SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Analysis and Application of Lie Group Theory on a certain type of PDEs in Finance

av

Elham Motiefar

2012 - No 4

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

Analysis and Application of Lie Group Theory on a certain type of PDEs in Finance

Elham Motiefar

Självständigt arbete i matematik 30 högskolepoäng, grundnivå

Handledare: Henrik Shahgholian

2012

Analysis and Application of Lie Group Theory on a certain type of PDEs in Finance

Elham Motiefar
Department of Mathematics at Stockholm University

Supervisor: Professor Henrik Shahgholian
Department of Mathematics
The Royal Institute of Technology

January 2012

Acknowledgements

First of all I would like to extend my sincerest appreciation and thanks to my supervisor Professor Henrik Shahgholian who was inspiring, guiding, and helping me for writing this thesis. I am grateful to Sadna Sajadini for acting as co-supervisor and for being always there to help me and gave me creative tips, incentives and criticism. Without her this thesis would not like be what it is.
I would like to thank all my teachers at the Department of Mathematics at Stockholm University who have guided me all through my studies during these years. I want also to thank my friends for their supports and helps.
My special thanks go to my lovely husband Hamid R. Lashgari who always believed in me, supported me and helped me to be where I am now.
I am grateful to my parents and my brother, specially my mother who always encouraged me, and believed in me through my entire life.

Abstract

The purpose of this study was to examine the connection between theory and practice in the vast mathematics world and to investigate the corresponding process in between. Another aim was to find out the connection between mathematical models and financial solutions. Because of this goal, in this thesis a financial problem was considered, the components of the problem were analysed and in order to solve such kind of problems, a method with the name "Lie group analysis" based on the symmetry analysis was chosen. Also different situation of this problem were examined in the study to obtain the reductions arising from Lie symmetry of a partial differential equation (PDE) to an ordinary differential equation (ODE), which was easier to solve in comparison with solving the PDE.

Contents

1 Introduction

The world of finance, which is one of the fastest growing areas in the modern banking and corporate world, consists of mathematical methods, formulas, models and systems. Therefore the financial mathematics is almost new and a developing area in science. In 1973, among the first scientists who obtained very important analytical results in this area were F. Black, M. Scholes and D. Merton. They described a model, known as Black-Scholes model. From the model, the Black-Scholes formula was generalized which leads to the value of an option.

In this thesis we will study a non-linear partial differential equation (PDE) arising from pricing asset related to one type of illiquid markets. In order to solve this type of equation, we will introduce a method "Lie group analysis", named after Sophus Lie, the Norwegian mathematician, who founded the theory of continuous transformation groups.
The idea behind applying Lie group theory on differential equations (DE) is as old as the theory itself, but this subject has been used quite recently, because of some *misconceptions* such as

I) *To find the symmetry group of an equation is as difficult as to solve the equation itself.*

II) *Lie group analysis provides randomly particular solution.*

III) *Lie group analysis is only for solving linear equations.*

Roughly speaking, the symmetry group of a system of differential equation is a group of transformation of all dependent and independent variables which maps each solution of equations to another solution of the same equation, i.e., leaves the set of solutions invariant. When a symmetry group of a system of an equation is obtained, then by using this, new solutions can be generated from the old one. There are some important application of the Lie group analysis, for example the symmetry reduction of a higher order ordinary differential equation (ODE) to a lower order ODE, the reduction of a partial differential equation to an ODE, which is the main subject of this thesis.
This thesis consists of six Sections. The First Section is introduction. In the Second Section, before describing any subject, we will study some fundamental definitions in mathematics and finance.
In Third Section we will analyse backgrounds in financial problems in mathematics, introduce some concepts and formulas which have an important role in pricing an asset, and we will show that how an asset price relates to a PDE. Then we will investigate three different illiquid market models and state non-linear PDEs arise from their pricing.

The aim of writing the Section Four is to understand the concept of symmetry and study definitions and examples informally which makes easier to understand the whole idea about applying Lie group analysis on a differential equation.
The contexts in Section Five, will give a broad view on the Lie group theory to the reader. We will study exact definitions and theorems about Lie symmetry analysis. They are important concepts to study the main PDE introduced in Section Three.
Section Six, under the title "Lie analysis on the obtained PDE from a Financial Market", is the main section of this research and consists of two parts. In the first part we will construct Lie algebra admitted by main equation. Then in the next part, we will study how we can find invariant and ordinary differential equations which are easier to solve in comparison with PDE.
In Section Seven which is the last one, we will discuss and represent the results. References are included.

2 Some Fundamental Definitions

Definitions have a fundamental role in understanding different subjects from the scientific aspect. Therefore here we state some important definitions in mathematics and finance, that we shall use them later in our discussions.

Definition 2.0.1. *A* ***Partial Differential Equation*** *is a kind of differential equation. An unknown function which depends on several independent variables and their partial derivatives with respect to them.*

Definition 2.0.2. $\sigma - algebra$. *Let X be a set, and let F be a non-empty subset of X. Then F is $\sigma - algebra$ if the following hold*

I) The set $X \in F$.

II) If a set $A \in F$, then the complement of $A \in F$ too.

III) If A_n is a sequence of elements of F, then $\bigcup A_n \in F$.

Definition 2.0.3. *A* ***Measurable space*** *is a set E together with a collection C of subsets of E which is a $\sigma - algebra$. The elements of C are called measurable sets.*

Definition 2.0.4. *A* ***Probability space*** *is a triple space (Ω, F, P) on the domain Ω, where (Ω, F) is a measurable space, F are the measurable subsets of Ω, and P is a measure on F with $P(\Omega) = 1$. Briefly, a probability space is a measure space such that the measure of the whole space is equal to one.*

Definition 2.0.5. *A* ***Stochastic Variable*** *is called random variable too. It is a measurable function X from a probability space (Ω, F, P) into a measurable space.*

Definition 2.0.6. ***Stochastic Process.*** *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let X be a collection of $X-$valued random variables on Ω, indexed by a set T (time). A stochastic process or random process F is a family F_t for all t belong T i.e.,*

$$\{F_t : t \in T\},$$

where F_t is a $X-$valued stochastic (random) variable on Ω.

Definition 2.0.7. ***Expected value*** *(Countable case)*
Let X be a discrete random variable, which takes values $x_1, x_2, ...$ with respective probabilities $p_1, p_2, ...$. Then the expected value of this random variable X, is the infinite sum

$$\mathbb{E}[\mathbb{X}] = \sum_{j=1}^{\infty} x_j p_j,$$

provided that this series converges absolutely (i.e., the sum must remain finite if we substitute x_j's with their absolute values.
In general, if X is a random variable defined on a probability space (Ω, F, P), then the expected value of this random variable, denoted by $\mathbb{E}[\mathbb{X}], \langle X\rangle$ or $\bar{X}$, is defined as follows

$$\mathbb{E}[\mathbb{X}] = \int_\Omega X dP = \int_\Omega X(\omega) P(d\omega).$$

Definition 2.0.8. *A **Variance** is a measure how far a set of numbers is distributed (spread out). If X be a discrete random variable with discrete probability distributions $x_1 \mapsto p_1, ..., x_n \mapsto p_n$, then the variance is defined as*

$$Var[x] = \sum_{j=1}^n p_j (x_j - \mu)^2,$$

where $\mu = \sum_{j=1}^n p_j x_j$. ($\mu$ also is called mean).

Definition 2.0.9. *A **Random walk** is a series of sequential movement in which the size and the direction of each move is randomly determined.*

Definition 2.0.10. ***Geometric Brownian motion.*** *Let X_t be a stochastic process. It is a Brownian motion if satisfies the following properties*

- *I) Continuously of sample paths: The map $t \mapsto X_t(w)$ is continuous for every w.*
- *II) Independent increments: $(X_{t_1} - X_{t_0})(X_{t_2} - X_{t_1}), ..., (X_{t_n} - X_{t_{n-1}})$ are independent for any collection times.*
- *III) Stationary of increments: The distribution of $X_t - X_S$ depends only on $(t - s)$.*

Definition 2.0.11. *A **Quadratic variation** is a kind of variation of a process. Consider a real valued stochastic process X_t defined on a probability space (Ω, F, P) for a real and non negative time value t. Its quadratic variation is the process, written as $[X]_t$ and defined as follows*

$$[X]_t = \lim_{\|P\|\to 0} \sum_{k=1}^n (X_{t_k} - X_{t_{k-1}})^2,$$

where P is the range over partition on interval $[0, t]$.

Definition 2.0.12. *A **Filtration** is a sequence of sets $A_1, A_2, ..., A_n$ with the property*

$$A_1 \subset A_2 \subset ... \subset A_n.$$

Definition 2.0.13. ***Adapted process.*** *Let t be time and X_t be a stochastic process defined on a probability space (Ω, F, P), we say that the process X_t is adapted to the filtration F_t, where $F_t \subseteq F$ if for all $t \geq t_0$, X_t is known at time t.*

Definition 2.0.14. *A* ***Martingale*** *is a sequence of random variables say X_t. We expect for this sequence, at a particular time t, the next value in the sequence is equal to the given knowledge of all prior observed value at the current time, i.e., X_t is martingale if the following hold for all t*

(i) $\mathbb{E}(|X_t|) < \infty$,

(ii) $\mathbb{E}(X_{t+1}|X_1, X_2, ..., X_t) = X_t$.

Definition 2.0.15. ***Cadlag.*** *A stochastic process X is said to be cadlag if it is almost surely a sample paths which are continuous to the right and has the left limits (In other words they are right continuous with the left limits).*

Definition 2.0.16. ***Semi-martingale.*** *For a real valued process X_t defined on the filtered probability space (Ω, F, P), we say that is semi-martingale if it can be decomposed as follows*

$$X_t = M_t + A_t,$$

where M is a local martingale and A is a Cadlag adapted process of locally bounded variation (a real value function whose total variation is bounded (finite).

Definition 2.0.17. ***Markov property*** *is a property that a set of stochastic process $X(t)$ may have. It describes that the past state is irrelevant because it does not matter how the present state was obtained, it is said that the process is memoryless. The stochastic process $X = (X(t) : t \in I)$ on some probability space (Ω, F, P) is Markovian if for any n and $t_1 < t_2 < ... < t_n$ we have*

$$P(X(t_n) \leq X_n \mid X(t_{n-1}), ..., X(t_1)) = P(X(t_n) \leq X_n \mid X(t_{n-1})).$$

Definition 2.0.18. *An* ***Ito process*** *is a stochastic process with normally distributed jumps and they are closed under functional transformations.*

Definition 2.0.19. *An* ***Ito Lemma (Ito-formula)*** *is used to find the differential of function where this function has a particular type of stochastic process. Consider the Ito precess*

$$dS_t = udt + vdW_t,$$

and for any twice differential function g(t,x) of two real variables t and x, will define a new Ito process

$$dg(t, S_t) = \frac{\partial g}{\partial t}(t, S_t)dt + \frac{\partial g}{\partial S}(t, S_t)dS_t + \frac{1}{2}\frac{\partial^2 g}{\partial S^2}(t, S_t)(dS_t)^2,$$

where $(dS_t)^2$ is computed by using

$$dt.dt = dt.dW_t = 0, \text{ and } (dS_t)^2 = dt,$$

where W_t is Brownian motion (See Definition 2.0.10).

Definition 2.0.20. ***Euler method*** *is a first order numerical formula, and is used to solve ordinary differential equations with a given initial value.*

Definition 2.0.21. *A* ***Heaviside step function*** *is a discontinuous function with these properties such as the function value is equal to zero for negative arguments and equal to one for positive arguments.*

Definition 2.0.22. *A* ***Lipschitz condition*** *is a restriction on the increasing of a function. It describes a property for a function which is a strong form of uniform continuity for a function, and if the function f satisfies the following condition at the point b in the neighbourhood of x*

$$|f(x) - f(b)| \leq C|x - b|^{\alpha},$$

where C is a constant, $0 \leq \alpha \leq 1$.

Definition 2.0.23. *A* ***Stock*** *is a type of security. It signifies ownership position in a corporation and represents a right on the corporation's assets, earnings and profits.*

Definition 2.0.24. *An* ***Option*** *is a contract between two groups or parties. It is the right to buy or sell an asset at a given price.*

Definition 2.0.25. *A* ***Call option*** *gives the right to buy an asset.*

Definition 2.0.26. *A* ***Put option*** *gives the right to sell an asset.*

Definition 2.0.27. *A* ***Maturity date*** *is a date at which the asset is converted to payment or a specific amount of money.*

Definition 2.0.28. *An* ***European option*** *is a kind of contract that can be exercised only at the maturity date.*

Definition 2.0.29. *An* ***American option*** *is a kind of contract that can be exercised at any date up to and including the maturity date.*

Definition 2.0.30. *A* ***Transaction*** *is an agreement between a buyer and a seller to exchange an asset for a certain amount of money (payment).*

Definition 2.0.31. *A* ***Bid*** *is the price for an asset that a buyer is willing to pay at a given time t.*

Definition 2.0.32. *An **Ask** is the price for an asset that a seller wants to get at a given time t.*

Definition 2.0.33. *A **Spread** is the price that is the difference between bid and ask.*

Definition 2.0.34. *A **Liquidation value** is the amount of money that the large trader receives if he liquidates his stock position.*

Definition 2.0.35. ***Illiquid** is a property which says that an asset cannot easily and quickly be converted into cash.*

Definition 2.0.36. ***Self-financing Strategy (Markovian Strategy).** Consider strategy which is a collection of pairs of stocks (a_t,b_t) and time t, $1 \leq t \leq T$, where T is maturity date. We say that the strategy is self-financing if the following equality is satisfied*

$$(a_t S_t) + (b_t B_t) = (a_{t+1} S_{t+1}) + (b_{t+1} B_{t+1}).$$

In another words this quantity is the same in the time t and $t + 1$.

Definition 2.0.37. ***Rate** is paid or charged for the use of money.*
Interest rate is often expressed as a yearly percentage of the amount of borrowed or unpaid money.

Definition 2.0.38. *An **Expiry date** is a date which a debt has to be paid.*

Definition 2.0.39. *A **Portfolio** is a collection of investment which all of them are owned by the same individual organization.*

Definition 2.0.40. ***Return** is change in asset price divided by original price, which often express in percentage.*

Definition 2.0.41. *A **Volatility** measures the standard deviation of the return of an asset. In other word it is the rate at which the price of a security moves up and down.*

Definition 2.0.42. *A **Risk-less asset** is an asset whose future return is certainly known.*

Definition 2.0.43. *An **Arbitrage** is a try to profit by exploiting form price difference of an identical or similar financial instrument on different markets.*

Definition 2.0.44. ***Drift** measures the average rate of an asset price.*

3 Backgrounds in Financial Mathematics

In the world of finance which is one of the fastest growing areas in the modern banking and corporate world, there are many mathematical methods, formulas, models and system, we shall discuss some of them here below and in the end we study three different illiquid market models and find their corresponding price as a mathematical expression.

