_1(\mathtt{1})$, by the uniqueness of liftings.

$\widetilde{f}_0 \simeq \widetilde{f}_1$ rel $\partial\mathsf{I}$, by the uniqueness of covering homotopies. □

Theorem 12. *If* G *acts as a group of homeomorphisms on a simply connected space, and if each point* $x \in \mathcal{X}$ *has a neighbourhood* $\mathcal{U}$ *which satisfies* $\mathcal{U} \cap g(\mathcal{U}) = \emptyset$ *for all* $g \in G \setminus \{e\}$ *then* $\pi_1(\mathcal{X}/G)$ *is isomorphic to* G.

(In the sequel π*:* $\mathcal{X} \longrightarrow \mathcal{X}/G$ *is the canonical projection.)*

Proof. Fix a point $x_0 \in \mathcal{X}$ and, given $g \in G$, join x_0 to $g(x_0)$ by a path γ. Then $\pi\gamma$ is a loop based at $\pi(x_0)$ in $\mathcal{X}/G$. Let $\pi\gamma'$ be another loop based at $\pi(x_0)$ in $\mathcal{X}/G$ such that $\pi\gamma' \simeq \pi\gamma$. By the path lifting lemma they will lift to γ' and γ, respectively, in $\mathcal{X}$, and $\gamma'(\mathtt{0}) = x_0 = \gamma(\mathtt{0})$. By the covering homotopy corollary $\gamma'(\mathtt{1}) = \gamma(\mathtt{1})$. Now, $g'(x_0) = \gamma'(\mathtt{1}) = \gamma(\mathtt{1}) = g(x_0)$.

$\because g' = g$ and $\gamma' \simeq \gamma$.

Given the canonical projection π: $\mathcal{X} \longrightarrow \mathcal{X}/G$, define the map

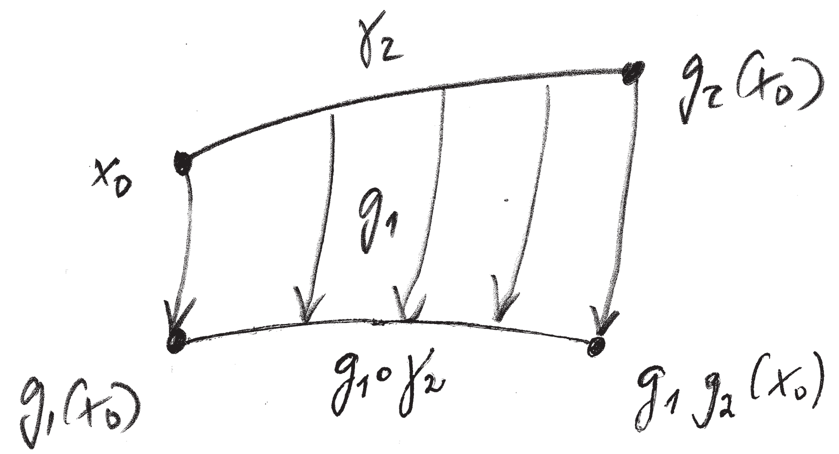
$$\varphi : G \to \pi_1(\mathcal{X}/G,\, \pi(x_0))$$

by $\varphi(g) = [[\pi\gamma]]$.

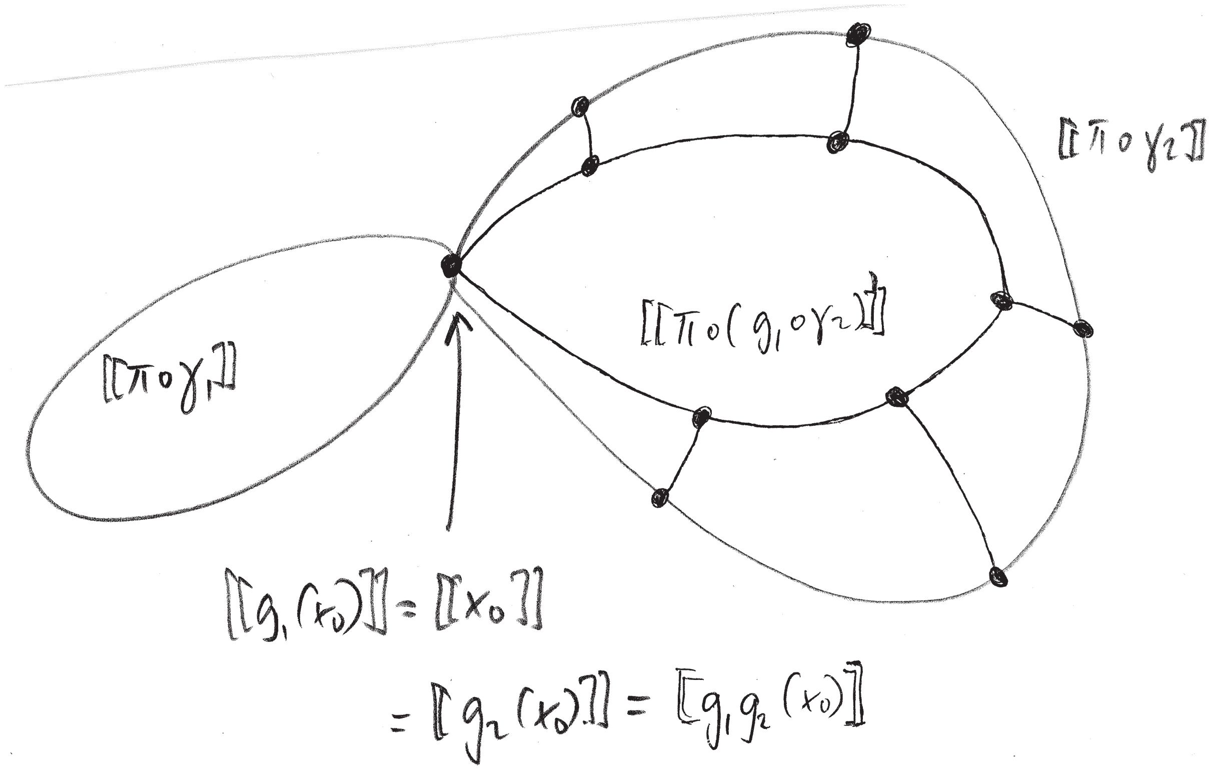
If $\pi\gamma'$ is another representative of the class $[[\pi\gamma]]$ then both γ and γ' must join x_0 to $g(x_0)$ in $\mathcal{X}$ and, since $\mathcal{X}$ is simply connected, $\gamma' \simeq \gamma$. Consequently, $[[\pi\gamma']] = [[\pi\gamma]]$ and φ is not affected by the choice of representatives.

$\because \varphi$ is well-defined

Let $g_1, g_2 \in G$ and join x_0 to $g_1(x_0)$ by a path γ_1 and to $g_2(x_0)$ by a path γ_2. The situation could be depicted like this:



$\gamma_1 g_1 \gamma_2$ joins x_0 to $g_1 g_2(x_0)$ in $\mathcal{X}$ and via π we get something like this in $\mathcal{X}/G$:



We can read from the diagram that $[[\pi\gamma_1 g_1\gamma_2]] = [[\pi\gamma_1]][[\pi g_1\gamma_2]] = [[\pi\gamma_1]][[\pi\gamma_2]]$, since, under π, $g_1\gamma_2$ becomes identified pointwise with γ_2, and so $[[\pi g_1\gamma_2]] = [[\pi\gamma_2]]$. By the definition of φ we have that $\varphi(g_1) = [[\pi\gamma_1]]$ and $\varphi(g_2) = [[\pi\gamma_2]]$. Consequently, $\varphi(g_1 g_2) = [[\pi\gamma_1 g_1\gamma_2]] = [[\pi\gamma_1]][[\pi\gamma_2]] = \varphi(g_1)\varphi(g_2)$.

$\because$ φ is a group homomorphism

Assume that $\varphi(g_1) = \varphi(g_2)$, that is $[[\pi\gamma_1]] = [[\pi\gamma_2]]$, where γ_i is a path in $\mathcal{X}$ joining x_0 to $g_i(x_0)$, $i = 1, 2$. By the path lifting theorem $[[\pi\gamma_1]]$ can be lifted to a unique path, which is in fact γ_1, in $\mathcal{X}$, and similarly $[[\pi\gamma_2]]$ will lift to the unique γ_2. At the same time, by the covering homotopy theorem, $\gamma_1(0) = x_0 = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$. But $g_1(x_0) = \gamma_1(1) = \gamma_2(1) = g_2(x_0)$, which entails $g_1 = g_2$.

$\because$ φ is a monomorphism

Let α be a loop in $\pi_1(\mathcal{X}/G, \pi(x_0))$, i.e. $\pi(x_0) = \alpha(0) = \alpha(1) = \pi(g(x_0))$ for some $e \neq g \in G$. Then, by the path lifting theorem, it can be lifted to a path in $\mathcal{X}$ such that $\pi^{-1}(\alpha(0)) = x_0$ and $\pi^{-1}(\alpha(1)) = g(x_0)$. Call it γ. But then γ joins x_0 to $g(x_0)$ in $\mathcal{X}$ and so $\alpha = [[\pi\gamma]] = \varphi(g)$.

$\because$ φ is an epimorphism

Combining all of the above we see that $\varphi : G \to \pi_1(\mathcal{X}/G, \pi(x_0))$ is a group isomorphism.

$\because \pi_1(\mathcal{X}/G, x_0) \cong G$ □

Thus

$$\pi_1(\mathsf{L}(\mathsf{p},\mathsf{q})) \cong \pi_1(\mathsf{S}^3/\mathbb{Z}_p) \cong \mathbb{Z}_p$$

An immediate consequence of this is that

$$\mathsf{L}(\mathsf{p},\mathsf{q}) \approx \mathsf{L}(\mathsf{p}',\mathsf{q}) \quad \text{if and only if} \quad \mathsf{p}' = \pm\mathsf{p} \quad \text{(homeomorphisms)}$$

$$\mathsf{L}(\mathsf{p},\mathsf{q}) \simeq \mathsf{L}(\mathsf{p}',\mathsf{q}) \quad \text{if and only if} \quad \mathsf{p}' = \pm\mathsf{p} \quad \text{(homotopy equivalences).}$$

A few more cases of group actions and orbit spaces:

- Consider the translation $x \longmapsto x+n$, $n \in \mathbb{Z}$, of the real line . This can be interpreted as the action of the group $\mathbb{Z}$ on the space $\mathbb{R}$. $\mathbb{R}$ is simply connected and the action is properly discontinuous. The orbit space of this action is $\mathbb{R}/\mathbb{Z} \approx \mathsf{S}^1$, so $\pi_1(\mathsf{S}^1) \cong \mathbb{Z}$, by the above. I should like to remind you here of the exponential map π: $\mathbb{R} \longrightarrow \mathsf{S}^1$ given by $x \longmapsto e^{2\pi i x}$. If $f : \mathsf{I} \longrightarrow \mathsf{S}^1$ is a loop at $1 \in \mathsf{S}^1$ and $\widetilde{f} : \mathsf{I} \longrightarrow \mathbb{R}$ is a lifting of f with $f(0) = 0$, then $\widetilde{f}\ (1) \in \pi^{-1}(\{1\}) = \mathbb{Z}$. Let $\widetilde{f}(1) = n$. This number depends only on the homotopy class $[[f]] \in \pi_1(\mathsf{S}^1)$ by the Corollary above. We shall call this integer the *degree* of f and write $n = \deg f$. It can be shown that deg: $\pi_1(\mathsf{S}^1) \longrightarrow \mathbb{Z}$ is an isomorphism and that the map $z \longmapsto z^n$ of $\mathsf{S}^1 \longrightarrow \mathsf{S}^1$ has degree n.
- Consider now the translation of the real plane of the type $(x, y) \longmapsto (x+m, y+n)$, $m, n \in \mathbb{Z}$. This can be interpreted as the action of the group $\mathbb{Z}\oplus\mathbb{Z}$ on the space $\mathbb{R}^2$. $\mathbb{R}^2$ is simply connected and the action is properly discontinuous.The orbit space of this action is $\mathbb{R}^2/(\mathbb{Z} \oplus\mathbb{Z}) \approx \mathsf{S}^1\times\mathsf{S}^1 \approx \mathsf{T}^2$, so $\pi_1(\mathsf{T}^2) \cong \mathbb{Z}\oplus\mathbb{Z}$.
- Consider the *antipodal* map on the unit sphere S^n, $n \geq 2$. We know that this map gives us the projective space P^n. If we denote the map by π we can consider the group of homeomorphisms generated by π, i. e. $\{id, \pi\} \cong \mathbb{Z}_2$, acting on S^n, $n \geq 2$. S^n is simply connected and the action is properly discontinuous, so $\pi_1(\mathsf{P}^n) \cong \mathbb{Z}_2$.

Nota bene. The same group can act in different ways on the same space. Let $\mathbb{Z}_2$ act on the torus $\mathsf{T}^2 \subseteq \mathbb{R}^3$ in the following ways:

- $(x, y, z) \longmapsto (x, -y, -z)$. We get the sphere. $\pi_1(\mathsf{S}^2) = \mathsf{0}$.
- $(x, y, z) \longmapsto (-x, -y, z)$. We get the torus. $\pi_1(\mathsf{T}^2) \cong \mathbb{Z} \oplus \mathbb{Z}$.
- $(x, y, z) \longmapsto (-x, -y, -z)$. We get the Klein bottle. $\pi_1(\mathsf{K}^2) = \{< a, b >| \ a^2 = b^2\}$. Incidentally, this is a non-abelian group.

