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Gottfried Wilhelm Leibniz— A Study of his Life, Ideas and the Development of Calculus

av

Lotta Rehn-Molin

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MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

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Lotta Rehn-Molin

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Abstract

This thesis aims to provide an overview of the life and ideas of the German philosopher and mathematician Gottfried Wilhelm Leibniz (1646-1716), with focus on his mathematics and the development of calculus.

Leibniz has played a very important role in the history of philosophy as well as in the history of mathematics, and he is known as one of the greatest thinkers of the 17th Century. In addition to his work on mathematics and philosophy, Leibniz also made rigorous contributions to physics, metaphysics, logic, epistemology, jurisprudence, history and geology.

Leibniz's philosophical view is mostly known for its optimism, including the idea that our Universe is the best of what God could possibly create. It was of great concern for Leibniz to investigate and give structure to the fundamental nature of being. He formulated his metaphysical view in terms of what he called simple substances, monads and a pre-established harmony.

During the 17th Century, mathematicians worked on advancing techniques for finding areas enclosed by curved lines (quadratures) and volumes enclosed by figures (cubatures). Leibniz argued that it is possible to transform figures whose equations include irrational numbers, so that the properties of those figures could be understood with infinite series of rational numbers. He found series to be useful for numerical approximations of areas, and as an example, he demonstrated that the area of a quarter of a circle with radius 1 can be expressed as

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Leibniz spent 1672-1676 in Paris and during these years he developed the infinitesimal calculus and its notation. He invented the notation $\int$ signifying a sum and d signifying a difference as well as rules for how to use these. He described differentiation as finding the difference between elements within a series, and summation as finding sums of such differences between elements.

After Leibniz had published his first results on calculus in 1684 he received some criticism, especially regarding the concept of infinitely small quantities. He also got accused of having taken the ideas from Newton who had formulated similar theories years before which were just not yet published.

Today, calculus is commonly applied within many fields including; physical science, computer science, statistics, engineering, economics, business and medicine. The notation Leibniz developed for calculus might be his greatest contribution to mathematics, and it is taught in schools and used all around the world more than 300 years after its invention.

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1 Introduction

This essay is the final thesis of a Bachelor degree in Mathematics and Philosophy at Stockholm University, written within the field of history of mathematics and philosophy. It aims to provide an overview of the life and ideas of Gottfried Wilhelm Leibniz, focusing on his mathematics and the development of calculus. The work is divided into seven chapters and a chapter overview is provided below.

Chapter 2 gives a summary of Leibniz life from when he was born until he died.

Chapter 3 explains the fundamental principles of his philosophy.

Chapter 4 briefly describes his metaphysical ideas.

Chapter 5 describes the background to, the invention of, and criticism against calculus.

Chapter 6 exemplifies what influence Leibniz's ideas have had after his death.

Chapter 7 provides my own thoughts about Leibniz and his ideas.

These interesting topics could evidently be described and discussed on a much deeper level than has been done here, and one of the biggest challenges with this work has been to limit it to a ten weeks project.

2 Biography

Gottfried Wilhelm Leibniz was born the 1st of July 1646 in Leipzig, Germany. His father, Friedrich Leibnütz, was a professor of moral philosophy at the University of Leipzig. Friedrich died at the age of 48 when Leibniz was six years old and from there on he got raised only by his mother, Catharina Schmuck. Leibniz also had a two years younger sister, Anna Catharina, who died at the age of 24.

Leibniz started his education in the Nicolai School, but was largely self-taught by all the reading he did in his father's library. At the age of 14, Leibniz started studying mathematics and philosophy at the University of Leipzig where he also got introduced to revolutionary men like Galileo Galilei, Francis Bacon, Thomas Hobbes, and René Descartes. When he was 20 years old, he applied for a doctor's degree in law at the University of Leipzig but got refused. He moved to Nürnberg where he at once got his doctor's degree from the University of Altdorfat, who also offered him a professorship which he declined. Instead, he took his first job as a secretary to a society of alchemists at Nürnberg. Shortly after, he got a job as a legal adviser to the Elector of Mainz.

Leibniz spent 1672-1676 in Paris, which was the main city of philosophical activity in Europe at that time. During these years he developed the infinitesimal calculus and its notation, which is taught and used all around the world today. In Paris, he also worked on physics and a number of technological ideas, among those a calculating machine. In 1673, he traveled to London for the first time to present his calculation machine for the Royal Society. This machine was far ahead for its time and was in use until the electronic calculators came about 300 years later.

In 1676, Leibniz returned to Germany and started working as a court councilor at Hanover. He was full-time employed there, but travelled a lot and continuously did part-time jobs in other cities too, among those; London, Vienna, Paris, Berlin, St. Petersburg. It is worth noticing that Leibniz, unlike many other great philosophers of his time, needed to work to make a living.

Leibniz did not just develop things in theory, but also in practice. As an example, he created a water pump run by windmills which improved the exploration of the mines of the Harz Mountain. In 1680-1685, he frequently worked in these mines as an engineer. Leibniz came up with the idea of that the Earth at first was molten, which is a hypothesis contributing to the fact that Leibniz is considered to be one of the creators of geology. During this time, Leibniz also developed a metaphysical system through research about the notion of a universal cause of all being, trying to reduce reasoning to algebra of thought. Furthermore, he worked on his mathematics and focused on the problem of finding the square having the same area as a given circle.

In 1686, he first described his ideas about that the predicate in a proposition is contained in the subject. This was to be the foundation and definition of his philosophy of Monadology, which was further developed later.

In 1687-1690, Leibniz was travelling around Germany, Austria and Italy. Wherever he went, he met scientists and continued to develop his scientific work and published essays on the duration of things as well as on the movement of celestial bodies.

Leibniz returned to Hanover in 1690. He kept on working on his theories of motion and in 1695 he published his dynamic theory of motion, which treated the relationship of substances and the pre-established harmony between the body and the soul. In 1697, he explained that the ultimate origin of things needs to be God.

In 1710, Leibniz published a work on his ideas about divine justice. The next year he moved to Vienna where he worked as an adviser of the empire and achieved the title Freiherr (“baron”). In 1714, he wrote *Monadologia*, which synthesized the philosophy of the Monadology.

The 14th of November in 1716, when Leibniz was 70 years old, he died in Hanover as a result of gout and colic. He was buried in the Neustädter Kirche in Germany. Figure 1 shows a portrait of Leibniz. [1][2]

Figure 1 *Portrait of Gottfried Leibniz.* [3]

Leibniz was one of the greatest thinkers of the 17th and 18th Centuries and he is known as the last “universal genius”. He made a rigorous contribution to the field of physics, metaphysics, logic, epistemology, jurisprudence, history, geology, philosophy of religion as well as mathematics.

