SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

On the Operad of 2-Gerstenhaber Algebras

av

Kaj Börjeson

2012 - No 8

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

On the Operad of 2-Gerstenhaber Algebras

Kaj Börjeson

Självständigt arbete i matematik 30 högskolepoäng, avancerad nivå

Handledare: Sergei Merkulov

2012

Abstract

The purpose of this thesis is to provide an introduction to the theory of operads with a focus on homological algebra and to analyze the homological properties of the operad of 2-Gerstenhaber algebras. With this view we present among other things the fundamentals of category theory, monoidal categories and operads with a focus on topological and algebraic operads. We define the cobar construction of an operad and the concept of Koszul operad. Koszulity of an operad provides a straightforward method of constructing a minimal resolution via the cobar construction. We analyze the operad 2-$\mathcal{G}erst$ of 2-Gerstenhaber algebras that was introduced by Etingof et al. in [EHKR] as the cohomology of the moduli space of stable genus zero real algebraic curves. Our main new result is a proof that this operad is a Koszul operad. This is done via the use of a distributive law as introduced by Markl in [Mar].

Contents

1 Introduction

Informally, operads consist of collections of "spaces of operations". These spaces can have extra structure; they are objects in some monoidal category, such as the category of sets, topological spaces, vector spaces or complexes of vector spaces. We speak about an operad $\mathcal{O}$ in a monoidal category C. Operads were first defined by May in the monoidal category of compactly generated Haussdorff spaces in [May]. This was easily generalized to operads in an arbitrary monoidal category. An algebra over an operad consists of a map of operads $a : \mathcal{O} \to Hom_C(V^{\otimes n}, V)$ where V is an object of C (we assume here that the category C has appropriate Hom-objects). This can be thought of as a space V together with operations $V^{\otimes n} \to V$ corresponding to elements of the operad. Furthermore, we can consider actions of the symmetric groups on operads; these correspond to permutations of inputs in the associated algebra structures. To support this extra structure we need to consider operads in a symmetric monoidal category. Elementary examples of algebras are topological monoids in the monoidal category of topological spaces, Lie algebras in the monoidal category of vector spaces and supercommutative algebras in the monoidal category of chain complexes of vector spaces. The philosophy is that every type of algebra has its own operad. In the symmetric monoidal category of differentially graded vector spaces we are interested in the homological algebra of our operads. We consider a generalization of the concept of a Koszul associative algebra to the operadic framework. Knowing that an operad is Koszul allows us to construct a minimal resolution in a straightforward manner. This is obtained by using a generalization of the cobar construction for associative algebras.

A monoidal functor between a monoidal category C and a monoidal category D allows us to transfer operadic structures. An operad in C yields an operad in D under the application of such a functor. The singular chains functor and the singular (co)homology functor are typical examples of monoidal functors used in this way. The singular chains functor allows us to obtain operads in the category of differentially graded vector spaces from operads in the category of topological spaces. Application of the singular (co)homology functor yields an operad in the category of graded vector spaces. The operad of little discs is an example of a topological operad. The objects are configuration spaces of disjoint discs in the plane. The homology of this operad is the operad of Gerstenhaber algebras. The operad of little discs is one of the most studied operads. It was invented by May for use in his recognition principle for loop spaces. Gerstenhaber algebras have two binary operations, one symmetric of degree 0 and a Lie bracket of degree 1 where the bracket is a derivation with respect to the other operation. If we instead consider the topological operad where the objects are moduli spaces of pointed stable real algebraic curves of genus zero and apply the singular cohomology functor we obtain the operad of 2-Gerstenhaber algebras. 2-Gerstenhaber algebras have two operations, one symmetric of degree 0 and a trinary antisymmetric "bracket" of degree -1 satisfying a generalized Jacobi identity. The trinary operation is a derivation with respect to the other operation. This operad was introduced in [EHKR] in this way.

In this thesis we provide in the first chapters an introduction to category theory, monoidal categories, operads and operadic homological algebra. These sections are to be viewed as an introduction of the concepts and to fix the notation and as such we do not provide proofs for all theorems, instead choosing to provide only proof sketches and/or appropriate references. In section 7 we define the operad 2-$\mathcal{G}erst$ of 2-Gerstenhaber algebras. Then we go on to prove that this operad is a Koszul operad. To the author's knowledge such a proof does not exist in the literature yet. The proof is using the concept of distributive law of [Mar] which is an application of the general concept for a monad introduced by Beck in [Bec].

2 Category Theory

Categories are fundamental to the description of modern algebra. This section presents the basic concepts with examples.

2.1 Categories

A category formalizes the notion of a collection of objects and a certain type of maps between them where the important concept is that of composition.

Definition 2.1.1. A category C consists of a class of objects $Ob(C)$, a class of morphisms $hom(C)$ and a notion of composition in $hom(C)$. Every f in $hom(C)$ has a unique source and target among $Ob(C)$ and we write $f : X \to Y$ for a morphism f with source X and target Y. We denote the class of such morphisms by $hom_C(X, Y)$ (omitting the C if it is clear from the context). Given objects X, Y and Z, there is a binary operation called composition (denoted by a circle in infix notation)

$$hom_C(X, Y) \times hom_C(Y, Z) \to hom_C(X, Z),$$

satisfying two axioms. Firstly, for any X, $hom(X, X)$ has a unique identity Id_X such that for f in $hom(X, Y)$ we have

$$f \circ Id_X = f = Id_Y \circ f.$$

Secondly, for objects X, Y, Z and W, morphisms f in $hom(X, Y)$, g in $hom(Y, Z)$ and h in $hom(Z, W)$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Definition 2.1.2. Given a category C, define the opposite category C^{op} as the category with the same objects as C, but for every f in $hom_C(X, Y)$ there is a unique f^{op} in $hom_{C^{op}}(Y, X)$. The composition law is also reversed so that if $g \circ f = h$ in C we have $f^{op} \circ g^{op} = h^{op}$ in C^{op}.

It is readily seen that this is a category and that $(C^{op})^{op} = C$.

2.2 Functors

A functor can be thought of as a function between categories which preserves the structure of the morphisms.

Definition 2.2.1. A (covariant) functor between a category C and a category D consists of an association $F : Ob(C) \to Ob(D)$, and an association of

$$F : hom_C(X, Y) \to hom_D(F(X), F(Y))$$

such that for f in $hom_C(X, Y)$ and g in $hom_C(Y, Z)$ we have

$$F(g \circ f) = F(g) \circ F(f).$$

By abuse of notation we use the same symbol F for the function on the objects and the morphisms.

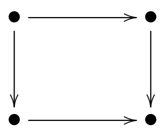
Definition 2.2.2. A contravariant functor between C and D is a covariant functor between C^{op} and D.

2.3 Diagrams

A commutative diagram in a category C is intuitively a collection of objects and a collection of morphisms written such that any two paths you take with the same source and the same target yields the same result. We could write an example to conway most of the point, but if we want to be rigorous we need to be more careful.

Definition 2.3.1. A diagram of type J in C is a functor from a category J to C.

Example 2.3.1. Diagram 1 shows a diagram of type J where J is category with four objects and four generating non-identity morphisms. If we want to just write a diagram type we would in this case write as follows.



This is also a notation for the corresponding category J.

2.4 Natural Transformations

A natural transformation is the appropriate notion of function between functors.

Definition 2.4.1. Suppose F and G are functors between categories C and D. A natural transformation η from F to G is an association for every object X in

C with a morphism $\eta_X : F(X) \to G(X)$ such that the for every f the following diagram commutes.

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \downarrow \eta_X & & \downarrow \eta_Y \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array} \tag{1}$$

2.5 Examples of Categories

The power of the categoric language is the abundance of examples of categories. Here we present some of the more common ones.

Example 2.5.1. Denote by **Set** the category whose objects consists of sets and the collection of morphisms from the set A to the set B is the class of functions from A to B. Composition of morphisms is given by composition of functions. The first axiom for being a category is satisfied since every object A has a unique identity morphism to itself; the identity function. The associativity axiom follows from the associativity of function composition.

Example 2.5.2. Denote by **Cat** the category where the objects consist of small categories (i.e., where the objects and $Hom(\bullet, \star)$ are sets) and the $Hom(A, B)$ are the functors between A and B with composition by composition of functors.

Example 2.5.3. The category **Ab** consists of abelian groups and morphisms that are group homomorphisms.

Example 2.5.4. The category $\mathbf{Vect}_K$ consists of vector spaces over a fixed field K. The morphisms are K-linear maps. If nothing else is explicitly stated we will assume that the characteristic of K is zero.

Example 2.5.5. The category $\mathbf{gVect}_K$ has as objects graded vector spaces. That is, a collection of vector spaces over $\{K_i\}_{i\in\mathbb{Z}}$. A morphism $f : V \to W$ of degree n is a family of K-linear maps $f_n : V_m \to W_{m+n}$. The morphisms of the category is the collection of morphisms of all degrees.

Example 2.5.6. The category $\mathbf{dgVect}_K$ has as objects differentially graded vector spaces. That is, graded vector spaces equipped with a distinguished degree -1 morphism ∂ such that $\partial^2 = 0$. A morphism $f : V \to W$ of degree n is a family of K-linear maps $f_n : V_m \to W_{m+n}$ such that $\partial \circ f = -(1)^n f \circ \partial$. The morphisms of the category is the collection of morphisms of all degrees. A morphism is called a quasi-isomorphism if the induced map on homology is an isomorphism.

Example 2.5.7. Denote the category of rings (unital but not necessarily commutative) by **Ring**. The morphisms consists of linear maps preserving products and identity.

Example 2.5.8. Denote the category of rings (not necessarily unital or commutative) by **Rng** (without i). The morphisms consists of linear maps preserving products.

Example 2.5.9. Denote the category of topological spaces by **Top**. The morphisms are the continuous maps.

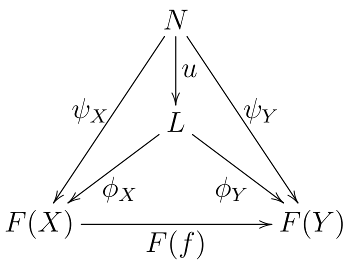
Example 2.5.10. Functor categories are categories where the objects consists of the functors between a fixed source category C and a fixed target category D. The morphisms consists of the natural transformation of functors. We denote this category by $\mathbf{Fun}(C, D)$.

Example 2.5.11. Denote by **FinSet** the category where the objects are the finite sets and the morphisms are all the bijections.

2.6 Limits and Colimits

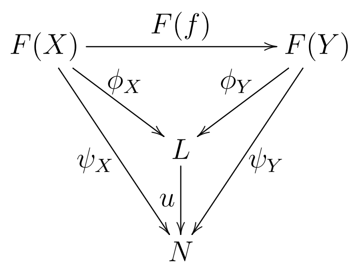
A lot of constructions can be formulated by a universal element that maps in or out of a diagram. This called a limit respectively a colimit of a diagram.

Definition 2.6.1. A limit of a diagram $F : J \to C$ consists of an object $L \in C$ and a morphism $\phi_X : L \to F(X)$ for every $X \in Ob(J)$ such that for any $F(f) : F(X) \to F(Y)$ we have $F(f) \circ \phi_X = \phi_Y$. Moreover, it has to satisfy the following universal property; for every object $N \in C$ with a collection of morphisms ψ_X satisfying the same conditions as for L there exists a morphism $u : N \to L$ such that the following diagram commutes.



Definition 2.6.2. A colimit of a diagram $F : J \to C$ consists of an object $L \in C$ and a morphism $\phi_X : F(X) \to L$ for every $X \in Ob(J)$ such that for any $F(f) : F(X) \to F(Y)$ we have $\phi_Y \circ F(f) = \phi_X$. Moreover, it has to satisfy the following universal property; for every object $N \in C$ with a collection of morphism ψ_X

satisfying the same conditions as for L there exists a morphism $u : L \to N$ such that the following diagram commutes.



2.7 Special Cases of Limits

2.7.1 Initial and Terminal Objects

Definition 2.7.1. A terminal object in a category C is a colimit of the diagram consisting of one object and no nontrivial morphisms.

Definition 2.7.2. An initial object in a category C is a limit of the diagram consisting of one object and no nontrivial morphisms.

Example 2.7.1. In the category **Set** there is a unique initial object; the empty set. A terminal object is given by a one-element set.

Example 2.7.2. In the category **Cat** the initial object is given by the empty category; the objects consist of the empty set. The category with one object and one morphism is a terminal object.

Example 2.7.3. In the categories **Ab** and $\mathbf{Vect}_K$ any trivial object is both an initial and a terminal object.

2.7.2 Products and Coproducts

Definition 2.7.3. A product in a category C is a limit of the diagram consisting of two objects and no nontrivial morphisms.

Definition 2.7.4. A coproduct in a category C is a colimit of the diagram consisting of two objects and no nontrivial morphisms.

Example 2.7.4. In the category **Set** the product of two sets A and B is given by the usual cartesian product $A \times B$. The coproduct of two sets A and B is given by the disjoint union $A \sqcup B$.

Example 2.7.5. In **Ab** and $\mathbf{Vect}_K$ the product is given by the cartesian product and the coproduct by the direct sum. Thus they coincide here; this is however not true for the generalization to infinite products and coproducts.

Example 2.7.6. In $\mathbf{gVect_K}$ and $\mathbf{dgVect_K}$ the products and coproducts are obtained by taking the product and coproduct at each degree.

Example 2.7.7. In the category **Top** the product is given by the ordinary product topology. The coproduct is given by the disjoint union.