Caveat! The problem here is that the torus is not simply connected to start with. Furthermore at least the first two actions are not properly discontinuous. We have already solved the probem for the sphere and the torus. For the Klein bottle we shall need a more recondite group. Let this group be generated by elements t and u subject to the relation $u^{-1}tut = e$ and let the action be given by $t(x, y) = (x + \mathsf{1}, y)$ and $u(x, y) = (-x + \mathsf{1}, y + \mathsf{1})$. We realize that the group G generated by $<t, u>$ defines a properly discontinuous action on the simply connected space $\mathbb{R}^2$. The orbit space is $\mathbb{R}^2/\mathsf{G}$.

Setting $a = tu$ and $b = u$ we get $a^2 = tutu = (tut)u = u^2 = b^2$, where we have used the relation between the generators. Thus $\mathsf{G} = \{< a, b >| \ a^2 = b^2\}$.

I shall not pursue the matter any further but the lesson is that we must be careful when considering group actions and orbit spaces. The conditions of Theorem 10 must hold before venturing to apply it! However, the results obtained on orbit spaces and product spaces may be very useful in determing the fundamental group of a topological space.

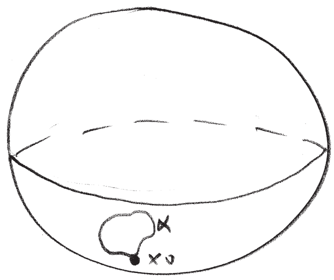
3 Homology

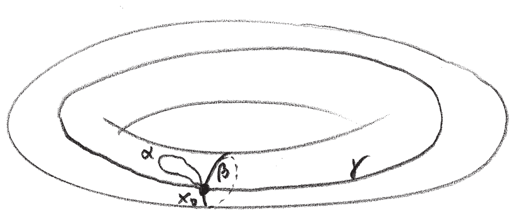
“ ’Can you do Addition? the White Queen asked. What’s one and one and one and one and one and one and one and one and one and one?’I don’t know,’ said Alice. ’I lost count.’ [...........] ’Can *you* do sums?’ Alice said, turning suddenly on the White Queen, for she didn’t like being found fault with so much. The Queen gasped and shut her eyes. ’I can do Addition,’ she said, ’if you give me time – but I can’t do Subtraction under *any* circumstances.’ ”

(Lewis Carroll: *Through the Looking Glass* , ch. 9 *Queen Alice*)

3.1 Homology Groups

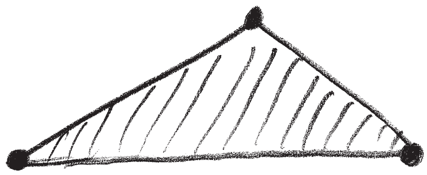
How can one distinguish between the sphere and the torus? A loop on the sphere can be shrunk to a point, the trivial loop. On the torus there are loops which cannot be shrunk to a point, nontrivial loops.

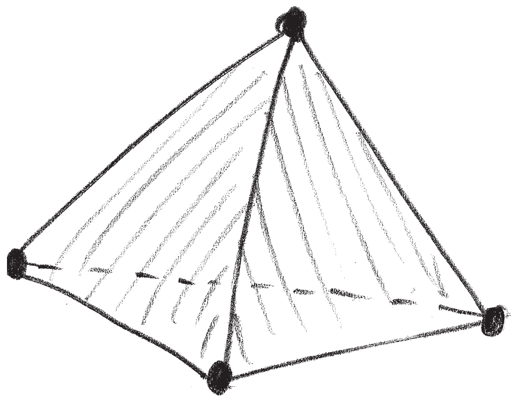




The fundamental group discloses properties that are two-dimensional but it cannot distinguish between, say, S^3 and S^4 . Indeed $\pi_1(\mathsf{S}^n) = 0$ for all $n \geq 2$. In simplicial homology we associate to a space a simplicial complex K through the process of triangulation, if the space is "nice" enough. We shall only consider spaces of this kind. A simplicial complex consists of simplices and these live in some Euclidian space. Some examples:

- 0-simplex = point/vertex
- 1-simplex = closed line segment
- 2-simplex = triangular patch
- 3-simplex = solid tetrahedron





The simplices have "faces". If A and B are simplices and the vertices of A form a subset of the vertices of B, then A is called a face of B. A finite collection of simplices in some Euclidean space $\mathbb{R}^n$ is called a simplicial complex if whenever a simplex lies in the collection then so does each of its faces and whenever two simplices of the collection intersect they do so in a common face. A space is triangulable if it is homeomorphic to the union of a finite collection of simplices fitting nicely together in some Euclidian space. A simplicial complex K can be viewed as a polyhedron $\mid \mathsf{K} \mid$, the "skeleton" of K. Let K be a simplicial complex. Then:

- $\mid \mathsf{K} \mid$ is a closed bounded subset of $\mathbb{R}^n$ and, hence, a compact space.
- Each point of $\mid \mathsf{K} \mid$ lies in the interior of exactly one simplex of K.
- If we take the simplices of K separately and give their union the identification topology, then we obtain exactly $\mid \mathsf{K} \mid$.
- If $\mid \mathsf{K} \mid$ is a connected space, then it is path-connected.

- Every simplex of K can be oriented in precisely two different ways (a vertex in only one way!).

We define now $\mathsf{C}_n(\mathsf{K})$ to be the free abelian group generated by the oriented n–simplices of K.

If we denote two such simplices by σ and τ, then $\sigma + \tau = 0$ only if they are the same simplices with opposite orientations. An element of $\mathsf{C}_n(\mathsf{K})$ is called an n–dimensional chain and $\mathsf{C}_n(\mathsf{K})$ is the nth chain group of K with rank equal to the number of n–simplices in K. Such a chain can be viewed as a linear combination $a_1\sigma_1 + a_2\sigma_2 + \cdots + a_r\sigma_r$ with integer coefficients of n-simplices of K, and $a(-\sigma) = (-a)\sigma$ while $-\sigma$ stands for the same simplex as σ but with opposite orientation. The boundary of an oriented n–simplex is the $(n-1)$–chain determined by the sum of its $(n-1)$–dimensional faces, each with the orientation induced from that of the whole simplex. An n–simplex can be denoted by the n–tuple of its vertices in some ordering, thus

$$(v_0, v_1, \dots, v_n) = \operatorname{sign}\theta(v_{\theta(0)}, v_{\theta(1)}, \dots, v_{\theta(n)})$$

and the boundary ∂ of this simplex is

$$\partial(v_0, v_1, \dots, v_n) = \Sigma_{i=0}^{n}(-1)^i(v_0, v_1, \dots, \hat{v}_i, \dots, v_n),$$

where $(v_0, v_1, \dots, \hat{v}_i, \dots, v_n)$ represents the $(n-1)$–simplex obtained by deleting vertex v_i.

∂ becomes thus a group homomorphism $\partial : \mathsf{C}_n(\mathsf{K}) \to \mathsf{C}_{n-1}(\mathsf{K})$. If $n = 1$ then we define the boundary of a single vertex to be zero, so $\mathsf{C}_{-1}(\mathsf{K}) = 0$. We shall denote the kernel of ∂ by $\mathsf{Z}_n(\mathsf{K}) =$ the group of n–cycles of K.

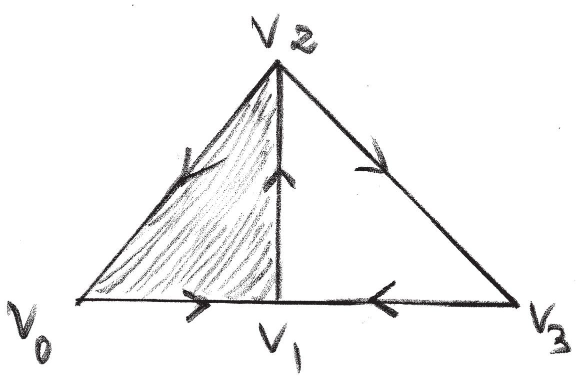
Lemma 13. *The composition* $\mathsf{C}_{n+1}(\mathsf{K}) \xrightarrow{\partial_{n+1}} \mathsf{C}_n(\mathsf{K}) \xrightarrow{\partial_n} \mathsf{C}_{n-1}(\mathsf{K})$ *is the zero homomorphism.*

Proof.

$$\begin{aligned}
\partial^2(v_0, v_1, \dots, v_{n+1}) &= \partial \sum_{i=0}^{n+1}(-1)^i(v_0, v_1, \dots, \hat{v}_i, \dots, v_{n+1}) \\
&= \sum_{i=0}^{n+1}(-1)^i \sum_{j=i+1}^{n+1}(-1)^{j-1}(v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_{n+1}) \\
&\quad + \sum_{i=0}^{n+1}(-1)^i \sum_{j=0}^{i-1}(-1)^j(v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_{n+1}).
\end{aligned}$$

Every oriented $(n-1)$–simplex $(v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_{n+1})$ appears twice, once with sign $(-1)^{i+j-1}$ and once with sign $(-1)^{i+j}$ so they cancel. The result is that $\partial^2 = \partial \circ \partial \equiv 0$, as desired. $\square$

We shall denote the image of ∂ by $\mathsf{B}_n(\mathsf{K})$ = the group of n–boundaries of K.



Both $\partial(v_0, v_1, v_2) = (v_0, v_1) - (v_0, v_2) + (v_1, v_2)$ and $(v_1, v_2) - (v_1, v_3) + (v_2, v_3)$ are cycles but only the former is a boundary, in the sense that it circumscribes a part of the space, the latter being there just for its own sake, it surrounds nothing, it surrounds a hole! We could somehow ignore the boundaries but not the other cycles since they are essential in themselves.

The *n–th homology group* of K is then defined to be

$$\mathsf{H}_n(\mathsf{K}) = \mathsf{Z}_n(\mathsf{K})/\mathsf{B}_n(\mathsf{K}) = \mathsf{ker}\partial_n/\mathsf{im}\partial_{n+1}$$

Two n–cycles whose difference is a boundary (a boundary n–cycle) are called homologous cycles. An element of $\mathsf{H}_n(\mathsf{K})$ will be a homology class. So $\mathsf{H}_n(\mathsf{K})$ is by definition a finitely generated abelian group and can be, as such, written as $\mathsf{H}_n(\mathsf{K}) = \mathsf{F}\oplus\mathsf{T}$, where F is a finitely generated free abelian group and can, in turn, be expressed as the sum of cyclic groups, and T is a finite abelian group. The elements of T are the elements of finite order, the torsion elements.

3.2 The Zeroth Homology Group

Theorem 14. *If $\mathcal{X}$ is path–connected then $\mathsf{H}_0(\mathcal{X}) \cong \mathbb{Z}$.*

Proof. Actually I am going to consider a triangulation of $\mathcal{X}$ and the corresponding complex K with skeleton $\mid \mathsf{K} \mid$.

If v and w are vertices in $\mid \mathsf{K} \mid$ then we can join them by an edge path $vv_1v_2......v_mw$, in which no two consecutive vertices are equal. This path can be viewd as the 1-simplex $(v,v_1) + (v_1,v_2) + ... + (v_m, w)$ and

$\partial((v,v_1) + (v_1,v_2) + ... + (v_m, w)) = w - v_m + v_m - v_{m-1} + ... - v_1 + v_1 - v = w - v$.

$\because v \sim w$, by the definition of $\sim$, for all vertices in $\mid \mathsf{K} \mid$.

Thus $\mathsf{C}_0(\mathsf{K}) = \mathsf{Z}_0(\mathsf{K}) \cong \mathbb{Z}$. Furthermore, no integer multiple of a vertex can be a boundary because then it would have to be the boundary of a 1-simplex which is impossible. So $\mathsf{B}_0(\mathsf{K}) = 0$.