Denis Diderot (1713-1784), who was a contemporary atheist and materialist, acclaimed Leibniz for his outstanding achievements and wrote in his Encyclopedia that *“Perhaps never has a man read as much, studied as much, meditated more, and written more than Leibniz…”*. Diderot further wrote that *“When one compares the talents one has with those of a Leibniz, one is tempted to throw away one's books and go die quietly in the dark of some forgotten corner.”*

Leibniz published two books during his life, the *Theodicy* (1710) and the *New Essays Concerning Human Understanding* (1765), but none of them contained a complete description of his philosophical core. Those trying to grasp Leibniz’s thoughts and ideas need to put pieces together from journals, unpublished work and his many letters. [4]

In fact, during his life he sent around 15000 letters to more than 1000 different recipients. It is not an easy task to summarize his ideas since Leibniz seems to have changed his views and refined his formulations on a number of issues along his career. [4][5]

3 Philosophy

Leibniz philosophy was inspired by ancients as Aristotle but also by the modern philosophers Descartes, Hobbes and Spinoza. His philosophical view is mostly known for its optimism, including the idea that our Universe is the best of what God could possibly create. He expressed that *"In my opinion, the greatest thing that a man can do naturally for himself is perfect his mind, which unifies him with God insofar as is possible by natural forces."* [6]

3.1 Fundamental Principles

In Leibniz work The *Monadology* (1714), he stated 90 short passages describing his later philosophical views. He formulated six principles which constitute the basis of his philosophical ideas and each one of them is shortly explained below.

The Principle of the Best, also known as the *Principle of Optimism*, means that God always acts for the best. Leibniz wrote that *"God is an absolutely perfect being, … power and knowledge are perfections, and, insofar as they belong to God, they do not have limits"*. [4]

Furthermore, Leibniz meant that human beings are limited to having only one view of a thing at the time, but that this view is not altogether different from *one* of God's views.*" For God understands things as we do but with this difference: that he understands them at the very same time in infinitely many ways, whereas we understand them in one way only."* [6]

The Predicate in Notation Principle explains Leibniz specific ideas about the truth. He wrote that "*in every true affirmative proposition, whether necessary or contingent, universal or particular, the notion of the predicate is in some way included in that of the subject. Praedicatum inest subjecto; otherwise I do not know what truth is."* [4]

As an example, saying that Caesar was a commander is to say that the concept 'commander' is contained in the individual concept 'Caesar'. [5]

The Principle of Contradiction describes Leibniz's logical view on the language and he simply put it as; *"a proposition cannot be true and false at the same time, and that therefore A is A and cannot be not A"*. This constitutes the basis for Leibniz's logic.

The Principle of Sufficient Reason explains Leibniz's view on cause and effect; that there is no effect without a cause, and that nothing is without a reason. Leibniz wrote that "*most of the time these reasons cannot be known to us".* He meant that the reason is always known to God, which relates this principle to the principle of the best.

The Principle of the Identity of Indiscernibles Leibniz described as *"it is not true that two substances can resemble each other completely and differ only in number"*. This means that if two things share all properties they must be identical, or $(\forall F)(Fx \leftrightarrow Fy) \rightarrow x = y$. Leibniz meant though, that certain kinds of properties are excluded from the list of possible properties for which this principle holds.

A related principle is the *Principle of the Indiscernibility of Identicals,* which says that if two things are identical, then they share all properties, or $x = y \rightarrow (\forall F)(Fx \leftrightarrow Fy)$.

When combining this principle with the original one, the so called "*Leibniz's Law*" is obtained: two things are identical if and only if they share all properties, or $x = y \leftrightarrow (\forall F)(Fx \leftrightarrow Fy)$.

The Principle of Continuity Leibniz described as *"Nothing takes place suddenly, and it is one of my great and best confirmed maxims that nature never makes leaps".* He believed that any change passes through an intermediate change and that there is an actual infinity in things. Furthermore, he used this principle to demonstrate that no motion can arise from a state of complete rest. [4]

This principle provided great value for Leibniz when he developed his calculus. [7]

4 Metaphysics

For Leibniz, it was of great concern to investigate and give structure to the fundamental nature of being. In order to understand his metaphysical ideas, three essential concepts will be explained; simple substances, monads and the pre-established harmony.

4.1 Simple Substances

Leibniz was concerned about understanding what the fundamental components of reality actually were and how reality was constituted. His viewpoint on this matter was that everything is composed of so called *simple substances* which are individual unities having perception and will. He described that each substance has a *"complete individual concept"* in which the past, present and future is contained. Because of this, the entire history of the universe could be read (possibly only by God) in the essence of any individual substance.

This is how Leibniz describes the nature of the simple substances:

1) No two substances can resemble each other completely and be distinct.
2) A substance can only begin in creation and end in annihilation.
3) A substance is not divisible.
4) One substance cannot be constructed from two.
5) The number of substances does not naturally increase and decrease.
6) Every substance is like a complete world and like a mirror of God or of the whole universe, which each expresses in its own way.

Furthermore, Leibniz meant that the substances could not causally interact with each other and this idea forms a premise of his argument for a pre-established harmony. [4]

4.2 Monads

The word *monad* has its origin in the Greek word *monas* which means *unit* or *one*. Leibniz described the monads to consist without parts and without extent in space. A monad can neither be "put together" nor "broken apart", since it is not constituted of parts. Monads can only be created from nothing and destroyed by disappearing completely. A monad does not exist in time or space since it is not material but spiritual. Each monad has its own set of qualities, different from all the others. A monad contains everything that will ever happen to it and it develops through its own energy and by its own laws. The monads are related to each other only in one way; namely though the state in with they occur at the same moment. Each monad reflects the whole universe in every moment. The reflection from one monad in a certain moment corresponds to the reflection from every other monad at the same moment. Leibniz meant that this correspondence is what relates the monads to each other and together they explain the real world – the world of phenomenon. This perception of the world is what he called the *pre-established harmony*. He meant that the monads will follow each other as synchronized watches, which shows the same time without being connected to each other.

Monads are bound together in aggregates so that some aggregates contain a big number of material-monads as well as one soul-monad. Leibniz meant that these aggregates are what constitute the phenomenon called human beings. Animals are constituted similarly but with a limited consciousness. According to Leibniz, body and mind are somewhat not two separate substances.

Everything is non-material and bodies in the world of phenomenon are just a way for the mind to appear. [5]

Leibniz said that *"I don't really eliminate body, but reduce it to what it is. For I show that corporeal mass, which is thought to have something over and above simple substances, is not a substance, but a phenomenon resulting from simple substances, which alone have unity and absolute reality."* [4]

4.3 Pre-established Harmony

The mind-body problem was of great concern for Leibniz. The problem is basically the following: If mind is thought and body is extension, then how do they interact and form a unity as in all human beings? In other words, how do thinking substance and extended substance unite? As Leibniz denied the possibility of causal interaction between the substances, he came to argue for a *pre-established harmony* which meant that each substance has a unique series of perceptions which makes it play in harmony with all other substances, and that these perceptions are set by God.

In his essay *A New System of Nature* (1695), he presented his arguments for the pre-established harmony:

1) There is no real influence of one created substance on another.
2) God originally created the soul (and any other unity) in such a way that everything must arise for it from its own depths, though perfect *spontaneity* relative to itself, and yet with a perfect *conformity* relative to external things.
3) This is what makes every substance represent the whole universe exactly and in its own way, from a certain point of view, and makes the perceptions or expressions of external things occur in the soul at a given time, in virtue of its own laws, as if in a world apart, and as if there existed only God and itself.
4) The organized mass, in which the point of view of the soul lies, being expressed more closely by the soul, is in turn ready to act by itself, following the laws of the corporeal machine, at the moment when the soul wills it to act, without disturbing the laws of the other – the spirits and blood then having exactly the motions that they need to respond to the passions and perceptions of the soul.
5) It is this mutual relation, regulated in advance in each substance of the universe, which produces what we call their *communication*, and which alone brings about *the union of soul and body*.