2.7.3 Equalizers and Coequalizers

Definition 2.7.5. A limit of a diagram of the following type is called an equalizer.

$$\bullet \rightrightarrows \bullet$$

Definition 2.7.6. A colimit of a diagram of the following type is called a coequalizer.

$$\bullet \rightrightarrows \bullet$$

Example 2.7.8. In the category **Set** the equalizer of a diagram $A \overset{f}{\underset{g}{\rightrightarrows}} B$ is given by the set $\{a \in A : f(a) = g(a)\}$. The coequalizer of the same diagram can be constructed as follows. Put an equivalence relation $\sim$ on B by putting $f(a) \sim g(a)$ for all a in A. The coequalizer is the set of equivalence classes.

Example 2.7.9. In the categories **Ab** and $\mathbf{Vect}_K$ the coequalizers is the ordinary quotient construction.

2.7.4 Pushouts

Definition 2.7.7. A pushout in a category C is a colimit of the diagram of type

$$\bullet \leftarrow \bullet \rightarrow \bullet.$$

Example 2.7.10. Suppose we are given the diagram following diagram in the category **Set** where the arrows are inclusion.

$$A \longleftarrow A \cap B \longrightarrow B$$

The pushout is the union $A \cup B$.

Example 2.7.11. Suppose we have the following diagram in **Ab**.

$$A \xleftarrow{f} C \xrightarrow{g} B$$

The pushout can be described explicitly as $(A \oplus B)/R$, where R is the subgroup generated by $(f(c), -g(c))$ for all $c \in C$.

Example 2.7.12. In **Top** the pushout can be seen as disjoint union followed by gluing. This is called an adjunction space.

2.7.5 Pullbacks

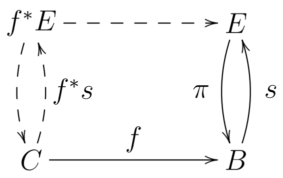
Definition 2.7.8. A pullback in a category C is a limit of the diagram of type $(\bullet \to \bullet \leftarrow \bullet)$.

Example 2.7.13. Suppose that we have the following diagram in **Set**.

$$A \xrightarrow{f} C \xleftarrow{g} B$$

The pullback is given by $\{(a, b) \in A \oplus B : f(a) = g(b)\}$

Example 2.7.14. In geometry there exists a notion named pullback; this is a precomposition with of a map between spaces. This is related to the categorical concept as follows. Suppose we have a fiber bundle (E, B, π, F) and a map between spaces $C \to B$. We obtain the following diagram where s is a section, f^*s is the usual geometric pullback and f^*E is the pullback in the category of topological spaces.



2.7.6 Inverse and Direct Limits

Definition 2.7.9. A category C is called a preorder if there is at most one morphism for every pair of objects. In addition, if the existence of arrows $A \rightleftarrows B$ implies $A = B$ we call the category a partial order.

Definition 2.7.10. An inverse limit is a limit of diagram of type J, where J is a partial order.

Definition 2.7.11. A direct limit is a colimit of diagram of type J, where J is a partial order.

Example 2.7.15. Consider the partial order on sets given by inclusion. The inverse limit is given by intersection and the direct limit as union.

Example 2.7.16. Consider the collection of rings $\mathbb{Z}_{p^n}$ (p prime) where there are morphisms from $\mathbb{Z}_{p^n}$ to $\mathbb{Z}_{p^m}$ if $n \geq m$ by reducing modulo p^m. This is a diagram of type J, where J is a total order. The inverse limit is called the p-adic integers.

Example 2.7.17. Stalks of a presheaf is formulated as an inverse limit. Given a topological space X, define a partial ordering on open sets by $U \subseteq V \to U \leq V$. This defines a category $X_{\text{inclusion}}$. The functor category $\mathbf{Fun}((X_{\text{inclusion}})^{op}, C)$ is called a C-valued presheaf on X. The subcategory of open sets containing a point x yields a diagram. The limit of this diagram, i.e., the inverse limit, is called the stalk at x. Take for example the sheaf of holomorphic functions on a Riemann surface, then the stalks are the germs of holomorphic functions.

2.8 Adjoints

Definition 2.8.1. Let A and B be categories with functors $F : A \to B$ and $G : B \to A$. Furthermore let $\{\phi_{a,b}\}$ be a family of set-bijections

$$\phi_{a,b} : Hom_B(Fa, b) \to Hom_A(a, Gb)$$

such that the following diagrams commute, where the vertical arrows are the ones induced by morphisms $a \mapsto a'$ and $b \mapsto b'$.

$$\begin{array}{ccc} Hom_B(Fa,b) & \xrightarrow{\phi} & Hom_A(a,Gb) \\ \downarrow & & \downarrow \\ Hom_B(Fa,b') & \xrightarrow{\phi} & Hom_A(a,Gb') \end{array} \qquad \begin{array}{ccc} Hom_B(Fa,b) & \xrightarrow{\phi} & Hom_A(a,Gb) \\ \downarrow & & \downarrow \\ Hom_B(Fa',b) & \xrightarrow{\phi} & Hom_A(a',Gb) \end{array}$$

A triple $\{F, G, \phi\}$ is called an adjunction and F left adjoint to G.

2.8.1 Free Objects

The main example of adjoint functors is in the construction of free objects.

Definition 2.8.2. If C is a category that has extra structure compared to D there is a functor $U : C \to D$ that "forgets" the extra structure. This is called a forgetful functor.

Definition 2.8.3. A left adjoint of a forgetful functor is called a free functor.

Example 2.8.1. Consider the functor $G : \mathbf{Vect} \to \mathbf{Set}$ which assigns the underlying set to a vector space. The left adjoint to this functor is the ordinary free vector space construction.

3 Monoidal Categories

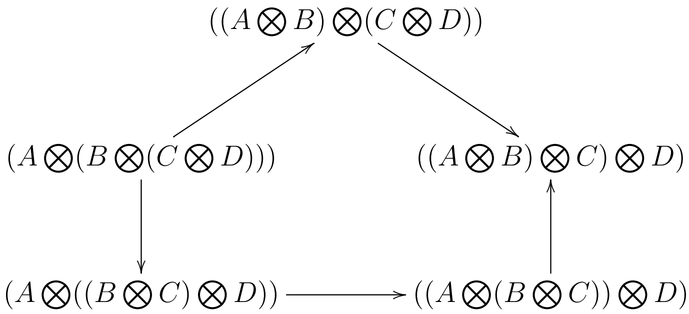
The notion of monoidal category is central to us since the definition of an operad involves a monoidal category. A monoidal category can loosely be thought of as category where we can define monoids in a generalized sense. Such a category is equipped with a "tensor" product. The category **Set** is the prototypical example with cartesian product as tensor product and ordinary monoids as monoids.

3.1 Semigroup Category

Definition 3.1.1. A semigroup category consists of a category C together with a functor (the product) $\bigotimes : C \times C \to C$ from the product category into the category together with a natural transformation (the associator)

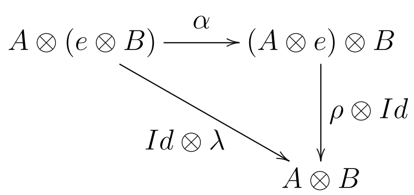
$$\alpha : (- \bigotimes (- \bigotimes -)) \to ((- \bigotimes) - \bigotimes -)$$

such that the following diagram (the pentagon) commutes, where the morphisms are the appropriate combination of α and identities.



3.2 Identity

Definition 3.2.1. A semigroup category $(M, \bigotimes, \alpha)$ together with an object e (two-sided unit) and natural isomorphisms $\lambda : e \otimes A \to A$ and $\rho : A \otimes e \to A$ such that the following diagram commutes is called a monoidal category.



3.3 Symmetric Monoidal Categories

Definition 3.3.1. Let $(M, \bigotimes, \alpha, e)$ be a monoidal category. Suppose we have a natural isomorphism $\tau : A \bigotimes B \to B \bigotimes A$ such that the following diagrams commutes.

$$\begin{array}{ccc}
(A \bigotimes B) \bigotimes C & \xrightarrow{\alpha} & A \bigotimes (B \bigotimes C) \\
\downarrow{\scriptstyle \tau \otimes Id} & & \downarrow{\scriptstyle \tau} \\
(B \bigotimes A) \bigotimes C & & (B \bigotimes C) \bigotimes A \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\
B \bigotimes (A \bigotimes C) & \xrightarrow{Id \otimes \tau} & B \bigotimes (C \bigotimes A)
\end{array} \tag{2}$$

$$\begin{array}{ccc}
e \otimes A & \xrightarrow{\tau} & B \otimes e \\
& \searrow{\scriptstyle \lambda} & \downarrow{\scriptstyle \rho} \\
& & A
\end{array} \tag{3}$$

$$A \otimes B \underset{\tau}{\overset{\tau}{\rightleftarrows}} B \otimes A \tag{4}$$

This is then called a symmetric monoidal category with twist map τ.

3.3.1 Braided Monoidal Categories

There is a weaker notion than symmetric monoidal category that is more natural in some instances.

Definition 3.3.2. The definition of a braided monoidal category is similar to the definition of symmetric monoidal category. If we omit diagram 4 and substitute diagram 2 with the following diagram we obtain the definition of a braided monoidal category.

$$
\begin{array}{ccc}
(A \bigotimes B) \bigotimes C & \xrightarrow{\alpha} & A \bigotimes (B \bigotimes C) \\
\downarrow{\scriptstyle \tau \otimes Id} & & \downarrow{\scriptstyle \tau} \\
(B \bigotimes A) \bigotimes C & & (B \bigotimes C) \bigotimes A \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\
B \bigotimes (A \bigotimes C) & \xrightarrow{Id \otimes \tau} & B \bigotimes (C \bigotimes A)
\end{array}
\qquad
\begin{array}{ccc}
A \bigotimes (B \bigotimes C) & \xrightarrow{\alpha^{-1}} & (A \bigotimes B) \bigotimes C \\
\downarrow{\scriptstyle Id \otimes \tau} & & \downarrow{\scriptstyle \tau} \\
A \bigotimes (C \bigotimes B) & & C \bigotimes (A \bigotimes B) \\
\downarrow{\scriptstyle \alpha^{-1}} & & \downarrow{\scriptstyle \alpha^{-1}} \\
(A \bigotimes C) \bigotimes B & \xrightarrow{\tau \otimes Id} & (C \bigotimes A) \bigotimes B
\end{array}
$$

3.4 Monoidal Functors

Definition 3.4.1. A functor $F : M \to N$ between monoidal categories $(M, \bigotimes, \alpha, e)$ and $(N, \bigotimes, \beta, f)$ together with natural transformations

$$\phi_{AB} : FA \bigotimes FB \to F(A \bigotimes B)$$

and a morphism $\psi : f \to F(e)$ is called monoidal if the following diagrams commute.

$$
\begin{array}{ccc}
(FA \bigotimes FB) \bigotimes FC & \xrightarrow{\beta} & FA \bigotimes (FB \bigotimes FC) \\
\downarrow{\scriptstyle \phi \otimes Id} & & \downarrow{\scriptstyle Id \otimes \phi} \\
F(A \bigotimes B) \bigotimes FC & & FA \bigotimes F(B \bigotimes C) \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\
F((A \bigotimes B) \bigotimes C) & \xrightarrow{F\alpha} & F(A \bigotimes (B \bigotimes C))
\end{array}
$$

$$
\begin{array}{ccc}
FA \bigotimes f & \xrightarrow{Id \otimes \psi} & FA \bigotimes Fe \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \phi} \\
FA & \xleftarrow{F\rho} & F(A \bigotimes e)
\end{array}
\qquad
\begin{array}{ccc}
f \bigotimes FA & \xrightarrow{\psi \otimes Id} & Fe \bigotimes FA \\
\downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \phi} \\
FA & \xleftarrow{F\lambda} & F(e \bigotimes A)
\end{array}
$$

Definition 3.4.2. A monoidal functor $F : M \to N$ between symmetric monoidal categories is called a symmetric monoidal functor if also the following diagram commutes.

$$\begin{array}{ccc} FA \bigotimes FB & \xrightarrow{\tau_B} & FB \bigotimes FA \\ \downarrow \phi & & \downarrow \phi \\ F(A \bigotimes B) & \xrightarrow{F\tau_A} & F(B \bigotimes A) \end{array}$$

3.5 Closed Monoidal Categories

Definition 3.5.1. A monoidal category is called closed if the tensor product functor $A \bigotimes -$ has a right adjoint $A \Rightarrow -$. That is, there are isomorphisms

$$\phi_{B,C} : Hom(A \bigotimes B, C) \to Hom(B, A \Rightarrow C),$$

that are natural in B and C.

3.6 Examples of Monoidal Categories and Functors

Example 3.6.1. The category **Set** form a monoidal category together with the ordinary cartesian product and the identity given by a one-element set. If given a map $\tau : A \times B \to B \times A$ which changes the order of the elements, this is a symmetric monoidal category.

Example 3.6.2. The functor category $\mathbf{Fun}(C, C) = \mathbf{End}(C)$ can be made into a monoidal category by taking composition of functors as product. The identity is given by the identity functor.

Example 3.6.3. We can put a product structure on the category **Ab** in the following way. To any two objects A and B, denote by F the free abelian group generated by the set $A \times B$. Denote the group generated by $(a, b+b') - (a, b) - (a, b')$, $(a + a', b) - (a, b) - (a', b)$ and $(an, b) - (a, nb)$ by R, where $n \in \mathbb{Z}$. Now let $A \bigotimes B = F/R$. This is called the tensor product of A and B. This turns **Ab** into a monoidal category.

Example 3.6.4. We can define a tensor product in $\mathbf{Vect}_K$ in almost the same way as in **Ab**. The construction is as above if we replace free abelian group with free vector space and $\mathbb{Z}$ with K. The category $\mathbf{Vect}_K$ is closed since

$$Hom(A \bigotimes B, C) \cong Hom(B, Hom(A, C))$$

in a natural way.