Notation

For simplicity here we make a list of the common notations coming in the next discussions.

- $\star$ S asset price.
- $\star$ t time.
- $\star$ T time to expiry date. (See Definition 2.0.38).
- $\star$ $V(S, t)$ price of an option. It is a function of stock price and time. (Definition 2.0.24 and Definition 2.0.23).
- $\star$ σ volatility of S.
  It measures the standard deviation of the return of an asset. (See also Definition 2.0.41).
- $\star$ μ drift.
  It measures the average rate (see Definition 2.0.37) of an asset price.
- $\star$ W Geometric Brownian motion. (See Definition 2.0.10).
- $\star$ r interest rate.

3.1 Mathematical Representation of an Asset Price

In this part we want to construct a mathematical representation about generating asset price.

We should know that there is always an observation about the asset price movement. Because of the efficiency of market hypothesis, it is usually stated that this movement is randomly.
There are two basic assumptions about this hypothesis

- The present price is totally reflected by the past history.
- Market reacts instantly to any new information about asset.

In fact making a model for the asset price is about making a model of new arrival information which effect on the stock price.
By considering of two above assumptions, unanticipated changes in the asset price are Markov process (is a time-varying random phenomenon for which Markov property (Definition 2.0.17) holds).
We note that instead of absolute change of asset price which is not a useful quantity, we associated return, defined as quotient of the change in the price divided by the principal price.

Now we aim to model the corresponding return on the asset $\frac{dS}{S}$, where S is the asset price at time t.
Consider dt as a small subsequence of time interval, in which the asset price changes from S to $S + dS$.

This model consists of the following two parts
• The first one is predictable and deterministic return. It is also related to the return on money invested in a risk-free bank

$$\mu dt,$$

where μ is the measure of the average rate of the asset price (also called drift).

• The second part related to return $\frac{dS}{S}$, presents random changes in asset price such as unexpected news. This part is

$$\sigma dW,$$

where σ is volatility which measures the standard deviation of the returns, and the term dW is known as Brownian motion (also called Wiener process), is a random variable and has normal distribution, where its mean value is equal to zero and its variance is equal to dt, i.e.,

$$dW = N(0, (\sqrt{dt})^2).$$

Finally by combination of these parts, we obtain the stochastic differential equation (SDE)

$$dS = \mu S dt + \sigma S dW, \tag{1}$$

which is mathematical representation for generating assets price. (All components in this formula have been introduced in Notation, in the previous page.)

Let us study some properties of equation (1). One of these properties is that this equation does not refer to past history of the corresponding asset price, i.e., the next price shown with $(S+dS)$ depends only on today's price. This independent property is called Markov property, "memoryless", (See also Definition 2.0.17).

The second property is that we consider the mean of dS as follows

$$\mathbb{E}[dS] = \mathbb{E}[\sigma S dW + \mu S dt] = \mu S dt,$$

therefore $\mathbb{E}[dW] = 0$.

The third property is the variance of dS is the following expression

$$Var[dS] = \mathbb{E}[dS^2] - \mathbb{E}[dS]^2 = \mathbb{E}[\sigma^2 S^2 dW^2] = \sigma^2 S^2 dt.$$

3.2 Ito's Lemma

As a matter of fact, asset prices are considered at discrete time interval. Thus there is a practical lower bound for the basic time-step "dt" of the random walk (1). If we had used these time-steps in practice, then we would have obtained an unmanageable large number of data. Instead we established a mathematical continuous model by taking $dt \to 0$. Ito lemma is the most important concept of using random variables and it is a version of Taylor's expansion.

Taylor Series: Let $f(x)$ be $\mathbb{C}^3$, (three times differentiable) function of (real or complex) number x in a neighbourhood of a. Then

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + o(x-a)^3.$$

Now we investigate that what happens to equation (1) when $t \to 0$.
We state that

$$(dW)^2 \to dt \text{ as } dt \to 0, \text{ with probability 1;}$$

for the proof of this statement see [15].

Let S be the asset price. Suppose $f(S)$ is a smooth function of S. If we change S by a small amount dS, then it is clear that f will also changes. Then from Taylor series we have

$$df = \frac{df}{dS}dS + \frac{1}{2}\frac{d^2 f}{dS^2}(dS)^2 + ...,$$

where the dots denote a remainder, which is smaller than the other retained term. Now by squaring (1), we obtain

$$(dS)^2 = (\mu^2 (dt)^2 + 2\mu\sigma dt dW + \sigma^2 (dW)^2) S^2.$$

Since

$$(dW)^2 \to dt \text{ as } dt \to 0,$$

$\sigma^2(dW)^2S^2$ dominate the above expansion, as dt become smaller.
Thus in order to approximate $(dS)^2$ as $dt \to 0$, we only use the term

$$\sigma^2S^2dt.$$

Then we have

$$\begin{aligned} df &= \frac{df}{dS}dS + \frac{1}{2}\frac{d^2f}{dS^2}(\sigma^2S^2)dt \\ &= \frac{df}{dS}(\mu Sdt + \sigma SdW) + \frac{1}{2}\frac{d^2f}{dS^2}(\sigma^2S^2)dt \\ &= \sigma S\frac{df}{dS}dW + (\mu S\frac{df}{dS} + \frac{1}{2}\frac{d^2f}{dS^2}\sigma^2S^2)dt. \end{aligned} \tag{2}$$

This is **Ito's lemma** relating a little change in a function of one stochastic variable, to a little change in the variable itself.
A version of Ito's lemma is for a function of several variables, for example in our case if f is a function of two variables S and t, then we have

$$df = \sigma S\frac{\partial f}{\partial S}dW + (\mu S\frac{\partial f}{d\partial S} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2S^2 + \frac{\partial f}{\partial t})dt. \tag{3}$$

3.3 Black-Scholes Mathematical Analysis

The Black-Scholes formula is used for calculating the price of European and American put and call options. We obtain option price by solving the Black-Scholes partial differential equation.
Before starting the analysis, we list the following assumptions

- The asset price satisfies the geometric Brownian motion (1).
- The asset volatility σ and interest rate r are known functions of time over the life of corresponding options.
- There are no transaction costs.
- The corresponding underlying asset pays no dividends.
- There is absence of arbitrage possibilities.
  It means that when there is no arbitrage opportunities, mean that all risk-free portfolios must pay off the same.
- The underlying asset can be traded continuously.
- Short selling is permitted and the assets are divisible.
  It says that there is an assumption that we can buy or sell any number of assets, and also we may sell assets that we do not own them.

The analysis of the Black-Scholes leads to the value of an option. Let $V(S,t)$ be a differentiable function value of an (call or put) option, where S is stock price and t is time. In order to find option value at an earlier time than the expire time T, it is necessary to study about how V arise as a function of S and t. Therefore by applying Ito lemma (3) on $V(S,t)$ we obtain

$$dV = \sigma S \frac{\partial V}{\partial S} dW + (\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\mu^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t})dt. \tag{4}$$

where μ is drift (see Definition 2.0.44) and σ is volatility (see Definition 2.0.41).
Now we construct a portfolio, which consist one option and a number $-\Delta$ of the underlying assets. Note that the value of this number is not specified yet. The value of this portfolio is

$$\Pi = V - \Delta S. \tag{5}$$

The jump of this value in one-time step is

$$d\Pi = dV - \Delta dS.$$

By putting (1), (4), (5), we obtain the following random walk of value of portfolio Π

$$d\Pi = \sigma S(\frac{\partial V}{\partial S} - \Delta)dW + (\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \Delta S)dt. \tag{6}$$

To eliminate the random component, we put

$$\Delta = \frac{\partial V}{\partial S}, \tag{7}$$

for more information about this equality, see [15] section 2.4.
Therefore we obtain random walk of Π as

$$d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt. \tag{8}$$

If the value of portfolio Π was invested in risk-less assets, then this value would be seen as a growth of $r\Pi dt$ in a interval of length dt.
Thus we should have the following expression for a fair price

$$rd\Pi dt = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt, \tag{9}$$

or equivalently

$$r(V - \frac{\partial V}{\partial S} S) = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}).$$

Here we substituted (5) and (7) into equation (9) and then divide the whole expression by dt to obtain the following **Black-Scholes partial differential equation**

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{10}$$

and for the American option the Black-Scholes partial differential equation (second order) is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \leq 0.$$

See [15] for the proof.
By solving these equations we are able to calculate the price of European and American put and call options.

3.4 Illiquid Market and Non-linear Black-Scholes Equations

Here we investigate three different illiquid market models and state non-linear PDEs arise from pricing them. The aim is applying Lie group analysis to the PDE. First we show that the solution of a non-linear standard Black-Scholes equation (which we use to evaluate the price of options) is the value of a self-financing strategy (usually called Markovian Strategy, see Definition 2.0.36). Then we introduce invariant variables and finally reduce the corresponding PDE to an ODE. These models contain two assets, a risk free money-market account B, and a risky asset S, called the stock (See Definition 2.0.23). Also we consider that they are modelled in a filtered probability space $(\Omega, F, \{F_t\}, P)$. Another assumption is that the money-market account is equal to one and the interest rate equal to zero.
We group these three illiquid market models as follows
I) Quadratic Transaction-Cost Model,
II) Reduced-Form Stochastic Differential Equation (SDE) Model,
III) Equilibrium or Reaction-Function Model.

Now we study each case briefly, for more detailed information see [13].

I) In this model we have a stock that its price S^0 satisfies the following geometric Brownian motion (As we demonstrated in Section 3.1)

$$dS_t^0 = \mu S_t^0 dt + \sigma S_t^0 dW_t, \tag{11}$$

where W is a standard Brownian motion, $\mu \in \mathbb{R}$ and is drift and $\sigma > 0$, is volatility, are constants.
We introduce the *transaction price* (See Definition 2.0.30) as

$$\bar{S}_t(\alpha) = e^{\rho\alpha} S_t^0, \tag{12}$$

where $\alpha \in \mathbb{R}$ is the number of shares, $\rho > 0$ is the liquidity parameter (See Definition 2.0.34) and t is time, that the price will be paid at this time for α. Considering a self-financing trading strategy (Φ_t, η_t), for the number of stocks and the position in the money market for predictable stochastic process, introduce paper value V_t which express the value of this strategy by $V_t = \Phi_t S_t^0 + \eta_t$. Then we can deduce that

$$dV_t = \Phi_t dS_t^0 - \rho S_t^0 d[\Phi]_t, \tag{13}$$

where $[\Phi]_t$ is quadratic variation of semi-martingale and is defined as follows

$$[\Phi]_t = \int_0^t (\phi_S(s, S_s^0)\sigma S_s^0)^2 \, ds,$$

see [4] and [1] for the details.
According to [13] after combination this expression by equation (13), and by considering the uniqueness property of semi-martingale, we apply Ito Lemma on the value of a self-financing strategy $u(t, S_t^0)$, and then we see $\phi \equiv u_S$, that implies $\phi_S \equiv u_{SS}$, hence we obtain the following non-linear PDE for u

$$u_t + \frac{1}{2}\sigma^2 S^2 u_{SS}(1 + 2\rho S u_{SS}) = 0. \tag{14}$$

with the final condition $u(T, S) = h(S)$, where $h : [0, \infty) \to \mathbb{R}$.
See [13] for detailed computations.

II) In this model we have large traders instead of considering transaction cost in the previous model and their trading activity affects the equilibrium stock price. Here it is assumed that the stock price S_t satisfies the the following stochastic differential equation

$$dS_t = \rho S_t d\Phi_t + \sigma S_t dW_t, \tag{15}$$

where ρ is the liquidity parameter, Φ is the semi-martingale (See Definition 2.0.16) and W_t is Brownian motion.

In this model there is a $\Delta\Phi$ number of stock which is bought or sold by the investor and a stock price which goes upward or downward by $\rho S_t - \Delta\Phi_t$. The strength of this price depends on the liquidity parameter ρ. Consider a portfolio (See Definition 2.0.39) of stock trading strategy Φ and value V; we say that it is self-financing, if $dV_t = \Phi_t dS_t$.
Now suppose $V_t = u(t, S_t)$ and $\Phi_t = \phi(t, S_t)$ for smooth functions u and ϕ. By applying Ito formula to the process $(u(t, S_t))$, it follows that $\phi \equiv u_S$, therefore $\phi_S \equiv u_{SS}$, and since semi-martingale is unique, it follows that u must satisfy

$$u_t + \frac{1}{2}(\upsilon)^2(t, S) S^2 u_{SS} = 0.$$

where $\upsilon(t, S)$ is called adjusted volatility. See [13] for a definition of adjusted volatility.
After some algebraic calculating, we get the following non-linear PDE

$$u_t + \frac{1}{2}\frac{\sigma^2}{(1-\rho S u_{SS})^2}S^2 u_{SS} = 0, \tag{16}$$

which should be considered with a suitable final condition.
For details, see [5], [6], [8],[9], [11].

III) In this model there is a *smooth reaction* function ψ. This function gives the equilibrium stock price S_t at time t as a function of fundamental value F_t and the stock position of a large trader, i.e.,

$$\psi \longmapsto S_t(F_t, \rho\Phi_t),$$

where ρ is a parameter that shows the size of trading.
More general, a reaction function can be seen as a representation of a reduced-form of an economic equilibrium point

$$D(F_t, S_t) + \rho\Phi_t = 1, \tag{17}$$

where $D(F_t, S_t)$ denotes the stock demand corresponding to the ordinary investors.

Because equation (17) for suitable D, has a unique solution hence S_t can be written as a function ψ which depends on F_t and $\rho\Phi_t$, in another word $S_t = \Psi(F_t, \rho\Phi_t)$.

According to the [10], by assuming $\Psi(f, \alpha) = fg(\alpha)$, for some increasing g, we obtain

$$S_t = F_t g(\rho\Phi_t) \quad \text{ and } \quad dS_t = g(\rho\Phi_t)dF_t.$$

Now we assume that the Markovian strategy as the trading strategy of the large trader has the form $\Phi_t = \phi(t, S)$, for a smooth function ϕ. By applying Ito formula on $S_t = F_t g(\rho\Phi_t)$, we get the following dynamics

$$dS_t = g(\rho\phi(t, S_t))dF_t + \rho F_t g_\alpha(\rho\phi(t, s_t))\phi_s(t, S_t)dS_t + b(t, S_t)dt,$$

where $b(t, S_t)$ is negligible because it is irrelevant to our aim.