$\because \mathsf{H}_0(\mathcal{X}) \cong \mathbb{Z}$ □

Corollary. *$\mathsf{H}_0(\mathcal{X}) \cong \bigoplus_i \mathbb{Z}$, $i = \#$ components of $\mathcal{X}$. In other words, $\mathsf{H}_0(\mathcal{X})$ is canonically isomorphic to the free abelian group based on the path–components of $\mathcal{X}$.*

3.3 The First Homology Group

Some notations:

$[\pi_1, \pi_1]$ is the commutator subgroup of π_1

$\widetilde{\pi}_1(\mathcal{X}, x_0) \cong \pi_1(\mathcal{X}, x_0)/[\pi_1, \pi_1]$

$[[f]]$ stands for the homotopy class of f

$f \simeq g$ means that f is homotopic to g

$[f]$ stands for the homology class of f

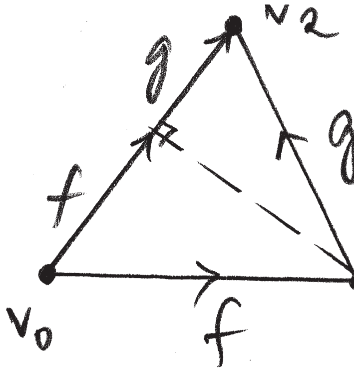
$f \sim g$ means that f is homologous to g

$f \star g$ is a product in $\widetilde{\pi}_1$

$f + g$ is a sum in H_1

Lemma 15. *If f and g are paths in $\mathcal{X}$ such that $f(1) = g(0)$ then the $1-$chain $f \star g - f - g$ is a boundary.*

Proof. I shall use methods of singular homology (see Bredon or Hatcher). Consider the standard 2-complex and put f on the edge (v_0, v_1) and g on the edge (v_1, v_2). Define the singular 2-simplex σ, which is a map, to be constant on the lines perpendicular to the edge (v_0, v_2). The situation is this:



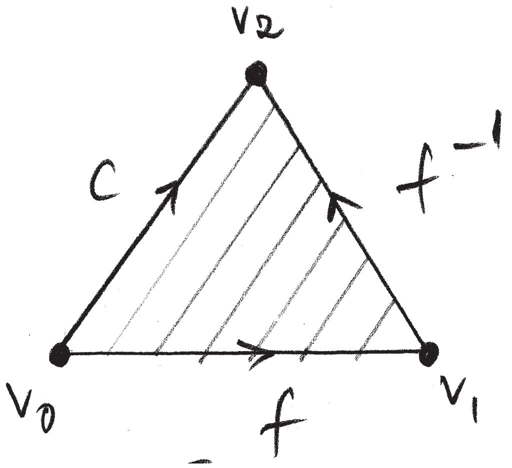
We realize that $f \star g$ lies on the edge $(v_0,\, v_2)$ and $\partial\sigma = f + g - f \star g$. This entails that $f \star g \sim f + g$ which means that we can reduce $f \star g$ considered as a 1−chain to the 1−chain $f + g$ modulo boundaries. □

Lemma 16. *If f is a path in $\mathcal{X}$ then $f + f^{-1}$ is a boundary. A constant path is a boundary.*

Proof. The boundary of a constant 2−simplex is a constant 1−simplex since two of the faces cancel. Thus a constant path is a boundary.

Consider the standard 2-complex and put f on the edge $(v_0,\, v_1)$.

Define the singular 2-simplex σ to be constant on lines parallel to the edge $(v_0,\, v_2)$.This flows f over to f^{-1} on the edge $(v_1,\, v_2)$. The situation is this:



Obviously $\partial\sigma = f + f^{-1} - c$, c constant. The conclusion is that $f + f^{-1}$ is a boundary, or, equivalently, $f^{-1} \sim -f$. □

Lemma 17. *If f and g are paths then $f \simeq g$ rel $\partial\mathsf{I} \Rightarrow f \sim g$.*

Proof. Let F: $\mathsf{I}\times\mathsf{I} \longrightarrow \mathcal{X}$ be a homotopy from f to g. The edge $\{0\}\times\mathsf{I}$ maps to a single point (as does the edge $\{1\}\times\mathsf{I}$). Then F factors through the map $\mathsf{I}\times\mathsf{I} \longrightarrow \Delta_2$ (= the standard 2−complex) which collapses that edge to the vertex v_0. We get a singular 2−simplex σ which is f on the edge (v_0, v_1) and g on the edge (v_0, v_2) while being constant on the edge (v_1, v_2).

$\partial\sigma = f - g + c$, c constant.

Hence, we conclude that $f - g$ is a boundary or, in other words, $f \sim g$, as desired. □

A loop f considered as a 1−chain is of course a cycle and so the function

$$\varphi\colon \pi_1(\mathcal{X}, x_0) \longrightarrow \mathsf{H}_1(\mathcal{X}) ,$$

given by $[[f]] \longmapsto [f]$, is well-defined.

$\varphi([[f]][[g]]) = \varphi(f \star g) = [f \star g] = [f + g] = [f] + [g]$. (Here we have used Lemma 15.)

$\varphi([[1]]) = [0]$

(*Nota bene*: $[[1]]$ = the class of trivial loops and $[0]$ = the class of trivial 1−chains.)

$\because$ φ is a group homomorphism

φ induces the homomorphism

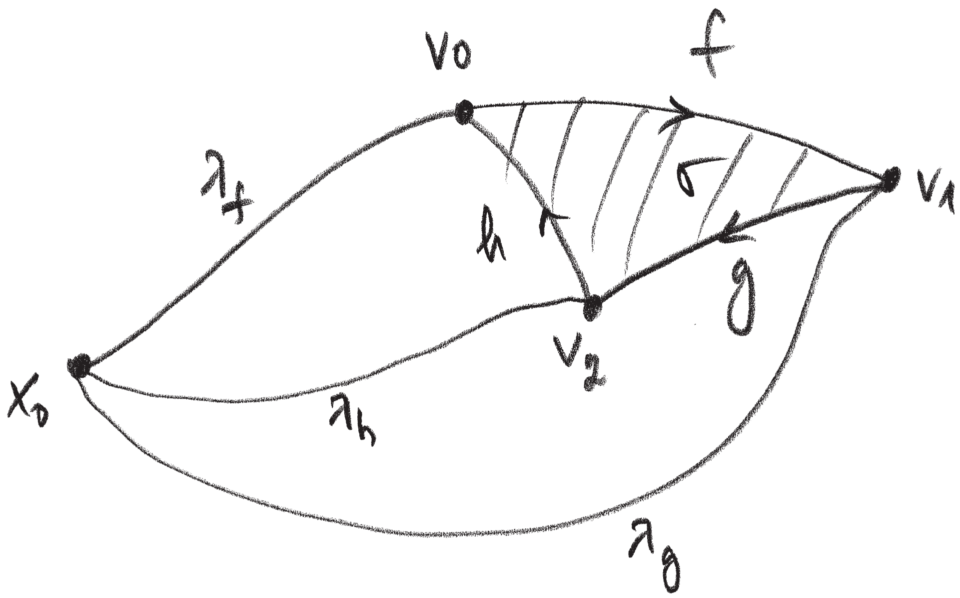
$$\varphi_\star\colon \widetilde{\pi}_1\ (\mathcal{X}, x_0) \longrightarrow \mathsf{H}_1(\mathcal{X})$$

Let f be an arbitrary 1−simplex and, as such, a path in $\mathcal{X}$. We want to make it a loop at x_0 so put $\widehat{f} = \lambda_{f(0)} \star f \star \lambda^{-1}_{f(1)}$. ($\lambda_{f(0)}$ is a path starting at $f(0)$). Now, this is a loop at x_0. Define $\psi(f) = [[\widehat{f}]] \epsilon\ \widetilde{\pi}_1\ (\mathcal{X}, x_0)$. This will give a homomorphism

$$\Delta_1(\mathcal{X}) \longrightarrow \widetilde{\pi}_1(\mathcal{X})$$

Lemma 18. *The map ψ takes the group $\mathsf{B}_1(\mathcal{X})$ of 1− boundaries in $\mathcal{X}$ into $[[1]]\ \epsilon\ \widetilde{\pi}_1(\mathcal{X})$.*

Proof. Let σ: $\Delta_2 \longrightarrow \mathcal{X}$ be a 2−simplex such that f lies on the edge (v_0,v_1), g lies on the edge (v_1,v_2), and h on the edge (v_2,v_0) as in the following figure:



Then:

$$\psi(\partial\sigma) = \psi(\ f\ +\ g - h^{-1}) = \psi(f)\psi(g)\psi(h^{-1})^{-1} =$$

$$[[\widehat{f}]][[\widehat{g}]][[\widehat{h^{-1}}^{-1}]] = [[\hat{f} \star \hat{g} \star \widehat{(h^{-1})}^{-1}]] =$$

$$[[\lambda_f \star f \star \lambda_g^{-1} \star \lambda_g \star g \star \lambda_h^{-1} \star \left(\lambda_f \star h^{-1} \star \lambda_h^{-1}\right)^{-1}]]$$

$$[[\lambda_f \star f \star \lambda_g^{-1} \star \lambda_g \star g \star \lambda_h^{-1} \star \lambda_h \star h \star \lambda_f^{-1}]] =$$

$$[[\lambda_f \star f \star g \star h \star \lambda_f^{-1}]] = [[constant]] = [[1]]$$

because $f \star g \star h \simeq$ constant (look at the above figure).

We can conclude that ψ induces a homomorphism $\psi_\star\colon \mathsf{H}_1(\mathcal{X}\ ) \longrightarrow \widetilde{\pi}_1(\mathcal{X})$. Furthermore,

$\psi_\star\varphi_\star([[f]]) = \psi_\star([f]) = [[\lambda_{x_0} \star f \star \lambda_{x_0}^{-1}]] = [[f]]$ because λ_{x_0} is constant.
$\because\ \psi_\star\varphi_\star = \mathrm{id}$ □

Lemma 19. *If σ is a $1-$simplex in $\mathcal{X}$ then the class $\varphi_\star\psi(\sigma)$ is represented by the cycle $\sigma + \lambda_{\sigma(0)} - \lambda_{\sigma(1)} = \sigma - \lambda_{\partial\sigma}$. Also, if c is a $1-$chain then $\varphi_\star\psi(c) = [c - \lambda_{\partial c}]$. If c is a $1-$ cycle then $\varphi_\star\psi(c) = [c]$*

Proof. $\varphi_\star\psi(\sigma) = \varphi_\star([[\lambda_{\sigma(0)} \star \sigma \star \lambda^{-1}_{\sigma(1)}]]) = [\lambda_{\sigma(0)} \star \sigma \star \lambda^{-1}_{\sigma(1)}] =$

$$[\lambda_{\sigma(0)} + \sigma + \lambda^{-1}_{\sigma(1)}] = [\lambda_{\sigma(0)} + \sigma - \lambda_{\sigma(1)}] \ ,$$

as desired.

The other statements follow immediately. □

Theorem 20. *[Hurewicz]*

$\mathsf{H}_1(\mathcal{X})$ *is isomorphic to* $\widetilde{\pi}_1(\mathcal{X}, x_0)$*, the abelianization of the fundamental group* $\pi_1(\mathcal{X}, x_0)$*. Equivalently, the homomorphism* $\varphi_\star$ *is an isomorphism if* $\mathcal{X}$ *is path-connected.*

Proof. If c is a $1-$cycle then $\varphi_\star\psi_\star([c]) = [c]$ by Lemma 19.

$\because \varphi_\star\psi_\star = \mathrm{id}$

By Lemma 18 we have that $\psi_\star\varphi_\star = \mathrm{id}$, so $\psi_\star^{-1} = \varphi_\star$.

$\because \varphi_\star$ is an isomorphism □

3.4 The Mayer–Vietoris Sequence

Remember that Construction V gave us the lens spaces as identication spaces of the type $\mathcal{V}_1 \cup_h \mathcal{V}_2$.

When computing homology groups in such cases it turns out to be quite useful to be able to use information about the homology groups of the ingredients of our identification. We shall encapsulate this in

Theorem 21. *[Mayer–Vietoris] Let* $\mathcal{A}, \mathcal{B} \subset \mathcal{X}$ *and suppose that* $\mathcal{X} = \mathrm{int}(\mathcal{A}) \cup \mathrm{int}(\mathcal{B})$.

Let $\iota^{\mathcal{A}} : \mathcal{A} \cap \mathcal{B} \hookrightarrow \mathcal{A}$*,* $\iota^{\mathcal{B}} : \mathcal{A} \cap \mathcal{B} \hookrightarrow \mathcal{B}$*,* $j^{\mathcal{A}} : \mathcal{A} \hookrightarrow \mathcal{A} \cup \mathcal{B}$*, and* $j^{\mathcal{B}} : \mathcal{B} \hookrightarrow \mathcal{A} \cup \mathcal{B}$ *be the inclusions.*

Then the sequence

$$0 \longrightarrow \mathsf{C}_n(\mathcal{A} \cap \mathcal{B}) \xrightarrow{\iota^{\mathcal{A}} \oplus \iota^{\mathcal{B}}} \mathsf{C}_n(\mathcal{A}) \oplus \mathsf{C}_n(\mathcal{B}) \xrightarrow{j^{\mathcal{A}} \ominus j^{\mathcal{B}}} \mathsf{C}_n(\mathcal{A} \cup \mathcal{B}) \longrightarrow 0$$

is exact and thus induces the long exact sequence

$$\cdots \longrightarrow H_n(\mathcal{A} \cap \mathcal{B}) \xrightarrow{\iota^{\mathcal{A}}_\star \oplus \iota^{\mathcal{B}}_\star} \mathsf{H}_n(\mathcal{A}) \oplus \mathsf{H}_n(\mathcal{B}) \xrightarrow{j^{\mathcal{A}}_\star \ominus j^{\mathcal{B}}_\star} \mathsf{H}_n(\mathcal{A} \cup \mathcal{B}) \longrightarrow H_{n-1}(\mathcal{A} \cap \mathcal{B}) \longrightarrow \cdots$$

called the "Mayer–Vietoris" sequence. If $\mathcal{A} \cap \mathcal{B} \neq \emptyset$ *then the reduced sequence is also exact.*

Proof. Let $\mathsf{C}_n(\mathcal{A}+\mathcal{B})$ be the subgroup of $\mathsf{C}_n(\mathcal{A}\cup\mathcal{B})$ whose elements are sums of chains in $\mathcal{A}$ and $\mathcal{B}$, respectively.