The main idea is that the body will follow its own laws, and the mind its own laws, and that they will not influence each other. According to Leibniz, the world can be described in terms of either set of laws. [4]

5 Mathematics

Leibniz is one of the most important mathematicians of all times. He is credited for inventing the infinitesimal calculus, which is a branch of mathematics focused on limits, functions, derivates, integrals and infinite series. In order to understand how and why Leibniz developed calculus, it is necessary to be familiar with the mathematical discussions of the time, what people and ideas that inspired him and what problems he faced which made him identify the need for such theory.

5.1 Background to Leibniz's Mathematical Career

The following three sections concern the mathematical discussion of the 17^{th} Century, Leibniz's strive to make complex mathematical computations possible as well as what people who inspired him in his mathematical career.

5.1.1 Controversies about the Concept of Infinity in the 17^{th} Century

During the 17^{th} Century, philosophers and mathematicians debated largely about how to approach the infinite and its constitution. The main discussions led to paradoxes within two different fields; the first was the general theory of magnitudes and the composition of continuous quantities, and the second was paradoxes related to the theory of space. As an example of the first, two lines of different length have infinitely many points and so it seem like one infinity can be greater than another. Different ways of approaching the infinite and its constitution were discussed by, among others, Evangelista Torricelli, Descartes and Leibniz. [8](Pages 118-119)

Torricelli came up with a fascinating discovery about a hyperbolic solid which made him one of the greatest European researchers in Geometry of his time. He showed how a hyperbolic solid of infinite length has a finite volume.

In order to understand the proof of Torricelli's statement, it is helpful to be familiar with the Archimedean proposition on the measurement of a circle. This states that the area of a circle is equal to the area of a right triangle whose legs are equal to the radius and the circumference of the circle, see Figure 2.

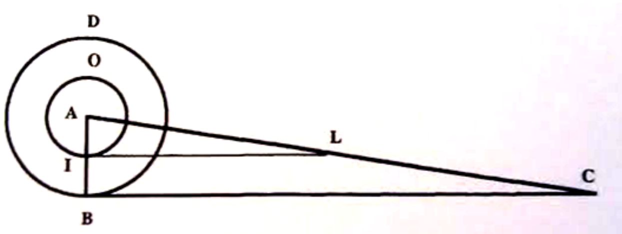

Figure 2 *Geometry of curved indivisibles.* [8]

The proof of the Archimedean proposition goes as follows:

Consider the circle BDB with radius AB and let I be an arbitrarily point on AB. Let BC equal the circumference BDB. Create a second circle with center in A and radius AI. Then the following relations hold

$$Circumferrence\ BDB : Circumference\ IOI = AB : AI = BC : IL$$

From this it follows that

$$Circumference\ BDB: BC = Circumference\ IOI: IL$$

and

$$Circumference\ IOI = IL$$

As I was chosen as an arbitrarily point, this is true for any point on AB. Torricelli concluded that *"all the peripheries taken together are equal to all straight lines taken together, that is, the circle BDB will be equal to the triangle ABC".*

This idea constitutes the basis for Torricelli's proof of the hyperbolic solid of infinite length and finite volume. Before the proof will be described, his definition of the figure of concern and his formulation of the theorem will be stated. The hyperbolic solid is illustrated in Figure 3.

Definition: *"If one rotates a hyperbola around an asymptote, as around an axis, one generates a solid infinite in length [longitudine infinitum] in the direction of the axis, which we call an acute hyperbolic solid."*

Theorem: *"An acute hyperbolic solid, infinitely long [infinite longum], cut by a plane [perpendicular] to the axis, together with a cylinder of the same base, is equal to that right cylinder of which the base is the latus transversum of the hyperbola (that is, the diameter of the hyperbola), and of which the altitude is equal to the radius of the base of this acute body."*

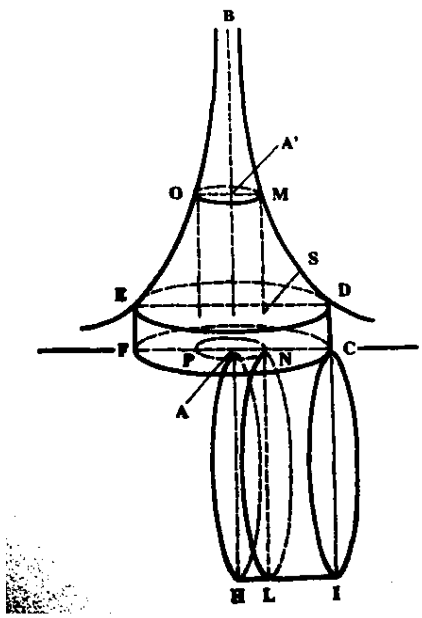

Figure 3 *A solid of Infinite length and finite volume.* [8]

The original proof of Torricelli's theorem goes as outlined below. It is unfortunately not easy to follow as some parts are left unsatisfactorily explained, for example the central concept of indivisibles. However, the formulation of the proof illustrates a mathematical argument of the 17th Century.

The idea of the proof is to show that the curved indivisibles of the infinitely long solid are equal to the curved indivisibles of the cylinder $ACIH$. Imagine that the infinitely long solid is constituted of all its cylindrical indivisibles, which means all surfaces of the type $POMN$. Furthermore, imagine that the cylinder $ACIH$ is constituted of all its circular indivisibles, which means the cross sections of diameter AH. Then, some arbitrarily point N determines an indivisible in the infinitely long solid and an associated indivisible in the cylinder $ACIH$. Each cylindrical indivisible of the infinitely long solid is also equal in area to the circle whose radius is AS, which means to an indivisible of $ACIH$ (since by construction $AH = 2AS$). As the indivisibles of the two figures are equal, the volumes of the two figures will also be equal. This means that the infinitely long solid is equal to the finite cylinder whose base is the circle with diameter AH and height AC, and the proof is completed. [8](Pages 131-134)

Torricelli's result concerned three topics which all were widely discussed in the 17th Century, namely the knowledge of infinity, the position of geometry in the web of knowledge, and the ontological status of mathematical objects. His result was not the only one to highlight the difficulty to grasp the infinity, but it got much attention since it was the first discovery of its kind. Not only mathematicians were interested in participating in the debate, but also great philosophers of this time.