Example 3.6.5. We can take the tensor product of objects in $\mathbf{gVect}_K$ and $\mathbf{dgVect}_K$ by defining

$$(V \bigotimes_{\mathbf{gVect}_K} W)_i = \bigoplus_{j+k=i} V_j \bigotimes_{\mathbf{Vect}_K} W_k.$$

The differential in the category $\mathbf{dgVect}_K$ of the product $V \bigotimes_{\mathbf{dgVect}_K} W$ is chosen such that it acts as $\partial(v \otimes w) = (\partial v) \otimes w + (-1)^{degv} v \otimes \partial w$ and extended by linearity. We let the twisting map be given by $\tau(\alpha \otimes \beta) = (-1)^{deg(a)*deg(b)} \beta \otimes \alpha$. We can take tensor products of morphisms as well; acting on tensor products of complexes they obey the Koszul sign rule

$$(f \otimes g)(a \otimes b) = (-1)^{deg(g)*deg(a)} f(a) \otimes g(b).$$

Example 3.6.6. Consider the free vector space functor, i.e., the functor which takes a set to a vector space given by taking the elements of the set as a basis. This is a monoidal functor $\mathbf{Set} \to \mathbf{Vect}_K$.

Example 3.6.7. We are going to define a monoidal functor from the category **Top** to the category $\mathbf{dgVect}_K$ called the cubical singular chains functor. Then we are going to consider the functor obtained after taking homology. This turns out to be equal to the ordinary singular homology functor (as defined using singular simplices instead of cubes).

Define a singular n-cube on a topological space X as a continuous map $\sigma : I^n \to X$ where I^n is the unit n-dimensional cube. We say that a singular cube is degenerate if it is independent of one of the variables. Consider the free vector space generated by the singular n-cubes where we consider elements equal if they only differ by degenerate n-cubes; call this space $C_n(X)$, the cubical singular n-chains. Now let $C_\bullet(X)$ be the corresponding graded vector space. Define a differential $\partial_n : C_n(X) \to C_{n-1}(X)$ by the formula

$$\begin{aligned}\partial_n(\sigma) = &\sum_i (-1)^i \sigma(t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}) \\ &- \sum_i (-1)^i \sigma(t_1, \ldots, t_{i-1}, 1, t_i, \ldots, t_{n-1}).\end{aligned}$$

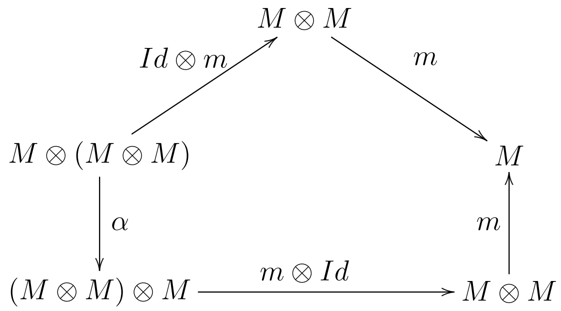
With this differential the we obtain a functor $Sing : \mathbf{Top} \to \mathbf{dgVect}_K$. To see that this is monoidal functor we need to construct a natural transformation $\eta : C_\bullet(X) \bigotimes C_\bullet(Y) \to C_\bullet(X \times Y)$. We construct it on individual singular n-cubes and extend by linearity. Suppose we have $\sigma : I^n \to X$ and $\tau : I^m \to Y$. This forms a singular $n+m$-cube $\sigma \otimes \tau : I^{n+m} \to X \times Y$ by letting σ act on the first n variables and τ on the rest. This construction is natural and therefore $Sing$ is a monoidal functor. If we compose the functor $Sing$ with the functor given by taking the homology we obtain another monoidal functor $H_\bullet \circ Sing : \mathbf{Top} \to \mathbf{dgVect}_K$. The Künneth theorem states that this functor actually induces natural isomorphisms

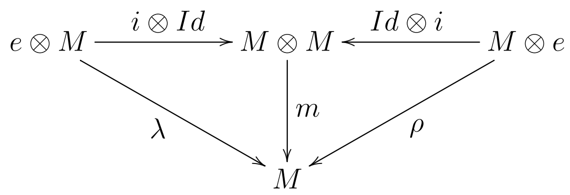
$$H_\bullet(X) \otimes H_\bullet(Y) \simeq H_\bullet(X \times Y).$$

A weaker notion does however hold for general coefficient rings. For a statement and proof of the Künneth theorem we refer the reader to [Mas].

3.7 Monoids

Definition 3.7.1. A monoid (M, m, i) in a monoidal category $(C, \bigotimes, e)$ is an object M together with morphisms $m : M \otimes M \to M$ and $i : e \to M$ such that the following diagrams commute.





Example 3.7.1. The monoids in the monoidal category **Set** is exactly the classical notion of monoids.

Example 3.7.2. A monoid in the category **Ab** is exactly a ring.

Example 3.7.3. A monoid in $\mathbf{Vect}_K$ is an associative K-algebra.

Example 3.7.4. A monoid in the category $\mathbf{End}(C)$ is called a monad over C.

3.8 Remark about Coherence Theorems

The definition of monoidal category may seem a little strange. The truth is that the pentagon axiom implies commutativity of "all" diagrams built with the morphisms given in the definition. Thus the "real" definition of monoidal categories should be that all diagrams of morphisms from the definition of monoidal category commute. A similar theorem holds for symmetric monoidal categories. Such a theorem is called a coherence theorem.

In the case of a braided monoidal category it is a bit more complicated. The coherence theorem for symmetric monoidal categories can be reformulated as follows. If we have two permutations of a single "tensor product word" there is a canonical map from one to the other satisfying some naturality conditions. The result for braided monoidal categories is analogous, with the braid group action replacing the symmetric group action. For a symmetric monoidal category every morphism built from the ones given in the definition has an underlying permutation and every permutation corresponds to a morphism. For a braided monoidal category there are underlying braids instead.

For details about the formal statements and proofs we refer the reader to [Mac]; sections VII.2, XI.1 and XI.5.

4 Operads

4.1 S-modules

Definition 4.1.1. A (symmetric) S-module is a functor $\mathcal{O} : \mathbf{FinSet} \to C$ where C is a symmetric monoidal category.

Remark 4.1.1. An S-module can be seen as a C-valued presheaf on $\mathbf{FinSet}^{op}$.

4.2 Operads

Our definition of an operad is what in some texts would be called a symmetric pseudo-operad. A discussion of the alternative definitions of an operad will be included in the end of the chapter.

Definition 4.2.1. An operad consists of an S-module together with maps for all finite sets X, Y and $i \in X$(called composition maps):

$$\circ_i^{X,Y} : \mathcal{O}(X) \bigotimes \mathcal{O}(Y) \to \mathcal{O}((X \setminus \{i\}) \sqcup Y),$$

that is natural in X, Y and i (note that $i \in X$). These morphisms has to satisfy the following axioms.

For finite sets X, Y, Z and distinct $i, j \in X$ the following diagram commutes.

$$\begin{array}{ccc} \mathcal{O}(X) \bigotimes \mathcal{O}(Y) \bigotimes \mathcal{O}(Z) & \xrightarrow{Id \otimes \tau} & \mathcal{O}(X) \bigotimes \mathcal{O}(Z) \bigotimes \mathcal{O}(X) \\ \downarrow \circ_i^{X,Y} \otimes Id & & \downarrow \circ_j^{X,Z} \otimes Id \\ \mathcal{O}((X \setminus \{i\}) \sqcup Y) \bigotimes \mathcal{O}(Z) & & \mathcal{O}((X \setminus \{j\}) \sqcup Z) \bigotimes \mathcal{O}(Y) \\ \downarrow \circ_j^{X \setminus \{i\} \sqcup Y, Z} & & \downarrow \circ_i^{X \setminus \{j\} \sqcup Z, Y} \\ \mathcal{O}((X \setminus \{i,j\}) \sqcup Y \sqcup Z) & \xrightarrow{Id} & \mathcal{O}((X \setminus \{i,j\}) \sqcup Z \sqcup Y) \end{array} \tag{5}$$

For finite sets X, Y, Z and elements $i \in X$, $j \in Y$ the following diagram commutes.

$$\begin{array}{ccc} \mathcal{O}(X) \bigotimes \mathcal{O}(Y) \bigotimes \mathcal{O}(Z) & \xrightarrow{Id \otimes \circ_j^{Y,Z}} & \mathcal{O}(X) \bigotimes \mathcal{O}((Y \setminus \{j\}) \sqcup Z) \\ \downarrow \circ_i^{X,Y} \otimes Id & & \downarrow \circ_i^{X,(Y \setminus \{j\}) \sqcup Z} \\ \mathcal{O}((X \setminus \{i\}) \sqcup Y) \bigotimes \mathcal{O}(Z) & \xrightarrow{\circ_j^{(X \setminus \{i\}) \sqcup Y, Z}} & \mathcal{O}((X \setminus \{i\}) \sqcup (Y \setminus \{j\}) \sqcup Z) \end{array} \tag{6}$$

Remark 4.2.1. The definition of an operad can be a mouthful. To help the intuition one may think of $\mathcal{O}(X)$ as the space of operations with inputs labelled by elements of X. The morphism $\circ_i^{X,Y}$ can in this context be thought of as taking functions with inputs Y and composing with a function of inputs X, where we insert the first function in the spot labeled i of the second function. The naturality conditions corresponds to exchanging labels. Diagram 5 can be interpreted as if we take a two functions as two different inputs of a third function the order of composition does not matter. Diagram 6 says that composition of operations is associative.

Remark 4.2.2. If the category C admits small colimits we can reformulate the definition of an S-module as a family of $\mathbb{S}_n$-modules $\mathcal{O}(n)$. The definition of an operad can then be formulated with composition morphisms

$$\circ_i^{[n],[m]} : \mathcal{O}(n) \bigotimes \mathcal{O}(m) \to \mathcal{O}(n+m-1).$$

We will sometimes use this formulation when it is suitable. We will also use the concept of $\mathbb{S}_n$-invariant morphism as a morphism of $\mathbb{S}_n$-modules. For details and proof of equivalence we refer the reader to, for example, [MSS]. Some authors call these objects Σ-modules; we will however not make the distinction and call these S-modules as well. See also section 4.9.

Remark 4.2.3. We will from this point on assume that the monoidal category has small colimits; this is not a severe restriction since this is true for the categories we are interested in, such as **Vect**, **dgVect** and **Top**.

Definition 4.2.2. A unital operad is an operad together with a morphism

$$u : 1_C \to \mathcal{O}(\bullet)$$

that is natural in $\bullet$ where $\{\bullet\}$ is any one-element set. It has to make the following compositions into identity maps for any set X, $i \in X$ and the unique isomorphism $\sigma : \mathcal{O}(\bullet) \to \mathcal{O}(i)$.

$$\mathcal{O}(X) \longrightarrow \mathcal{O}(X) \bigotimes 1_C \xrightarrow{Id \otimes \sigma u} \mathcal{O}(X) \bigotimes \mathcal{O}(i) \xrightarrow{\circ_i^{X,i}} \mathcal{O}(X) \tag{7}$$

$$\mathcal{O}(X) \longrightarrow 1_C \bigotimes \mathcal{O}(X) \xrightarrow{\sigma u \otimes Id} \mathcal{O}(i) \bigotimes \mathcal{O}(X) \xrightarrow{\circ_i^{i,X}} \mathcal{O}(X) \tag{8}$$

4.3 Elementary Examples of Operads

Example 4.3.1. Consider the S-module in the category **Set** such that $\mathcal{O}(\bullet) = \{\mathbf{1}\}$ for any one-element set $\{\bullet\}$ and $\mathcal{O}(X) = \emptyset$ otherwise. This has an operadic structure by the unique map

$$\mathcal{O}(\bullet) \bigotimes \mathcal{O}(\bullet) \to \mathcal{O}(\bullet).$$

This operad is denoted by **1**.

Remark 4.3.1. It is called **1** since it is the initial object in the category of operads to be defined.

Example 4.3.2. Consider the S-module such that $\mathcal{O}(X) = \{\mathbf{1}\}$ for any set X. This forms an operad with the unique maps

$$\circ_i^{X,Y} : \mathcal{O}(X) \bigotimes \mathcal{O}(Y) \to \mathcal{O}(X \setminus \{i\} \sqcup Y).$$

This operad is called the commutative operad $\mathcal{C}om$.

Remark 4.3.2. It is called $\mathcal{C}om$ since the algebras over it (defined in section 4.5) are the commutative semigroups. It is the terminal object in the category of operads.

Example 4.3.3. Consider an S-module in a closed symmetric monoidal category C, where $\mathcal{O}(X) = hom_C(A^{\otimes|X|}, A)$ (the $|X|$ copies of A are indexed by the set X) for a fixed $A \in Ob(C)$. The hom-sets can be considered as objects in C since C is closed. The morphisms $\circ_i^{X,Y}$ are constructed by composition of morphisms as follows.

$$\begin{array}{ccc} hom_C(A^{\otimes|X|}, A) \bigotimes_C hom_C(A^{\otimes|Y|}, A) & \longrightarrow & hom_C(A^{\otimes|X+Y-1|}, A) \\ f(x_1, \ldots, x_i, \ldots, x_{|X|}) \otimes g(y_1, \ldots, y_{|Y|}) & \longmapsto & f(x_1, \ldots, g(y_1, \ldots, y_{|Y|}), \ldots, x_{|X|}) \end{array}$$

This is called the endomorphism operad and is denoted by End_X.

4.4 Morphisms of Operads

Definition 4.4.1. A morphism of S-modules in a category C is a natural transformation of functors.

Remark 4.4.1. This is not a surprising definition since an S-module is an element of the functor category $\mathbf{Fun}(\mathbf{FinSet}, C)$ whose morphisms are the natural transformations.

Definition 4.4.2. A morphism of operads is a morphism of S-modules such that it is compatible with the composition maps $\circ_i^{X,Y}$ and bijections of **FinSet**.