According to [13], by assuming that the fundamental value process F_t follows a geometric Brownian motion i.e., $dF_t = \mu F_t dt + \sigma F_t dW_t$, and by introducing variables $S_t = S^2$ and $dW_t = u_{SS}$, we get the following PDE for the value function $u(t, S)$ for a self-financing strategy

$$u_t + \frac{1}{2}\frac{(1-\rho u_S)^2}{(1-\rho u_S - \rho S u_{SS})^2}\rho^2 S^2 u_{SS} = 0. \tag{18}$$

For more detailed information we refer to [12], [1], [7],[10].

The non-linear PDEs (14), (16) and (18) in the above cases are of the form

$$u_t + \frac{1}{2}\sigma^2 S^2 \upsilon(\rho u_S, \rho S u_{SS}) u_{SS} = 0, \tag{19}$$

where υ could be any function with property that $\upsilon(0,0) = 1$.

We often consider the liquidity parameter ρ as a very small value. Now by first order Taylor approximation for $\upsilon(\rho u_S, \rho u_{SS})$ around $\rho = 0$, equations (16) and (18) lead us to (14). Therefore we aim to solve (14), by applying Lie group analysis, later in this thesis.

4 Symmetry Methods for Differential Equations

In this section we will study informally some basic concepts and ideas related to symmetry methods for solving differential equations (DEs). We will solve some elementary examples which help to clarify the concepts later on in this thesis (solving a PDE by applying Lie group methods). For study more about basic concepts and simple examples, see [16] and [18].

4.1 Symmetry Concept

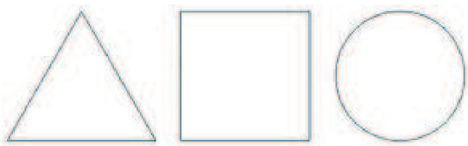
In order to understand the meaning of symmetry related to a DE, it is useful to look at symmetry concept in other context. The symmetry concept exists almost everywhere, in nature such as in some sort of planets, crystals, art etc. We have a well developed sense for symmetry in daily circumstances. The word symmetry comes from a Greek word "symmertos" that means "well-proportioned" or "harmonic".

4.1.1 What does symmetry mean?

Generally it is easy to see if a geometric object has symmetry or not, but to explain about how we distinguish this symmetry, is not easy. Thus we should systematize the symmetry concept. In Mathematics we have the following definition:

An object is symmetric if there exist a transformation which leaves the object unchanged (or invariant). A transformation which leaves an object invariant is called symmetry for the object.

A symmetry is a transformation or a mapping, which maps an object on itself. Consider for example the following figures



The equilateral triangle can be mapped onto itself in many ways, for example by rotation angel $2\pi/3$, counter-clockwise, reflection or by a mirror image on one of the three sides. Totally a equilateral triangle has six symmetries, which these symmetries make a group.
Similarly a square has eight symmetries (more than a triangle). The symmetry of these two figures are discrete. On the other hand a circle has a continuous and infinite symmetry, for every rotation around the center of

the circle maps circle on itself. For more information see [16].

4.1.2 Symmetry and physical laws

In many physical theories, symmetry is a fundamental concept. A symmetry in this context is a transformation which leaves physical laws unchanged (invariant). The relation between physical law and symmetry was observed by Galileo and Newton. Specially Newton realized that these symmetries arising from movement of particles in nature are even more important than their movements. Generally many physical theories and connections are consequences of some sort of symmetries.
From a mathematical point of view we can put some questions that how symmetry for a physical system can be determined in differential equations. In order to answer these questions we should first explain exactly what we mean by a symmetry in differential equation.

4.1.3 Mathematics formulation of the symmetry

A symmetry of an object is a transformation with the following properties

i) It is a mapping of an object to itself.

ii) It preserves the structure.

Furthermore it is required that the number of all symmetries of an object make a group (See also Definition 5.1.4) which should satisfies the following properties

I) There exist a neutral element i.e., the identical transformation.

II) Every symmetry has an inverse.

III) Combination of two symmetries should be a symmetry.

IV) Combination of symmetries should be associative.

Group theory is a mathematical tool for studying the symmetry. We can say that the group theory constitutes the abstract version of symmetry concept. Here we mention that group theory was invented in 1832 by 21-years-old French mathematician E. Galois, in order to describe symmetries of some expressions of roots to an algebraic equation. Later, Sophus Lie substituted algebraic equations with differential equations and roots with solutions to differential equation.

4.2 Introduction to Symmetries and Differential Equations

A common method to solve differential equations is to introduce new variables to obtain an easier equation. For example a DE of the following type (called homogeneous ODE)

$$y' + f(x)y = g(x)y^{\alpha}, \quad (\alpha \neq 0, \alpha \neq 1),$$

transforms to a linear DE by substituting

$$z = y^{1-\alpha}.$$

Then by some easy calculations we obtain

$$z' = (1-\alpha)y^{-\alpha}y',$$

which implies

$$\frac{dz}{dx} + (1-\alpha)f(x)z = (1-\alpha)g(x).$$

This is a linear equation which is satisfied by new variable z. Solving this equation gives $y = z^{\frac{1}{1-\alpha}}$, $\alpha > 1$.

The question is, how we can find a successful variable substitution for general differential equations. To answer to this question Lie introduced the method of symmetry groups. The transformation that Lie introduced and used to study ODE and PDE were a group of continuous transformation, which are known as Lie group.
Lie started from the following problems:

1. Is there a general method to determine a solution to a DE?
2. How can we classify DEs?

The theory that Lie developed to answer the above question, consist of two main parts:

1. Lie symmetry concept for DE.
2. Lie invariant theory for DE.

4.2.1 Symmetry and differential equation

A symmetry for a differential equation is a transformation of dependent and independent variables of the equation, which maps each solution of the equation to a solution (generally another solution) of the same equation.
If a transformation is symmetry for an equation, thus the equation keeps its structure invariant. Conversely if a DE is invariant under a transformation,

thus the transformation is symmetry to this DE. An important application of the discussion above is, if we know a solution to an equation and symmetry to this equation then we can generate several solutions.
If we find a symmetry for a PDE, then we can find solutions which are functions of some combination of the main variables. For example a PDE with two independent variables reduces to an ordinary differential equation. Some of these symmetries such as translation, scaling and rotation, can be obtained by an easy calculation. In this thesis we focus on continuous symmetry.

4.2.2 One-parameter transformation

As mentioned before, a symmetry for a DE is a transformation which leaves the solutions to the equation invariant. According to Lie's investigation these transformations make a group which always depends on one or several variables. One type of these transformations is called "point-transformation", which acts on the space of independent and dependent variables. Here we consider only this type which is called one-parameter *Lie group* of transformation.

Example 4.2.1. *Let*

$$\frac{dy}{dx} = \frac{y^3 + x^2 y - y - x}{xy^2 + x^3 + y - x},$$

and consider the rotation for (x, y) *as*

$$(\tilde{x}, \tilde{y}) = (x \cos\varepsilon - y \sin\varepsilon, x \sin\varepsilon + y \cos\varepsilon).$$

The collection of solution curves (sketched in figure 1), shows that this rotation form a one-parameter of Lie symmetries around the origin.

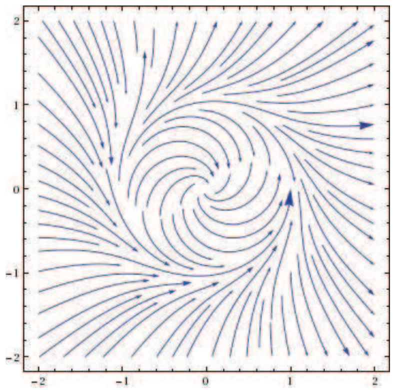


Figure 1: Symmetry around origin

■

Example 4.2.2. *Let*

$$u_t - u_{xx} = 0.$$

This equation is invariant under family of transformations, denoted by $\{\Gamma_\varepsilon\}$*,*

$$\begin{aligned} \tilde{x} &= x + 2\varepsilon t, \\ \tilde{t} &= t, \\ \tilde{u} &= u\exp(-(\varepsilon x + \varepsilon^2 t)), \end{aligned} \tag{20}$$

since $\tilde{u}_{\tilde{t}} - \tilde{u}_{\tilde{x}\tilde{x}} = 0$.

Note that every transformation Γ_ε *in family (20) is determined by a specific parameter* ε*.*

■

Generally every family $\{\Gamma_\varepsilon\}$ of invertible transformations $\Gamma_\varepsilon(x, t, u) = (\tilde{x}, \tilde{t}, \tilde{u})$, where $\varepsilon \in \mathbb{R}$, and

$$\begin{aligned} \tilde{x} &= X(x, t, u, \varepsilon), \\ \tilde{t} &= T(x, t, u, \varepsilon), \\ \tilde{u} &= U(x, t, u, \varepsilon), \end{aligned}$$

is said to be a one-parameter *Lie group* of transformation if it satisfies the following conditions:

I) There exists an identity element $\varepsilon = 0$, such that $\Gamma_0 = I$, where I is the identity mapping.

II) Every transformation Γ_ε has an inverse $\Gamma_\varepsilon^{-1} = \Gamma_{-\varepsilon}$, so it follows that $\Gamma_{-\varepsilon}(\tilde{x}, \tilde{t}, \tilde{u}) = (x, t, u)$.

III) Composition of two transformation in the family also belongs to the family, i.e., $\Gamma_{\varepsilon_1}\Gamma_{\varepsilon_2} = \Gamma_{\varepsilon_1+\varepsilon_2}$.

Example 4.2.3. *Consider the ODE*

$$\frac{dy}{dx} = \frac{2y}{x}, \tag{21}$$

which has the general solution

$$y = cx^2, \tag{22}$$

where c is an arbitrary constant.
Let us consider the region $x > 0$, $y > 0$, in which the family of solution curve is determined by a specific value of the constant $c > 0$.
One of the symmetries to the ODE (21) is one-parameter Lie group of scaling

$$(\tilde{x}, \tilde{y}) = (e^{\varepsilon}x, c_1 e^{-\varepsilon}x^2). \tag{23}$$

By solving for variable x, we obtain $x = e^{-\varepsilon}\tilde{x}$, thus the transformed solution becomes

$$\tilde{y} = c_1 e^{-3\varepsilon}\tilde{x}^2. \tag{24}$$

Hence the action of the symmetry (23) maps the solution $y = cx^2$, to the solution $\tilde{y} = c_1 e^{-3\varepsilon}\tilde{x}^2$, (sketched in the figure 2).

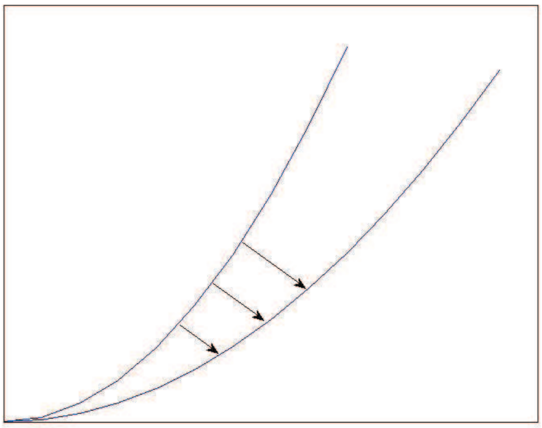


Figure 2: The action of symmetry (23) on ODE (21).

■

Remark. Sometimes it happens that the composition of the transformations is just defined for the parameters which lies near to identity element ε_0. In this case the family of transformation Γ_ε is called *local* one-parameter Lie group.

The Lie's first theorem states that we can parametrize a group such that

I) $\varepsilon = 0$ gives identical mapping.

II) The inverse is obtained by changing the sign of the parameter.

III) The group composition for the parameters are common addition.

It follows that we can re-parametrize a one-parameter Lie group of transformation such that these three conditions are satisfied.

4.2.3 Infinitesimal generator

Some type of symmetries, for example rotation to a DE can be obtained by an easy calculation, i.e., in Example 4.2.1, where by $\varepsilon = 0$, we obtain $(\tilde{x}, \tilde{y}) = (x, y)$. But to determine other types of symmetries, their definitions lead to a complicated non-linear function of transformations.

Lie realized that there are some invariant conditions, based on a infinitesimal (linearised) version of the problem. Instead of determining the global (finite) form of the group, can use the linearised version by determining Taylor expansion of the function which defines transformation around $\varepsilon = 0$. Lie called this *infinitesimal form*.
Then the global form of the group can be obtained by either solving a certain kind of equation called *Lie equation*, or by using the group generators, which is called *infinitesimal generator*.

Consider a Lie group of the type one-parameter group of transformations

$$\begin{aligned} \tilde{x} &= X(x, t, u, \varepsilon), \\ \tilde{t} &= T(x, t, u, \varepsilon), \\ \tilde{u} &= U(x, t, u, \varepsilon), \end{aligned} \tag{25}$$

where by $\varepsilon = 0$, identity transformation is obtained.
The Taylor expansion around $\varepsilon = 0$ is

$$\begin{aligned} \tilde{x} &= x + \varepsilon\xi(x, t, u) + o(\varepsilon^2), \qquad (= X(x, t, u, \varepsilon)), \\ \tilde{t} &= t + \varepsilon\tau(x, t, u) + o(\varepsilon^2), \qquad (= T(x, t, u, \varepsilon)), \\ \tilde{u} &= u + \varepsilon\phi(x, t, u) + o(\varepsilon^2), \qquad (= U(x, t, u, \varepsilon)), \end{aligned} \tag{26}$$

where

$$\xi = \frac{\partial \tilde{x}}{\partial \varepsilon} \,|_{\varepsilon=0}, \qquad \tau = \frac{\partial \tilde{t}}{\partial \varepsilon} \,|_{\varepsilon=0}, \qquad \phi = \frac{\partial \tilde{u}}{\partial \varepsilon} \,|_{\varepsilon=0} \,. \tag{27}$$

The functions ξ, τ, ϕ are called the group's *infinitesimals*. The differential operator

$$V = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \phi \frac{\partial}{\partial u}, \tag{28}$$

is called *infinitesimal generator*. This is one of the fundamental concepts in symmetry analysis of differential equations.