Descartes meant that *"we call infinite that thing whose limits we have not perceived and so by that word we do not signify what we understand about a thing, but rather what we do not understand."* Furthermore, he thought that *"Since we are finite, it would be absurd for us to determine anything concerning the infinite; for this would be to attempt to limit it and grasp it. So we shall not bother to reply to those who ask is half an infinite line would itself be infinite, or whether an infinite number is odd or even, and so on."*

Leibniz's reply to this was the following: *"Even though we are finite, we can yet know many things about the infinite: for example, about asymptotic lines. . . about spaces which are infinite in length but not greater than an area than a given finite space, and about the sums of infinite series. Otherwise we should also know nothing with certainty about God. However, it is one thing to know something about a matter and another to comprehend the matter, that is, to have within out power all that is hidden in it."*

Regarding the figures of infinite length whose areas are finite, Leibniz meant that there is nothing more extraordinary about this than about the sums of infinite series, for example:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+etc.=1$$

Leibniz defined the unlimited and the infinite as: *"Thus I call unlimited that in which no last point can be taken, if not on one side. But by infinite I understand a quantity either limited or unlimited greater than any quantity that can be assigned by us or that can be designated by numbers."* He meant that infinity consists of limited and unlimited quantities, where the limited can be objects of measurement but the unlimited not. [8]

5.1.2 The Calculating Machine

Before Leibniz's mathematical breakthrough in 1772-1776, he worked on inventing other technologies that would advance mathematics. At this time, the idea of using machines for mathematics and problem-solving was central. Galileo had invented a compass and Descartes had created curve-producing machines. Leibniz created new instruments and machines, with which he both aimed to solve mathematical problems exactly as well as producing sufficiently accurate approximations. However, one of Leibniz most well known innovations came to be the mechanical calculating machine, see Figure 4. This machine made use of the binary system and could perform multiplication, division as well as extracting square roots. [3][6]

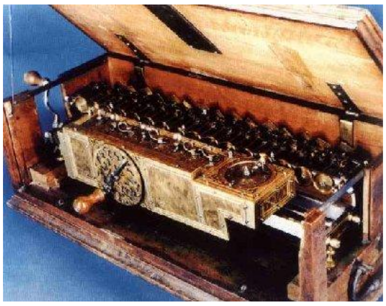

Figure 4 *Leibniz's calculating machine.* [1]

Leibniz was optimistic about the power of numerical sequences and he meant that if a method of using sequences to handle fractions and root in general were to be developed, *"every figure could be squared, … and every median proportion could be found, and geometry could be perfected".* [6]

5.1.3 Sources of Inspiration

During the 17th Century, mathematicians worked on advancing techniques for finding areas enclosed by curved lines (quadratures) and volumes enclosed by figures (cubatures), as well as for determining the centers of gravity of surfaces and bodies.

Leibniz was very interested in the ideas of Bonaventura Cavalieri as well as those of John Wallis, and the criticism they both received from Hobbes. Leibniz studied this closely and at an early stage he realized the serious limitations with their respective methods: Cavalieri's method depended on geometrical figures and Walli's method was based on the use of induction on a certain sequence of numbers. In an article Leibniz published in the *Journal de Sçavans*, he formulated it as follows: *"It is this which he calls the Analysis of infinites, which is entirely different from the Geometry of indivisibles of Cavalieri and the Arithmetic of infinites of Mr Wallis. For that geometry of Cavalieri, which moreover is very restricted, is attached to figures where it seeks the sums of ordinates. And Mr Wallis, in order to facilitate this investigation, gives us by means of induction the sums of certain classes of numbers, whereas the new analysis of infinites considers neither figures nor numbers, but magnitudes in general, as does algebra."* Leibniz read Hobbes' writings on law, logic, and mechanical philosophy as well as on mathematics carefully and he got very inspired by it. When reading this, Leibniz got introduced to the discussion concerning geometrical rigor and the foundation of the method of indivisibles. He seemed to have really admired Hobbes, but this has often been rejected by scholars because of Hobbes' bad mathematical reputation. However, Leibniz found two things in Hobbes' mathematics that really caught his interest; the fact that Hobbes doubted the Pythagorean Theorem and the fact that he failed in his attempts on squaring the circle. [7](Pages 31-76)

5.2 Infinite Series & Squaring the Circle

Leibniz argued that it is possible to transform figures whose equations include irrational numbers, so that the properties of those figures could be understood with infinite series of rational numbers. He meant that such transformations had previously been discovered by chance, but that they now could be performed with the methodology he had developed. According to himself, too many mathematicians had refused to accept the need for the kind of mathematics where drawings are made of written expressions.

Leibniz found series to be useful for numerical approximations of areas and he demonstrated that the area of a quarter of a circle with radius 1 can be expressed as

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

He meant that the magnitude of a circle *"can most simply be expressed by this series, that is, the aggregate of fractions alternately added and subtracted."* He continued, *"…but this, as I said, is to be considered primarily for the exercising of the intelligence"*.

Unlike many of his contemporaries, Leibniz thought that a symbolic expression could offer a very wide range and depth of knowledge. He claimed that in mathematics we reason, not on *"the thing itself, but on the characters that we have substituted in place of the thing"*. [6]

5.2.1 Proof of Squaring the Circle

Squaring a circle means to construct a square with equal (or proportional) area to a given circle, just by using a compass and a straightedge. Normally, attempts to square the circle had involved dividing the circle with parallel lines, but Leibniz divided the circle from a single point of view and produced an infinite number of triangles. Figure 5 and Figure 6 illustrate Leibniz method.

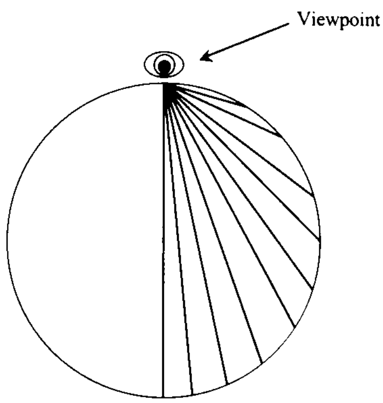

Figure 5 *Perspectival intuition behind Leibniz's quadrature.* [6]

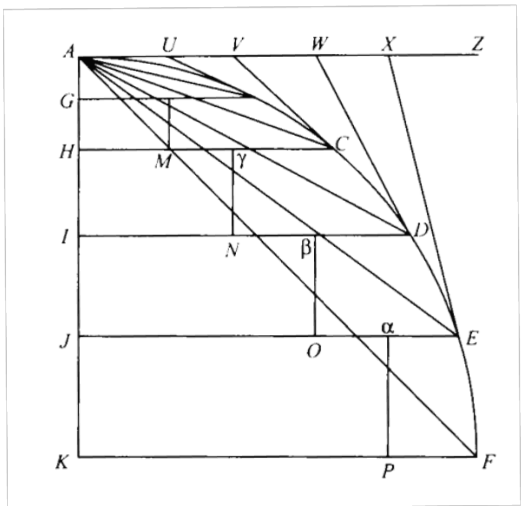

Figure 6 *Transmutation of circle (not to scale).*

The rectangles are formed by copying the distances AX, AW, AV, AU, … (formed by the tangents), onto the segments KF and JE, JE and ID, …, to form KP and Jα, JO and Iβ, …, (Each rectangle has twice the area of its corresponding triangle.) Thus, the circle is transmuted into a curve defined using the tangents to the circle. [6]

The following proof of how Leibniz squared the circle is taken from [6] and [9]. None of these sources provided all steps of the proof explicitly which leave some bits unclear to the reader. However, the following is what these sources did provide:

- Divide the circle into triangles intersecting at a point on the circle (*A*)
- For each curvilinear triangle (*AFE, AED, ADC,…*), construct a rectangular area (*JKPα, IJOβ, HINγ,…*) using tangents drawn from the circle. The quarter of the circle has area *AFK* + (all the triangles), which approximately equals *AKF* + ½ (all the rectangles). The sector of the circle is thereby "transmuted" (*γ, β, α,…*), when the number of triangles becomes infinite. The area of the quarter circle equals *AFK* + 1/2 (the area under the curve formed by *γβα…*).
- Find the area of this new curve, described by the equation
$$x = \frac{2az^2}{a^2 + z^2}$$
where a is the radius of the circle, x is a variable corresponding to the sequence of values *AG, AH, AI,…*, and z is a variable corresponding to *AU, AV, AW,…*