The boundary map ∂: $\mathsf{C}_n(\mathcal{A}\cup\mathcal{B}) \longrightarrow \mathsf{C}_{n-1}(\mathcal{A}\cup\mathcal{B})$ takes $\mathsf{C}_n(\mathcal{A}+\mathcal{B})$ to $\mathsf{C}_{n-1}(\mathcal{A}+\mathcal{B})$, thus the $\mathsf{C}_n(\mathcal{A}+\mathcal{B})$ form a chain complex. The inclusions $\mathsf{C}_n(\mathcal{A}+\mathcal{B}) \hookrightarrow \mathsf{C}_n(\mathcal{A}\cup\mathcal{B})$ induce isomorphisms on homology groups, so the Meyer – Vietoris sequence is the long exact sequence associated with the short exact sequence of chain complexes given by

$$0 \longrightarrow \mathsf{C}_n(\mathcal{A}\cap\mathcal{B}) \xrightarrow{\varphi} \mathsf{C}_n(\mathcal{A}) \oplus \mathsf{C}_n(\mathcal{B}) \xrightarrow{\psi} \mathsf{C}_n(\mathcal{A}+\mathcal{B}) \longrightarrow 0,$$

where $\varphi(x) = (x, -x)$ and $\psi(x, y) = x + y$.

A chain in $\mathcal{A}\cap\mathcal{B}$ that is zero in $\mathcal{A}$ and in $\mathcal{B}$ must simply be the zero chain. $\varphi(x) = (x, -x) = (0, 0) \Longrightarrow x = 0$.

$\because$ ker $\varphi = 0$

This shows that φ is a monomorphism and that the sequence is exact at $\mathsf{C}_n(\mathcal{A}\cap\mathcal{B})$.

$\psi\varphi(x) = \psi(x, -x) = 0$.

$\because \mathsf{im}\ \varphi \subseteq \mathsf{ker}\ \psi$.

Assume that $\psi(x, y) = 0$, i. e. $x + y = 0$, which, of course, is equivalent to $x = -y$.

Hence, x is a chain in both $\mathcal{A}$ and $\mathcal{B}$ and, consequently, $x \in \mathsf{C}_n(\mathcal{A}\cap\mathcal{B})$; $(x, y) = (x, -x) \in \mathsf{im}\ \varphi$.

$\because \mathsf{ker}\ \psi \subseteq \mathsf{im}\ \varphi$.

This shows that $\mathsf{ker}\ \psi = \mathsf{im}\ \varphi$ and so the sequence is exact at

$\mathsf{C}_n(\mathcal{A}) \oplus \mathsf{C}_n(\mathcal{B})$. By the definition of $\mathsf{C}_n(\mathcal{A}+\mathcal{B})$ we realize that ψ is an epimorphism, so the sequence is exact at $\mathsf{C}_n(\mathcal{A}+\mathcal{B})$.

The conclusion is that we have indeed a short exact sequence.

□

3.5 The Euler Characteristic

The Euler characteristic is a space invariant defined in different ways. I shall mention two. If K is a simplicial complex of dimension n, we define its Euler characteristic to be

$$\chi(\mathsf{K}) = \sum_{i=0}^{n} (-1)^i \alpha_i \ ,$$

where α_i is the number of i–simplices of K.

For a combinatorial surface this is the number of vertices minus the number of edges plus the number of faces, i.e. the well–known $v - e + f$.

Furthermore if A and B are simplicial complexes which intersect in a common subcomplex, then

$$\chi(\mathsf{A} \cup \mathsf{B}) = \chi(\mathsf{A}) + \chi(\mathsf{B}) - \chi(\mathsf{A} \cap \mathsf{B})$$

The second definition will be

$$\chi(\mathsf{K}) = \sum_{i=0}^{n} (-1)^i \operatorname{rank} \mathsf{H}_i(\mathsf{K}) = \sum_{i=0}^{n} (-1)^i \beta_i$$

$\beta_i = \operatorname{rank} \mathsf{H}_i(\mathsf{K})$ are the Betti numbers or the rank of the free abelian part of $\mathsf{H}_i(\mathsf{K})$. This is actually called the Euler–Poincaré formula.

The two definitions are of course equivalent:

Let K be a finite simplicial complex (or, a finite CW-complex) and let the α_i be as above, the number of i–simplices of K (or, the number of i–cells of K).

Set $\alpha_i = rank\ \mathsf{C}_i$. Let $\mathsf{Z}_i \subseteq \mathsf{C}_i$ be the group of i–cycles, $\mathsf{B}_i = \partial \mathsf{C}_{i+1} \subseteq \mathsf{C}_i$ be the group of i–boundaries and $\mathsf{H}_i = \mathsf{Z}_i/\mathsf{B}_i$, the ith homology group of K.

The exact sequence

$$0 \longrightarrow \mathsf{Z}_i \hookrightarrow \mathsf{C}_i \overset{\partial}{\twoheadrightarrow} \mathsf{B}_{i-1} \longrightarrow 0$$

shows that

$$rank\ \mathsf{C}_i = rank\ \mathsf{Z}_i + rank\ \mathsf{B}_{i-1}.$$

The exact sequence

$$0 \longrightarrow \mathsf{B}_i \hookrightarrow \mathsf{Z}_i \twoheadrightarrow \mathsf{H}_i \longrightarrow 0$$

shows that

$$rank\ \mathsf{Z}_i = rank\ \mathsf{B}_i + rank\ \mathsf{H}_i.$$

Hence:

$$\sum_i (-1)^i \alpha_i = \sum_i (-1)^i rank \mathsf{C}_i = \sum_i (-1)^i (rank\ \mathsf{Z}_i + rank\ \mathsf{B}_{i-1})$$

$$\sum_i (-1)^i (rank\ \mathsf{B}_i + rank\ \mathsf{H}_i + rank\ \mathsf{B}_{i-1}) = \sum_i (-1)^i\ rank\ \mathsf{H}_i\ ,$$

because the terms in $\mathsf{B}_\star$ cancel.

Consider the projection $\pi : \mathsf{S}^3 \to \mathsf{S}^3/\mathbb{Z}_p$. A point $[x] \in \mathsf{S}^3/\mathbb{Z}_p$ is an equivalence class

$$\left\{x, \tau x, \tau^2 x, \ldots, \tau^{p-1}x : x \in \mathcal{X}, \tau \in \mathbb{Z}_p\right\}.$$

Examining a triangulation of $\mathsf{S}^3/\mathbb{Z}_p$ we realize that for every vertex $[x]$ we have p vertices in the associated triangulation of S^3. In other words, the simplices in the triangulation of $\mathsf{S}^3/\mathbb{Z}_p$ lift in p different ways to a triangulation of S^3, and so, an i–simplex in $\mathsf{S}^3/\mathbb{Z}_p$ means p different i–simplexes in S^3. Thus we conclude that $\chi(\mathsf{S}^3) = p \cdot \chi(\mathsf{S}^3/\mathbb{Z}_p)$.

It is known known that $\mathsf{H}_0(\mathsf{S}^0) \approx \mathbb{Z} \oplus \mathbb{Z}$ and that , for $n > 0$,

$$\mathsf{H}_k(\mathsf{S}^n)\begin{cases} \mathbb{Z} & , k = 0, n \\ 0 & , 0 < k < n \end{cases}$$

Thus we realize that $\chi(\mathsf{S}^{2n}) = 2$ while $\chi(\mathsf{S}^{2n+1}) = 0$. In our case the net result is

$$\chi(\mathsf{L}(\mathsf{p},\mathsf{q})) = \chi(\mathsf{S}^3/\mathbb{Z}_p) = 0.$$

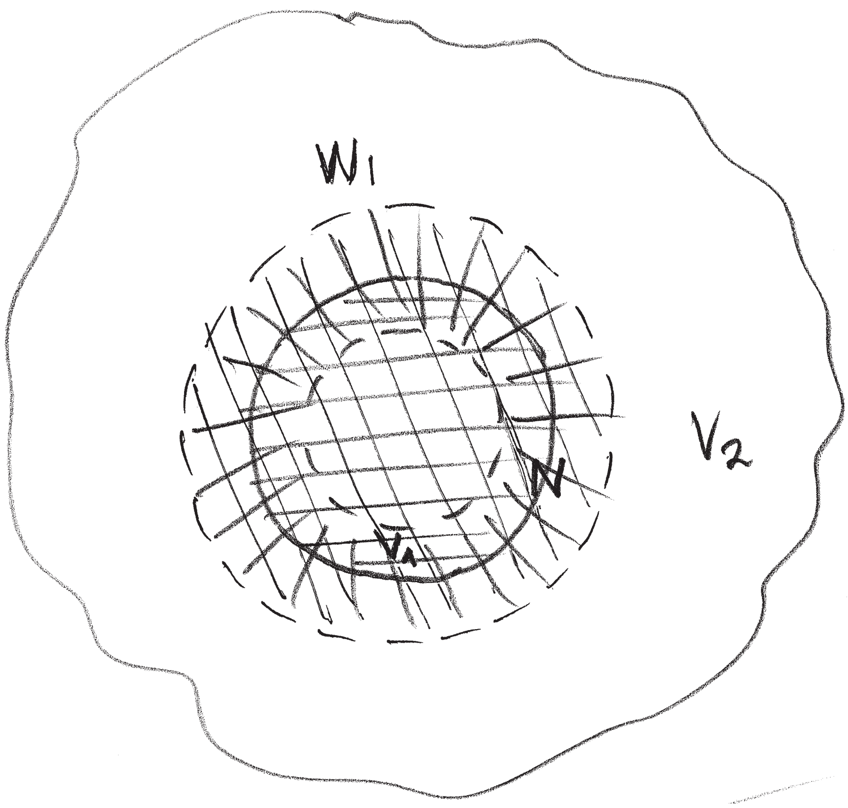
3.6 Computations

We shall cheat at the computations of the homology groups of lens spaces using different sorts of information that we have by now.

- $\mathsf{H}_0(\mathsf{L}(\mathsf{p},\mathsf{q})) \cong \mathbb{Z}$, because of path–connectedness
- $\mathsf{H}_1(\mathsf{L}(\mathsf{p},\mathsf{q})) \cong \mathbb{Z}_p$, because it is the abelianization of $\pi_1(\mathsf{L}(\mathsf{p},\mathsf{q}))$ which, incidentally, is already abelian, viz. $\mathbb{Z}_p$
- $\mathsf{H}_{n\geq 4}(\mathsf{L}(\mathsf{p},\mathsf{q})) = 0$, because $\dim(\mathsf{L}(\mathsf{p},\mathsf{q})) = 3$

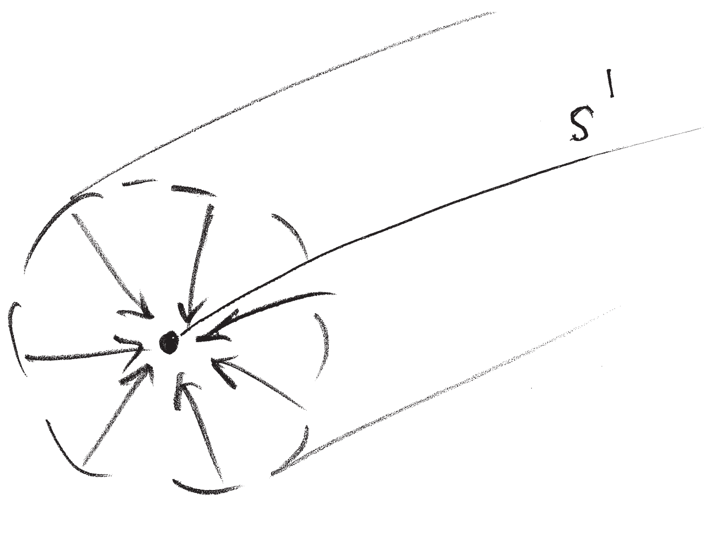
What about the second and the third homology groups?

Remember that $\mathsf{L}(\mathsf{p},\mathsf{q}) \approx \mathcal{V}_1 \cup_h \mathcal{V}_2$, the $\mathcal{V}_i$ being solid tori and $h : \partial\mathcal{V}_2 \to \partial\mathcal{V}_1$ being a homeomorphism. Let $\mathcal{N}$ be some neighbourhood of $\partial\mathcal{V}_2 \overset{h}{\approx} \partial\mathcal{V}_1$ (for instance, a tubular "*collar*") and set $\mathcal{W}_1 = \mathcal{V}_1 \cup \mathcal{N}$ and $\mathcal{W}_2 = \mathcal{V}_2 \cup \mathcal{N}$. In other words the $\mathcal{W}_i$ are open solid tori which look something like this in a meridinal section:

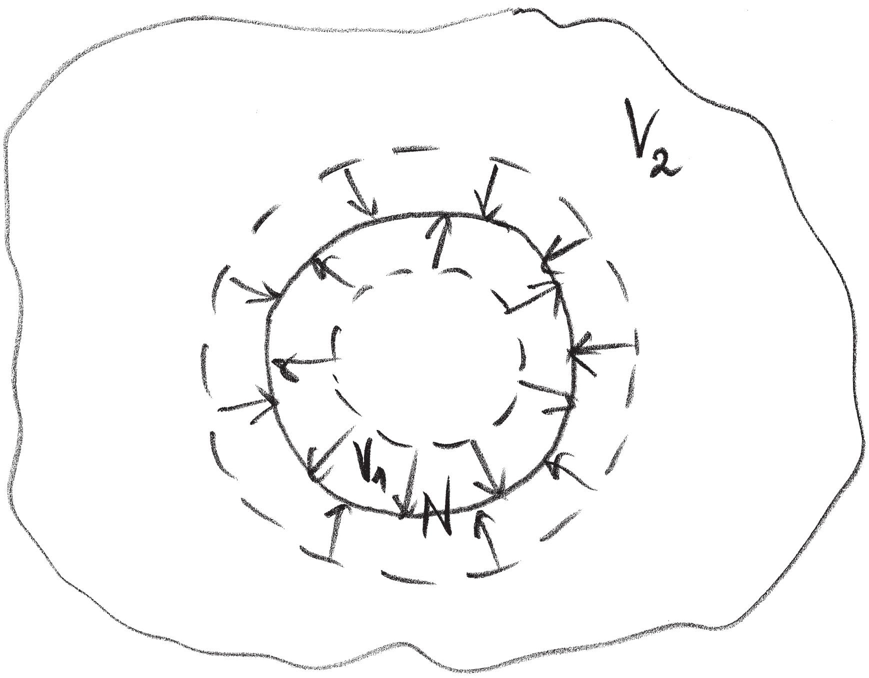


and similarly for $\mathcal{W}_2$. We have now that $\mathcal{N} = \mathcal{W}_1 \cap \mathcal{W}_2$ and, since they are open, the $\mathcal{W}_i$ can retract to a circle, i.e.

$$\mathcal{W}_i \simeq \mathsf{S}^1, \quad i = 1, 2.$$



From within $\mathcal{V}_1$ $\mathcal{N}$ retracts to $\partial\mathcal{V}_1$, a torus; likewise, from within $\mathcal{V}_2$ $\mathcal{N}$ retracts to $\partial\mathcal{V}_2$, as it happens the same torus under the identification induced by h. So $\mathcal{N} \approx \mathsf{T}^2$.