4.5 Algebras over Operads

Definition 4.5.1. An algebra over an operad $\mathcal{O}$ (also called an $\mathcal{O}$-algebra) consists of an object $X \in Ob(C)$ together with a morphism $\alpha_X : \mathcal{O} \to End_X$, where End_X was defined in Example 4.3.3. If the underlying category has finite colimits we can form the following product.

$$\mathcal{O}(n) \bigotimes_{\mathbb{S}_n} X^{\otimes n} = \underset{\sigma \in \mathbb{S}_n}{\text{coequalizer}} \left\{ \sigma^{-1} \otimes \sigma : \mathcal{O}(n) \bigotimes X^{\otimes n} \to \mathcal{O}(n) \bigotimes X^{\otimes n} \right\}.$$

Then an algebra over an operad can equivalently be described as a family of morphisms $\alpha_X(n) : \mathcal{O}(n) \bigotimes_{\mathbb{S}_n} X^{\otimes n} \to X$.

Definition 4.5.2. A morphism of $\mathcal{O}$-algebras consist of a map $\phi : X \to Y$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{O}(n) \bigotimes_{\mathbb{S}_n} X^{\otimes n} & \xrightarrow{\alpha_X(n)} & X \\ \Big\downarrow {\scriptstyle Id \otimes \phi^{\otimes n}} & & \Big\downarrow {\scriptstyle \phi} \\ \mathcal{O}(n) \bigotimes_{\mathbb{S}_n} Y^{\otimes n} & \xrightarrow{\alpha_Y(n)} & Y \end{array}$$

Definition 4.5.3. The collection of $\mathcal{O}$-algebras together with the morphisms of $\mathcal{O}$-algebras is called the category of $\mathcal{O}$-algebras.

Example 4.5.1. An algebra over the **Set**-operad $\mathcal{C}om$ is a commutative semigroup; any permutation of inputs yields the same output.

Definition 4.5.4. Suppose C is a symmetric monoidal category such that $\bigotimes$ is distributive over coproducts, that is $A \bigotimes (\coprod_i B_i) = \coprod_i (A \bigotimes B_i)$, and likewise from the other side. Define the Schur functor associated to an operad $\mathcal{O}$ as

$$\mathcal{S}_{\mathcal{O}} = \coprod_{n \geq 1} \mathcal{O}(n) \bigotimes_{\mathbb{S}_n} X^{\otimes_n}.$$

Definition 4.5.5. Given a Schur functor, define operations α as follows.

$$\alpha(n; m_1, \ldots, m_n) = \mathcal{O}(n) \bigotimes ((\mathcal{O}(m_1) \bigotimes_{\mathbb{S}_{m_1}} X^{\otimes m_1}) \bigotimes \cdots \bigotimes, (\mathcal{O}(m_n) \bigotimes_{\mathbb{S}_{m_n}} X^{\otimes m_n})) \cong$$

$$\mathcal{O}(n) \bigotimes (\mathcal{O}_{m_1} \bigotimes \cdots \bigotimes \mathcal{O}_{m_n}) \bigotimes_{\mathbb{S}_{m_1} \times \cdots \times \mathbb{S}_{m_n}} X^{\otimes (m_1 + \cdots + m_n)} \longrightarrow$$

$$\mathcal{O}(m_1 + \cdots + m_n) \bigotimes_{\mathbb{S}_{m_1} \times \cdots \times \mathbb{S}_{m_n}} X^{\otimes (m_1 + \cdots + m_n)},$$

where we have used the operadic composition in all n inputs simultaneously.

Theorem 4.5.1. *The operations α gives $\mathcal{S}_{\mathcal{O}}(X)$ the structure of an $\mathcal{O}$-algebra.*

Proof. By careful consideration and by using the equivariance axiom of the operad structure. □

Theorem 4.5.2. *The Schur functor is isomorphic to the free $\mathcal{O}$-algebra functor, i.e., the left adjoint functor to the forgetful functor taking an $\mathcal{O}$-algebra to its underlying space.*

Proof. We sketch the proof; for full details we refer the reader to [MSS]. Let $U_{\mathcal{O}}$ denote the forgetful functor taking a $\mathcal{O}$-algebra to its underlying space. There is a natural transformation $Id \to \mathcal{U}_{\mathcal{O}}\mathcal{S}_{\mathcal{O}}$ given on an object X by

$$X \cong Id \otimes X \to \mathcal{O}(1) \otimes X \to \coprod_{n \geq 1} \mathcal{O}(n) \bigotimes_{\mathbb{S}_n} X^{\otimes n}.$$

There is also a natural transformation $\mathcal{S}_{\mathcal{O}}\mathcal{U}_{\mathcal{O}} \to Id$ given on an object X by using the $\mathcal{O}$-algebra structure. □

Remark 4.5.1. In the category $\mathbf{Vect}_K$ (also $\mathbf{dgVect}_K$ and similar categories) we can obtain the operad up to isomorphism if we are only given the Schur functor (see for example [MSS]). Thus if we are only interested about operads in this category we could use Schur functors from the beginning and build the same theory. This approach is emphasized in [LV]. See section 6 for details.

4.6 Operads and Monoidal Functors

A monoidal functor $F : C \to D$ can be used to construct an operad in a category D from an operad in a category C. The S-module is given by composition $F \circ \mathcal{O}$. An operadic morphism $\circ_i^{X,Y} : \mathcal{O}(X) \bigotimes \mathcal{O}(Y) \to \mathcal{O}(X \setminus \{i\} \sqcup Y)$ in the category C is taken to a morphism in the category D and since F is monoidal we obtain a morphism

$$\circ_i^{X,Y} F(\mathcal{O}(X)) \bigotimes F(\mathcal{O}(Y)) \to F(\mathcal{O}(X \setminus \{i\} \sqcup Y)),$$

which induces an operadic structure in the category D.

Example 4.6.1. By applying the free vector space functor (see example 3.6.6) to the operad $\mathcal{C}om$ of example 4.3.2 we obtain an operad in the category of vector spaces. The category of algebras over this operad is exactly the category of commutative algebras.

4.7 Graphs and Trees

Definition 4.7.1. A graph G consists of a triple $(Flag(G), V, e)$, where $Flag(G)$ is a finite set, V is a function $V : Flag(G) \to V(G)$ onto some set $V(G)$ and e is an involutive bijection of $Flag(G)$ with itself. The set $V(G)$ is called the set of vertices. The fiber $V^{-1}(v)$ of an element $v \in V(G)$ is called the set of legs attached to v, denoted $Leg(v)$. Consider the orbits of $Flag(G)$ under e; the two-element orbits are called the internal edges $E(G)$ and the remaining form the legs of the graph, $Legs(G)$.

Remark 4.7.1. Our definition of graph does not coincide with the classical one since we allow free legs of the graph. However if we only consider graphs where $Legs(G) = \emptyset$ we obtain the classical finite graphs (without edges with both ends attached to a single vertex).

Definition 4.7.2. The geometrical realization of G is a topological space constructed as follows. For every element of $Flag(G)$ take a closed unit interval labelled by that element and consider the disjoint union of these intervals. Glue intervals whose labels are contained in a common set $Leg(v)$ for some v at the point 0. Furthermore, if two flags are contained in the same orbit under e we glue them at the point 1. The space obtained with the gluing topology is called the geometric realization of G.

Definition 4.7.3. Consider a graph G. If its geometrical realization is connected we call G connected and likewise with simply connected and other topological properties.

Definition 4.7.4. A tree is a graph that is connected and simply connected.

Definition 4.7.5. An isomorphism of graphs consists of a bijection $b : Flag(G) \to Flag(H)$ such that the partitions of flags are preserved and such that it commutes with the involutions. The category of graphs together with isomorphisms is denoted **IsoGraph**.

Remark 4.7.2. An isomorphism of graphs induces bijections $V(G) \leftrightarrow V(H)$ and $Leg(G) \leftrightarrow Leg(H)$.

Definition 4.7.6. A tree T with one leg distinguished as the root is called a rooted tree. The legs are then partitioned as $Leg(T) = Leaf(T) \sqcup Root(T)$. Every vertex v has a leg such that it is closest to the root (since every point of the geometrical realization is connected to the root by a unique path). Denote this leg $Root(v)$. This partitions $Leg(v)$ as $Leg(v) = Root(v) \sqcup In(v)$. An isomorphism of rooted trees is a graph isomorphism that takes the root of the first tree to the root of the second tree. The corresponding category is denoted by **IsoTree**.

Definition 4.7.7. An I-labelled tree $T(I)$ is a rooted tree T together with a bijection $b_T : I \to Leaf(T)$. An isomorphism of I-labelled trees is an isomorphism f of trees such that it respects the labelling, that is $b_T = b_S \circ f$. Denote the corresponding category with $\mathbf{IsoTree}(I)$.

Definition 4.7.8. Given an I-labelled tree T and a J-labelled tree S we can define the grafting of S to the leaf $i \in I$ and produce a tree $T \circ_i S$ with labelling $(I \setminus i) \sqcup J$ as follows. First, let $Flag(T \circ_i S) = Flag(T) \sqcup Flag(S)$. Let the vertices be the disjoint union. However, the involution is modified a bit. For all flags except the one labelled by i and the root of S it will be the same. Finally let $e(i) = Root(S)$ and $e(Root(S)) = i$ (by abuse of notation we denote the flag labelled by i also by i).

4.8 Free Operads

The construction of a free operad in a symmetric monoidal category C from an S-module A is done in several steps. First we associate to every isomorphism class of trees an object of C, thought of as the space of operations obtained by composing elements of A as indicated by the tree. Technically this is the space of decorations of vertices in the tree. Secondly we form the colimit over all trees to obtain our S-module. Thirdly we use graftings of trees to give this an operad structure. We will assume that the underlying category have finite colimits.

Definition 4.8.1. Consider a set X together with a bijection $f : X \to \{1, \dots, n\}$ and an object A_x in a symmetric monoidal category C for every $x \in X$. Define

$$\bigotimes_f A_f = A_{f^{-1}(1)} \otimes A_{f^{-1}(2)} \otimes \cdots \otimes A_{f^{-1}(n)}.$$

Furthermore, for every $\sigma \in S_n$ we have a morphism

$$\bar{\sigma} : \bigotimes A_f \to \bigotimes A_{\sigma \circ f},$$

given by permuting the factors. Now define the unordered tensor product as follows.

$$\bigotimes_X A_x = \underset{\sigma \in \mathbb{S}_n}{\text{coequalizer}} \left\{ \bar{\sigma} : \coprod_{f \in \text{bijections } X \to \{1,\dots,n\}} \bigotimes_f A_f \to \coprod_{f \in \text{bijections } X \to \{1,\dots,n\}} \bigotimes_f A_f \right\},$$

where $\coprod$ denotes the coproduct.

Definition 4.8.2. Let A be an S-module in a category C and let $T(I)$ be a labelled tree. Define using the unordered tensor product:

$$A(T(I)) = \bigotimes_{v \in V(T)} A(V^{-1}(v)).$$

Theorem 4.8.1. *The above construction gives for a fixed S-module A a covariant functor* **IsoTree** $\to C$.

Proof. By checking the induced maps carefully to see that the construct is indeed functorial. For more details we refer the reader to [MSS]. □

Definition 4.8.3. Define an S-module $Free(A)$ as

$$Free(A)(I) := \underset{T(I) \in \mathbf{IsoTree}(I)}{\operatorname{colim}} A(T(I)).$$

There is a natural operad structure on $Free(A)$ given as follows. Suppose we have decorated labelled trees $T(I)$ and $S(J)$. By grafting the root of S to the leaf i of T we obtain a decorated tree with labeling $(I \setminus \{i\}) \sqcup J$. This induces a structure on the colimit. Call this operad the free operad generated by A and denote it by $\Psi(A)$.

Theorem 4.8.2. *The free operad construction is a covariant functor that is left adjoint to the forgetful functor taking an operad to its underlying S-module.*

Proof. There are quite many axioms to check. As such the proof is rather long and technical and we omit it. We refer the reader to [MSS]. □

4.9 Related Definitions of Operads

As we have indicated there are several definitions of operads that are not equivalent to ours but almost are.

Perhaps the most common approach is to define compositions

$$\mathcal{O}(I) \bigotimes \left(\bigotimes_{i \in I} \mathcal{O}(X_i) \right) \to \mathcal{O}(\bigsqcup X_i)$$

satisfying similar axioms as in definition 4.2.1. This can be seen as compositions where we insert operations in every input, in contrast with our definition where we compose at one input at a time. This definition is often accompanied by the requirement that there exists a unit. In that case it is equivalent to our definition of unital operad. This can be easily seen since a composition where all but one of the inputs are identities corresponds to a composition of definition 4.2.1.

Another variant is to use Σ-modules instead of S-modules. A Σ-module is defined as a functor from the category **FinOrd** (finite ordinals with automorphisms as morphisms) into a symmetric monoidal category C. Under the assumption that C has small colimits and that the functor $X \bigotimes -$ preserves colimits these approaches are equivalent; see theorem 1.60 in [MSS]. The categories we are interested in this text all satisfy this, thus we will not distinguish between these approaches.

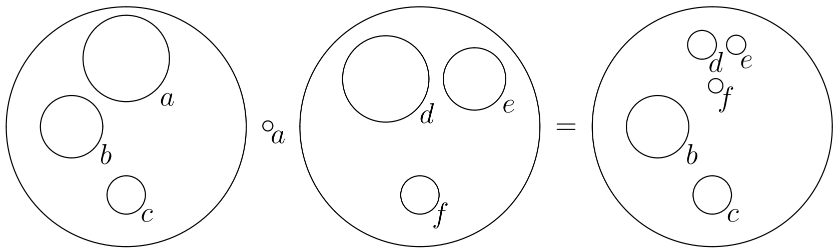
5 Topological Operads

A topological operad is an operad in the symmetric monoidal category **Top**.

5.1 Little Discs

The operad of little discs appear in a lot of places. It is without doubt one of the most important operads.