Leibniz made use of what he had learnt from the German mathematician Nicolaus Mercator's *Logarithmotechnia* and performed long division and integrated term by term and arrived to the expression

$$Area\ of\ circle\ sector = az - \frac{z^3}{3a} + \frac{z^5}{5a^3} - \frac{z^7}{7a^5} + \cdots$$

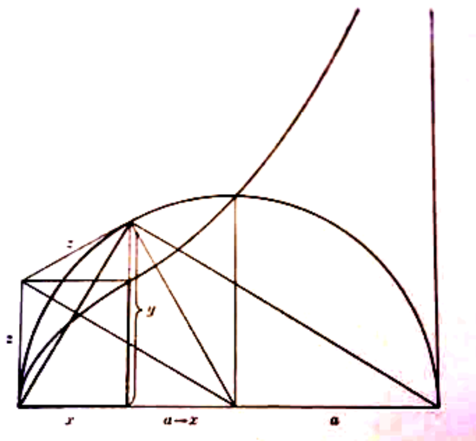

Figure 7 *Geometry of circle sector.* [9]

The geometry of a, x and z are shown in Figure 7. It can be seen that when $z = a$, the area of the circle sector equals a forth of the area of the whole circle. By letting $z = a$ and by letting the radius be $x = 1$, Leibniz obtained the area of a quarter of the circle as

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Leibniz's described his result as an expression, one as exact as one desire. He meant that *"a value can be expressed exactly, either by a quantity or a progression of quantities whose nature and way of continuing are known"*.

To Leibniz, it was more important that he demonstrated how he had discovered the proof than demonstrating the proof itself. "*I have found a general method of usefully resolving every figure into an infinity of little triangles ending at a single point, by means of convergent ordinates."*, he said. Furthermore, he explained this procedure to be among *"the most general and most useful that exists in geometry"*. He called the procedure for converting curves *transmutation*.

It was of great importance for Leibniz to use tangents to find quadratures, and the implications of this approach became apparent when he made the breakthrough in the development of calculus. [6](Pages 171-172) [9](Pages 59-60)

5.3 The Whole & its Parts

In Paris, Leibniz had been studying the Euclid's axiom: "*The whole is always greater than the part"*. This was a widely discussed axiom because it seemed to fail when applied to the angle of contact, which is the angle between a circle-arc and its tangent. This issue was clarified first when the angle was declared to be the measure of rotation and the angle of contact was zero. Leibniz thought he found the crucial difficulty with this axiom, namely that the angle of contact lack the quality of being a magnitude. He believed there are two types of truths: definitions and identities. This axiom did not fit any of these so Leibniz was concerned to find the constituent parts on which the axiom was built.

He concluded the following (extracted from page 13-14 in [9]):

1) If of two objects one is part of another, then the first is called smaller and the second larger, and this is a definition
2) Everything that is affected with magnitudes is equal to itself, and this is a statement of identity.
3) A magnitude which is equal to a part of another is smaller than this (by definition).
4) The part is equal to a part of the whole.
5) Therefore every part of a magnitude is smaller than the whole.

These ideas constituted the basis of his study of differences, which was the beginning of his development of calculus. [9]

5.4 Techniques for Finding Sums of Infinite Lengths

Leibniz developed different techniques for finding sums of infinitely many terms, and three of these techniques are demonstrated below.

5.4.1 Rewrite Sums of Numbers as Sums of their Differences

To solve problems of infinite or finite summations, Leibniz rewrote sums of numbers as sums of their differences. As an example, suppose one wanted to find the sum of

$$b_1 + b_2 + b_3 + \cdots + b_n \,.$$

If there is a a_i such that

$$b_i = a_i - a_{i+1}$$

then

$$b_1 + b_2 + b_3 + \cdots + b_n = (a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \cdots + (a_{n-1} - a_{n+1}) = a_1 - a_{n+1}$$

[6]

As another example of how Leibniz rewrote sums of numbers as sums of their differences, consider the infinite series

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} \dots$$

Leibniz realized that each term can be written as

$$\frac{2}{t(t+1)}$$

For $t > 0$, the difference between two terms can be expressed as

$$\frac{2}{t} - \frac{2}{t+1}$$

When the series has *n* terms, the sum could be written as

$$\sum_{t=1}^{n} \frac{2}{t(t+1)} = 2 - \frac{2}{n+1}$$

The second term on the right-hand side becomes infinitely small as n grows to infinity, so the sum of the series equal to 2.

These studies led Leibniz to his first major mathematical discovery: he believed it should be possible to derive the sum of any series whose terms can be described by some rule, when the sum consists of infinitely many terms but the sum approaches a finite limit. [8]

5.4.2 Geometrical Procedure

Leibniz continued investigating his discovery and looked more closely into demonstrations of geometrical progression, which had already been studied by the mathematician Grégorie de Saint-Vincent. The idea was to use an intuitive geometrical procedure when finding sums of infinitely many terms. In order to find the sum of a geometrical progression of line-segments, the whole line was divided into parts as is shown in Figure 8.

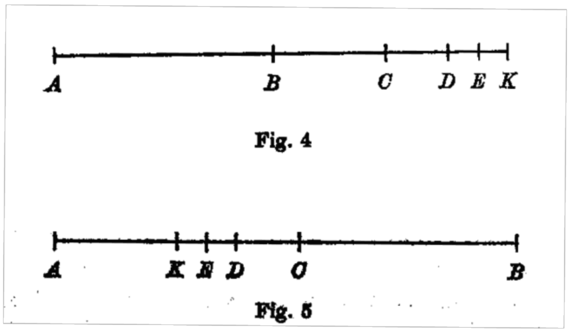

Figure 8 *Intuitive geometry for finding sums of infinitely many terms.* [9]

The distances in Figure 8 are related to each other as

$$AB\colon BC = BC\colon CD = CD\colon DE = \ \ldots$$

Furthermore, the point K is such as $AB\colon BC = AK\colon BK$ and so

$$AB\colon BK = BC\colon CK = CD\colon DK = DE\colon EK = \cdots$$

AK is then larger than any finite number of terms and not less than the sum of the complete infinite series. Since AK cannot be larger than the sum of the infinite series, it follows that it needs to be exactly equal to the sum.

Figure 9 is constructed so that the line-segments BC, CD, DE … are placed end to end from B towards A.