We use Lie group of type (25) later in a PDE case;

$$\Delta(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0. \tag{29}$$

It means that equation (29) transforms to itself, i.e., the new variables satisfies the same equation;

$$\Delta(\tilde{x}, \tilde{t}, \tilde{u}, \tilde{u_x}, \tilde{u_t}, \tilde{u_{xx}}, \tilde{u_{xt}}, \tilde{u_{tt}}) = 0,$$

and every solution $u = \Phi(x, t)$ maps on a solution to the same equation.

Other related concept in Lie theory is *orbit*. By orbit for a one-parameter Lie group of transformation through a point (x, t, u), we mean the amount of all points generated by (25).
The orbit for a point that does not map to itself, determines a curve and the vector $\vec{v} = (\xi, \tau, \phi)$ is tangent vector to the curve in the point (x, t, u). It means that the generator to a one-parameter Lie group of transformation can be considered as a vector field.
The vector $\vec{v} = (\xi, \tau, \phi)$, characterizes the group entirely (not only locally), since the global form i.e., (25), can be reconstruct from vector $\vec{v}$. This is the meaning of *Lie equations*.

Theorem 4.2.4. ***Lie equations:*** *The global form of (26) is obtained by determining the solution to the system*

$$\frac{d\tilde{x}}{d\varepsilon} = \xi(\tilde{x}, \tilde{t}, \tilde{u}),$$
$$\frac{d\tilde{t}}{d\varepsilon} = \tau(\tilde{x}, \tilde{t}, \tilde{u}),$$
$$\frac{d\tilde{u}}{d\varepsilon} = \phi(\tilde{x}, \tilde{t}, \tilde{u}),$$

which satisfies initial conditions $\tilde{x} \mid_{\varepsilon=0}= x$, $\tilde{t} \mid_{\varepsilon=0}= t$, $\tilde{u} \mid_{\varepsilon=0}= u$.

For the proof of this theorem we refer to [17].

Example 4.2.5. *Consider the ODE* $y = y(x)$*, and a Lie group of the type*

$$\tilde{x} = X(x, y, \varepsilon),$$
$$\tilde{y} = Y(x, y, \varepsilon),$$

and the rotation group

$$\tilde{x} = x \cos\varepsilon - y \sin\varepsilon,$$
$$\tilde{y} = x \sin\varepsilon + y \cos\varepsilon.$$

The infinitesimals are

$$\xi = -y, \quad \phi = x.$$

It follows that the infinitesimal generator becomes

$$V = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

The global form is obtained by solving the system

$$\frac{d\tilde{x}}{d\varepsilon} = -\tilde{y}, \quad \frac{d\tilde{y}}{d\varepsilon} = -\tilde{x},$$

with the initial conditions $\tilde{x}\mid_{\varepsilon=0}= x$, $\tilde{y}\mid_{\varepsilon=0}= y$.

■

4.2.4 Prolongation

In order to explain the concept of prolongation, we first study symmetry conditions in higher order ODE such as

$$y^{(n)} = f(x, y, y^{(1)}, ..., y^{(n-1)}), \tag{30}$$

where

$$y^{(k)} = \frac{d^k y}{dx^k}, \qquad k = 0, 1, ..., n.$$

The symmetries of (30) are diffeomorphism (See Definition 5.1.3) that map the family solution of this ODE to itself. Each diffeomorphism

$$\Gamma : (x, y) \longmapsto (\tilde{x}, \tilde{y}),$$

preserves the structure of the object and maps smooth plane curve to smooth plane curve.

A transformation of (x, y), which in fact is the action of Γ, induces a transformation of the derivatives, which is the following map

$$\Gamma : (x, y, y^{(1)}, ..., y^{(n)}) \longmapsto (\tilde{x}, \tilde{y}, \tilde{y}^{(1)}, ..., \tilde{y}^{(n)}).$$

This mapping is called *n-th prolongation* of Γ. We determine the function

$$\tilde{y}^{(k)} = \frac{d^k\tilde{y}}{d\tilde{x}^k},$$

by using the chain rule and calculating recursively, i.e.,

$$\tilde{y}^{(k)} = \frac{d\tilde{y}^{k-1}}{d\tilde{x}} = \frac{D_x\tilde{y}^{k-1}}{D_x\tilde{x}}, \tag{31}$$

where $\tilde{y}^{(0)} = \tilde{y}$.

Here D_x signifies the *total derivative* operator, with respect to x defined by

$$\begin{aligned} D_x &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y^{(1)}}\frac{dy^{(1)}}{dx} + ... \\ &= \partial_x + \partial_y y^{(1)} + \partial_{y^{(1)}} y^{(2)} + ... \end{aligned} \tag{32}$$

(See also Definition 5.1.25).
The following expression is the symmetry condition for ODE (30)

$$\tilde{y}^{(n)} = f(\tilde{x}, \tilde{y}, \tilde{y}^{(1)}, ..., \tilde{y}^{(n-1)}), \tag{33}$$

where $\tilde{y}^{(k)}$ is determined by (31).

For the most of DEs, symmetry condition given by (33) is non-linear. In order to obtain Lie symmetry, we linearise symmetry condition (33) by Taylor expansion around $\varepsilon = 0$, and obtain the following prolonged Lie symmetry

$$\begin{aligned} \tilde{x} &= x + \varepsilon\xi + o(\varepsilon^2), \\ \tilde{y} &= y + \varepsilon\phi + o(\varepsilon^2), \\ \tilde{y}^{(k)} &= y^{(k)} + \varepsilon\phi^{[k]} + o(\varepsilon^2), \end{aligned} \tag{34}$$

where $k \geqslant 1$.
Note the the superscript $\phi^{[k]}$, is only an index and it does not denote a derivative.
By substituting (34) into the symmetry condition (33), we obtain the linearised symmetry condition

$$\phi^{[n]} = \xi f_x + \phi f_y + \phi^{[1]} f_{y^{(1)}} + ... + \phi^{n-1} f_{y^{(n-1)}}. \tag{35}$$

We calculate functions $\phi^{[k]}$ recursively from (31) as follows. Consider $k = 1$, then

$$\tilde{y}^{(1)} = \frac{D_x \tilde{y}}{D_x \tilde{x}} = \frac{y^{(1)} + \varepsilon D_x \phi + o(\varepsilon^2)}{1 + \varepsilon D_x \xi + o(\varepsilon^2)} = y^{(1)} + \varepsilon(D_x\phi - y^{(1)} D_x \varepsilon) + o(\varepsilon^2).$$

By substituting (34) into this equation we obtain

$$\phi^{[1]} = D_x\phi - y^{(1)} D_x \xi. \tag{36}$$

Similarly

$$\tilde{y^{(k)}} = \frac{D_x \tilde{y}}{D_x x} = \frac{y^{(k)} + \varepsilon D_x \phi^{[k-1]} + o(\varepsilon^2)}{1 + \varepsilon D_x \xi + o(\varepsilon^2)},$$

and therefore

$$\phi^{[k]}(x, y, y^{(1)}, ..., y^{(k)}) = D_x \phi^{[k-1]} - y^{(k)} D_x \xi. \tag{37}$$

Now consider (35), the right hand side of this expression is Vf, where V is infinitesimal generator

$$V = \xi \partial_x + \phi \partial_y.$$

As we mentioned before, infinitesimal generator is the vector field with the tangent vectors ξ, ϕ, through the point (x, y). We need to deal with the action of Lie symmetries of the $n - th$ order differential equations, therefore we introduce

$$pr^{(n)}V = \xi\partial_x + \phi\partial_y + \phi^{[1]}\partial_{y^{(1)}} + ... + \phi^{[n]}\partial_{y^{(n)}}, \tag{38}$$

which is called *prolonged infinitesimal generator*.
Similarly we can determine prolonged infinitesimal generator for a PDE, that we will define exactly and study this subject in the next section.

5 Some Basics of Lie Groups

Lie groups were studied by the Norwegian mathematician Marius Sophus Lie (17 December 1842 - 18 February 1899), at the end of the 19th century. His interest was solving the equations. At that time mathematicians used to solve the equations by using a lot of tricks. A typical method to solve an equation was changing the variables on a way that one of the variable drops off the equation.

Lie groups lie between two fundamental fields of mathematics: algebra and geometry. We should know that first of all the Lie group is a group; secondly is a smooth manifold where this manifold is a special sort of geometric object. It can be said that Lie group is a group of continuous symmetries. Lie groups are present in mathematics and all area of science. Lie groups are associated to any system which has a continuous group of symmetries.

5.1 Lie Groups in PDE

In this section we introduce applying the Lie group analysis on non-linear PDEs such as (14), (16), (18), and study all invariant solutions. Then by using the symmetry group and its invariants, we can reduce a PDE to an ODE. In [2] and [12] they have studied symmetry group to equations (14) and (18). Here we will study the symmetry group and its invariant solution to the (16).

5.1.1 Lie Group Analysis of Differential Equations

The PDEs that will be studied here consist of two independent variables and one dependent variable.

In order to apply Lie symmetry group to the PDE, we start by introducing and defining some important concepts, where most of definitions are taken from [14].
X : a space of independent variables (S, t), i.e., $(S, t) \in X$ and X is isomorphic to space $\mathbb{R}^2$.
U : a space of one dependent variable $u \in U$ and U is isomorphic to $\mathbb{R}$.
$U_{(1)}$: a space that its coordinates are the first derivatives of u with respect to (w.r.t.) S and t, i.e.,

$$U_{(1)} = \left(\frac{\partial u}{\partial S}, \frac{\partial u}{\partial t} \right).$$

$U_{(2)}$: a space that its coordinates are the second derivatives of u w.r.t. S and t, i.e.,

$$U_{(2)} = \left(\frac{\partial^2 u}{\partial S^2}, \frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 u}{\partial S \partial t} \right).$$

(We can continue and introduce a space of type $U_{(n)}$, for $n > 2$, but because we study the second order equation, it is enough with $n = 2$.)

Definition 5.1.1. ***Cartesian product*** *is a direct product of two sets A and B with these properties*

- *not necessarily commutative $A \times B \neq B \times A$.*
- *not necessarily associative$(A \times B) \times C \neq A \times (B \times C)$.*
- *nice with intersection $(A \cap B) \times (C \cup D) = (A \cap C) \times (B \cup D)$.*
- *for union and intersection hold $A \times (B \cap C) = (A \times B) \cap (A \times C)$.*
- *$A \times (B \cup C) = (A \times B) \cup (A \times C)$.*

Definition 5.1.2. *A n-dimension **Manifold** M is a set that contains a number of countable subset U_α and some one-to-one functions F_α, which map the U_α to connected open subset of Euclidean space V_α, called local coordinate maps, which satisfy the following properties:*

I) The coordinates charts cover M, i.e.,

$$\textstyle\bigcup_\alpha U_\alpha = M.$$

II) For all $U_\alpha \bigcap U_\beta$

$$F_\beta \circ F_\alpha^{-1} : F_\alpha(U_\alpha \bigcap U_\beta) \to F_\beta(U_\alpha \bigcap U_\beta)$$

is a smooth function i.e., infinitely differentiable function.

III) If $x \in U_\alpha$ and $\hat{x} \in U_\beta$, then there exist two open subsets W and $\hat{W}$, such that $W \subset V_\alpha$, and $\hat{W} \subset V_\beta$, with $F_\alpha(x) \in W, F_\beta(\hat{x}) \in \hat{W}$ which satisfy

$$F_\beta^{-1}(W) \bigcap F_\beta^{-1}(\hat{W}) = \emptyset.$$

Definition 5.1.3. ***Diffeomorphism.*** *Let M and N be two manifolds. Let f be a bijective function from M to N. Then f is called a diffeomorphism function if both*

$$f : M \to N$$

and it's inverse

$$f^{-1} : N \to M$$

are smooth. If these functions are $k-$times continuously differentiable, the function f if called a C^k-diffeomorphism.

Definition 5.1.4. *A **Group** is a set G which has a binary operation (called multiplication) between the elements of the group. Furthermore the following axiom for arbitrary elements g, h and k, of the group must be satisfied by the group operation:*

I) *Closure: If g, h in G then the result of operation $g \cdot h$ is also in G.*

II) *Associativity: $h \cdot (g \cdot k) = (h \cdot g) \cdot k$.*

III) *Identity element: there is an identity element, e, in a group which has the following behaviour under group operation $e \cdot g = g \cdot e = g$.*

IV) *Inverse: For each element h in the group G there is an inverse h^{-1} which under the group multiplication returns the identity element e, i.e., $h \cdot h^{-1} = h^{-1} \cdot h = e$.*

Definition 5.1.5. *A **Lie group** is a group G which carries the structure of a smooth manifold that both the following group operations*

$$m : G \times G \longrightarrow G, \quad m(h, g) = h \circ g, \quad h, g \in G,$$

and the inversion

$$i : G \longrightarrow G \quad i(g) = g^{-1}, \quad g \in G.$$

are smooth maps between manifolds.

Lie groups are not often defined on the whole manifold, but they are defined only on some part of the manifold. Therefore we consider a so-called local Lie group.

Definition 5.1.6. *A **Local Lie group** consists of connected open subsets $V_0 \subset V \subset \mathbb{R}^r$ that contains origin and smooth map as the group operation*

$$m : V \times V \longrightarrow \mathbb{R}^r.$$

and a smooth map

$$i : V_0 \longrightarrow V,$$

as the group inversion, with the following properties

I) If $x, y, z \in V$ and $m(x, y), m(y, z) \in V$ then $m(x, m(y, z)) = m(m(x, y), z)$,

II) For all $x \in V \quad m(0, x) = x = m(x, 0)$,

III) For all $x \in V_0 \quad m(x, i(x)) = 0 = m(i(x), x)$.

Usually Lie groups arise as group of transformations on some manifold L. Lie groups will be represented here as a group of transformation of some manifold. It is not necessary for these transformation groups to be defined for all of the elements of the group or for all of the points of the manifold, i.e., it is enough to act locally.

Definition 5.1.7. ***Local group of transformation.*** *Let M be a smooth manifold. A local group of transformations acting on M is a (local) Lie group G, an open subset U, with the domain of the definition of the group action $\{e\} \times M \subset U \subset G \times M$, and a smooth function $\Gamma : U \longrightarrow M$ with the properties*

I) If $(h, x) \in U, (g, \Gamma(h, x)) \in U$, and besides $(g.h, x) \in U$, then $\Gamma(g, \Gamma(h, x)) = \Gamma(g.h, x)$,

II) For all $x \in L, \Gamma(e, x) = x$,

III) If $(g, x) \in U$, then $(g^{-1}, \Gamma(g, x)) \in U$ and $\Gamma(g^{-1}, \Gamma(g, x)) = x$.