Definition 5.1.1. The S-module underlying the little disc operad $\mathcal{D}_2$ can be described as follows. For a one-element set $\{\bullet\}$ let $\mathcal{D}_2(\bullet) = \bullet$. Composition with this is taken to be the identity. For a set X with $|X| \geq 2$ we define $\mathcal{D}_2(X)$ as the configuration space of disjoint discs labelled by elements of X contained in the unit disc in the complex plane. An isomorphism $X \cong Y$ induces an isomorphism $\mathcal{D}_2(X) \cong \mathcal{D}_2(Y)$ by changing the labels. The operadic composition consists of affine scaling of a configuration and substituting it into a circle of another configuration. This is indicated in the following diagram.



Remark 5.1.1. The spaces $\mathcal{D}_2(X)$ are homotopy equivalent to the configuration space of $|X|$ points in $\mathbb{R}^2$ labelled by elements of the set X. Monoidal functors into a monoidal category D that are homotopy invariant yield the same operads in D. This can be used to obtain an operadic structure on the (co)homology of configuration spaces of points. Since the fundamental group of a configuration space of distinct labelled points corresponds to the braid groups we can make a similar analysis there.

Remark 5.1.2. We can of course consider spaces of higher dimension and define operads $\mathcal{D}_n$ where the objects are configurations of disjoint n-dimensional balls with operadic composition defined similarly.

5.2 Pointed Stable Real Algebraic Curves of Genus Zero

The operad of pointed stable real algebraic curves of genus zero where introduced in [EHKR]. The cohomology of this operad is the main operad of study in this thesis and is treated in section 7.

Definition 5.2.1. A real stable curve of genus 0 consists of a finite union of real projective lines $\{\mathbb{R}\mathbf{P}^1_i\}_i$ with labelled points z_j such that the following axioms are satisfied.

i) Each z_i belongs to one and only one of the $\mathbb{R}\mathbf{P}^1_i$.

ii) Every pairwise intersection of projective lines are either empty or consist of a single point. Intersections has to be transversal.

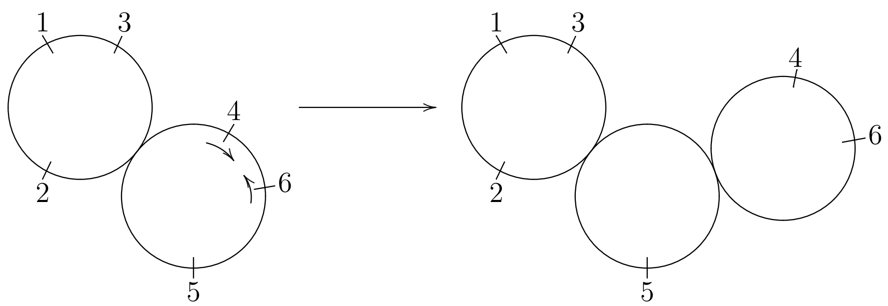
iii) The component graph is a tree, i.e., it is simply connected.

iv) The number of special points of each component has to be at least 3. A special point is either an intersection point or a point z_j.

Two curves are equivalent if there exists an isomorphism of algebraic curves such that it takes each marked point to the corresponding marked point.

Definition 5.2.2. The real locus of the Deligne-Mumford compactification of the moduli space of genus 0 curves with n marked points is denoted by M_n.

Remark 5.2.1. M_n as a set is the equivalence classes of stable n-pointed real genus 0 algebraic curves as in definition 5.2.1. The topology of the moduli space is given such that two curves are close if the points marked points and the intersection points are close. Moreover, when marked points approach each other they "blow up" and a new copy of $\mathbb{R}\mathbf{P}^1$ is inserted where the points collide and the marked points move on the newly inserted copy instead. This is indicated in figure 9. With this topology M_n has a manifold structure for $n \geq 3$. For more details and rigor we refer the reader to [DM] and [Dev].



(9)

Definition 5.2.3. The spaces M_n provides an S-module structure $\mathcal{M}$ as follows. Let $\mathcal{M}(1)$ be a one point space. Let $\mathcal{M}(n) = M_{n+1}$ for $n \geq 2$. The $\mathbb{S}_n$-actions are given by permuting the points $(2, \ldots, n+1)$. Furthermore we have an operad structure on $\mathcal{M}$ by thinking of the first labelled point as the output and the other as inputs and thereby gluing the surfaces accordingly.

6 Algebraic Operads

An algebraic operad is an operad in the category $\mathbf{Vect}_K$ or related operads such as differentially graded operads (these are operads in $\mathbf{dgVect}_K$ with extra assumptions of commutativity with the differential; this is defined in section 6.2).

6.1 Schur Functors

Recall the definition of a Schur functor (definition 4.5.4).

$$\mathcal{S}_{\mathcal{O}} = \coprod_{n \geq 1} \mathcal{O}(n) \bigotimes_{\mathbb{S}_n} X^{\otimes n}$$

For an algebraic operad we can recover the operad from its Schur functor. This is accomplished as follows.

Theorem 6.1.1. *Suppose $\mathcal{S}_{\mathcal{O}}$ is a Schur functor for some operad $\mathcal{O}$. Let X_n be the free vector space on generators $\{x_1, \ldots, x_n\}$. Let $\mathcal{S}_{\mathcal{O}}(n)$ be the subspace of $\mathcal{S}_{\mathcal{O}}(X_n)$ spanned by all elements with x_i appearing once. This space has a natural action of the symmetric group by permuting the x_i. The morphisms $\mathcal{O}(n) \to \mathcal{S}_{\mathcal{O}}(n)$ taking α to $\alpha \otimes x_1 \otimes \cdots \otimes x_n$ are equivariant isomorphisms of vector spaces.*

Furthermore, the operadic composition of $\mathcal{O}$ is related to the $\mathcal{O}$-algebra composition as follows.

$$(\alpha \circ_i \beta)(x_1 \otimes \cdots \otimes x_{m+n-1}) = \alpha(x_1 \otimes \ldots, \otimes x_{i-1} \otimes \beta(x_i \otimes x_{m+i-1}) \otimes x_{m+i} \otimes \cdots \otimes x_{m+n-1})$$

Proof. The idea is that $\mathcal{S}_{\mathcal{O}}(n)$ consists of all words written by operations of $\mathcal{O}$ and placeholder variables. It is clear that it is onto since every element can be written $\alpha \otimes x_1 \otimes \cdots \otimes x_n$ by using the symmetric action. It is injective since $\alpha \otimes x_1 \otimes \cdots \otimes x_n = 0$ implies that $\alpha = 0$. The rest of the proof is just an application of the equivariance and composition axioms. For slightly more details we refer the reader to [MSS]. □

Remark 6.1.1. This proposition shows a correspondence between operads and Schur functors. In fact, for algebraic operads we could have started by defining Schur functors and then defined an operad associated to a Schur functor (for example the book [LV] emphasizes Schur functors). Schur functors also constitutes a monad. Algebras over this monad are exactly the operad algebras.

6.2 Differentially Graded Operads

We would like to consider the homological algebra of our operads. That is, we would like to work with operads in the category $\mathbf{dgVect}_K$. However, if we just define operads in $\mathbf{dgVect}_K$ there is no a priori compatibility between the differential and the operadic maps and symmetries.

Definition 6.2.1. A differentially graded S-module (also called a dg S-module) is a functor **FinSet** $\to$ $\mathbf{dgVect}_K$ such that the differentials commute with the isomorphisms induced by isomorphisms in **FinSet**.

Definition 6.2.2. A differentially graded operad (also called a dg operad) is a dg S-module with operadic structure. The structure maps are morphisms in $\mathbf{dgVect}_K$. A dg operad morphism is an operad morphism that respect the differentials. Denote this category by **dgOp**.

6.3 Ideals and Quotient Operads

Definition 6.3.1. A left module $\mathcal{M}$ over an operad $\mathcal{O}$ is an S-module together with maps

$$\circ_i^{X,Y} : \mathcal{M}(X) \bigotimes \mathcal{O}(Y) \to \mathcal{M}((X \setminus \{i\}) \sqcup Y).$$

A right module $\mathcal{M}$ over an operad $\mathcal{O}$ is an S-module together with maps

$$\circ_i^{X,Y} : \mathcal{O}(X) \bigotimes \mathcal{M}(Y) \to \mathcal{M}((X \setminus \{i\}) \sqcup Y).$$

These has to satisfy commutative diagrams analogous to the ones in definition 4.2.1.

Definition 6.3.2. An $\mathcal{O}$-module is an S-module that is both a left and a right $\mathcal{O}$-module. The left and right structures has to be compatible in the sense analogous to a bimodule in the case of an R-module where R is a ring.

Definition 6.3.3. An S-submodule of an S-module A is given by an S-module B and natural inclusions $B(X) \to A(X)$.

Definition 6.3.4. An operadic ideal is an S-submodule that is also an $\mathcal{O}$-module.

Definition 6.3.5. Given an operad $\mathcal{O}$ and an S-submodule A we can define the ideal generated by A in the obvious way.

Definition 6.3.6. Given an algebraic operad $\mathcal{O}$ and an ideal $\mathcal{I}$ we can define the quotient operad $\mathcal{O}/\mathcal{I}$ as follows by taking the quotients $\mathcal{O}(X)/\mathcal{I}(X)$ for every labeling set X. This clearly yields an S-module. The operadic structure is induced from the one in $\mathcal{O}$; the maps are well defined since $\mathcal{I}$ is closed under compositions.

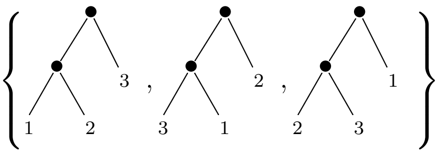
6.4 Lie

Using the machinery in section 6.3 we can define the operad $\mathcal{L}ie$. The algebras over this operad are the ordinary Lie algebras.

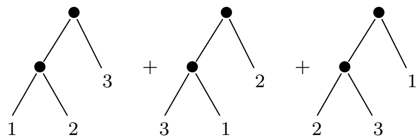
Definition 6.4.1. Consider the S-module A defined as follows.

$$A(n) = \begin{cases} k[\mu] & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases}$$

The functorial behavior is given such that a transposition of the elements in $\{1, 2\}$ induces multiplication by -1. This is also called the sign representation. Consider the free operad $\Psi(A)$. Now $\Psi(A)(3)$ is a 3-dimensional representation of $\mathbb{S}_3$ spanned by the following elements.



Since we only have one type of label at the vertices we do not write it out and just indicate with a dot. The action of $\mathbb{S}_3$ interchanges the labels on the leaves. There is an S-submodule R spanned by the following element.



Let $\mathcal{I}$ be the ideal generated by R. Finally let $\mathcal{L}ie = \Psi(A)/\mathcal{I}$.

Remark 6.4.1. The element μ corresponds to the Lie bracket. The sign representation codes the anti-symmetry. That we quote out by R corresponds to the Jacobi identity.

Remark 6.4.2. We could also define this operad by use of theorem 6.1.1 and the free Lie algebra functor. The machinery of quotient operads is however very useful, especially to define quadratic operads; see section 6.6.

6.5 The Cobar Complex

There are different approaches to constructing the cobar complex of an operad. We follow [GK] and [MSS] with the exception of allowing higher arities than two in the definition of the quadratic dual.

Definition 6.5.1. Suppose that $\mathcal{O}$ is a dg operad. Assume furthermore that $\mathcal{O}(X)$ is finite-dimensional for all X. Define the dual dg S-module $\mathcal{O}^*$ as the dual vector spaces $\mathcal{O}^*(X)$ with the $\mathbb{S}_n$-actions induced by the actions on $\mathcal{O}(X)$. The dual differential is given by $d^*(\alpha) = (-1)^{deg(\alpha)}\alpha(d)$ where d is the differential of $\mathcal{O}$. The operadic compositions

$$\circ_i^{X,Y} : \mathcal{O}(X) \bigotimes \mathcal{O}(Y) \to \mathcal{O}(X \setminus \{i\} \sqcup Y)$$

dualizes to give maps

$$\Delta_i^{X,Y} : \mathcal{O}^*(X \setminus \{i\} \sqcup Y) \to \mathcal{O}^*(X) \bigotimes \mathcal{O}^*(Y)$$

by setting

$$\Delta_i^{X,Y}(\lambda)(\alpha \otimes \beta) = \lambda(\alpha \circ_i^{X,Y} \beta).$$

Remark 6.5.1. The maps $\Delta_i^{X,Y}$ satisfies axioms dual to those of definition 4.2.1 (turn all arrows around). Such a structure is called a cooperad. Note also that the tree construction in section 4.8 extends to the case of dg operads (since the differential commutes with reorderings) giving us a construction of a free dg operad from a dg S-module.

Definition 6.5.2. Suppose $A \in \mathbf{dgVect}_K$ and define sA such that $(sA)^i = A^{i-1}$. Likewise, define s^{-1} such that $(s^{-1}A)^i = A^{i+1}$.

Definition 6.5.3. Given a finite set X, let $Det(X) \in \mathbf{dgVect}_K$ as $\bigwedge^{|X|}(sK^X)$, where $\bigwedge$ is the exterior product. Given a labelled rooted tree T define $Det(T)$ as $Det(E(T))$. Recall that $E(T)$ is the set of internal edges.