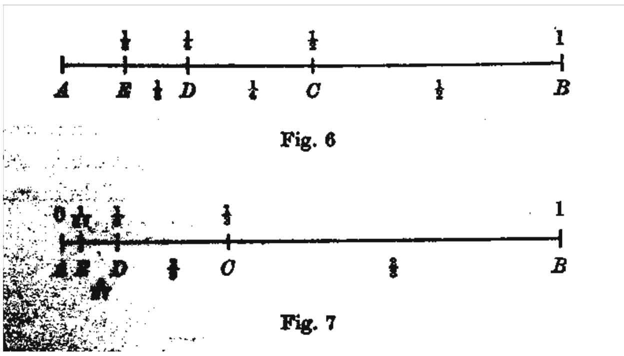

Figure 9 *Intuitive geometry for finding sums of infinitely many terms.* [9]

Arguing like this, the following sums can be constructed:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=1$$

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots=\frac{1}{2}$$

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\cdots=\frac{1}{3}$$

When Leibniz studied the work of Grégorie, he grasped the essence of it to be that the line-segments were not to be placed end to end, but must all start from one and the same point. The figures above visualize the results and it can be concluded that more generally it holds that

$$\frac{1}{t}+\frac{1}{t^2}+\frac{1}{t^3}+\cdots=\frac{1}{(t-1)}$$

[8]

5.4.3 Additive Process

The arithmetic triangle is a triangular array consisting of the binomial coefficients. Leibniz demonstrated how the terms in it could be generated by summing terms from the row above. The "hooks" in Figure 10 and Figure 11 indicate an additive process.

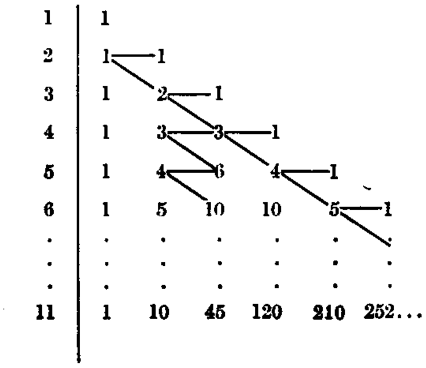

Figure 10 *Generation of the arithmetic triangle.* [9]

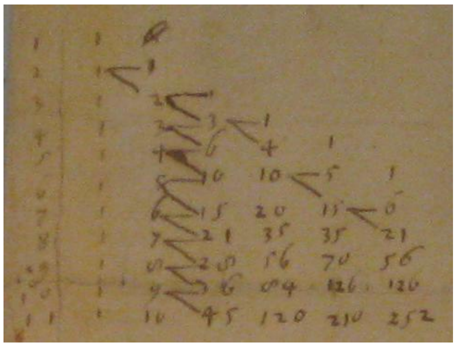

Figure 11 *Leibniz original drawing of the generation of the arithmetic triangle, found in a letter written to Oldenburg the 3rd of February 1672 or 1673.* [My own photo]

Leibniz knew that the arithmetic triangle had been studied for centuries, but he didn't know that the additive process of generating the terms also had been studied before. When he first wrote about his results, in a letter to the secretary Henry Oldenburg of The Royal Society, Leibniz was neither aware of that Pascal accurately had explained the same thing. [9]

5.5 The Invention of Calculus

When Leibniz developed calculus, he invented the notations $\int$ signifying a sum and d signifying a difference. He described differentiation as finding the difference between elements within a series, and summation as finding sums of such differences between elements.

For Leibniz, differentiation and summation were operations that operate on a sequence of variables and produce another sequence of variables. In order to solve problems with tangents and quadratures, Leibniz created a set of operations upon series of differences which made it possible to easily move among different expressions of the same series. Differentiation and summation are inverse operations and to illustrate the procedure he constructed a table of sequences connected by addition and subtraction, see Table 1.

Table 1 *Illustration of how differentiation and summation are inverse operations.*

Diffs.		1		2		3		4		5			…	dx
Series	0		1		3		6		10		15		…	x
Sums		0		1		4		10		20		35	…	$\int x$

The terms of the series are the sums of the differences, or $\int dx$, so 3=1+2 and 6=1+2+3 etc. The differences of the sums of the series are terms of the series, or $\int x = x$, so 3=4-1 and 6=10-4 etc. [6]

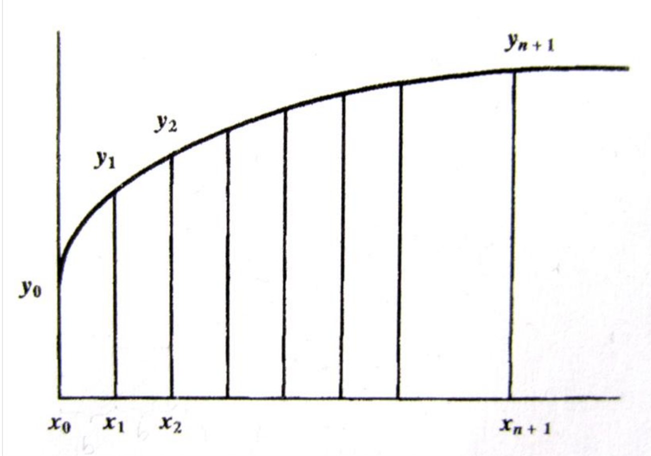

Figure 12 *Visualization of the segments dx and dy.* [8]

Figure 12 visualizes Leibniz idea of how the area between the graph and the axis can be found. The distance $x_{i+1} - x_i = dx$ is a constant variable, and the distance $y_{i+1} - y_i = dy$ is a non-constant variable. The variable dy approximates the slope of the tangent to the curve between y_i and y_{i+1}. The smaller dx is, the more accurate the approximated area $dx \sum y_i (0 < i \leq n+1)$ becomes. Leibniz argued that by letting the distance dx be infinitely small, the determination of the areas and tangents of the curve could be obtained without error. [8]

For quadratures, Leibniz expressed that *"Finding the areas of figures is reduced to this: given a series, to find sums, or (to explain this better) given a series, to find another one whose differences coincide with the terms and the given series."* Letting dx be infinitely small segments, the integral expression $\int ydx$ would for be expressed as

$$\int ydx = y_n(x_{n+1} - x_n) + y_{n-1}(x_n - x_{n-1}) + \cdots + y_0(x_1 - x_0)$$

For tangents, he meant that *"Finding the tangents to curves is reduced to the following problem: to find the differences of series".* The segment ds of an infinitely sided polygon can be extended and then constitute a line tangent to the curve with the slope dx/dy. He formulated rules for how to compute dx and dy which are provided in the next section. [6]

5.5.1 Rules for Differentiation

Leibniz's first publication on calculus was a short essay in 1684, called *Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas nec irrationales quantitates moratur et singulare pro illis calculi genus*. In this essay, he states the rules for differential of constants, sums, differences, products and quotients as well as for powers and roots. However, he did not explain how he had come up with these results or gave any proofs for them.

Leibniz's rules for differentiation were described as follows:

- If a constant, then $da = 0$ and $d(ax) = a\,dx$.
- If $v = y$, then $dv = dy$.
- If $v = z - y + w + x$, then $dv = d(z - y + w + x) = dz - dy + dw + dx$.
- If $z = v/y$, then $dz = d(v/y) = [-vdy + ydv]/yy$.
- If $z = x^a$, then $dz = d(x^a) = ax^{a-1}dx$.
- If $z = \sqrt[b]{x^a}$, then $dz = d\left(\sqrt[b]{x^a}\right) = \frac{a}{b} \cdot dx\sqrt[b]{x^{a-b}}$.