Briefly, we can denote $\Gamma(g, x)$ by (g, x), where $g, h \in G$ and $x \in M$, so the conditions above of the definition take a simpler form

(i) $g \cdot (h \cdot x) = (g \cdot h) \cdot x$,

(ii) $(e.x) = x$,

(iii) $g^{-1} \cdot (g \cdot x) = x$.

In order to present the main tool of Lie group analysis of differential equations, called *infinitesimal generator*, we need at first to explain the concept of a vector field.

Definition 5.1.8. ***Tangent vector.*** *Let M be a manifold, and let I be a subinterval of $\mathbb{R}$. Suppose C is a smooth curve in M, which is parametrized by $\phi : I \to M$. The curve C, for a real value ϵ, is given by smooth functions $\phi(\epsilon) = (\phi^1(\epsilon), ..., \phi^m(\epsilon))$, in local coordinates $x = (x^1, ..., x^m)$. At each point $x = \phi(\epsilon)$ of C the curve has a* ***tangent vector,*** *that is to say the derivative $\dot{\phi}(\epsilon) = \dfrac{d\phi}{d\epsilon} = (\dot{\phi}^1(\epsilon), ..., \dot{\phi}^m(\epsilon))$. So for the tangent vector to curve C at $x = \phi(\epsilon)$, we denote*

$$V\mid_x=\dot{\phi}(\epsilon)=\dot{\phi}^1(\epsilon)\frac{\partial}{\partial x^1}+\dot{\phi}^2(\epsilon)\frac{\partial}{\partial x^2}+...+\dot{\phi}^m(\epsilon)\frac{\partial}{\partial x^m}.$$

Definition 5.1.9. ***Vector field.*** *Let U be an open subset on manifold M, and let x be a point on M. A vector field on $U\subset M$ is a family of tangent vectors X_x at each point x such that for every for every differentiable function f on an open subset $V\subset U$, the function $x\longmapsto X_x(f)$ is differentiable.*

Definition 5.1.10. *A **Tangent space** is the collection of all tangent vectors which pass through the given point x on a manifold M, is called tangent space to M at x. It is denoted by $TM\mid_x$.*

Definition 5.1.11. ***Infinitesimal generator.*** *Let V be a vector field on M. Then this vector field assigns a tangent vector $V\mid_x,\in TM\mid_x$ to every point $x\in M$. The tangent vector $V\mid_x$ in local coordinate has the following form*

$$V\mid_x=\xi^1(x)\frac{\partial}{\partial x^1}+\xi^2(x)\frac{\partial}{\partial x^2}+\xi^3(x)\frac{\partial}{\partial x^3},$$

*where it is called **Infinitesimal generator**.*

Here we denote an infinitesimal generator by

$$V=\xi(S,t,u)\frac{\partial}{\partial S}+\tau(S,t,u)\frac{\partial}{\partial t}+\phi(S,t,u)\frac{\partial}{\partial u},$$

where ξ,τ,ϕ are smooth functions.

Definition 5.1.12. ***Integral curve.*** *Let V a vector field. An integral curve of V is a smooth parametrized curve $x=\phi(\epsilon)$ whose tangent vector at any point coincide with the value of vector field V at the same point. In other words for all ϵ*

$$\dot{\phi}(\epsilon)=V\mid_{\phi(\epsilon)}.$$

Definition 5.1.13. ***Flows.*** *Let V be a vector field and M be a manifold. Let $\Gamma(\epsilon,x)$ be a notation of the parametrized maximal integral curve (by maximal means manifold does not contain any longer integral curve), passing through $x\in M$. We call Γ the flow generated by V. Thus for each $x\in M$, and ϵ in some interval $I_x\in\mathbb{R}$ which contains 0, $\Gamma(\epsilon,x)$ will be a point on the integral curve passing through $x\in M$. The flow of a vector field, for all x inM, and $\delta,\epsilon\in\mathbb{R}$, has the following basic properties*

- $\Gamma(\delta,\Gamma(\epsilon,x))=\Gamma(\delta+\epsilon,x)$,
- $\Gamma(0,x)=x$,
- $\frac{d}{d\epsilon}\Gamma(\epsilon,x)=V\mid_\Gamma(\epsilon,x)$.

Definition 5.1.14. ***Lie algebra*** *is a vector space V over a field F together with a binary operation*

$$[.,.] : V \times V \to V, \text{ which is called Lie bracket or Lie product,}$$

and with the operation

$$[X, Y] = XY - YX,$$

on the vector space V that satisfies the following axiom

a)Bi-linearity

$$[aX + bY, Z] = a[X, Z] + b[Y, Z], \quad [Z, aX + bY] = a[Z, X] + b[Z, Y],$$

where $a, b \in F$, and $X, Y, Z \in V$.

b)Alternating on V

$$[X, X] = 0.$$

c)Alternating and Bi-linearity imply Skew-symmetry

$$[X, Y] = -[Y, X].$$

d) Jacobian identity holds

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 \ .$$

Definition 5.1.15. ***Sub-algebra.*** *Let g be a Lie algebra, and let h be a vector space. Then $h \subseteq g$ is a sub-algebra if h is closed under Lie bracket operation, or h itself is a Lie algebra under the same bracket operation as Lie algebra g.*

Definition 5.1.16. *An* ***Ideal Lie algebra*** *g is a subspace h for which whenever $x \in h$ or $y \in h$ then $[x, y] \in h$.*

Definition 5.1.17. ***Solvable Lie algebra.*** *Let $L_j, j < \infty$, be a Lie algebra. It is said to be solvable if there is a sequence*

$$L_1 \subset L_2 \subset ... \subset L_{j-1} \subset L_j,$$

of sub-algebras of the dimension $j, j-1, ..., 1$ such that L_k is an ideal in L_{k+1} where $1 \leq k \leq j-1$.

Definition 5.1.18. *A* ***Jet bundle*** *is a space $M^{(n)} = M \times U \times U_{(1)} \times ... \times U_{(n)}$, which its coordinates represent independent and dependent variables and derivatives of dependent variables up to order n, are called the $n-th$ order jet bundle of the base space M.*

Now consider a system of differential equation which has p independent variables $x = (x^1, ..., x^p)$, and q dependent variables $u = (u^1, ..., u^q)$. The solution to the differential equation is denoted as $u^\alpha = f^\alpha(x^1, ..., x^p)$ where $\alpha = 1, ..., q$. We take the multi-index notation, defined below; by

$$\partial_J f(x) = \frac{\partial^k f(x)}{\partial x^{j_1}, ..., \partial x^{j_k}}.$$

Definition 5.1.19. ***Multi-index.*** *Let $J = (j_1, ..., j_k)$ be an unordered $k-$tuple of integers, with entries $1 \leqslant j_k \geqslant p$. We denote the order of such a multi index by $\#J =\mid J \mid$. It indicates how many derivative are being taken.*

Definition 5.1.20. ***Invariant function.*** *Let G be a local group of transformations acting on a manifold M, and N be another manifold. A function $F : M \to N$ is called a $G-$invariant function if for all $x \in M$ and for all $g \in G$ such that $g \cdot x$ is defined,*

$$F(g \cdot x) = F(x).$$

A real-valued $G-$invariant function $\Phi : M \to \mathbb{R}$ is called an invariant of G.

Another important concept that we define it below is symmetry group. It is a local group of transformation. It has a property that transforms a solution of a equation to another solution. By knowing the symmetry group we will be able to reduce the differential equation to a simpler model.

Definition 5.1.21. ***Symmetry group.*** *Let S be a system of differential equations, and let M be an open subset of the space of independent and dependent variables for the system. A symmetry group of the system S is a local group of transformations G acting on M withe the property that whenever $u = f(x)$ is a solution of the system S, and whenever for $g \in G, g \cdot f$ is defined, then $u = g.f(x)$ is a solution of the system S.*

By *solution* we mean any smooth solution $u = f(x)$, which is defined on any subset domain $\Omega \in X$.

Definition 5.1.22. ***Prolongation of differential equation.*** *Let $u = f(x)$ be a smooth function, so $f : X \to U, x \in X$. There is an induced function $u^{(n)} = pr^{(n)} f(x)$, called $n-th$ prolongation of the function f, which is defined by the equations*

$$u_J^\alpha = \partial_J f^\alpha(x).$$

Therefore $pr^{(n)} f(x)$ is a function from space X to space $U^{(n)}$.

A system of $n-$th order differential equations is given as the following system of equations

$$\Delta_{\upsilon}(x,u^{(n)})=0, \quad \upsilon=1,...,l.$$

with this assumption that all $\Delta(x,u^{(n)})=\big(\Delta_1(x,u^{(n)}),...,\Delta_l(x,u^{(n)})\big)$ are smooth functions in their arguments, therefore Δ can be viewed as the following smooth map

$$\Delta : X\times U^{(n)}\to \mathbb{R}^l,$$

where $X\times U^{(n)}$ is a jet space and $\mathbb{R}^l$ is a $l-$dimension Euclidean space.

Definition 5.1.23. ***Solution manifold.*** *The equality $\Delta=0$, determines the system of differential equations solution manifold L_Δ, (is called subvariety) which is defined by*

$$L_\Delta=\{(x,u^{(n)}):\Delta(x,u^{(n)})=0\}\subset X\times U^{(n)}, \tag{39}$$

of the jet space. It is called solution manifold.

We set $U^{(2)}=U_1\times U_2$ as a vector which coordinates are all derivatives up to order two of function $f(x)$.

Definition 5.1.24. ***Maximal rank.*** *Let*

$$\Delta_\nu(x,u^{(n)})=0, \quad \nu=1,...,l,$$

be a system of differential equations. Let p and q be the number of corresponding independent and dependent variables in the system. Then the system is said to be of maximal rank if the $l\times(p+qp^{(n)})$ Jacobian matrix

$$J_\Delta(x,u^{(n)})=\left(\frac{\partial\Delta_\nu}{\partial x^i},\frac{\partial\Delta_\nu}{\partial u^\alpha_J}\right),$$

of Δ with respect to all the variables $(x,u^{(n)})$ is of rank l when $\Delta(x,u^{(n)})=0$.

Theorem 5.1.1. ***Fundamental theorem of Lie group analysis.***
Let

$$\Delta_\nu(x,u^{(n)})=0 \quad , \quad \nu=1,...,l,$$

be a system of partial differential equations of maximal rank which is defined over a space which is a subset of Cartesian product $M\subset X\times U$. If G is a local group of transformations acting on M, and

$$pr^{(2)}V[\Delta_\nu(x,u^{(n)})]=0 \quad , \quad \nu=1,...,l \quad whenever \quad \Delta(x,u^{(n)})=0, \tag{40}$$

for every infinitesimal generator V of G, then G is a symmetry group admitted by the the system. The equation (40) is called ***determining equation.***

Definition 5.1.25. ***Total derivative.*** *The i-th total derivative of function $\varphi(x, u^{(n)})$ is a unique smooth function $D_i\varphi(x, u^{(n+1)})$, defined on M^{n+1}. This smooth function is obtained by the following formula*

$$D_i\varphi = \frac{\partial \varphi}{\partial x^i} + \sum_J u_{J,i}\frac{\partial \varphi}{\partial u_J},$$

where for $J = (j_1, j_2, ..., j_k)$, it is

$$u_{J,i} = \frac{\partial u_j}{\partial x^i} = \frac{\partial^{(k+1)} u}{\partial x^i, \partial x^{j_1} ... \partial x^{j_k}}.$$

Theorem 5.1.2. ***General Prolongation formula.***
Let

$$V = \sum_{i=1}^{n} \xi^i(x, u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi_\alpha(x, u)\frac{\partial}{\partial u^\alpha},$$

be a vector field defined on an open subset $M \subset X\times\ U$. Then n-the prolongation of the vector V is the vector field

$$pr^{(n)}V = V + \sum_{\alpha=1}^{p} \sum_J \varphi_\alpha^J(x, u^{(n)})\frac{\partial}{\partial u_J^\alpha},$$

defined on the jet space $M^{(n)} = X \times U^{(n)}$, the second summation being over all (unordered) multi-indices $J = (j_1, ..., j_k)$ with $1 \leq j_i \leq p,\quad l \leq k \leq n$. The coefficient functions φ_α^J of $pr^{(n)}V$ are given by the following formula

$$\varphi_\alpha^J(x, u^{(n)}) = D_J(\varphi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha) + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha \tag{41}$$

where

$$u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i} \qquad \text{and} \qquad u_{J,i}^\alpha = \frac{\partial u_J^\alpha}{\partial x^i}.$$

The idea behind prolongation procedure is introducing an equivalent relation for all the smooth functions involve in prolongation formula. Consider a point x that belongs to an open subset of X. If all the derivative of these smooth functions up to order n coincide in the point x, then we call the functions equivalent in the point. This point is called $n - jet$. In this way we prolonged our space.

Theorem 5.1.3. ***Symmetry Group.***
Let M be an open subset of $X\times U$. Suppose $\Delta(x, u^{(n)})$ is an n-th order system of a differential equation which is defined over space M, with its corresponding manifold $L_\Delta \subset M^{(n)}$, where $M^{(n)} \subset X\times U^{(n)}$. Suppose G is a local group

of transformations acting on M whose prolongation leaves L_Δ invariant, in the meaning that whenever $(x, u^{(n)}) \in L_\Delta$, we have $pr^{(n)}g.(x, u^{(n)}) \in L_\Delta$ for all $g \in G$. Then G is a symmetry group of the system of differential equations.

6 Lie Analysis on the obtained PDE from a Financial Market

In this section we shall discuss certain result in finance, using methods from PDE and Lie group. We applying the methods discussed before, on our equation (16).

6.1 Lie algebraic structure of the main equation

In other to show how we can solve equation (16), we consider a more general equation from [3], as follows

$$u_t + \frac{\sigma^2 S^2}{2} \frac{u_{SS}}{(1-\rho\lambda(S)Su_{SS})^2} = 0, \tag{42}$$

where $\lambda : (0,\infty) \to (0,\infty)$ is a continuous function , $\rho > 0$ is liquidity parameter (see Definition 2.0.34), and u is a smooth function of stock $S > 0$, (see Definition 2.0.23) and time $t \in (0,T]$, where T denotes maturity date (see Definition 2.0.27).
For $\lambda \equiv 1$ equation (42) becomes equation (16).