Since$\mathsf{L}(\mathsf{p},\mathsf{q}) = \mathcal{W}_1 \cup \mathcal{W}_2$, $\mathcal{W}_i \simeq \mathsf{S}^1$ $(i = 1, 2)$ and $\mathcal{W}_1 \cap \mathcal{W}_2 = \mathcal{N} \approx \mathsf{T}^2$ the Mayer–Vietoris sequence becomes:

$$\begin{aligned} 0 \longrightarrow & \mathsf{H}_3(\mathsf{T}^2) \longrightarrow \mathsf{H}_3(\mathsf{S}^1) \oplus \mathsf{H}_3(\mathsf{S}^1) \longrightarrow \mathsf{H}_3(\mathsf{L}(\mathsf{p},\mathsf{q})) \longrightarrow \\ & \mathsf{H}_2(\mathsf{T}^2) \longrightarrow \mathsf{H}_2(\mathsf{S}^1) \oplus \mathsf{H}_2(\mathsf{S}^1) \longrightarrow \mathsf{H}_2(\mathsf{L}(\mathsf{p},\mathsf{q})) \longrightarrow \\ & \mathsf{H}_1(\mathsf{T}^2) \longrightarrow \mathsf{H}_1(\mathsf{S}^1) \oplus \mathsf{H}_1(\mathsf{S}^1) \longrightarrow \mathsf{H}_1(\mathsf{L}(\mathsf{p},\mathsf{q})) \longrightarrow \\ & \mathsf{H}_0(\mathsf{T}^2) \longrightarrow \mathsf{H}_0(\mathsf{S}^1) \oplus \mathsf{H}_0(\mathsf{S}^1) \longrightarrow \mathsf{H}_0(\mathsf{L}(\mathsf{p},\mathsf{q})) \longrightarrow 0 \end{aligned}$$

and, hence, we get:

$$\begin{aligned} &\mathsf{0} \longrightarrow \mathsf{0} \longrightarrow \mathsf{0} \longrightarrow \mathsf{H}_3(\mathsf{L}(\mathsf{p},\mathsf{q})) \longrightarrow \mathbb{Z} \longrightarrow \mathsf{0} \longrightarrow \mathsf{H}_2(\mathsf{L}(\mathsf{p},\mathsf{q})) \\ &\qquad \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}_p \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathsf{0} \end{aligned}$$

And so, a first result is:

$$\mathsf{0} \longrightarrow \mathsf{H}_3(\mathsf{L}(\mathsf{p},\mathsf{q})) \xrightarrow{\cong} \mathbb{Z} \longrightarrow \mathsf{0}$$

whereas

$$\mathsf{0} \longrightarrow \mathsf{H}_2(\mathsf{L}(\mathsf{p},\mathsf{q})) \xrightarrow{\text{mono}} \mathbb{Z} \oplus \mathbb{Z}$$

more modestly, can mean one of three things, namely:

$$\mathsf{H}_2(\mathsf{L}(\mathsf{p},\mathsf{q})) \cong \mathbb{Z}\oplus\mathbb{Z} \text{ or } \mathsf{H}_2(\mathsf{L}(\mathsf{p},\mathsf{q})) \cong \mathbb{Z} \text{ or } \mathsf{H}_2(\mathsf{L}(\mathsf{p},\mathsf{q})) = 0.$$

On the other hand

$$\begin{aligned}\chi(\mathsf{L}(\mathsf{p},\mathsf{q})) &= \operatorname{rank}\mathsf{H}_0(\mathsf{L}(\mathsf{p},\mathsf{q})) - \operatorname{rank}\mathsf{H}_1(\mathsf{L}(\mathsf{p},\mathsf{q})) + \operatorname{rank}\mathsf{H}_2(\mathsf{L}(\mathsf{p},\mathsf{q})) \\ &\quad - \operatorname{rank}\mathsf{H}_3(\mathsf{L}(\mathsf{p},\mathsf{q})) \\ &= 1 - 0 + \operatorname{rank}\mathsf{H}_2(\mathsf{L}(\mathsf{p},\mathsf{q})) - 1 = 0\end{aligned}$$

$\because$ *rank* $\mathsf{H}_2(\mathsf{L}(\mathsf{p},\mathsf{q})) = 0$

Comparing this with the above information about $\mathsf{H}_2(\mathsf{L}(\mathsf{p},\mathsf{q}))$ we may conclude that $\mathsf{H}_2(\mathsf{L}(\mathsf{p},\mathsf{q})) = 0$.

Summarizing:

- $\mathsf{H}_0(\mathsf{L}(\mathsf{p},\mathsf{q})) \cong \mathbb{Z}$
- $\mathsf{H}_1(\mathsf{L}(\mathsf{p},\mathsf{q})) \cong \mathbb{Z}_p$
- $\mathsf{H}_2(\mathsf{L}(\mathsf{p},\mathsf{q})) = 0$
- $\mathsf{H}_3(\mathsf{L}(\mathsf{p},\mathsf{q})) \cong \mathbb{Z}$
- $\mathsf{H}_{n\geq 4}(\mathsf{L}(\mathsf{p},\mathsf{q})) = 0$

Let us look afresh at the whole matter.

Staying within simplicial homology if we had an explicit triangulation we could read out the generators of the different homology groups. In CW–homology we need to decide which are the n–cells instead. And we may recall the definition

$$\mathsf{H}_n(\mathcal{X}) = \mathsf{Z}_n(\mathcal{X})/\mathsf{B}_n = \mathsf{ker}\partial_n/\operatorname{im}\partial_{n+1}$$

where we have the sequence

$$\mathsf{C}_{n+1}(\mathcal{X}) \xrightarrow{\partial_{n+1}} \mathsf{C}_n(\mathcal{X}) \xrightarrow{\partial_n} \mathsf{C}_{n-1}(\mathcal{X})$$

In our case we have explicitly:

$$\begin{aligned}0 \xrightarrow{\partial_4} \mathsf{C}_3(\mathsf{L}(\mathsf{p},\mathsf{q})) \xrightarrow{\partial_3} \mathsf{C}_2(\mathsf{L}(\mathsf{p},\mathsf{q})) \xrightarrow{\partial_2} \mathsf{C}_1(\mathsf{L}(\mathsf{p},\mathsf{q})) \\ \xrightarrow{\partial_1} \mathsf{C}_0(\mathsf{L}(\mathsf{p},\mathsf{q})) \xrightarrow{\partial_0} 0\end{aligned}$$

If we consider the construction L we realize that we can choose any point x on the equator as the unique 0–cell.

The equivalents of this point x divide the equator in p segments. Take one such segment as the unique $1-$cell a.

$$\partial_1 a = x - x = 0.$$

The upper hemisphere which will be glued to the lower hemisphere can be taken as the unique $2-$cell b with boundary p times the $1-$cell a.

$$\partial_2 b = pa \qquad (\text{or } -pa).$$

The whole upper B^3 can be taken as the unique $3-$cell c.

$\partial_3 c = \pm(b - b) = 0$ or $\partial_3 c = \pm(b + b) = \pm\, 2b$. But $\pm\, 2b$ is not a cycle so $\partial_3 c = 0$.

$$\begin{aligned} \mathsf{C}_0(\mathsf{L}(\mathsf{p},\mathsf{q})) &= \langle x \rangle. \\ \ker\partial_0 &= \mathsf{C}_0(\mathsf{L}(\mathsf{p},\mathsf{q})) = \langle x \rangle \\ \operatorname{im}\partial_1 &= 0 \end{aligned}$$

$\because \mathsf{H}_0(\mathsf{L}(\mathsf{p},\mathsf{q})) = \ker\partial_0/\operatorname{im}\partial_1 = \langle x \rangle/0 \cong \mathbb{Z}$

$$\begin{aligned} \mathsf{C}_1(\mathsf{L}(\mathsf{p},\mathsf{q})) &= \langle a \rangle. \\ \ker\partial_1 &= \mathsf{C}_1(\mathsf{L}(\mathsf{p},\mathsf{q})) = \langle a \rangle \\ \operatorname{im}\partial_2 &= \pm pa \end{aligned}$$

$\because \mathsf{H}_1(\mathsf{L}(\mathsf{p},\mathsf{q})) = \ker\partial_1/\operatorname{im}\partial_2 = \langle a \rangle/ < pa > \cong \mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}_p$

$$\begin{aligned} \mathsf{C}_2(\mathsf{L}(\mathsf{p},\mathsf{q})) &= \langle b \rangle. \\ \ker\partial_2 &= 0 \\ \operatorname{im}\partial_3 &= 0 \end{aligned}$$

$\because \mathsf{H}_2(\mathsf{L}(\mathsf{p},\mathsf{q})) = \ker\partial_2/\operatorname{im}\partial_3 = 0/0 = 0$

$$\begin{aligned} \mathsf{C}_3(\mathsf{L}(\mathsf{p},\mathsf{q})) &= \langle c \rangle. \\ \ker\partial_3 &= \mathsf{C}_3(\mathsf{L}(\mathsf{p},\mathsf{q})) = \langle c \rangle \\ \operatorname{im}\partial_4 &= 0 \end{aligned}$$

$\because \mathsf{H}_3(\mathsf{L}(\mathsf{p},\mathsf{q})) = \ker\partial_3/\operatorname{im}\partial_4 = \langle c \rangle/0 \cong \mathbb{Z}$

This gives another approach to computing homology groups.

4 Cohomology

“ ‘I didn’t know that Cheshire cats always grinned; in fact, I didn’t know that cats *could* grin.’‘They all can’, said the Duchess; ‘and most of ’em do.’‘I don’t know of any that do,’ Alice said very politely, feeling quite pleased to have got into a conversation.‘You don’t know much, said the Duchess; ‘and that’s a fact.’ “

(*Alice′s Adventures in Wonderland* ”, ch. 6 *Pig and Pepper*)

4.1 Cohomology Groups

What is cohomology?

Consider a 1–dimensional simplicial complex $\mathsf{C}(\mathcal{X})$ and a fixed abelian group G. Denote the set of all functions from the vertices of $\mathcal{X}$ to G by $\mathsf{C}^0(\mathcal{X};\mathsf{G})$. Similarly the set of all functions from the edges of $\mathcal{X}$ to G will be denoted by $\mathsf{C}^1(\mathcal{X};\mathsf{G})$.

We can define a homomorphism

$$\delta : \mathsf{C}^0(\mathcal{X};\mathsf{G}) \to \mathsf{C}^1(\mathcal{X};\mathsf{G})$$

and consider the sequence

$$0 \longrightarrow \mathsf{C}^0(\mathcal{X};\mathsf{G}) \xrightarrow{\delta} \mathsf{C}^1(\mathcal{X};\mathsf{G}) \longrightarrow 0$$

The homology groups of this chain complex will be the cohomology groups of $\mathcal{X}$, by definition:

$$\mathsf{H}^0(\mathcal{X};\mathsf{G}) = \ker\delta \subseteq \mathsf{C}^0(\mathcal{X};\mathsf{G}) \quad \text{and} \quad \mathsf{H}^1(\mathcal{X};\mathsf{G}) = \mathsf{C}^1(\mathcal{X};\mathsf{G})/\operatorname{im}\delta.$$

On the other hand

$$\mathsf{C}^0(\mathcal{X};\mathsf{G}) \cong \mathrm{Hom}(\mathsf{C}_0(\mathcal{X}),\mathsf{G}) \quad \text{and} \quad \mathsf{C}^1(\mathcal{X};\mathsf{G}) \cong \mathrm{Hom}(\mathsf{C}_1(\mathcal{X}),\mathsf{G})$$

so we could generalize to

$$\mathsf{C}^n(\mathcal{X};\mathsf{G}) \cong \mathrm{Hom}(\mathsf{C}_n(\mathcal{X}),\mathsf{G})$$

and, given $\partial : \mathsf{C}_{n+1}(\mathcal{X}) \to \mathsf{C}_n(\mathcal{X})$, we could define

$$\delta : \mathsf{C}^n(\mathcal{X};\mathsf{G}) \to \mathsf{C}^{n+1}(\mathcal{X};\mathsf{G})$$

such that for every $\varphi \in \mathsf{C}^n(\mathcal{X};\mathsf{G})$ we have

$$\begin{aligned}\delta\varphi(v_0, v_1, \ldots, v_n) &= \sum_{i=0}^{n} (-1)^i \varphi(v_0, v_1, \ldots, \hat{v}_i, \ldots, v_n)\\ &= \varphi\partial(v_0, v_1, \ldots, v_n),\end{aligned}$$

in other words, such that $\delta\varphi = \varphi\partial$.