Leibniz introduced the differential dx as a fixed finite segment and seems to have, on purpose, avoided referring to it as in infinitely small quantity since that could lead to foundational objections. It is important to realize that dx also is a variable, a variable ranging over differences. It is therefore possible to use the operator d on dx and obtain ddx, which is a variable too ranging over differences of differences. This can be generalized as a variable $d^n x$ on which operator d repeatedly can be applied to obtain the n^{th} order of differentials as $d^{n+1} = dd^n x$. As for all constant variables, when dx is constant (not saying dx needs to be constant), the differential $d(dx)$ equals zero. [8]

5.6 Criticism against Calculus

After Leibniz had published his first results on calculus in 1684, other mathematician became familiar with its techniques and applications. This led to Leibniz receiving criticism on his new invention, especially regarding the concept of infinitely small quantities. He also got accused of having taken the ideas from Newton who had formulated similar theories years before which were just not published yet. Both these topics are briefly discussed below.

5.6.1 Infinitely Small Quantities

Leibniz mentioned Nieuwentijt & Clüver as being the first ones to criticize his calculus. Clüver put his critique as follows: *"I think that your method in the differential calculus is not sufficient to obtain ultimate precision in Geometry. The source of every imperfection is that you take the ratio between the unit and an infinite number to be equal to nothing, i.e. 1/N=0, which is… an impossible supposition."* Clüver was just one out of many mathematicians and philosophers who criticized Leibniz for his use of infinitely small quantities. [8]

However, Leibniz described his position regarding infinitely small quantities as follows: *"The things we have said up to now about infinite and infinitely small quantities will appear obscure to some, as does anything new; nevertheless, with a little reflection they will be easily comprehended by everyone, and whoever comprehends them will recognize their fruitfulness. Nor does it matter whether there are such quantities in the nature of things, for it suffice that they be introduced by fiction, since they allow economies of speech and though in discovery as well as in demonstration."* [7]

In 1701, Leibniz stated that the calculus should be granted only as an approximation method and not as a rigorous science. In 1702, he summarized his point of view as follows (extracted from page 172 in [8]):

- There is no need to base mathematical analysis on metaphysical assumptions.
- We can nonetheless admit infinitesimal quantities, if not as real, as well-founded fictitious entities, as one does in algebra with square roots of negative numbers. Arguments for this position depended on a form of the metaphysical principle of continuity.
- One could organize the proofs so that the error will be always less than any assigned error.

Leibniz was not concerned by the fact whether infinitesimal quantities exists or not, but whether the use of infinitesimal small quantities in calculus were reliable. He also pointed out that debates and criticism are important factors in helping science acquire better foundations. [8]

5.6.2 The Publicity War between Leibniz & Newton

Leibniz and Isaac Newton (1642-1727) independently formulated the infinitesimal calculus around the same time, but Leibniz published his results first in 1684 in his short essay *Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas nec irrationales quantitates moratur et singulare pro illis calculi genus*. As mentioned above, Leibniz here invented the notation d for differentiation as well as stated the ruled for differentiation. He also exemplified the use of the calculus as finding maxima and minima, finding tangents and solving inverse tangent problem. However, he did not explain how he had come up with these results and did not give any proofs for them. Two years later, n 1686, Leibniz publishes his first article on the foundation of the integral calculus called *De geometria recondita et analysi indivisibilium atque infinitorium.* [8]

It was not until 1687 Newton published his first results on calculus in the *Principia Mathematica*. In the same book, Newton also described his discoveries in physics. Once Leibniz had read about the results Newton had published, he printed his own three essays in *Acta Eruditorum*, one on optics (*De lineis otics*), one on motion in a resisting medium (*Schediasma*) and one on the causes of planetary motion (*Tentamen*). Leibniz claimed that he had, at this stage, never seen Newton's *Principia* in its original version consisting of hundreds of pages but only read a review about it published in *Acta Eruditorum*. He meant that he had seen *Principia* for the first time in Rome in late 1689, when his own work *Tentamen* was already published. Furthermore, he also meant that the review he had read about *Principia* earlier had not change his theories even slightly. However, Leibniz did receive several objections since it seemed unlikely that his own twelve pages in *Tentamen* would not have had anything taken from the hundreds Newton had published on the same topic earlier. [10]

The main differences between Leibniz's and Newton's views are about the conceptions of proof, the utility of symbolism and about how mathematics is related to the physical world. Leibniz never treated mathematics as an isolated subject, he rather viewed it a part of the development of his symbolic language leading to a better understanding of the universe as a whole. Newton on the other hand, meant his calculus was to be used mainly for problems of velocity and acceleration in physics. [7](Page 7) [13]

Newton expressed strong criticism towards Leibniz's calculus, meaning it was an unrigorous symbolic method of discovery that did not meet the standard of rigorous proofs required in geometry. He claimed Leibniz had plagiarized his own work, just covering it in new fancy symbols. Newton meant his own method of fluxions was deeply true since it originated in "real genesis of things" and did not depend on the superposition of infinitesimals as Leibniz's method did. Leibniz himself accepted many of Newton's ideas but questioned Newton's understanding of orders of the infinitely small. [7]

The Royal Society decided to credit Newton for first discovering calculus and to credit Leibniz for the first publication. Later on though, when Newton was the president of the Royal Society, their point of view changed and Leibniz got accused of plagiarism which damaged his reputation. However, it was Leibniz's mathematics that triumphed in the end and it is his notation, not Newton's, which is still used in mathematics today. [3]

5.6.2.1 Leibniz's & Newton's Notation

Leibniz and Newton used different notations for differentiation and integration, see Table 2.

Leibniz used dx and dy to indicate infinitesimal increments in the independent and dependent variables. Newton's notation to indicate derivative was dot over the variable, and two dots to indicate second derivative, etc.

Leibniz notation for integration is an extension of the letter S, signifying the sum of infinitely many infinitely small quantities $f(x)$ for each infinitesimal increment dx, between the limits a and b. Newton did not use a consistent notation for integration, but sometimes he used a bar above the variable. [3]

Table 2 *Leibniz' and Newton's notation for differentiation and integration.*

	Leibniz	**Newton**
Differentiation	$\frac{d(f(x))}{dt}$ or $\frac{dy}{dx}$	$\dot{y} = \frac{dy}{dt}$
Integration	$\int_a^b f(x)dx$	$\bar{x}$

Continental mathematicians preferred Leibniz version of calculus before Newton's already when it was first published, and today Leibniz's notation has become the standard notation for calculus world-wide. [12]

Newton's notation is also still in use, but mainly within physics.

6 Leibniz's Influence after His Death

Leibniz has played a very important role in the history of philosophy as well as in the history of mathematics, and he is known as one of the greatest thinkers of the 17th Century. It is not an easy task to quantify the full scope of influence Leibniz's ideas have had since it has been so expansive, but some examples are presented below.