Claim 6.1.1. *The differential equation (42) for $\lambda(S)$ as an arbitrary function admits a three dimension Lie algebra spanned by the following infinitesimal generators*

$$V_1 = \frac{\partial}{\partial t}, \quad V_2 = \frac{\partial}{\partial u}, \quad V_3 = S\frac{\partial}{\partial u}.$$

If we have a special form of function $\lambda(S)$ such as $\lambda(S) \equiv \omega S^k$, where $\omega, k \in \mathbb{R}$, then equation (42) admits four dimension Lie algebra spanned by the following generators

$$V_1 = \frac{\partial}{\partial t}, \quad V_2 = S\frac{\partial}{\partial u}, \quad V_3 = S\frac{\partial}{\partial u} \quad V_4 = S\frac{\partial}{\partial S} + (1-k)u\frac{\partial}{\partial u}.$$

Proof: Consider $\lambda(S)$ as an arbitrary function in PDE (42).

We want to study the Lie group analysis admitted by equation

$$\Delta(S,t,u,u_S,u_t,u_{SS},u_{St},u_{tt}) = u_t + \frac{\sigma^2 S^2}{2} \frac{u_{SS}}{(1-\rho\lambda(S)Su_{SS})^2} = 0. \tag{43}$$

Therefore we should find infinitesimal generator of the corresponding Lie algebra. Consider the infinitesimal operator

$$V = \xi(S,t,u)\frac{\partial}{\partial S} + \tau(S,t,u)\frac{\partial}{\partial t} + \phi(S,t,u)\frac{\partial}{\partial u}. \tag{44}$$

According to the Theorem 5.1.2, in order to find the coefficient ξ, τ, ϕ, we should calculate the prolongation of the infinitesimal generator (44), since in our case, PDE (42) is second order, hence we determine second prolongation $pr^{(2)}V$, as follows

$$\begin{aligned} pr^{(2)}V &= \xi(S,t,u)\frac{\partial}{\partial S} + \tau(S,t,u)\frac{\partial}{\partial t} + \phi(S,t,u)\frac{\partial}{\partial u} \\ &+ \phi^S(S,t,u)\frac{\partial}{\partial u_S} + \phi^t(S,t,u)\frac{\partial}{\partial u_t} \\ &+ \phi^{SS}(S,t,u)\frac{\partial}{\partial u_{SS}} + \phi^{St}(S,t,u)\frac{\partial}{\partial u_{St}} + \phi^{tt}(S,t,u)\frac{\partial}{\partial u_{tt}}, \end{aligned} \tag{45}$$

where the coefficients $\phi^S, \phi^t, \phi^{SS}, \phi^{St}, \phi^{tt}$ according to the formula (41), are given by

$$\begin{aligned} \phi^S &= D_S(\phi - \xi u_S - \tau u_t) + \xi u_{SS} + \tau u_{tS}, \\ \phi^t &= D_t(\phi - \xi u_S - \tau u_t) + \xi u_{St} + \tau u_t, \\ \phi^{SS} &= D_S^2(\phi - \xi u_S - \tau u_t) + \xi u_{SSS} + \tau u_{SSt}, \\ \phi^{St} &= D_S D_t(\phi - \xi u_S - \tau u_t) + \xi u_{SSt} + \tau u_{SSt}, \\ \phi^{tt} &= D_t^2(\phi - \xi u_S - \tau u_u) + \xi u_{Stt} + \tau u_{ttt}. \end{aligned} \tag{46}$$

Because in our case the PDE dose not depend on u_{St}, u_{tt}, t, u, so we skip the terms $\phi^{St}\frac{\partial}{\partial u_{St}}, \phi^{tt}\frac{\partial}{\partial u_{tt}}, \tau\frac{\partial}{\partial t}, \phi\frac{\partial}{\partial u}$, so we consider the following form of prolongation

$$pr^{(2)}V = \xi\frac{\partial}{\partial S} + \phi^t\frac{\partial}{\partial u_t} + \phi^{SS}\frac{\partial}{\partial u_{SS}}, \tag{47}$$

where the coefficient from corresponding expression in (46) are

$$\phi^t(S,t,u) = \phi_t + u_t\phi_u - u_S\xi_t - u_S u_t \xi_u - u_t\tau_t - (u_t)^2\tau_u, \tag{48}$$

and

$$\begin{aligned} \phi^{SS}(S,t,u) &= \phi_{SS} + 2u_S\phi_{Su} + u_{SS}\phi_u \\ &+ (u_S)^2\phi_{uu} - 2u_{SS}\xi_S - u_S\xi_{SS} - 2(u_S)^2\xi_{Su} \\ &- 3u_S u_{SS}\xi_u - (u_S)^3\xi_{uu} - 2u_{St}\tau_{SS} \\ &- 2u_S u_t\tau_{Su} - (u_t u_{SS} + 2u_S u_{St})\tau_u - (u_S)^2 u_t\tau_{uu}, \end{aligned} \tag{49}$$

and partial derivative of PDE (42), in corresponding prolonged formula (47) are

$$\begin{aligned}
\frac{\partial \Delta}{\partial S} &= \sigma^2 S \frac{u_{SS}}{(1-\rho\lambda(S)Su_{SS})^2} + \frac{\sigma^2 S^2}{4}\frac{u_{SS}}{\rho u_{SS}(\lambda'(S)S-\lambda(S))(-1+2\rho\lambda(S)Su_{SS})},\\
\frac{\partial \Delta}{\partial u_t} &= 1,\\
\frac{\partial \Delta}{\partial u_{SS}} &= \frac{\sigma^2 S^2}{2}\frac{\big(1-\rho\lambda(S)Su_{SS})^2 - u_{SS}(2\rho\lambda(S)S(-1+2\rho\lambda(S)Su_{SS})\big)}{(1-\rho\lambda(S)Su_{SS})^4}. \qquad (50)
\end{aligned}$$

In order to find functions $\xi(S,t,u), \tau(S,t,u), \phi(S,t,u)$, we use determining equation (40). This equation arises from the action of the prolonged formula $pr^{(2)}V$ on the equation $\Delta = 0$.
In our case the determining equation takes the following form (see Theorem 5.1.1)

$$\begin{aligned}
pr^{(2)}V(\Delta) &= \xi\big(\sigma^2 S \frac{u_{SS}}{(1-\rho\lambda(S)Su_{SS})^2}\\
&+ \frac{\sigma^2 S^2}{4}\frac{u_{SS}}{\rho u_{SS}(\lambda'(S)S-\lambda(S))(-1+2\rho\lambda(S)Su_{SS})}\big)\\
&+ \phi^t\\
&+ \phi^{SS}\big(\frac{\sigma^2 S^2}{2}\frac{\big(1-\rho\lambda(S)Su_{SS})^2 - u_{SS}(2\rho\lambda(S)S(-1+2\rho\lambda(S)Su_{SS})\big)}{(1-\rho\lambda(S)Su_{SS})^4}\big)\\
&= 0. \qquad (51)
\end{aligned}$$

By inserting all of the partial derivatives (50), the coefficients (48) and (49) in equation (51), we obtain the following expression

$$\begin{aligned}
pr^{(2)}V(\Delta)\mid_{\Delta=0} &= \xi\big(\sigma^2 S \frac{u_{SS}}{(1-\rho\lambda(S)Su_{SS})^2}\\
&+ \frac{\sigma^2 S^2}{4}\frac{u_{SS}}{\rho u_{SS}(\lambda'(S)S-\lambda(S))(-1+2\rho\lambda(S)Su_{SS})}\big)\\
&+ \phi_t + u_t\phi_u - u_S\xi_t - u_S u_t\xi_u - u_t\tau_t - (u_t)^2\tau_u\\
&+ \phi_{SS} + 2u_S\phi_{Su} + u_{SS}\phi_u\\
&+ (u_S)^2\phi_{uu} - 2u_{SS}\xi_S - u_S\xi_{SS} - 2(u_S)^2\xi_{Su}\\
&- 3u_S u_{SS}\xi_u - (u_S)^3\xi_{uu} - 2u_{St}\tau_{SS}\\
&- 2u_S u_t\tau_{Su} - (u_t u_{SS} + 2u_S u_{St})\tau_u - (u_S)^2 u_t\tau_{uu}\\
&\big(\frac{\sigma^2 S^2}{2}\frac{\big(1-\rho\lambda(S)Su_{SS})^2 - u_{SS}(2\rho\lambda(S)S(-1+2\rho\lambda(S)Su_{SS})\big)}{(1-\rho\lambda(S)Su_{SS})^4}\big)\\
&= 0. \qquad (52)
\end{aligned}$$

Substituting $u_t = -\frac{\sigma^2 S^2}{2}\frac{u_{SS}}{(1-\rho\lambda(S)Su_{SS})^2}$ in equation (52), makes sure that we remain on solution manifold (Definition 5.1.23) of PDE (42).

After this substitution and some transformation we should solve the system of equations $pr^{(2)}V(\Delta)\mid_{\Delta=0}\equiv 0$, by assuming that $S, t, u, u_S, ...$ all are independent variables in $M^{(2)}$. The next step is demanding that all coefficients of every monomial which contains all derivatives $u_S, u_{SS}, u_t...$, must be equal to zero. For similar detailed calculations see [2], [12].
After some calculations and relative integrations, we get a set of equations:

$$\xi(S,t,u) = a_1 S, \quad \tau(S,t,u) = a_2, \quad \phi(S,t,u) = a_3 S + a_4 + a_5 u, \tag{53}$$

where $a_1, ..., a_5$ are arbitrary constants and ξ, τ, ϕ are the same as in equation (44).

According to [3], we obtain

$$a_1 S \lambda_s(S) - (a_1 - a_5)\lambda(S) = 0. \tag{54}$$

By satisfying this equation for all S, we obtain $a_1 = a_5 = 0$ which implies that

$$\begin{aligned} \xi(S,t,u) &= 0, \\ \tau(S,t,u) &= a_2, \\ \phi(S,t,u) &= a_3 S + a_4. \end{aligned} \tag{55}$$

In the end, by considering these equations and substituting them in expression (44), Lie algebra admits the following infinitesimal generators

$$V_1 = \frac{\partial}{\partial t}, \quad V_2 = S\frac{\partial}{\partial u}, \quad V_3 = \frac{\partial}{\partial u}, \tag{56}$$

and the relation $[V_1, V_2] = [V_1, V_3] = [V_2, V_3] = 0$,
(i.e., commutator relation $[V_1, V_2] = V_1 \cdot V_2 - V_2 \cdot V_1 = \frac{\partial}{\partial t} \cdot S\frac{\partial}{\partial t} - S\frac{\partial}{\partial t} \cdot \frac{\partial}{\partial t} = 0$).

Now consider $\lambda(S) = \omega S^k, \quad \omega, k \in \mathbb{R}$, in PDE (42).
By the same calculation as we did for the case $\lambda(S)$ as an arbitrary function, here we obtain

$$\begin{aligned} \xi(S,t,u) &= a_1 S, \\ \tau(S,t,u) &= a_2, \\ \phi(S,t,u) &= (1-k)a_1 u + a_3 S + a_4. \end{aligned} \tag{57}$$

Therefore the generators in this case are

$$V_1 = \frac{\partial}{\partial t}, \quad V_2 = S\frac{\partial}{\partial u}, \quad V_3 = \frac{\partial}{\partial u} \quad V_4 = S\frac{\partial}{\partial S} + (1-k)u\frac{\partial}{\partial u}, \tag{58}$$

and moreover the commutator relation, are as follow

$$[V_1,V_2]=[V_1,V_3]=[V_1,V_4]=[V_2,V_3]=0,$$
$$[V_2,V_4]=-kV_2,\quad [V_3,V_4]=(1-k)V_3.$$

■

6.2 The symmetry group admitted by the main equation

In this part we will show that how we can find the symmetry group and we will use the symmetry group to create the invariant solution to the main PDE (42).

Claim 6.2.1. *The action of the symmetry group of equation (42), for $\lambda(S)$ as an arbitrary function is given by*

$$\begin{aligned} S_\epsilon &= S, \\ t_\epsilon &= t + a_1 \\ u_\epsilon &= u + Sa_2 + a_3, \end{aligned} \tag{59}$$

where a_1, a_2, a_3 are arbitrary constants.
For the case $\lambda(S)=\omega S^k$, where $\omega, k \in \mathbb{R}$, equation (42) becomes the following special form

$$u_t + \frac{\sigma^2 S^2}{2}\frac{u_{SS}}{1-\omega\rho S^{k+1}u_{SS}} = 0. \tag{60}$$

Hence the symmetry group corresponding to this form of PDE has the following structure

$$\begin{aligned} S_\epsilon &= Se^{a_1\epsilon}, \\ t_\epsilon &= t + a_2\epsilon, \\ u_\epsilon &= u + \frac{a_3}{a_1}S(e^{a_1\epsilon}-1) + a_4\epsilon, \quad k=1, \\ u_\epsilon &= ue^{a_1\epsilon} + a_3 S\epsilon e^{a_1}\epsilon + \frac{a_4}{a_1}(e^{a_1\epsilon}-1) \quad k=0, \\ u_\epsilon &= ue^{a_1(1-k)\epsilon} + \frac{a_3}{a_1 k}S\epsilon e^{a_1\epsilon}(1-e^{-a_1 k\epsilon}) \\ &+ \frac{a_4}{a_1(1-k)}(e^{a_1(1-k)\epsilon}-1), \quad k\neq 0, k\neq 1. \end{aligned} \tag{61}$$

with this assumption that $a_1 \neq 0$.

***Proof*:** In order to show how we can obtain symmetry groups (59) and (61), we consider different types for $\lambda(S)$, as follows

- $\lambda(S)$ as an arbitrary function.

- $\lambda(S) = \omega S^k$,

where the last type divides to three cases

- $k = 1 \Rightarrow \lambda(S) = \omega S$.
- $k \in \mathbb{R} \Rightarrow \lambda(S) = \omega S^k$.
- $k = 0 \Rightarrow \lambda(S) = \omega$.

We study the mentioned types separately here below.