We might consider δ as a dual of ∂ so that to the chain complex of free abelian groups

$$\cdots \longrightarrow \mathsf{C}_{n+1}(\mathcal{X}) \xrightarrow{\partial_{n+1}} \mathsf{C}_n(\mathcal{X}) \xrightarrow{\partial_n} \mathsf{C}_{n-1}(\mathcal{X}) \xrightarrow{\partial_{n-1}} \cdots$$

and its homology groups we can associate the cochain complex

$$\cdots \longleftarrow \mathsf{C}^{\star}{}_{n+1}(\mathcal{X}) \xleftarrow{\delta_{n+1}} \mathsf{C}^{\star}{}_{n}(\mathcal{X}) \xleftarrow{\delta_{n}} \mathsf{C}^{\star}{}_{n-1}(\mathcal{X}) \xleftarrow{\delta_{n-1}} \cdots$$

where $\mathsf{C}^{\star} = \mathrm{Hom}(\mathsf{C}_n, \mathsf{G})$ is a cochain group, δ is the coboundary map $\partial_n^{\star} : \mathsf{C}^{\star}{}_{n-1} \to \mathsf{C}^{\star}{}_{n}$, the elements of $\ker\delta$ are cocycles and those of $\mathrm{im}\,\delta$ are coboundaries. We can now define the nth cohomology group as

$$\mathsf{H}^n(\mathcal{X};\mathsf{G}) = \ker\delta_{n+1}/\,\mathrm{im}\,\delta_n.$$

I will not pursue the matter further but I shall introduce the following object:

$$\mathsf{Ext}(\mathsf{H},\mathsf{G}) = \text{the set of isomorphism classes of extensions of } \mathsf{G} \text{ by } \mathsf{H},$$

G and H being abelian groups. This is called the extension functor and we shall need the following facts for our computations:

$$\begin{aligned} \mathsf{Ext}(\mathsf{H},\mathsf{G}) &= 0, && \text{,H is free} \\ \mathsf{Ext}(\mathbb{Z}_n,\mathsf{G}) &\cong \mathsf{G}/n\mathsf{G} && \text{,in particular } \mathsf{Ext}(\mathbb{Z}_n,\mathbb{Z}) \cong \mathbb{Z}_n \\ \mathsf{Ext}(\mathbb{Z}_n,\mathbb{Z}_n) &\cong \mathbb{Z}_n. \end{aligned}$$

Furthermore I will state the following theorem:

The universal coefficient theorem for cohomology. *If a chain complex* C *of free abelian groups has homology groups* $\mathsf{H}_n(\mathsf{C})$*, then the cohomology groups* $\mathsf{H}^n(\mathsf{C};\mathsf{G})$ *of the cochain complex* $\mathrm{Hom}(\mathsf{C}_n,\mathsf{G})$ *are determined by split exact sequences*

$$0 \longrightarrow \mathsf{Ext}(\mathsf{H}_{\mathsf{n}-1}(\mathsf{C}),\mathsf{G}) \longrightarrow \mathsf{H}^n(\mathsf{C};\mathsf{G}) \longrightarrow \mathrm{Hom}(\mathsf{H}_n(\mathsf{C}),\mathsf{G}) \longrightarrow 0$$

For a proof I refer to Hatcher (2002) or Bredon (1993).
And finally:

$$\begin{aligned} \mathsf{H}^0(\mathcal{X};\mathsf{G})) &\cong \mathrm{Hom}(\mathsf{H}_0(\mathcal{X}),\mathsf{G}) \\ \mathsf{H}^1(\mathcal{X};\mathsf{G})) &\cong \mathrm{Hom}(\mathsf{H}_1(\mathcal{X}),\mathsf{G}) \\ \mathsf{H}^n(\mathcal{X};\mathsf{G})) &\cong (\mathsf{H}_n(\mathcal{X})/\mathsf{T}_n) \oplus \mathsf{T}_{n-1}, \end{aligned}$$

if $\mathsf{H}_n(\mathcal{X})$ and $\mathsf{H}_{n-1}(\mathcal{X})$ are finitely generated with torsion subgroups T_n and T_{n-1}, respectively.

4.2 Computations

$$\mathsf{H}^0(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}) \cong \mathrm{Hom}(\mathsf{H}_0(\mathsf{L}(\mathsf{p},\mathsf{q}),\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z},\mathbb{Z})$$

φ: $\mathbb{Z} \longrightarrow \mathbb{Z}$

These homomorphisms are totally determined by $\varphi(1)$ since $\mathbb{Z} =< 1 >$. There are $\aleph_0$ possibilities.

$\because$ $\mathsf{H}^0(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z})) \cong \mathbb{Z}$

$$\mathsf{H}^1(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}) \cong \mathrm{Hom}(\mathsf{H}_1(\mathsf{L}(\mathsf{p},\mathsf{q}),\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}_p,\mathbb{Z})$$

φ: $\mathbb{Z}_p \longrightarrow \mathbb{Z}$

The only possible homomorphism in this case is the trivial one because suppose $\varphi(a) = b \neq 0$, where $< a >= \mathbb{Z}_p$. Then:

$$\varphi(u) = \varphi(na) = n\varphi(a) = nb$$
$$\varphi(v) = \varphi(ma) = m\varphi(a) = mb$$

and say that $u + v = p \equiv \mathsf{0}$

$$\varphi(u + v) = \varphi(p) = \varphi(\mathsf{0}) = \mathsf{0}$$

Yet, $\varphi(u) + \varphi(v) = nb + mb = (n + m)b \neq 0$ is possible.

$\because$ $\mathsf{H}^1(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z})) = 0$

$$\mathsf{H}^2(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}) \cong (\mathsf{H}_2(\mathsf{L}(\mathsf{p},\mathsf{q})/\mathsf{T}_2) \oplus \mathsf{T}_1$$

$\mathsf{H}_1(\mathsf{L}(\mathsf{p},\mathsf{q})) \cong \mathbb{Z}_p$ entails that $\mathsf{T}_1 \approx \mathbb{Z}_p$. On the other hand we have $\mathsf{H}_2(\mathsf{L}(\mathsf{p},\mathsf{q})) = \mathsf{0}$ so $(\mathsf{H}_2(\mathsf{L}(\mathsf{p},\mathsf{q}))/\mathsf{T}_2 = \mathsf{0}$

$\because$ $\mathsf{H}^2(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}) \cong \mathbb{Z}_p$.

$$\mathsf{H}^3(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}) \cong (\mathsf{H}_3(\mathsf{L}(\mathsf{p},\mathsf{q})/\mathsf{T}_3) \oplus \mathsf{T}_2$$

$\mathsf{H}_2(\mathsf{L}(\mathsf{p},\mathsf{q})) = \mathsf{0}$ entails that $\mathsf{T}_2 = \mathsf{0}$. $\mathsf{H}_3(\mathsf{L}(\mathsf{p},\mathsf{q})) \cong \mathbb{Z}$ entails that $\mathsf{T}_3 = \mathsf{0}$.

$\because$ $\mathsf{H}^3(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}) \cong \mathbb{Z}$.

Let us compute even the cohomology groups with coefficients modulo p.

$$\mathsf{H}^0(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p) \cong \mathrm{Hom}(\mathsf{H}_0(\mathsf{L}(\mathsf{p},\mathsf{q}),\mathbb{Z}_p) = \mathrm{Hom}(\mathbb{Z},\mathbb{Z}_p)$$

φ: $\mathbb{Z} \longrightarrow \mathbb{Z}_p$
There are p possibilities for $\varphi(1)$.
$\because$ $\mathsf{H}^0(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p) \cong \mathbb{Z}_p$

$$\mathsf{H}^1(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p) \cong \mathrm{Hom}(\mathsf{H}_1(\mathsf{L}(\mathsf{p},\mathsf{q}),\mathbb{Z}_p) = \mathrm{Hom}(\mathbb{Z}_p,\mathbb{Z}_p)$$

Of course, even here, there are p possibilities.
$\because$ $\mathsf{H}^1(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p) \cong \mathbb{Z}_p$

$$\mathsf{H}^2(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p)$$

Consider the long exact sequence:

$$0 \longrightarrow \mathsf{Ext}(\mathsf{H}_1(\mathsf{L}(\mathsf{p},\mathsf{q})),\mathbb{Z}_p) \longrightarrow \mathsf{H}^2(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p) \longrightarrow \mathrm{Hom}(\mathsf{H}_2(\mathsf{L}(\mathsf{p},\mathsf{q})),\mathbb{Z}_p) \longrightarrow 0$$

$$\mathsf{Ext}(\mathsf{H}_1(\mathsf{L}(\mathsf{p},\mathsf{q})),\mathbb{Z}_p) = \mathsf{Ext}(\mathbb{Z}_p,\mathbb{Z}_p) \cong \mathbb{Z}_p$$
$$\mathrm{Hom}(\mathsf{H}_2(\mathsf{L}(\mathsf{p},\mathsf{q})),\mathbb{Z}_p) = \mathrm{Hom}(0,\mathbb{Z}_p) = 0$$

Hence we get the short exact sequence:

$$0 \longrightarrow \mathbb{Z}_p \xrightarrow{\cong} \mathsf{H}^2(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p) \longrightarrow 0$$

$\because$ $\mathsf{H}^2(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p) \cong \mathbb{Z}_p$.

$$\mathsf{H}^3(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p)$$

Consider the long exact sequence:

$$0 \longrightarrow \mathsf{Ext}(\mathsf{H}_2(\mathsf{L}(\mathsf{p},\mathsf{q})),\mathbb{Z}_p) \longrightarrow \mathsf{H}^3(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p) \longrightarrow \mathrm{Hom}(\mathsf{H}_3(\mathsf{L}(\mathsf{p},\mathsf{q})),\mathbb{Z}_p) \longrightarrow 0$$

$$\mathsf{Ext}(\mathsf{H}_2(\mathsf{L}(\mathsf{p},\mathsf{q})),\mathbb{Z}_p) = \mathsf{Ext}(0,\mathbb{Z}_p) = 0$$
$$\mathrm{Hom}(\mathsf{H}_3(\mathsf{L}(\mathsf{p},\mathsf{q})),\mathbb{Z}_p) = \mathrm{Hom}(\mathbb{Z},\mathbb{Z}_p)$$

but we have already seen that $\mathrm{Hom}(\mathbb{Z},\mathbb{Z}_p)\cong\mathbb{Z}_p$.

We get the following short exact sequence:

$$0 \longrightarrow \mathsf{H}^3(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p) \xrightarrow{\cong} \mathbb{Z}_p \longrightarrow 0$$

$\because\ \mathsf{H}^3(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p)\cong\mathbb{Z}_p$

Furthermore it goes without saying that both

$$\mathsf{H}^{n\geq 4}(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}) = 0 \quad \text{and} \quad \mathsf{H}^{n\geq 4}(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p) = 0,$$

since $\dim \mathsf{L}(\mathsf{p},\mathsf{q}) = 3$.

Using Poincaré's Duality Theorem by which $\mathsf{H}^k(\mathcal{X};\mathsf{G})\cong\mathsf{H}_{n-k}(\mathcal{X};\mathsf{G})$, where $n=\dim\mathcal{X}$, we can determine even the homology groups of $\mathsf{L}(\mathsf{p},\mathsf{q})$ with coefficients modulo p.

A summary:

$$\begin{array}{rlrl}
\mathsf{H}_0(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}) \cong \mathbb{Z} & & \mathsf{H}^0(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}) \cong \mathbb{Z} & \\
\mathsf{H}_1(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}) \cong \mathbb{Z}_p & & \mathsf{H}^1(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}) = 0 & \\
\mathsf{H}_2(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}) = 0 & & \mathsf{H}^2(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}) \cong \mathbb{Z}_p & \\
\mathsf{H}_3(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}) \cong \mathbb{Z} & & \mathsf{H}^3(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}) \cong \mathbb{Z} & \\
\mathsf{H}_{n\geq 4}(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}) = 0 & & \mathsf{H}^{n\geq 4}(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}) = 0 &
\end{array}$$

$$\begin{array}{rlrl}
\mathsf{H}_0(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p) \cong \mathbb{Z}_p & & \mathsf{H}^0(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}) \cong \mathbb{Z}_p & \\
\mathsf{H}_1(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p) \cong \mathbb{Z}_p & & \mathsf{H}^1(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}) \cong \mathbb{Z}_p & \\
\mathsf{H}_2(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p) \cong \mathbb{Z}_p & & \mathsf{H}^2(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}) \cong \mathbb{Z}_p & \\
\mathsf{H}_3(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p) \cong \mathbb{Z}_p & & \mathsf{H}^3(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}) \cong \mathbb{Z}_p & \\
\mathsf{H}_{n\geq 4}(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p) = 0 & & \mathsf{H}^{n\geq 4}(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}) = 0 &
\end{array}$$

None of the above involves in any way the parameter q but, surely, q must be essential too. When Tietze introduced the three-dimensional lens spaces $\mathsf{L}(\mathsf{p},\mathsf{q})$ in 1908 they were the first known examples of $3-$manifolds which were not entirely determined by their fundamental group and by their homology neither was their homeomorphism type determined by their homotopy type. There are lens spaces with the same homology but different homotopy type and lens spaces with the same homotopy type but different homeomorphism type. The complete classification of three-dimensional lens spaces is given by Reidermeister torsion which we shall not discuss though.