Definition 6.5.4. Given a dg operad $\mathcal{O}$, define the cobar bicomplex as follows. Its component in degree i, j is given by

$$(\mathcal{C}(\mathcal{O}))^{i,j} := \operatorname*{colim}_{T \in \mathbf{IsoTree}\ ,\ |E(T)|=i+1} (\mathcal{O}^*(T))^j \otimes Det(T).$$

The i-grading is over the positive integers and the j-grading over all integers. All objects are considered given some labeling of leaves. However, we choose not to write it out. The differential raising the j-degree by one is called the internal differential $d_{\mathcal{O}}$ and is given by the operad structure as defined in definition 6.5.1. The differential raising the i-degree by one is called the tree differential d_{Tree}. The ith component $d^i_{Tree} : (\mathcal{C}(\mathcal{O}))^{i,j} \to (\mathcal{C}(\mathcal{O}))^{i+1,j}$ is the colimit of the components $d_{T,T'}$ which are defined as

$$d_{T,T'} = \begin{cases} \Delta_e^{X,Y} \otimes e \wedge - & \text{if } T' = T \sqcup e \\ 0 & \text{otherwise} \end{cases}$$

where e is an internal edge inserted to give a tree T' with one extra internal edge. $\Delta_e^{X,Y}$ is the cooperad map (since adding an extra internal edge corresponds to splitting an operation into a composition of two operations).

Remark 6.5.2. The tree differential can be seen as a sum over all ways of inserting an extra edge. The associativity of the cooperad structure and the fact that the exterior algebra is anti-symmetric ensures that $d_{Tree}^2 = 0$.

We would like to reduce this to a complex with one grading, therefore we use the total complex construction as follows.

Definition 6.5.5. Given a dg operad $\mathcal{O}$, define the cobar complex $\mathcal{C}(\mathcal{O})^\bullet$ (in contrast to bicomplex) as follows. Start with the cobar bicomplex defined above and give it the grading $i + j$. Let the differential be given by $d_{Tree} + (-1)^{i-1} d_{\mathcal{O}}$. Since the cobar construction is made with colimits over trees as in the case of the free operad we can give it the operad structure of the free operad, namely the one given by grafting of trees.

Theorem 6.5.1. *The cobar complex construction is a contravariant functor*

$$\mathcal{C} : \mathbf{dgOp} \to \mathbf{dgOp}.$$

This functor preserves quasi-isomorphisms.

Proof. The proof is quite elaborate and we refer the reader to [MSS]. □

Definition 6.5.6. Given an operad $\mathcal{O}$, define the operadic suspension $\mathfrak{s}\mathcal{O}$ as the operad with dg S-module given in arity n by

$$s^{n-1}\mathcal{O}(n) \otimes sgn_n,$$

where sgn_n is the sign representation of the symmetric group, that is, we multiply by -1 if the permutation is odd. The operad structure is inherited from the operad structure of $\mathcal{O}$. The desuspension $\mathfrak{s}^{-1}\mathcal{O}$ is given by

$$s^{1-n}\mathcal{O}(n) \otimes sgn_n.$$

Definition 6.5.7. Define the dual dg operad as

$$\mathcal{D}(\mathcal{O}) := \mathfrak{s}^{-1}\mathcal{C}(\mathcal{O}).$$

Remark 6.5.3. Note that the dual dg operad differs from the quadratic dual operad defined in section 6.6. They are related though.

Definition 6.5.8. Given an operad $\mathcal{O}$, define the S-module $\mathcal{O}^+$ as follows.

$$\mathcal{O}^+(n) = \begin{cases} \mathcal{O}(n) & \text{if } n \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

This inherits an operad structure from $\mathcal{O}$.

Theorem 6.5.2. *There is a canonical quasi-isomorphism*

$$\mathcal{D}(\mathcal{D}(\mathcal{O})) \to \mathcal{O}^+.$$

Proof. The proof is quite long; we refer the reader to [MSS]. □

Remark 6.5.4. This theorem provides us with a canonical free resolution of the operad $\mathcal{O}$. However, the resulting operad is often large. We would prefer a smaller resolution. This is one of the motivations behind introducing the concept of quadratic operads and duals in section 6.6. More precisely we would like the resolution to be a minimal operad (see definition 6.7.2).

6.6 Quadratic Operads and Duals

A quadratic operad is an operad that is the quotient of a free operad such that the generators of the quotient ideal consists of sums of trees decorated with 2 operations. More precisely it is formulated like this.

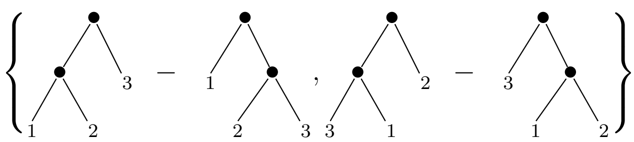
Definition 6.6.1. Suppose that E is an S-module. Suppose that R is an S-submodule of $\Psi(E)$ such that it is concentrated in tree-degree 2, that is, rooted trees with 1 internal edge. Then let $\mathcal{I}$ be the ideal generated by R and let

$$\mathcal{O}(E, R) := \Psi(E)/\mathcal{I}.$$

An operad admitting a presentation like this is called a quadratic operad.

Example 6.6.1. It is clear from the construction in section 6.4 that the operad $\mathcal{L}ie$ is quadratic.

Example 6.6.2. The operad $\mathcal{C}om$ which we defined earlier admits a quadratic presentation. Let $E(2)$ be the one-dimensional vector space generated by one element, the representation is taken to be the trivial representation, i.e., the $\mathbb{S}_n$ action changes nothing. Let $R(3)$ be the subspace generated by the following elements.

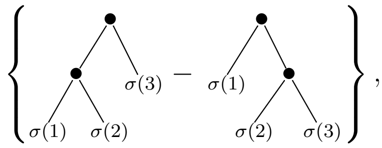


The resulting quadratic operad is exactly the operad $\mathcal{C}om$.

Example 6.6.3. The operad $\mathcal{A}ss$ coding associative algebras is also quadratic; its presentation can be given as follows. Consider the S-module A given by

$$A(n) = \begin{cases} k[\mathbb{S}_2] & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases}$$

where $k[\mathbb{S}_2]$ is the regular representation, i.e., it is a two-dimensional vector space where the permutation (12) permutes the basis elements. Now $\Psi(A)(3)$ is a 12-dimensional space spanned by all the possible compositions. There is a unique basis spanned by trees with two vertices where every vertex is labelled by the first of the two basis elements of $k[\mathbb{S}_2]$. Consider the subspace R which is spanned by elements of the form



for every $\sigma \in \mathbb{S}_3$. The resulting quadratic operad obtained by taking the quotient by the ideal generated by R is the operad $\mathcal{A}ss$.

Definition 6.6.2. Given a quadratic operad $\mathcal{O}(E, R)$ we define the quadratic dual data as follows. First consider $\Psi(E^\vee)$ where $E^\vee$ is defined as

$$E^\vee(n) = s^{-1}(Hom(k[s^{-1}]^{\otimes n}, k[s^{-1}]) \otimes E^*(n)).$$

Consider the subspace of trees decorated with 2 elements of $E^\vee$. These act on the subspace of $\Psi(E)$ consisting of trees decorated with 2 elements of E by the following pairing. By applying the induced permuting action to the trees we can transform them such that a tree of $\Psi(E)$ looks the same as a tree from $\Psi(E^\vee)$. Then we can apply the functionals one by one to the corresponding elements of the other tree and then multiply together. Extending this linearly we see that the subspace of trees decorated with 2 elements of $E^\vee$ acts on on the subspace of $\Psi(E)$ consisting of trees decorated with 2 elements of E. Let $R^\perp$ be the annihilator of $R \subseteq \Psi(E)$ under the pairing just described. Define the dual operad $\mathcal{O}^!$ as

$$\mathcal{O}^!(n) := \mathcal{O}^!(E^\vee, R^\perp).$$

Remark 6.6.1. In the case where E is concentrated on arity 2 there is an alternative characterization of $E^\vee$. Namely, we have $E^\vee = E^* \otimes sgn_2$ where sgn_2 is the sign representation. This is seen easily since $Hom(k[s^{-1}]^{\otimes 2}, k[s^{-1}])$ is in degree 1 with the sign representation.

The definition of dual operad may seem arbitrary but the following proposition shows that it is not.

Theorem 6.6.1. *There is a natural transformation*

$$\Theta_{\mathcal{O}} : \mathcal{D}(\mathcal{O}) \to \mathcal{O}^!.$$

Furthermore this induces an isomorphism

$$H^0(\mathcal{D}(\mathcal{O}(n)), d_{Tree}) \cong \mathcal{O}^!(n).$$

Proof. The proof is rather long and complicated. See for example [MSS]. □

Now a natural question to ask is when this natural transformation induces a quasi-isomorphism of the whole complex. If it does, the transformation given by first applying the quadratic dual construction and then the dual dg construction induces a quasi-isomorphism. This question is the seed of the concept of Koszul operad that is treated in section 6.7.

Example 6.6.4. The quadratic dual gives correspondences between the operads we have defined. We see that $\mathcal{L}ie^! \cong \mathcal{C}om$ and that $\mathcal{C}om^! \cong \mathcal{L}ie$. Furthermore we have that $\mathcal{A}ss^! \cong \mathcal{A}ss$.

The involutivity seen above is actually a more general theorem, also motivating the word dual.

Theorem 6.6.2. *Suppose $\mathcal{O}$ is a quadratic operad where $\mathcal{O}(n)$ is finite dimensional. Then*

$$(\mathcal{O}^!)^! = \mathcal{O}.$$

Proof. It is obvious by writing out the definition in full; the finite dimension hypothesis ensures that $(E^*)^* \cong E$. □

6.7 Koszul Operads

Definition 6.7.1. A quadratic operad is called Koszul when the natural transformation of theorem 6.6.1 induces a quasi-isomorphism of the dg operads.

Theorem 6.7.1. *A quadratic operad is Koszul if and only if there is a quasi-isomorphism*

$$\mathcal{D}(\mathcal{O}^!) \cong \mathcal{O}.$$

Proof. See for example [MSS]. □

Remark 6.7.1. Many natural quadratic operads turns out to be Koszul. For example $\mathcal{L}ie$, $\mathcal{C}om$, $\mathcal{A}ss$ are Koszul. The operad of Gerstenhaber algebras is defined and proved to be Koszul in section 6.9.1. For a lot more examples we refer the reader to [LV].

Definition 6.7.2. Let E be an S-module and let $\Psi(E)$ be the free operad on this S-module. Suppose that $\Psi(E)$ is differentially graded with differential δ. Suppose furthermore that $\delta(E)$ consists of decomposables, i.e., generated by trees with more than one vertex. Then $(\Psi(E), \delta)$ is called a minimal operad.

Definition 6.7.3. A minimal model of a dg operad $\mathcal{O}$ is a minimal operad $(\Psi(E), \delta)$ together with with a quasi-isomorphism

$$\mathcal{O} \cong (\Psi(E), \delta).$$

Theorem 6.7.2. *If there exists a minimal model it is unique up to isomorphism. Every differentially graded $\mathcal{O}$ operad such that $H(\mathcal{O})(1) = K$ admits a minimal model.*

Proof. The proof is quite long and technical; we refer the reader to [MSS]. □

Remark 6.7.2. Suppose that the quadratic dg operad $\mathcal{O}$ is Koszul, then $\mathcal{D}(\mathcal{O}^!)$ is a minimal model of $\mathcal{O}$. That $\mathcal{D}(\mathcal{O}^!)$ is minimal is seen directly from the construction of the differential. By the Koszulity assumption a quasi-isomorphism is given by the map $\Theta_{\mathcal{O}}$. This means that in the case of Koszul operads we have a concrete construction of the minimal model.

Definition 6.7.4. Given an operad $\mathcal{O}$ with a minimal model $\mathfrak{M}_{\mathcal{O}}$, a strong homotopy $\mathcal{O}$-algebra is defined as an algebra over $\mathfrak{M}_{\mathcal{O}}$.

Remark 6.7.3. Any $\mathcal{O}$-algebra is in particular a strong homotopy $\mathcal{O}$-algebra.

6.8 Distributive Laws

Definition 6.8.1. Given two S-modules A and B, define the tensor product as

$$(A \bigotimes B)(n) = \bigoplus_{i+j=n} Ind_{\mathbb{S}_i \times \mathbb{S}_j}^{\mathbb{S}_n} A(i) \otimes B(j) = \bigoplus_{i+j=n} (A(i) \otimes B(j)) \otimes_{k[\mathbb{S}_i \times \mathbb{S}_j]} k[\mathbb{S}_n],$$

where Ind is the induced representation.

Definition 6.8.2. Given two S-modules A and B, define $A \circ B$ as

$$A \circ B = \bigoplus_{k \geq 0} A(k) \otimes_{\mathbb{S}_k} B^{\otimes k}.$$

Remark 6.8.1. We can view $A \circ B$ in the same way as in the construction of the free operad. An element of $A \circ B$ is represented by a formal sum of isomorphism classes of trees with two "levels", where the first level vertex is colored by an element of A and the others are colored by elements of B.

Theorem 6.8.1. *The category of S-modules form a monoidal category with respect to the product $\circ$. A monoid in this category is exactly a unital operad. The composition morphisms corresponds to the morphism $\mathcal{O} \circ \mathcal{O} \to \mathcal{O}$.*

Proof. See for example [MSS] or [LV]. □

We would like to take the composition product of two operads and obtain a new operad; a priori there is however no canonical way of doing this.