- If $\lambda(S)$ is an arbitrary function in PDE (42)

According to Claim 6.1.1 we have obtained the corresponding Lie algebra (56) for the main PDE. According to the fundamental theorem of the Lie group analysis, i.e., Theorem 5.1.1, in order to find the global representation of the symmetry group admitted by the main PDE (42), we have to solve the following system of ordinary differential equations with the corresponding initial values

$$\begin{aligned}\frac{dS_\epsilon}{d\epsilon} &= 0 \quad S_\epsilon \mid_{\epsilon=0} = S,\\ \frac{dt_\epsilon}{d\epsilon} &= a_2, \quad t_\epsilon \mid_{\epsilon=0} = t,\\ \frac{du_\epsilon}{d\epsilon} &= a_3 S + a_4, \quad u_\epsilon \mid_{\epsilon=0} = u.\end{aligned} \tag{62}$$

We will consider the following cases of the system (62)

1. $a_2 \neq 0, \quad a_3 \neq 0, \quad a_4 \neq 0,$
2. $a_2 = 0, \quad a_3 \neq 0, \quad a_4 \neq 0,$
3. $a_2 \neq 0, \quad a_3 = 0, \quad a_4 \neq 0,$
4. $a_2 \neq 0, \quad a_3 \neq 0, \quad a_4 = 0.$

Now our aim is to study the results of these different cases in the system (62), their corresponding subgroup and their invariants. Then by using the obtained invariants variables we can reduce PDE (42) to corresponding ordinary differential equations.

1. $a_2 \neq 0, \quad a_3 \neq 0, \quad a_4 \neq 0$, in ODE system (62).

Here we study the symmetry group admitted by equation (43).
By solving the first equation in system (62), we obtain

$$S_\epsilon = C_1$$

where C_1 is an arbitrary constant.
The corresponding initial condition will define the constant value as $C_1 = S$. Therefore the transformation of the variable S under the action of the symmetry group has the following form

$$S_\epsilon = S. \tag{63}$$

We solve the second equation in system (62)

$$t_\epsilon = a_2\epsilon + C_2,$$

By inserting the corresponding initial condition, the value of the arbitrary constant C_2 becomes as follows

$$\begin{aligned} a_2\epsilon + C_2 \mid_{\epsilon=0} &= t \\ C &= t \end{aligned}$$

Thus the transformation of the variable t under the action of the corresponding symmetry group has the following form

$$t_\epsilon = a_2\epsilon + t. \tag{64}$$

The third equation of the system (62) defines the transformation of u which is a dependent variable, under the action of the symmetry group.
By solving this equation we get

$$u_\epsilon = a_3 S\epsilon + a_4\epsilon + C_3,$$

We insert the corresponding initial value in order to find the value of the constant C_3, i.e.,

$$(a_3 S\epsilon + a_4\epsilon + C_3) \mid_{\epsilon=0} = u$$
$$C_3 = u.$$

It follows that the transformation of the variable u under the action of the symmetry group has the following form

$$u_\epsilon = a_3 S\epsilon + a_4\epsilon + u. \tag{65}$$

Thus by calculations above, we obtained symmetry groups (63), (64), (65), hence we can identify the invariance of these symmetry groups.

In general any function of invariant is an invariant, therefore the invariant is not unique and we represent one of the possible invariants of symmetry group.

In order to construct the invariant solution, we define invariant variables by excluding ϵ from the symmetry groups (63), (64), (65) i.e., from equation (64) we obtain

$$\epsilon = \frac{t_\epsilon - t}{a_2}. \tag{66}$$

By substituting this in equation (65), we obtain

$$u_\epsilon = a_3 S(\frac{t_\epsilon - t}{a_2}) + a_4(\frac{t_\epsilon - t}{a_2}) + u,$$

which is equivalent with

$$u_\epsilon - a_3 S t_\epsilon - a_4 t_\epsilon = -a_3 S t - a_4 t + a_2 u. \tag{67}$$

In this case, by considering equations (63) and (67), we can obtain the following invariants

$$\begin{aligned} inv_1 &= S, \\ inv_2 &= u - \frac{(a_3 S\epsilon + a_4\epsilon)t}{a_2}. \end{aligned} \tag{68}$$

These two invariant variables are unacceptable. Because they do not lead us to any reduced ODE of PDE (42), in the case $\lambda(S)$ as an arbitrary function.

For the Cases 2, 3 and 4, i.e., $\lambda(S)$ as an arbitrary function in PDE (42) and considering different cases for constants a_2, a_3, a_4, in system (62); after some calculation with the same way as we did in the case 1, we will obtain the same result i.e., unacceptable invariant variables, therefore in the case $\lambda(S)$ as an arbitrary function, we should solve PDE equation with other methods, and we do not discuss about them in this thesis.

- Case $\lambda(S) = \omega S^k$, in PDE (42)

We have found equations ξ, τ, ϕ as in (57) and the corresponding Lie algebra (58). In order to find the global representation of the symmetry group according to Theorem 5.1.1, we solve the following system of ordinary differential equations with corresponding initial value

$$\begin{aligned} \frac{dS_\epsilon}{d\epsilon} &= a_1 S_\epsilon, \qquad S_\epsilon \mid_{\epsilon=0} = S, \\ \frac{dt_\epsilon}{d\epsilon} &= a_2, \qquad t_\epsilon \mid_{\epsilon=0} = t, \\ \frac{du_\epsilon}{d\epsilon} &= (1-k)a_1 u_\epsilon + a_3 S_\epsilon + a_4, \quad u_\epsilon \mid_{\epsilon=0} = u. \end{aligned} \tag{69}$$

The result of solving the first equation in system (69) is

$$S_\epsilon = C_1 e^{a_1 \epsilon},$$

where C_1 is an arbitrary constant.
By inserting the corresponding initial value we obtain the constant value $C_1 = S$, thus the transformation for variable S under action of the symmetry group becomes

$$S_\epsilon = S e^{a_1 \epsilon}. \tag{70}$$

We solve the second equation in ODE system (69) and we obtain

$$t_\epsilon = a_2 \epsilon + C_2,$$

where C_2 is an arbitrary constant.
The given initial value $t_\epsilon \mid_{\epsilon=0} = t$, determines the constant value $C_2 = t$, thus the transformation for the variable t under action of the symmetry group becomes

$$t_\epsilon = a_2 \epsilon + t. \tag{71}$$

In order to solve the third equation of system (69) and get transformation for the variable u, we should consider three different cases

I) $k = 1$.

II) $k = 0$.

III) $k \neq 0$ and $k \neq 1$.

Case I) $k = 1$ in the third equation in ODE system (69) obtained by putting $\lambda(S) = \omega S^k, k \in \mathbb{R}$, in PDE (42):

For $k = 1$ in the third equation of system (69) becomes

$$\frac{du_\epsilon}{d\epsilon} = a_3 S_\epsilon + a_4, \tag{72}$$

and after solving this equation we obtain

$$u_\epsilon = a_3 S_\epsilon \epsilon + a_4 \epsilon + C, \tag{73}$$

where C is an arbitrary constant.

Now we use the method of variation of the parameters and represent constant C as a function of ϵ, i.e., $C = C(\epsilon)$.
By inserting $C(\epsilon)$ and $S_\epsilon = S e^{a_1 \epsilon}$, in equation (73) and satisfying equation (72), we obtain

$$C'(\epsilon) = -(a_1 a_3 \epsilon S) e^{a_1 \epsilon},$$

and by integrating this equation with respect to ϵ, we obtain the following expression for $C(\epsilon)$

$$C(\epsilon) = -e^{a_1\epsilon}(a_3\epsilon S) + \frac{a_3}{a_1}Se^{a_1\epsilon} + D,$$

where D is an arbitrary constant.
Thus the expression for u_ϵ takes the following form

$$u_\epsilon = a_3 S_\epsilon \epsilon + a_4\epsilon - e^{a_1\epsilon}(a_3\epsilon S) + \frac{a_3}{a_1}Se^{a_1\epsilon} + D.$$

By inserting the initial value $u_\epsilon \mid_{\epsilon=0} = u$, we obtain the value of the constant D as

$$D = u - \frac{a_3}{a_1}S.$$

Finally we obtain the transformation of the dependent variable u under the action of the symmetry group

$$u_\epsilon \quad = \quad u + \frac{a_3}{a_1}S(e^{a_1\epsilon} - 1) + a_4\epsilon, \quad k = 1. \tag{74}$$

For the cases II and III, i.e., for different $k-$value in ODE system (69); the transformation of variables S and t are the same as (70) and (71). In order to find the corresponding transformation of u, we use the same method as in case I and obtain

$$\begin{aligned} u_\epsilon \quad &= \quad ue^{a_1(1-k)\epsilon} + \frac{a_3}{a_1 k}S\epsilon e^{a_1\epsilon}(1 - e^{-a_1 k\epsilon}) \\ &+ \quad \frac{a_4}{a_1(1-k)}(e^{a_1(1-k)\epsilon} - 1), \quad k \neq 0, k \neq 1. \end{aligned} \tag{75}$$

and

$$u_\epsilon \quad = \quad ue^{a_1\epsilon} + a_3 S\epsilon e^{a_1}\epsilon + \frac{a_4}{a_1}(e^{a_1\epsilon} - 1), \quad k = 0. \tag{76}$$

The next step is to construct invariant variables. Since invariants are not unique hence we can obtain the following invariants variables in the case $\lambda(S) = \omega S^k, \quad k \in \mathbb{R}$. From equation (71) we can find that

$$\epsilon = \frac{t_\epsilon - t}{a_2}. \tag{77}$$

By substitution this expression in equation (70), we obtain

$$S_\epsilon = Se^{a_1(\frac{t_\epsilon - t}{a_2})}$$

which implies

$$S_\epsilon e^{-\frac{a_1}{a_2}t_\epsilon} = Se^{-\frac{a_1}{a_2}t}.$$

This expression is remaining in an original state (unaltered) under the action of the symmetry group, that means this expression is an invariant of the symmetry group. In another word,

$$inv_1 = lnS - \frac{a_1}{a_2}t. \tag{78}$$

and according to [13], we get following expression for inv_2

$$inv_2 = uS^{(k-1)} \tag{79}$$

Now we aim, with help of the invariant variables (78) and (79), to reduce the number of the independent variables in PDE from two to one which implies an ordinary differential equation which depends on one variable. The solutions of this ODE are the group of invariant solutions of the main PDE. We should consider different cases for $\lambda(S)$.

1. $\lambda(S) = \omega S^k, \quad \omega, k \in \mathbb{R}, \quad k \neq 1, k \neq 0,$ in PDE (42)

We start with studying the original PDE (42), for $\lambda(S) = \omega S^k, k \in \mathbb{R}$, that takes the following form

$$u_t + \frac{\sigma^2 S^2}{2} \frac{u_{SS}}{(1 - \rho\omega S^{k+1} u_{SS})^2} = 0, \tag{80}$$

where $\omega \neq 0$, $\rho \in (0, 1)$, $S > 0$.
Let us introduce two new variables, one independent say z

$$z = inv_1 = lnS + at, \quad a = \frac{-a_1}{a_2}, a_2 \neq 0, \tag{81}$$

and one dependent variable $v(z)$,

$$v(z) = inv_2 = uS^{(k-1)}. \tag{82}$$

Thus representation of the dependent variable u by using (z, v) turns to the form

$$u = vS^{(1-k)}.$$

The presentation of partial derivatives of u are

$$\begin{aligned}
u_t &= S^{1-k} v_z a, \\
u_S &= S^{-k}(((1-k)v + S v_S), \\
u_{SS} &= -k S^{-k-1}((1-k)v + S v_S) + S^{-k}((1-k)v_S + v_S + S v_{SS}),
\end{aligned} \tag{83}$$

Because v is a function of z, we obtain

$$\begin{aligned}
v_S &= v_z z_S, \\
v_{SS} &= v_{zz}\frac{1}{S} - v_z \frac{1}{S^2},
\end{aligned}$$

by substitution these expressions in (83), we obtain new expression which by rewriting PDE (80) with them, it turns to have a reduced PDE to a second order ODE with the form

$$a v_z + \frac{\sigma^2}{2}\frac{v_{zz} + (1-2k)v_z - k(1-k)v}{(1 - b(v_{zz} + (1-2k)v_z - k(1-k)v))^2} = 0, \tag{84}$$

where $a, b \neq 0$.

We should always consider that the dominator is not equal to zero i.e.,

$$(1 - b(v_{zz} + (1-2k)v_z - k(1-k)v))^2 \neq 0.$$

We can obtain one of the solutions to the second order ODE (84) simply if we assume that $v =$ constant or $v_z = 0$.
This equation has trivial solution $v = 0$ for any k and $v_z = 0$ for $k \neq 1, k \neq 0$. Another solution is $v =$ non-zero constant and $v_z =$ non-zero constant, for $k = 0$, $k = 1$.

Let us introduce another set of solutions to second order ODE (84). Since all of the coefficient are constant, hence we can reduce the order of this ODE from two to one by a suitable substitution. Therefore we introduce a new dependent variable

$$x(v) = v_z(z). \tag{85}$$

Now we express v_{zz} as follows

$$v_{zz} = \frac{d}{dz}(v_z(z)) = \frac{d}{dz}(x(v)) = \frac{d}{dv}(x(v))\frac{dv}{dz} = (x(v))_v x(v)) = x_v x.$$

By substitution this in equation (84) we obtain a first order ODE as a reduction of PDE (80)

$$a x + \frac{\sigma^2}{2}\frac{x x_v + (1-2k)x - k(1-k)v}{(1 - \rho\omega(x x_v + (1-2k)x - k(1-k)v))^2} = 0,$$

where its solutions are

$$\begin{aligned} x &= 0, \quad k=0, k=1, \\ x_v &= -1+2k-\frac{\sigma^2}{4a(\rho\omega)^2x^2}+\frac{1}{(\rho\omega)x}+\frac{k(1-k)v}{x}-\frac{\sqrt{\sigma^2(\sigma^2-8a\rho\omega x)}}{4a(\rho\omega)^2x^2}. \end{aligned} \tag{86}$$

where in the last expression $x, a, \rho, \omega \neq 0$.
These equations are easier to solve compared with the main PDE (42). The solutions of these equations are the invariant solutions to the main PDE. This work is excluded from this thesis.

■

2. $\lambda(S) = \omega S$, $(k=1)$, $\omega \in \mathbb{R}$, in PDE (42)

As a shortcut, we put $k=1$ in equation (84), then divide the whole expression by $a \neq 0$, then we obtain the following second order ODE

$$v_z + \frac{\sigma^2}{2a}\frac{v_{zz}-v_z}{(1-\rho\omega(v_{zz}-v_z))^2} = 0, \tag{87}$$

where the dominator is not equal to zero, i.e.,

$$v(z) \neq -\frac{z}{\rho\omega} + c_1 e^z + c_2,$$

where c_1, c_2 are arbitrary constant.