5 *Equivalences*

“ ’And who are *these*?’ said the Queen pointing to the three gardeners who were lying round the rose-tree; for, you see, as they were lying on their faces, and the pattern on their back was the same as the rest of the pack, she could not tell whether they were gardeners, or soldiers, or courtiers, or three of her own children.”

(Lewis Carroll: *Alice's Adventures in Wonderland* , ch. 8 *The Queen's Croquet – Ground*)

5.1 Cohomology Products

Trying to solve the question of homotopy equivalences and to understand the importance of the parameter q requires some more advanced results in cohomology, more specifically two types of products that I shall define below without being prepared to go into technicalities.

The Kronecker product

Given $\alpha = [f] \in \mathsf{H}^n(\mathcal{X}; \mathsf{G})$ and $\gamma = [c] \in \mathsf{H}_n(\mathcal{X}; \mathsf{G})$ we define the Kronecker product of α and γ by $< \alpha, \gamma >= f(c) \in \mathsf{G}$.

$\alpha = [f]$ is a cohomology class of functions $f : \mathsf{C}_n \to \mathsf{G}$ which assign values in G to the nth simplices of $\mathcal{X}$. $\gamma = [c]$ is a homology class in $\mathsf{H}_n(\mathcal{X}; \mathsf{G})$. It is then natural to call the Kronecker product evaluation of a cohomology class on a homology class.

We are going to need later the following property of the Kronecker product:

$$(\psi^\star(\alpha))(b) =< \psi^\star(\alpha), b >=< \alpha, \psi_\star(b) >= \alpha(\psi_\star(b))$$

where $\psi_\star$ represents an induced map on homology and $\psi^\star$ represents an induced map on cohomology.

The idea is that given a homology class b in $\mathsf{H}_n(\mathcal{X})$ and a cohomology class α in $\mathsf{H}^n(\mathcal{X};\Lambda)$ evaluating α at $\psi_\star(b)$ should be the same as evaluating $\psi^\star(\alpha)$ at b. For a proper proof of this I refer to Bredon, 1993.

The cup product

We would like to define a product of cohomology classes. So let us start tentatively with cochains $\alpha \epsilon\ \mathsf{C}^k(\mathcal{X};\Lambda)$ and $\beta\ \epsilon\ \mathsf{C}^l(\mathcal{X};\Lambda)$ where Λ is some ring and define a *cup product* $\alpha \smile \beta\ \epsilon\ \mathsf{C}^{k+l}(\mathcal{X};\Lambda)$ as the cochain whose value on a $(k + l)-$simplex σ is given by the formula

$$(\alpha \smile \beta)(\sigma) = \alpha(\sigma|[v_0, v_1, ..., v_k]) \cdot \beta(\sigma|[v_k, v_{k+1}, ..., v_{k+l}])$$

where the product on the right–hand side is the product in Λ. But this is only a product of cochains and not of cohomology classes yet. We need to relate it to the coboundary map δ.

Theorem 22. *Given $\alpha \epsilon\ \mathsf{C}^k(\mathcal{X};\Lambda)$ and $\beta\ \epsilon\ \mathsf{C}^l(\mathcal{X};\Lambda)$*

$$\delta(\alpha \smile \beta) = \delta\alpha \smile \beta + (-1)^k \alpha \smile \delta\beta.$$

Proof. Let σ be a $(k+l+1)-$simplex.

$$(\delta\alpha\smile\beta)(\sigma) =$$

$$\Sigma_{i=0}^{k+1}(-1)^i\alpha(\sigma|[v_0, v_1, ..., \hat{v}_i, ..., v_{k+1}])\cdot\beta(\sigma|[v_{k+1}, v_{k+2}, ..., ..., v_{k+l+1}])$$

$$(-1)^k(\alpha\smile\delta\beta)(\sigma)$$

$$\Sigma_{i=k}^{k+l+1}(-1)^i\alpha(\sigma|[v_0, v_1, ..., ..., v_k])\cdot\beta(\sigma|[v_k, v_{k+1}, ..., \hat{v}_i, ..., v_{k+l+1}]).$$

Adding the two expressions we realize that the last term of the first sum cancels the first term of the second sum and we recognize the rest of the right-hand side as $(\alpha\smile\beta)(\partial\sigma) = \delta(\alpha\smile\beta)(\sigma)$

$\therefore\delta(\alpha\smile\beta) = \delta\alpha\smile\beta + (-1)^k\alpha\smile\delta\beta.$

$\square$

Examining the formula $\delta(\alpha\smile\beta) = \delta\alpha\smile\beta \pm \alpha\smile\delta\beta$ we draw the conclusion that the cup product of two cocycles is again a cocycle.

Suppose $\delta\alpha = 0$ which means that α is a coboundary. Then we get $\delta(\alpha\smile\beta) = \pm\alpha\smile\delta\beta$ or, equivalently, $\alpha\smile\delta\beta = \pm\delta(\alpha\smile\beta)$. Similarly, we get $\delta\alpha\smile\beta = \pm\delta(\alpha\smile\beta)$ if $\delta\beta = 0$.

Hence, we conclude that the cup product of a cocycle and a coboundary must be a coboundary. Consequently our cochain cup product induces a cup product of cohomology classes

$$\smile: \mathsf{H}^k(\mathcal{X};\Lambda) \times \mathsf{H}^l(\mathcal{X};\Lambda) \longrightarrow \mathsf{H}^{k+l}(\mathcal{X};\Lambda)$$

5.2 The Invariant $\mathtt{t}_q$

Consider the short exact sequence of coefficient groups:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\star p} \mathbb{Z} \longrightarrow \mathbb{Z}_p \longrightarrow 0$$

It induces the exact sequence:

$$\begin{aligned} \cdots \longrightarrow \mathsf{H}^n(\mathcal{X};\mathbb{Z}) &\xrightarrow{\star p} \mathsf{H}^n(\mathcal{X};\mathbb{Z}) \xrightarrow{\rho} \mathsf{H}^n(\mathcal{X};\mathbb{Z}_p) \\ &\xrightarrow{\beta_0} \mathsf{H}^{n+1}(\mathcal{X};\mathbb{Z}) \longrightarrow \cdots \end{aligned}$$

Define $\beta = \rho \circ \beta_0 : \mathsf{H}^n(\mathcal{X};\mathbb{Z}_p) \to \mathsf{H}^{n+1}(\mathcal{X};\mathbb{Z}_p)$. β_0 and β are the Bockstein homomorphisms.

Consider now the exact sequence:

$$\begin{aligned} \mathsf{H}^1(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}) \longrightarrow \mathsf{H}^1(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p) &\xrightarrow{\beta_0} \mathsf{H}^2(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}) \\ \xrightarrow{\star 0} \mathsf{H}^2(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}) &\longrightarrow \mathsf{H}^2(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p) \end{aligned}$$

Since we have already computed the cohomology groups in question this becomes

$$0 \longrightarrow \mathbb{Z}_p \xrightarrow{\beta_0} \mathbb{Z}_p \xrightarrow{\star 0} \mathbb{Z}_p \xrightarrow{\rho} \mathbb{Z}_p,$$

and, finally, this shows that β_0 is a monomorphism. But an injective map from a finite set to itself is always surjective so β_0 is an isomorphism.

$\mathbb{Z}_p \xrightarrow{\star 0} \mathbb{Z}_p \xrightarrow{\rho} \mathbb{Z}_p$ shows that $\mathsf{ker}\rho = 0$ and so ρ is a monomorphism. Again, we may conclude that ρ is an isomorphism, and finally we have shown that $\beta = \rho \circ \beta_0$ is an isomorphism.

Let us consider

$$\beta : \mathsf{H}^1((\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p) \to \mathsf{H}^2((\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p).$$

Let $a \in \mathsf{H}^1(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p)$ be a generator. Then $\beta(a)$ is a generator for $\mathsf{H}^2(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p)$.

The cup product $a \smile \beta(a)$ is an element of $\mathsf{H}^3(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p)$ by the definition of $\smile$. A corollary of the Poincaré Duality Theorem (see Bredon, 1993) guarantees that $a \smile \beta(a)$ is a generator of $\mathsf{H}^3(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p)$.

Let $[\mathsf{L}(\mathsf{p},\mathsf{q})] \in \mathsf{H}^3(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p)$ be the modulo p reduction of a generator of $\mathsf{H}^3((\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z})$. This is unique up to sign. Since $a \smile \beta(a)$ is a generator of $\mathsf{H}^3(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p) \cong \mathbb{Z}_p$, $< a \smile \beta(a),\ [\mathsf{L}(\mathsf{p},\mathsf{q})] >$ will be a generator of $\mathbb{Z}_p$.

Let b be another generator of $\mathsf{H}^1(\mathsf{L}(\mathsf{p},\mathsf{q});\mathbb{Z}_p)$. Then $b = na$ with $\gcd(n,p) = 1$ and

$$\begin{aligned} < b \smile \beta(b), \pm[\mathsf{L}(\mathsf{p},\mathsf{q})] > &= < na \smile \beta(na), \pm[\mathsf{L}(\mathsf{p},\mathsf{q})] > \\ &= \pm n^2 < a \smile \beta(a), [\mathsf{L}(\mathsf{p},\mathsf{q})] > . \end{aligned}$$

Therefore the generator $< a \smile \beta(a), [\mathsf{L}(\mathsf{p},\mathsf{q})] >$ is essentially unique independently of the choice of $a \in \mathsf{H}^1(\mathsf{L}(\mathsf{p},\mathsf{q}); \mathbb{Z}_p)$.

Define a relation $\triangleq$ on $\mathbb{Z}_p$ by $i \triangleq j$ if and only if $i \equiv \pm n^2 \pmod{p}$

We define now the invariant t_q by

$$\mathsf{t}_q \triangleq < a \smile \beta(a), [\mathsf{L}(\mathsf{p},\mathsf{q})] >$$

The equivalence $\mathsf{t}_q \triangleq q\mathsf{t}_\mathbf{1}$

Consider the maps $\varphi : \mathsf{S}^3 \to \mathsf{S}^3$ and $\tau_{(p,q)} : \mathsf{S}^3 \to \mathsf{S}^3$ given by

$$\varphi(u,v) = (u, v^q) \quad \text{and} \quad \tau_{(p,q)}(u,v) = (\varepsilon u, \varepsilon^q v)$$

respectively, and ε as before.

We have that

$$\varphi\tau_{(p,1)}(u,v) = \varphi(\varepsilon u, \varepsilon v) = (\varepsilon u, \varepsilon^q v^q)$$

and

$$\tau_{(p,q)}\varphi(u,v) = \tau_{(p,q)}(u, v^q) = (\varepsilon u, \varepsilon^q v^q),$$

so $\varphi\tau_{(p,1)} = \tau_{(p,q)}\varphi$, showing thus that φ carries the action of $\tau_{(p,1)}$ on S^3 to that of $\tau_{(p,q)}$. As a consequence φ induces a map $\psi : \mathsf{L}(\mathsf{p},1) \to \mathsf{L}(\mathsf{p},\mathsf{q})$. $\pi_1(\mathsf{L}(\mathsf{p},1))$ is generated (by covering space theory) by the loop which is the image of any path in S^3 from, say x_0 to $\tau_{(p,1)}(x_0)$, and similarly for $\pi_1(\mathsf{L}(\mathsf{p},\mathsf{q}))$ and $\tau_{(p,q)}$. This means that φ carries the path for $\tau_{(p,1)}$ to that for $\tau_{(p,q)}$ and we may conclude that the induced homomorphism

$$\psi_\sharp : \pi_1(\mathsf{L}(\mathsf{p},1)) \to \pi_1(\mathsf{L}(\mathsf{p},\mathsf{q}))$$

is an isomorphism.

This induces another isomorphism (Hurwicz) in homology

$$\psi_\star : \mathsf{H}_1(\mathsf{L}(\mathsf{p},1)) \to \mathsf{H}_1(\mathsf{L}(\mathsf{p},\mathsf{q}))$$

which in turn induces a third isomorphism in cohomology:

$$\psi^\star : \mathsf{H}^1(\mathsf{L}(\mathsf{p},1); \mathbb{Z}_p) \to \mathsf{H}^1(\mathsf{L}(\mathsf{p},\mathsf{q}); \mathbb{Z}_p)$$

Considering the diagram:

$$\begin{array}{ccc} \mathsf{H}_3(\mathsf{S}^3) & \xrightarrow{\varphi_\star} & \mathsf{H}_3(\mathsf{S}^3) \\ \downarrow & & \downarrow \\ \mathsf{H}_3(\mathsf{L}(\mathsf{p},1)) & \xrightarrow{\psi_\star} & \mathsf{H}_3(\mathsf{L}(\mathsf{p},\mathsf{q})) \end{array}$$

we realize that $\deg\psi = \deg\varphi = q$ since the verticals must both have degree p being p–fold coverings.