Definition 6.8.3. Suppose A and B are algebraic unital operads. Suppose furthermore that we are given a map $\tau : A \circ B \to B \circ A$. If the following diagrams commute we call τ a distributive law.

$$\begin{array}{ccc}
A \circ B \circ B & \xrightarrow{Id \circ \gamma_B} & A \circ B \\
\downarrow{\scriptstyle \tau \circ Id} & & \\
B \circ A \circ B & & \downarrow{\scriptstyle \tau} \\
\downarrow{\scriptstyle Id \circ \tau} & & \\
B \circ B \circ A & \xrightarrow{\gamma_B \circ Id} & B \circ A
\end{array}$$

$$\begin{array}{ccc}
A \circ A \circ B & \xrightarrow{\gamma_A \circ Id} & A \circ B \\
\downarrow{\scriptstyle Id \circ \tau} & & \\
A \circ B \circ A & & \downarrow{\scriptstyle \tau} \\
\downarrow{\scriptstyle \tau \circ Id} & & \\
B \circ A \circ A & \xrightarrow{Id \circ \gamma_A} & B \circ A
\end{array}$$

$$\begin{array}{ccc}
A & \xrightarrow{Id \circ i_B} & A \circ B \\
& {\scriptstyle i_B \circ Id} \searrow & \downarrow{\scriptstyle \tau} \\
& & B \circ A
\end{array}
\qquad
\begin{array}{ccc}
B & \xrightarrow{i_A \circ Id} & A \circ B \\
& {\scriptstyle Id \circ i_A} \searrow & \downarrow{\scriptstyle \tau} \\
& & B \circ A
\end{array}$$

Theorem 6.8.2. *If there is a distributive law τ there is an operad structure on $A \circ B$ given by the composition of the following maps (composition of morphisms not operadic composition).*

$$(A \circ B) \circ (A \circ B) \xrightarrow{Id \circ \tau \circ Id} A \circ A \circ B \circ B \xrightarrow{\gamma_A \circ \gamma_B} A \circ B$$

Call this operad $(\mathcal{A} \circ \mathcal{B})_\tau$.

Proof. By inspection and drawing all the relevant diagrams. Some more details appear in [LV]. □

Theorem 6.8.3. *Suppose* $\mathcal{A}$ *and* $\mathcal{B}$ *are operads with a distributive law* τ. *Then* $(\mathcal{A} \circ \mathcal{B})_\tau$ *is Koszul if and only if both* $\mathcal{A}$ *and* $\mathcal{B}$ *are Koszul.*

Proof. See [LV]. □

One way of obtaining a distributive law is by using the notion of rewriting rule.

Definition 6.8.4. Suppose we have two quadratic operads $\mathcal{P}(E, R)$ and $\mathcal{Q}(F, S)$. Denote by $E \circ_1 F$ the S-submodule of $\Psi(E \oplus F)$ represented by trees with 2 vertices, where the vertex closest to the root is labelled by elements of E and the other by elements of F. A morphism $\lambda : F \circ_1 E \to E \circ_1 F$ is called a rewriting rule. The submodule of $\Psi(E \oplus F)$ spanned by elements of the form $x - \lambda(x)$ is denoted by D_λ. The quadratic operad $\mathcal{O}(E \oplus F, R \oplus S \oplus D_\lambda)$ is denoted by $\mathcal{P} \vee_\lambda \mathcal{Q}$.

A rewriting rule does not always imply a distributive law. The following theorem states a sufficient condition.

Theorem 6.8.4. *Let the notation be chosen as in definition 6.8.4. There is a surjective map* $p_\lambda : \mathcal{P} \circ \mathcal{Q} \to \mathcal{P} \vee_\lambda \mathcal{Q}$ *and a surjective map* $q_\lambda : \mathcal{Q} \circ \mathcal{P} \to \mathcal{P} \vee_\lambda \mathcal{Q}$. *If moreover* p_λ *is an isomorphism of* S*-modules it induces a distributive law and is an isomorphism of operads.*

Proof. The map p_λ is induced by the map

$$\Psi(E) \circ \Psi(F) \to \Psi(E \oplus F) \to \Psi(E \oplus F)/(R \oplus S \oplus D_\lambda)$$

and similarly for q_λ. The distributive law is given by $p_\lambda^{-1} \circ q_\lambda$. For the rest of the proof we refer the reader to [LV]. □

Remark 6.8.2. Note that theorem 6.8.4 together with theorem 6.8.3 yields a method of proving that quadratic operads is Koszul by decomposing them into simpler pieces.

The following lemma weakens the assumptions of theorem 6.8.4 such that it is easier to check.

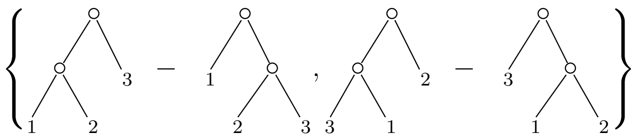
Lemma 6.8.1. *Suppose we have two quadratic operads* $\mathcal{P}(E, R)$ *and* $\mathcal{Q}(F, S)$. *Suppose moreover that we have a rewriting rule* $\lambda : F \circ_1 E \to E \circ_1 F$. *If the induced map* p_λ *is injective on the restriction to trees with* 3 *vertices the conclusion of theorem 6.8.4 holds.*

Proof. See [LV]. □

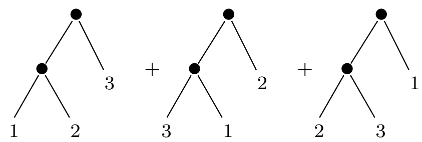
6.9 The Gerstenhaber Operad

As an example of an operad given by a distributive law we introduce the operad $\mathcal{G}erst$.

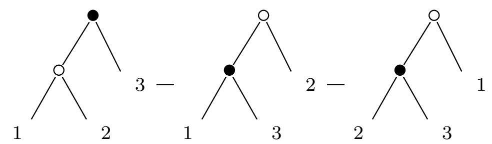
Definition 6.9.1. Consider the S-module E which is generated in arity 2 by one trivial representation in degree 0 (denoted $\circ$) and one trivial representation in degree 1 (denoted $\bullet$). Consider the S-submodule $R_{\mathcal{C}om}$ of $\Psi(E)$ spanned by the following elements.



Consider also the S-submodule R_{Jac} generated by the following element.



Finally consider the S-submodule R_{Der} generated by the following element.

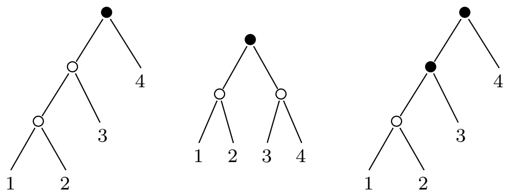


The quadratic operad obtained by taking the quotient of $\Psi(E)$ with the ideal generated by $\{R_{\mathcal{C}om}, R_{Jac}, R_{Der}\}$ is called the operad $\mathcal{G}erst$.

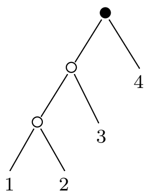
Remark 6.9.1. The topological operad of little disks defined in section 5.1.1 is fundamentally connected with the operad $\mathcal{G}erst$. Since the singular chains functor is symmetric monoidal we can apply it to the little disc operad and then take the homology. The resulting operad is isomorphic to the operad $\mathcal{G}erst$. This is proved in [Coh]. The generators of the operad $\mathcal{G}erst$ can be seen as coming from explicit chains. Since the S-module of the little disc operad is homotopy equivalent to the configuration space of centers of discs we can compute the homology of the configuration space instead. In arity 2 the configuration space is homotopy equivalent to the circle. The generator of degree 0 of $\mathcal{G}erst$ corresponds to the single homology generator in dimension 0. The degree 1 generator corresponds to the homology generator of dimension 1. The actions are determined by the transformation of cochains when we relabel the points in the configuration space.

Theorem 6.9.1. *The operad $\mathcal{G}erst$ can be obtained by a distributive law. The first operad in the composition is generated by the operation $\bullet$ together with the relations generated by $R_{\mathcal{C}om}$. The second operad is generated by the operation $\circ$ together with relations generated by R_{Jac}. The rewriting rule of definition 6.8.4 is given by R_{Der}.*

Proof. The idea is to use lemma 6.8.1. Let $\mathcal{P}$ be the quadratic suboperad generated by the operation $\bullet$ and let $\mathcal{Q}$ be the suboperad generated by $\circ$. We pick a basis of $\mathcal{P} \circ \mathcal{Q}$ and check that when rewriting the basis elements using the relations in different order we obtain equivalent elements of $\mathcal{Q} \circ \mathcal{P}$. Thus p_λ is injective and induces a distributive law. There are three critical cases corresponding to the following generators (all other elements are either spanned by those elements or contain only labels from one operad).

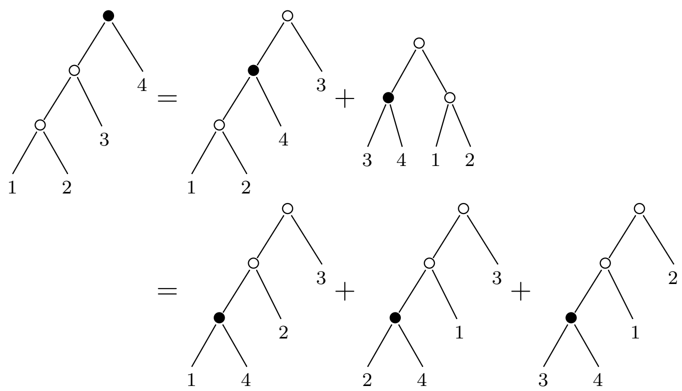


Case 1:

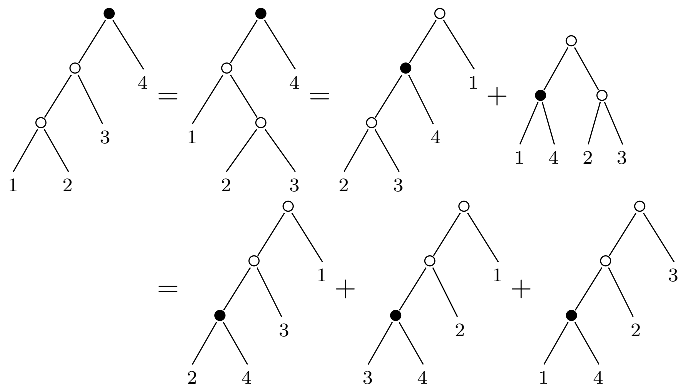


There are two different ways of rewriting this, either we use the rewriting law first or the associativity.

Rewriting law first:

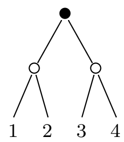


Associativity first:



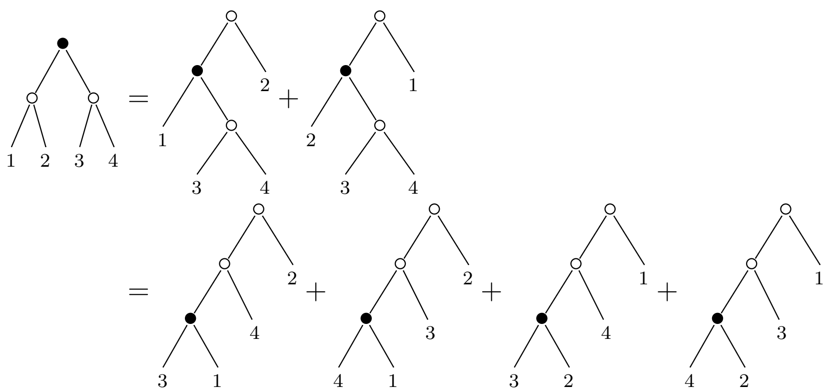
We see that the two expressions are already equivalent in $\mathcal{Q} \circ \mathcal{P}$, thus we do not introduce any new relations in this case.

Case 2:

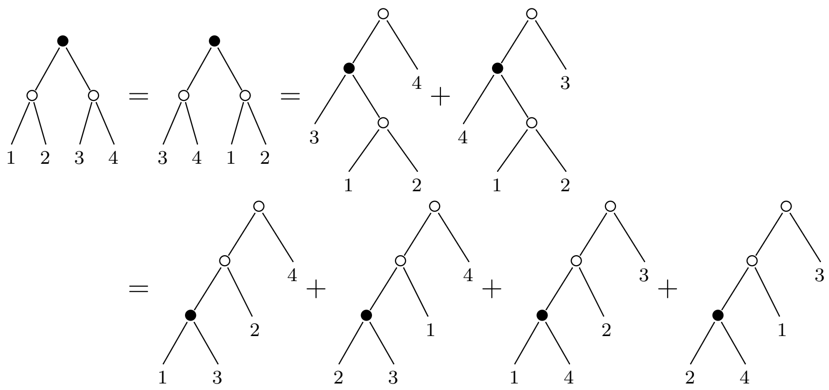


There are two different ways of rewriting this, either we apply the rewriting law directly or we use the symmetry and apply the rewriting law to the second argument first.