We can reduce the order of ODE (87) from two to one, similarly to the previous section by a suitable variable substitution as (85).
One of the solutions to equation (87) is found by a simple method. Suppose that v_z =constant which implies $v_{zz} = 0$, by inserting them in equation (87) we obtain a second order polynomial which has the roots

$$v_z = \frac{1}{\rho\omega}\left(-1 \pm \sqrt{\frac{\sigma^2}{2a}}\right), \quad b \neq 0. \tag{88}$$

Consequently we can present one of the solution $u(S,t)$ to PDE (80) for $k=1$ on the following way: by integrating equation (87) we obtain

$$v(z) = \frac{1}{\rho\omega}\left(-1 \pm \sqrt{\frac{\sigma^2}{2a}}\right) z + c, \tag{89}$$

where c is an arbitrary constant. And for $k = 1$ in (82) we obtain $v(z) = u$, by a suitable substitution it follows that

$$u(S,t) = \frac{1}{\rho\omega}\left(-1 \pm \sqrt{\frac{\sigma^2}{2a}}\right)(logS + at) + c. \tag{90}$$

In order to find the set of solutions to the second order ODE (87), we introduce a new dependent variable

$$y(z) = v_z(z), \tag{91}$$

by substitution them in equation (87) we obtain the following fist order ODE

$$yy_z^2 - 2\left(y^2 + \frac{1}{\rho\omega}y - \frac{(\sigma^2/2a^2)}{2(\rho\omega)^2}\right)y_z + \left(y^2 + \frac{2}{\rho\omega}y + \left(\frac{1-(\sigma^2/2a^2)}{(\rho\omega)^2}\right)\right)y = 0, \tag{92}$$

where $\rho\omega \neq 0$.

The set of solutions to this ODE are the following equations

$$\begin{aligned}
y &= 0,\\
y &= (-1 \pm \sqrt{(\sigma^2/2a)})/\rho\omega, \quad \rho\omega \neq 0,\\
y_z &= \left(y^2 + \frac{1}{\rho\omega}y - \frac{(\sigma^2/2a)}{2(\rho\omega)^2} - \sqrt{\frac{\sigma^2}{2a(\rho\omega)^3}}\left(\frac{\sigma^2/2a}{4\rho\omega} - y\right)\right)\frac{1}{y},\\
y_z &= \left(y^2 + \frac{1}{\rho\omega}y - \frac{(\sigma^2/2a)}{2(\rho\omega)^2} + \sqrt{\frac{\sigma^2}{2a(\rho\omega)^3}}\left(\frac{\sigma^2/2a}{4\rho\omega} - y\right)\right)\frac{1}{y}, \qquad (93)
\end{aligned}$$

where in the two last equations $y \neq 0$.

Consider the two last equations of the set of solutions (93). We put the right hand side of these equations equal to $f(y)$, i.e., $y_z = f(y)$. If in all points of domain, $\frac{\partial f}{\partial y}$ exists and is bounded then the Lipschitz condition is satisfied. Therefore we can have the unique solution for the last two equations in the set (93). As we see the Lipschitz is satisfied everywhere except the lines

$$y = 0, \quad y = \frac{\sigma^2}{8a\rho\omega}, \quad y = \infty. \tag{94}$$

Studying the behaviour of the solution in the neighbourhood of lines (94) is achieved in the complex plane which is excluded form this thesis. For more study in this subject we refer to [3].
All listed equations in (93) are more simple to solve compared with the main

PDE (42). The solutions of these equations give rise to the invariant solutions of the PDE (42). But this project is not included in this thesis.

■

3. $\lambda(S) = \omega$, $(k = 0)$, $\omega \in \mathbb{R}$, in PDE (42)

In this case for $\omega = 1$, we will obtain the PDE (16), then according to the explanations below, we can get the reduced form of this PDE to an ODE and its corresponding solutions.
The methods are very similar to the case $k = 1$, thus we just mention the results of our calculation.
PDE (84) for $k = 0$ becomes a second order ODE on the form

$$v_z + \frac{\sigma^2}{2a}\frac{v_{zz} + v_z}{(1 - \rho\omega(v_{zz} + v_z))^2} = 0, \tag{95}$$

where $\rho\omega \neq 0, a \neq 0$.

In order to study set of solutions to PDE (42), we start by introducing a new dependent variable

$$g(z) = v_z(z). \tag{96}$$

By substituting (96) in equation (95) and assuming that the dominator in (95) is not equal to zero, we obtain the following first order ODE

$$gg_z^2 + 2\left(g^2 - \frac{1}{\rho\omega}g + \frac{(\sigma^2/2a^2)}{2(\rho\omega)^2}\right)g_z + \left(g^2 - \frac{2}{\rho\omega}g + \left(\frac{1 + (\sigma^2/2a^2)}{(\rho\omega)^2}\right)\right)g = 0, \tag{97}$$

where $\rho\omega \neq 0$.

The set of solution to equation (97) are

$$\begin{aligned}
g &= 0,\\
g &= (-1 \pm \sqrt{(\sigma^2/2a)}/\rho\omega, \qquad \rho\omega \neq 0,\\
g_z &= \left(-g^2 + \frac{1}{\rho\omega}g - \frac{(\sigma^2/2a)}{2(\rho\omega)^2} - \sqrt{\frac{\sigma^2}{2a(\rho\omega)^3}}\left(\frac{\sigma^2/2a}{4\rho\omega} - g\right)\right)\frac{1}{g},\\
g_z &= \left(-g^2 + \frac{1}{\rho\omega}g - \frac{(\sigma^2/2a)}{2(\rho\omega)^2} + \sqrt{\frac{\sigma^2}{2a(\rho\omega)^3}}\left(\frac{\sigma^2/2a}{4\rho\omega} - g\right)\right)\frac{1}{g}, \qquad (98)
\end{aligned}$$

where in two last equations $g \neq 0$.

The two last equations in the collection of solution (98) do not satisfy Lipschitz condition in

$$g = 0, \quad g = \frac{\sigma^2}{8a\rho\omega}, \quad g = \infty. \tag{99}$$

See [13], in order to study the behaviour of these equations in the neighbourhood of lines (99).
In order to obtain invariant solution to PDE (42) where $\lambda(S) = \omega$, we solve equations listed in (98) which can be solved much easier than PDE (42), but this work is out of the frame of this thesis.

■

7 Conclusions

In this thesis our aim has been to show how we can solve a PDE, in order to calculate the asset price related to one type of an illiquid market, called "Reduced-Form SDE models".

First we have introduced three formulas about pricing an asset, such as:
• Mathematical representation for generating asset price as the following expression

$$dS = \mu S dt + \sigma S dW,$$

where S, μ, t, σ and W, are stock price, drift, time, volatility and Brownian motion, respectively.

• Ito lemma (for the function f which depends on two variables)

$$df = \sigma S \frac{\partial f}{\partial S} dW + (\mu S \frac{\partial f}{d\partial S} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 + \frac{\partial f}{\partial t})dt,$$

to find differential of the smooth function of a stochastic variable.

• Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Note that the equality sign in the expression above, for America options becomes less than or equal to zero.
In this expression V is a differentiable function of stock price S and time t, volatility is denoted by σ, and r is interest rate.
This formula is used for calculating the price of European and American put and call options. We obtain option price by solving the Black-Scholes partial differential equation.

We have discussed about three different illiquid markets with the relevant PDEs for the value function $u(S, t)$, and considered about one of the models "Reduced-Form SDE Model" which rises to the following PDE

$$u_t + \frac{1}{2}\frac{\sigma^2}{(1-\rho S u_{SS})^2} S^2 u_{SS} = 0,$$

where u is a differentiable function of stock price S and time t, $\rho > 0$ is liquidity parameter and σ is volatility.
This is a non-linear PDE and is difficult to solve. Thus we have chosen a method, called "Lie group analysis" to solve it.

In order to understand how to apply Lie analysis method on PDEs and its related definitions and theorems, we have studied informally this subject and demonstrated some elementary examples.

In order to solve the above PDE, we have considered a more general equation from [3]

$$u_t + \frac{\sigma^2 S^2}{2} \frac{u_{SS}}{(1 - \rho\lambda(S)Su_{SS})^2} = 0,$$

where we have considered two cases, $\lambda(S)$ as an arbitrary function and $\lambda(S) = \omega S^k$.
By applying Theorem 5.1.2, we have obtained corresponding Lie algebra (56) for the main PDE where $\lambda(S)$ was an arbitrary function and Lie algebra (58) for the main PDE with $\lambda(S) = \omega S^k$.
Then by applying Theorem 5.1.1 we have solved the system of ordinary differential equation (62), where we have considered different cases for constant a_2, a_3, a_4, and calculated in detailed only for one of these cases and have obtained symmetry groups (63),(64),(65). Then with the help of these symmetry groups we have calculated invariant variables (68), where these variables have been useless, because they did not lead us to any reduced ODE of the main PDE.
The corresponding Lie algebra for the the case $\lambda(S) = \omega S^k$, in the main PDE, by considering different $k-$values and by applying Theorem 5.1.2 have been obtained as (58). The corresponding symmetry groups are (70), (71) and (74),(75),(76).
We have construct invariant variables

$$\begin{aligned} inv_1 &= lnS + at, \quad a = -a_1/a_2, \\ inv_2 &= uS^{(k-1)}. \end{aligned}$$

By inserting these variables in PDE (42), for the case $\lambda(S) = \omega S^k$, $k \in \mathbb{R}$, we have obtained the second order differential equation (84), where by a relevant variable substitution (85), we have reduced our PDE to the following first order ODE

$$ax + \frac{\sigma^2}{2} \frac{xx_v + (1-2k)x - k(1-k)v}{(1 - \rho\omega(xx_v + (1-2k)x - k(1-k)v))^2} = 0,$$

where its solutions are (86).
The solutions of this ODE give rise to the invariant solution of PDE (42).

For the case $\lambda(S) = \omega S, (k = 1)$ in PDE (42), with the help of the same invariant variables as above, we have obtained the following second order ODE

$$v_z + \frac{\sigma^2}{2a}\frac{v_{zz} - v_z}{(1 - \rho\omega(v_{zz} - v_z))^2} = 0.$$

We have reduced this equation to a first order ODE by the following variable substitution

$$y(z) = v_z(z),$$

we have obtained the following first order ODE to PDE (42)

$$yy_z^2 - 2\left(y^2 + \frac{1}{\rho\omega}y - \frac{(\sigma^2/2a^2)}{2(\rho\omega)^2}\right)y_z + \left(y^2 + \frac{2}{\rho\omega}y + \left(\frac{1 - (\sigma^2/2a^2)}{(\rho\omega)^2}\right)\right)y = 0,$$

where $\rho\omega \neq 0$. The solutions of this equation are (93).
These set of equations are easier to solve than the main PDE. The solutions of these equations lead us to invariant solution.

At the end we have considered the last case $\lambda(S) = \omega$, $(k = 0)$ in main PDE (42). We have obtained the second order ODE

$$v_z + \frac{\sigma^2}{2a}\frac{v_{zz} + v_z}{(1 - \rho\omega(v_{zz} + v_z))^2} = 0.$$

We have made it more simple to solve by variable substitution $g(z) = v_z(z)$, then we have obtained a first order ODE as follows

$$gg_z^2 + 2\left(g^2 - \frac{1}{\rho\omega}g + \frac{(\sigma^2/2a^2)}{2(\rho\omega)^2}\right)g_z + \left(g^2 - \frac{2}{\rho\omega}g + \left(\frac{1 + (\sigma^2/2a^2)}{(\rho\omega)^2}\right)\right)g = 0,$$

which has the set of solutions (98). By solving this equation we can obtain invariant solutions.

References

[1] P. Bank and D. Baum, Hedging and portfolio optimization in financial markets with a large trader, Math. Finance 14 (2004), 1-18 .

[2] M.V. Bobrov, The fair price valuation in illiquid markets, Master's Thesis in Financial Mathematics, Technical report IDE0738, Halmstad University, Sweden, 2007.

[3] L. A. Bordag, On option-valuation in illiquid markets: invariant solutions to a non-linear model, in A. Sarychev A. Shiryaev, M. Guerra and M. R. Grossinho (eds): Mathematical Control Theory and Finance, Springer, (2008), 71-94.

[4] U. Cetin, R. Jarrow and P. Protter, Liquidity risk and arbitrage pricing theory, Finance Stoch. 8 (2004), 311-341.

[5] R. Frey, *Market illiquidity as a source of model risk in dynamic hedging*, in "Model Risk" (R. Gibson, ed.), 125-136, Risk Publications, London, 2000.

[6] R. Frey and P. Patie, *Risk management for derivatives in illiquid markets: a simulation study*, in "Advances in Finance and Stochastics" (K. Sandmann and P. Schonbucher, eds), Springer, Berlin, 2002.

[7] R. Frey and A. Stremme, Market volatility and feedback effects from dynamic hedging, Math. Finance 7 (1997), 351-374.

[8] M. Jandacka and D. Sevcovic, On the risk-adjusted pricing-methodology-based valuation of vanilla options and explanation of the volatility smile, *J. Appl. Math.* 3 (2005), 253-258.

[9] H. Liu and J. Yong, Option pricing with an illiquid underlying asset market, *J. Economic Dynamics Control,* 29 (2005), 2125-2156.

[10] E. Platen and M. Schweizer, On feedback effects from hedging derivatives, *Math.Finance* 8 (1998), 67-84.

[11] U. Polte, *On Hedging and Pricing of Derivatives in Incomplete Markets - a PDE Approach,* PhD-thesis, Universitat Leipzig, 2007.

[12] N. Sergeeva, *The feedback effects in illiquid markets, hedging strategies of large traders,* Master's Thesis in Financial Mathematics, Technical report IDE0741, Halmstad University, Sweden, 2007.

[13] L. Bordag and R. Frey. *Pricing Options In Illiquid Markets: symmetry Reduction and Exact Solution*

[14] Peter J. Oliver (1993) Applications of lie groups to differential equations. Springer-Verlag, New York

[15] Paul Wilmott, Sam Howison, Jeff Dewynne (1995) The mathematical of financial derivatives, USA

[16] Some studies within Applied mathematics with Focus on conditional symmetries of partial differential equations and Bending waves in plates (2005)

[17] G.W. Bluman and S.C. Acno. Symmetry and Integration Method for Differential Equations. Springer-Verlag, New York, 2002.

[18] P.E. Hydon, Symmetry Methods for differential Equations. Cambridge University Press, Cambridge, 2000.