The philosopher and enlightenment writer Voltaire (1694-1778) wrote a satire called Candide which was based on Leibniz work *Théodicée,* which mocked the Leibnizian optimism. Voltaire was a big fan of Newton and did not mind to smear Leibniz for his eventual plagiarism of Newton's calculus. Voltaire's description of Leibniz's philosophical ideas in the written satire influenced many and made people believe that *all* the ideas described were the original ideas of Leibniz, which harmed his reputation. [11]

For 13 years, Leibniz had a letter correspondence with Johann Christian Wolff (1679-1754), a Rationalist philosopher whose philosophy was dominant in Germany during the 18th Century. The correspondence involved detailed mathematical discussions, and Wolff was the first to formally teach calculus in Germany. Wolff adopted some of Leibniz's fundamental ideas, namely; the view of metaphysics as a demonstrative *a piori* science and the extensive use of the *Principle of Sufficient Reason*. Later on, Wolff got credited for linking the philosophical systems of Leibniz and the German philosopher Immanuel Kant. [12][15]

The mathematicians and brothers Jacob (1654-1705) and Johann (1667-1748) Bernoulli from Switzerland generalized calculus and formulated the so called "calculus of variations". This generalized calculus is used to find the path, curve, surface etc., for which a given function has a stationary value. This theory got further developed by the mathematicians Leonhard Euler (1707-1783) and Joseph Louis Lagrange (1736-1813). Lagrange also formulated the *Mean Value Theorem* which states that for a given section of a smooth differentiable curve, there is at least one point on that section at which the derivative of the curve is equal to the mean derivate of the section. [16]

Both Leibniz and Newton made use of the concept of *infinitesimals* when developing their calculus, which bothered many mathematicians as these did not exist in nature. The Irish philosopher Bishop Berkley (1685-1753) strongly criticized calculus and expressed his views in his work *The Analyst* (1734). This work represented a direct attack on the foundations and principals of the infinitesimal calculus and he referred to infinitesimals as *"the ghosts of departed quantities".* However, Berkeley's criticism highlighted some important aspects and contributed to mathematicians focusing on a logical clarification of the calculus. About 100 years later, calculus got reformulated by the French mathematician Augustin-Louis Cauchy (1789-1857), the German mathematician Bernhard Riemann (1826-1866) and the German mathematician Karl Weierstrass (1815-1897). This version of calculus was based on finite but small distances ε and δ, and did no longer involve infinitesimals. [17]

Today, calculus is commonly applied within many fields including; physical science, computer science, statistics, engineering, economics, business and medicine. In fact, it is useful for any problem which is mathematically modeled where it is desired to compute an optimal solution. The notation Leibniz developed for calculus might be his greatest contribution to mathematics, and it is taught in schools and used all around the world more than 300 years after its invention.

In addition to calculus, Leibniz also contributed to the development of liner algebra. He re-discovered a thousand year old Chinese method of arranging linear equations into a matrix, which could be manipulated to find the solution. This constituted the foundation for later works on matrices by Carl Friedrich Gauss (1777-1855) in the 18^{th} and 19^{th} Century. [3]

Furthermore, Leibniz is credited for being one of the early developers of the binary number systems and the calculating machine he invented could be seen as a simple computer. In some working drafts, Leibniz explained the basic logical principals of what is now called conjunction, disjunction, negation, identity, set inclusion and the empty set. Even though he never published anything on this, he is known as one of the most important logicians between Aristotle in Ancient Greece and George Boole (1815-1864) in the 19^{th} Century. Boole continued to develop Leibniz's binary system, where 1 and 0 represented "true and false" or "on and off". Boolean algebra was the start of modern mathematical logic and it led to the development of computer science. [3][17]

In the late 19^{th} Century, the British philosopher and mathematician Bertrand Russell (1872-1970) read Leibniz work on metaphysics. He published a study of it in 1900 called *A Critical Exposition of the Philosophy of Leibniz*, which contributed to a re-discovery of Leibniz ideas and made Leibniz more respectable among other philosophers of the 20^{th} Century. [12][18]

7 Afterword

When I had decided to let the topic for this thesis be Leibniz, I thought I should try and find some of his original writings in order to get to know him and his style better. At the Royal Society in London there is an archive where writings from 350 years back are kept and preserved. I went there and got to see a number of original letters Leibniz had written, in particular to the first secretary of the Royal Society; Henry Oldenburg. I also got to have a look at the original manuscript of Newton's *Principia*, shown in Figure 13 and Figure 14.

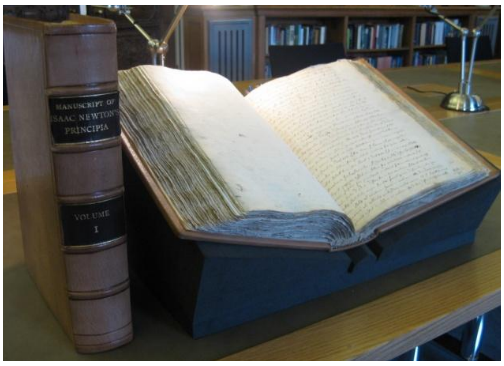

Figure 13 *Newton's Principia at the Royal Society.*
[My own photo]

Figure 14 *Me and Newton's Principia at the Royal Society.*
[My own photo]

Leibniz's letters as well as Newton's *Principia* were written in Latin so I could unfortunately not understand much, but they were still fascinating to see. Being there, having items in my hands which have been held by Leibniz and Newton felt surreal.

Leibniz has amazed me in many ways. I think it is fascinating how someone knowing so much within so many fields, still could come up with things as revolutionary as he did. Perhaps his broad base of knowledge was a necessity for him to reach all those unique ideas. In fact, his studies within the different topics were often related to each other. By studying geometry, he ended up with the philosophical infinity problem which also became essential in his calculus. If I would have had more time to work on this thesis, it would have been interesting to look deeper into how Leibniz utilized his broad base of knowledge and how this knowledge was interconnected between the different fields. However, it is an interesting thought to imagine what he would have come up with if he had spent all his life focusing only on mathematics.

Today it seems like we don't have any such "experts in everything" similar to what Leibniz was of his time. The easy access to huge amounts of knowledge we have today makes the competition harder. We might also be learning in a different way now than then, since we do not need to remember details as much anymore as we can easily retrieve them when needed. However, the lack of availability to knowledge and other's thought might have affected the developed of ideas in the 17th Century. Maybe it was easier to develop individual and unique ideas when not getting influences by other's thoughts on the same topic.

Along the development of this thesis, I have been giving a lot of thought to *why* Leibniz was concerned about things he was at this time, and what triggers scientific revolutions in general. Discovery is a process that takes time and competition is essential for progress. Different scientists can come up with parts of a final solution, so determining who will get the honor for it is not always easy. Furthermore, it is of great importance for scientist to be questioned. Leibniz seemed to have been open for criticism and willing to be questioned whilst that seems to not have been the case for Newton. Some scientific facts get discovered by accident and that could partly have been the case with calculus. What if Leibniz, who needed to work for living, would not have gotten the opportunity to focus on mathematics for four years in Paris?

No apparatus were needed for the development of calculus so in that sense it was not limited by the technology of the time. Math is different from normal science and does not depend on physical experiments, so maybe calculus should be seen as an invention rather than a discovery. Calculus provides an approach to view curves, areas and volumes as being constituted of infinitely small segments, as well as rules for how to analyze these. New approaches can be formulated and existing approaches can be improved, but calculus as it is cannot really be falsified.

Philosophy is a subject with no really right or wrong and it involves issues which can neither be proven nor disproven. It seems like the philosophical issues of concern today, to a great extent, are the same as in the 17^{th} Century. Some examples are the relationship between mind and body, between cause and effect as well as the role of, and relationship to, God. Though, when it comes to the concept of infinity, I think a difference in attitude can be observed. The concept of infinity is of great importance in calculus but it gets very little attention when calculus is taught in school today. It seems to have become more accepted and is nowadays widely used as an imaginary concept. With today's computers we can, for example, easily let dx become "as small as one desires". Hence, Leibniz was right when he claimed that infinitely small quantities might appear ungraspable to some, but will after a bit of reflection appear very fruitful.

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