Nota bene. Recall that $\varphi(u,v) = (u, v^q)$, thus $\deg\varphi = q$.

Now we can conlude that $\psi_\star([\mathsf{L}(\mathsf{p},1)]) = \pm q\,[\mathsf{L}(\mathsf{p},\mathsf{q})]$. Consider a new diagram:

$$\begin{array}{ccc} \mathsf{H}^1(\mathsf{L}(\mathsf{p},1)) & \xrightarrow{\psi^\star} & \mathsf{H}^1(\mathsf{L}(\mathsf{p},\mathsf{q})) \\ {\scriptstyle\beta}\big\downarrow & & {\scriptstyle\beta}\big\downarrow \\ \mathsf{H}^2(\mathsf{L}(\mathsf{p},1)) & \xrightarrow{\psi^\star} & \mathsf{H}^2(\mathsf{L}(\mathsf{p},\mathsf{q})) \end{array}$$

For $a \in \mathsf{H}^1(\mathsf{L}(\mathsf{p},1);\mathbb{Z}_p)$, a generator, and $b = \psi^\star(a)$, we have that

$$\beta(b) = \beta(\psi^\star(a)) = \psi^\star(\beta(a)).$$

This entails that $b \smile \beta(b) = \psi^\star(a) \smile \psi^\star(\beta(a)) = \psi^\star(a \smile \beta(a))$ and so, finally, we get the following:

$$\begin{aligned} \mathsf{t}_1 &\triangleq < b \smile \beta(b), [\mathsf{L}(\mathsf{p},1)] > \\ &= < \psi^\star(a \smile \beta(a)), [\mathsf{L}(\mathsf{p},1)] > \\ &= < a \smile \beta(a), \psi_\star([\mathsf{L}(\mathsf{p},1)]) > \\ &= < a \smile \beta(a), \pm q([\mathsf{L}(\mathsf{p},\mathsf{q})]) > \\ &= \pm q < a \smile \beta(a), [\mathsf{L}(\mathsf{p},\mathsf{q})] > \\ &\triangleq q\mathsf{t}_q \end{aligned}$$

This means, by the definition of $\triangleq$, that $q\mathsf{t}_1 \equiv \pm n^2 \cdot q^2\mathsf{t}_q = \pm m^2\mathsf{t}_q$.

$\because\ q\mathsf{t}_1 \triangleq \mathsf{t}_q$

5.3 Homotopy Equivalences

Theorem 23. $\mathsf{L(p,q')} \simeq \mathsf{L(p,q)} \iff \mathsf{qq'} \equiv \pm\mathsf{n}^2 \pmod{\mathsf{p}}$

Proof. ($\Longrightarrow$): Let $h : \mathsf{L(p,q')} \to \mathsf{L(p,q)}$ be a homotopy equivalence. This means that it has degree ±1 and, hence, $h_\star([\mathsf{L(p,q')}]) = \pm[\mathsf{L(p,q)}]$. Suppose $b \in \mathsf{H}^1(\mathsf{L(p,q)};\mathbb{Z}_p)$ is a generator and $a = h^\star(b)$.

We get:

$$\begin{aligned}
\mathsf{t}_{q'} &\triangleq< a \smile \beta(a), [\mathsf{L(p,q')}] > \\
&=< h^\star(b) \smile \beta(h^\star(b)), [\mathsf{L(p,q')}] > \\
&=< h^\star(b \smile \beta(b)), [\mathsf{L(p,q')}] > \\
&=< b \smile \beta(b), h_\star([\mathsf{L(p,q')}]) > \\
&=< b \smile \beta(b), \pm[\mathsf{L(p,q)}] > \\
&= \pm < b \smile \beta(b), [\mathsf{L(p,q)}] > \\
&\triangleq \mathsf{t}_q
\end{aligned}$$

so $q'\mathsf{t}_1 \triangleq \mathsf{t}_{q'} \triangleq \mathsf{t}_q \triangleq q\mathsf{t}_1$ which leads to $qq'\mathsf{t}_1 \triangleq q^2\mathsf{t}_1$ which entails that $qq' \triangleq q^2 \triangleq 1$ and this means, by the definition of the relation $\triangleq$ that $\mathsf{qq'} \equiv \pm\mathsf{n}^2 \bmod \mathsf{p}$ as desired.

($\Longleftarrow$): Assume that $\mathsf{qq'} \equiv \pm\mathsf{n}^2 \bmod \mathsf{p}$.

This is equivalent to $n^2(qq')^{-1} \equiv \pm1 \pmod{p}$, in other words, $n^2(qq')^{-1} + mp = \pm1$.

Consider the maps $\theta : \mathsf{S}^3 \to \mathsf{S}^3$ and $\tau_{(p,q)} : \mathsf{S}^3 \to \mathsf{S}^3$ given by $\theta(u,v) = (u^n, v^{(qq')^{-1}n})$ and $\tau_{(p,q)}(u,v) = (\varepsilon u, \varepsilon^q v)$ respectively, where $\varepsilon = e^{\frac{2\pi i}{p}}$.

(*Nota bene*: $(qq')^{-1}$ is a positive integer modulo p.)

Then the equalities

$$\theta\tau_{(p,q)}(u,v) = \theta(\varepsilon u, \varepsilon^q v)$$

$$(\varepsilon^n u^n,\ \varepsilon^{qq^{-1}(q')^{-1}n} v^{(qq')^{-1}n}) =$$

$$(\varepsilon^n u^n,\ \varepsilon^{(q')^{-1}n} v^{(qq')^{-1}n})$$

and

$$\tau^n_{(p,q')}\theta(u,v) = \tau^n_{(p,q')}(u^n, v^{(qq')^{-1}n}) =$$

$$(\varepsilon^n u^n,\ \varepsilon^{(q')^{-1}n} v^{(qq')^{-1}n})$$

show that $\theta\tau_{(p,q)} = \tau^n_{(p,q')}\theta$ which means that θ carries the action generated by $\tau_{(p,q)}$ to the action generated by $\tau^n_{(p,q')}$.

Recalling the fact that for $\varphi : \mathsf{S}^1 \to \mathsf{S}^1$, given by $u \longmapsto u^n$, $\deg\varphi = n$ (since you can lift u^n in n different ways) we realize that $\deg\theta = n \cdot (qq')^{-1}n = n^2(qq')^{-1}$ (since you can lift $(u^n, v^{(qq')^{-1}n})$ in $n \cdot (qq')^{-1}n$ different ways).

Consider once again the model for the construction S of $\mathsf{L(p,q)}$ from $\mathsf{S}^3 \subseteq \mathbb{C} \times \mathbb{C}$. We know that $\tau_{(p,q)}$ permutes the fundamental regions determined by a pair of consecutive vertices on the unit circle in the first factor $\mathbb{C}$. (Every such region is homeomorphic to B^3.)

Take p disjoint disks in S^3 and pinch their boundaries to p distinct points. One such operation on one disk gives an S^3. Construct the space $\mathsf{W} = \mathsf{S}^3_0 \vee \mathsf{S}^3_1 \vee \cdots \mathsf{S}^3_p$ out of these p copies of S^3 plus one more, S^3_0.

W is a one-point union at different points. Consider the map $\mathsf{S}^3 \to \mathsf{W}$ where S^3 and W have the $\tau_{(p,q)}$–action, whereas the other S_i, $1 \leq i \leq p$, are simply permuted by $\tau_{(p,q)}$.

Consider then the map $\mathsf{W} \to \mathsf{S}^3$ where θ above acts on S^3_0 and maps of degree m act on the other S_i, $1 \leq i \leq p$.

Then the composition $\Phi : \mathsf{S}^3 \to \mathsf{W} \to \mathsf{S}^3$ has degree $\deg\theta + mp = n^2(qq')^{-1} + mp = \pm\mathsf{1}$ and carries the action of $\tau_{(p,q)}$ to the action of $\tau^n_{(p,q')}$.

Since $\deg\Phi = \pm\mathsf{1}$ it induces isomorphisms $\pi_i(\mathsf{S}^3) \to \pi_i(\mathsf{S}^3)$ for all i (See Bredon). The induced map $\Psi : \mathsf{L(p,q')} \to \mathsf{L(p,q)}$ on the orbit spaces gives isomorphisms

$$\Psi_\sharp : \pi_i(\mathsf{L(p,q')}) \to \pi_i(\mathsf{L(p,q)}) \qquad \text{for all } i.$$

A result from homotopy theory guarantees now that Ψ is a homotopy equivalence, i.e. $\mathsf{L(p,q')} \simeq \mathsf{L(p,q)}$ as desired. $\square$

5.4 Concluding Computations

The lens spaces $\mathsf{L}(\mathsf{p},\mathsf{q}) \approx \mathsf{S}^3/\mathbb{Z}_p$ are $\mathsf{3}-$dimensional topological manifolds, also called spherical $\mathsf{3}-$manifolds. They are closed, compact, path$-$connected and orientable.

Given p we seem to have the following candidates for lens spaces:

$$\mathsf{L}(\mathsf{p},1), \mathsf{L}(\mathsf{p},2), \ldots, \mathsf{L}(\mathsf{p},\mathsf{q}), \ldots, \mathsf{L}(\mathsf{p},\mathsf{p}-1).$$

We must of course eliminate from the list those cases for which $\gcd(\mathsf{p},\mathsf{q}) \neq 1$. Among the remainder we identify all those for which $\mathsf{q}' \equiv \pm \mathsf{q}^{\pm 1} \bmod \mathsf{p}$.

Here are a few examples:

- $\mathsf{L}(0,1) \approx \mathsf{S}^1 \times \mathsf{S}^2$
- $\mathsf{L}(1,\mathsf{q}) \approx \mathsf{S}^3$
- $\mathsf{L}(2,1) \approx \mathbb{R}\mathsf{P}^3$
- $\mathsf{L}(3,1)$ and $\mathsf{L}(3,2)$

  $2 \cdot 1 \overset{3}{\equiv} -1$

  $\because \mathsf{L}(3,2) \approx \mathsf{L}(3,1)$
- $\mathsf{L}(4,1)$ and $\mathsf{L}(4,3)$

  $3 \cdot 1 \overset{4}{\equiv} -1$

  $\because \mathsf{L}(4,3) \approx \mathsf{L}(4,1)$
- $\mathsf{L}(5,1)$ and $\mathsf{L}(5,4)$

  $4 \cdot 1 \overset{5}{\equiv} -1$

  $\because \mathsf{L}(5,4) \approx \mathsf{L}(5,1)$
- $\mathsf{L}(5,2)$ and $\mathsf{L}(5,3)$

  $3 \cdot 2 \overset{5}{\equiv} 1$

  $\because \mathsf{L}(5,3) \approx \mathsf{L}(5,2)$
- $\mathsf{L}(6,1)$ and $\mathsf{L}(6,5)$

  $5 \cdot 1 \overset{6}{\equiv} -1$

  $\because \mathsf{L}(6,5) \approx \mathsf{L}(6,1)$

- $\mathsf{L}(7,1)$ and $\mathsf{L}(7,6)$

  $6 \cdot 1 \overset{7}{\equiv} -1$

  $\because \mathsf{L}(7,6) \approx \mathsf{L}(7,1)$

- $\mathsf{L}(7,2)$ and $\mathsf{L}(7,4)$

  $4 \cdot 2 \overset{7}{\equiv} 1$

  $\because \mathsf{L}(7,4) \approx \mathsf{L}(7,2)$

- $\mathsf{L}(7,3)$ and $\mathsf{L}(7,5)$

  $5 \cdot 3 \overset{7}{\equiv} 1$

  $\because \mathsf{L}(7,5) \approx \mathsf{L}(7,3)$

$\mathsf{L}(7,1)$ and $\mathsf{L}(7,2)$ are not homeomorphic but since $1 \cdot 2 \overset{7}{\equiv} 3^2$ it follows that

$$\mathsf{L}(7,1) \simeq \mathsf{L}(7,2).$$

A curiosity:

$\mathsf{L}(14,1) \simeq \mathsf{L}(14,3)$ because $1 \cdot 3 \overset{14}{\equiv} -5^2$. Yet, it turns out that the connected sum of three copies of $\mathbb{R}\mathsf{P}^2$ can be embedded in $\mathsf{L}(14,3)$ but not in $\mathsf{L}(14,1)$ (See Bredon and Wood, 1969).

Bibliography

M.A. Armstrong. 1983. *Basic Topology.* Springer Verlag.

G.E. Bredon. 1993. *Topology and Geometry.* Springer Verlag.

G.E. Bredon and J.W. Wood. 1969. *Non − orientable 3 − manifolds*, in *Inventiones Mathematicae*, 7 , pp 83–110.

E.J.Brody. 1960. *The topological classification of lens spaces*, in *Annals of Mathematics* , (2), 71, pp 163–184

M.M. Cohen. 1973 *A Course in Simple-Homotopy Theory.* Springer Verlag.

A. Hatcher. 2002. *Algebraic Topology.* Cambridge University Press.

D. Rolfsen. 1990. *Knots and Links*, Publish or Perish.