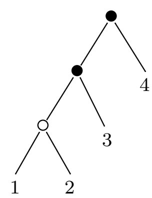
First argument first:



Second argument first:

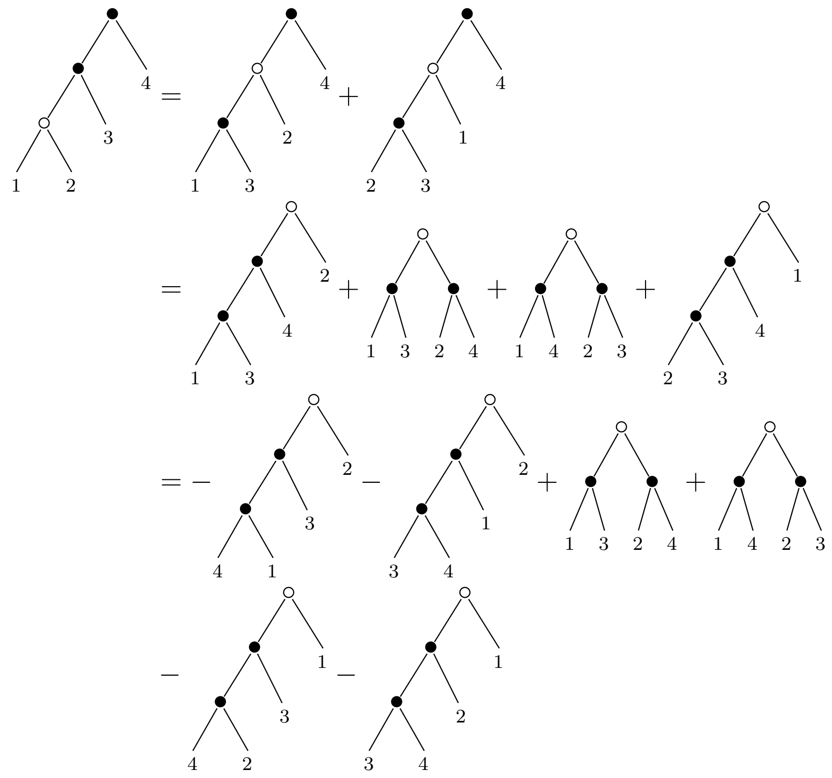


We see that these are equivalent because of the associativity of $\circ$.
Case 3:

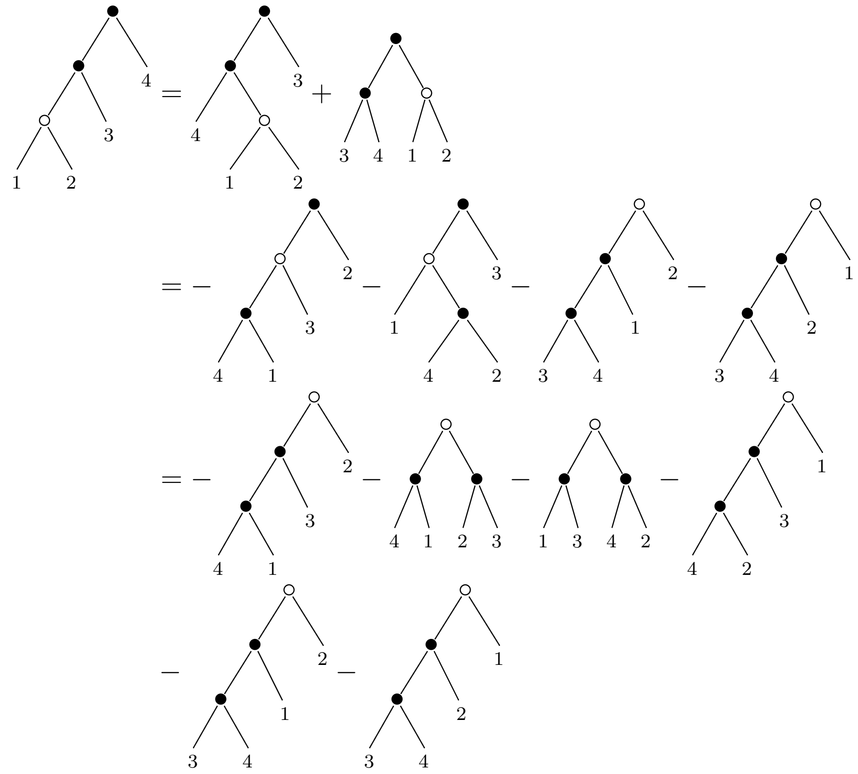


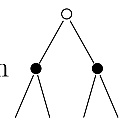
This tree can be rewritten in two ways as well. Either we use the rewriting rule first or we use the Jacobi rule first.

Rewriting rule first:



Jacobi rule first:



We see that these expressions are equivalent since the terms of the form  cancel because $\bullet$ has odd degree, causing minus signs from the Koszul sign rule. Thus p_λ is injective and $\mathcal{G}erst$ satisfy a distributive law.

□

7 The 2-Gerstenhaber Operad

7.1 The Graded Commutative Operad

Definition 7.1.1. Define the operad of graded commutative algebras by considering the operad $\mathcal{C}om$ (see example 6.6.2) to be an operad in $\mathbf{gVect}_K$ concentrated in degree 0 of a graded vector space. The algebras over this operad are the graded commutative ones, that is, the multiplication $\bullet$ satisfies

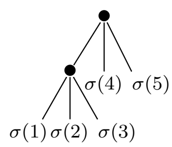
$$a \bullet b = (-1)^{deg(a)*deg(b)} b \bullet a.$$

Theorem 7.1.1. *The operad $\mathcal{C}om$ is Koszul.*

Proof. A standard result which is proven in most texts on operadic Koszul duality. See for example [MSS] or [LV]. □

7.2 The Operad 2-$\mathcal{L}ie$

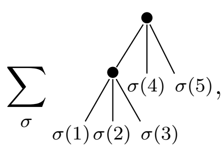
Definition 7.2.1. Consider an S-module E which as a vector space is a 1-dimensional space concentrated in arity 3 and degree -1. Let the representation be the sign representation, that is, the action of $\mathbb{S}_3$ is given such that if the permutation is odd the vectors are multiplied by -1. The space $\Psi(E)(5)$ is 10-dimensional. It has a basis spanned by elements of the form



where σ runs over a set of even permutations such that there is exactly one permutation for every way of assigning $\sigma(4)$ and $\sigma(5)$ (representative elements of the cosets of $\mathbb{S}_3$ in $\mathbb{S}_5$). An example of such a set of permutations is

$$\{12345, 14235, 13425, 43215, 12534, 15324, 52314, 14523, 54213, 34512\}.$$

Consider the subspace R spanned by



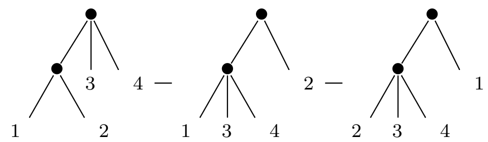
where σ runs over the permutations described above. The quadratic operad obtained by taking the quotient with the ideal generated by R we call the operad 2-$\mathcal{L}ie$.

Theorem 7.2.1. *The operad 2-$\mathcal{L}ie$ is Koszul.*

Proof. We refer the reader to [DK]. This is also mentioned but not proven explicitly in [HW] where 2-Lie algebras where introduced. These references considers a suspended version of ours. This change does however not change Koszulity. Another straight-forward but arduous way to check it is to use the "rewriting method" of [LV] to find a PBW basis. □

7.3 The Operad 2-$\mathcal{G}erst$

Definition 7.3.1. Consider the S-module E such that $E(2)$ is the trivial representation in degree 0 and such that $E(3)$ is the sign representation in degree -1. Let $\Psi(E)^{(2)}$ be the S-submodule given by trees with 2 vertices. Define submodules of $\Psi(E)^{(2)}$, denoted $R_{\mathcal{C}om}$ and $R_{2\text{-}\mathcal{L}ie}$, constructed as in the definition of $\mathcal{C}om$ and 2-$\mathcal{L}ie$. Let R_{Der} be the submodule spanned by following element.



It is clear by the arities which elements decorate which vertices. Remember that the permutation used to put the numbers in their place before using σ will cause some sign changes in the algebra. The quadratic operad obtained by taking the quotient by the ideal generated by

$$R = R_{\mathcal{C}om} + R_{2\text{-}\mathcal{L}ie} + R_{Der}$$

is called the operad 2-$\mathcal{G}erst$ and algebras over this operad are called 2-Gerstenhaber algebras.

Remark 7.3.1. Algebras over the operad 2-$\mathcal{G}erst$ consist of a graded vector space V equipped with a symmetric binary operation $\mu(x, y)$ of degree 0 and a skew-symmetric trinary operation $\tau(x, y, z)$ of degree -1. They satisfy the identities

$$\begin{aligned}
&\mu(x, y) = (-1)^{\bar{x}\bar{y}}\mu(y, x) \\
&\tau(x, y, z) = -(-1)^{\bar{x}\bar{y}}\tau(y, x, z) = -(-1)^{\bar{y}\bar{z}}\tau(x, z, y) \\
&\mu(\mu(x, y), z) = \mu(x, \mu(y, z)) \\
&\sum_{\sigma}(-1)^{sgn(\sigma)}\tau(\tau(\sigma(x), \sigma(y), \sigma(z)), \sigma(v), \sigma(w)) = 0 \\
&\tau(\mu(x, y), z, v) = (-1)^{\bar{y}(\bar{z}+\bar{w})}\mu(\tau(x, z, w), z) + (-1)^{\bar{x}(\bar{y}+\bar{z}+\bar{w})}\mu(\tau(y, z, w), x),
\end{aligned}$$

where $\bar{x}$ denotes the degree of x and $\sum_{\sigma}$ runs over all permutations in the set specified in the definition of 2-$\mathcal{L}ie$.

Remark 7.3.2. Applying the symmetric monoidal singular chains functor to the moduli space operad defined in 5.2.3 we obtain a differentially graded operad. Taking the cohomology of this yields exactly the operad 2-$\mathcal{G}erst$. The proof is quite complicated and we refer the reader to [EHKR]. However, we can indicate where the generators come from. The generator in arity 2 and degree 0 corresponds simply to the only 0-chain in M_3 since M_3 is just a point. The generator in arity

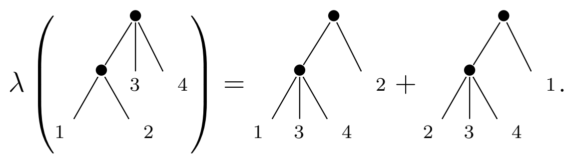
3 is represented by a 1-chain in M_4. M_4 is topologically just a circle and thus the cohomology is generated by a single element in degree -1. The relabeling action on the moduli space reverses the orientation of the 1-chain, thus this generator has the sign representation in the dg operad. The difficult part of the proof is to show that these two actually generates the whole cohomology operad.

Theorem 7.3.1. *The operad 2-$\mathcal{G}erst$ satisfies a distributive law such that it is composed of the operads $\mathcal{C}om$ and 2-$\mathcal{L}ie$.*

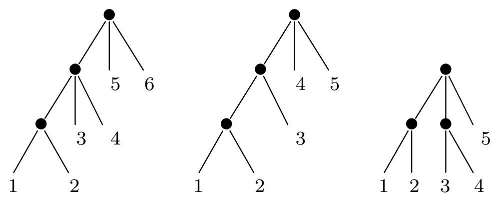
Proof. The proof idea is to use lemma 6.8.1 since by definition

$$2\text{-}\mathcal{G}erst \cong 2\text{-}\mathcal{L}ie \vee_\lambda \mathcal{C}om,$$

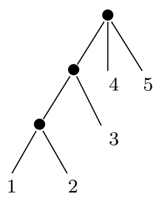
where λ is defined by



To prove that the map p_λ is injective on the restriction to trees of 3 vertices we consider generators of 2-$\mathcal{L}ie \circ \mathcal{C}om$ and check that if we rewrite them using the operadic relations we do not induce any extra relations in $\mathcal{C}om \circ$ 2-$\mathcal{L}ie$. There are 3 critical cases corresponding to the following generators (all other elements are either spanned by those elements or contain only labels from one operad).

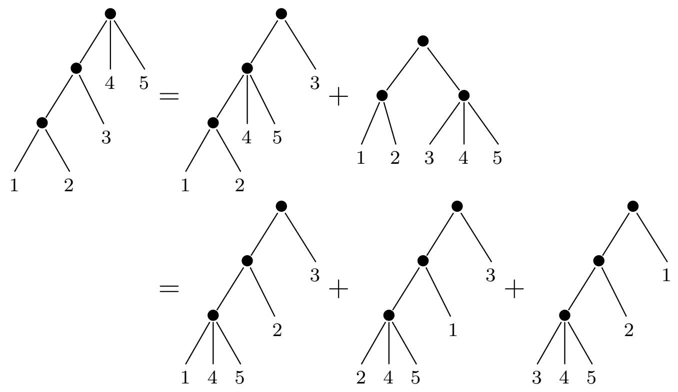


Case 1:

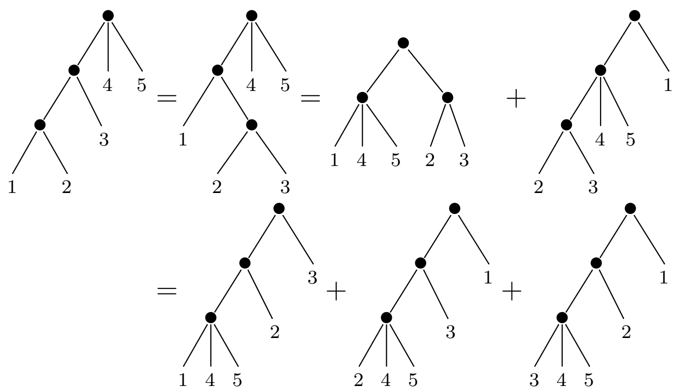


There are two ways of rewriting this tree, either we use the rewriting rule first or the rules coming from $R_{\mathcal{C}om}$. We have to check that the results are equivalent in $\mathcal{C}om \circ$ 2-$\mathcal{L}ie$.

Rewriting law first:

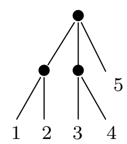


Associativity first:



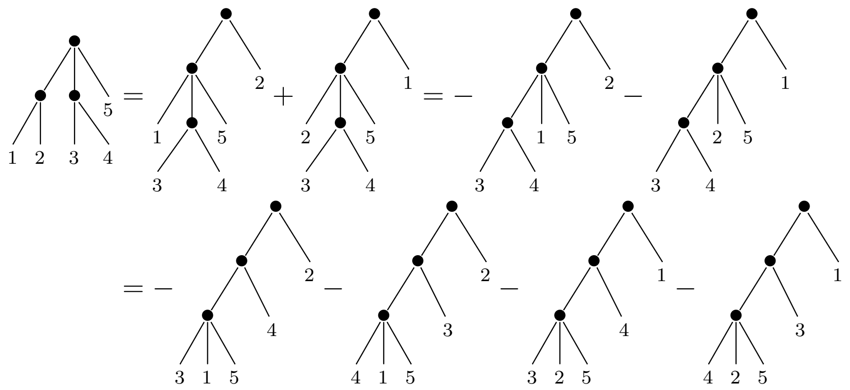
We see that the two expressions are already equivalent in $\mathcal{C}om \circ 2\text{-}\mathcal{L}ie$, thus we do not introduce any new relations in this case.

Case 2:

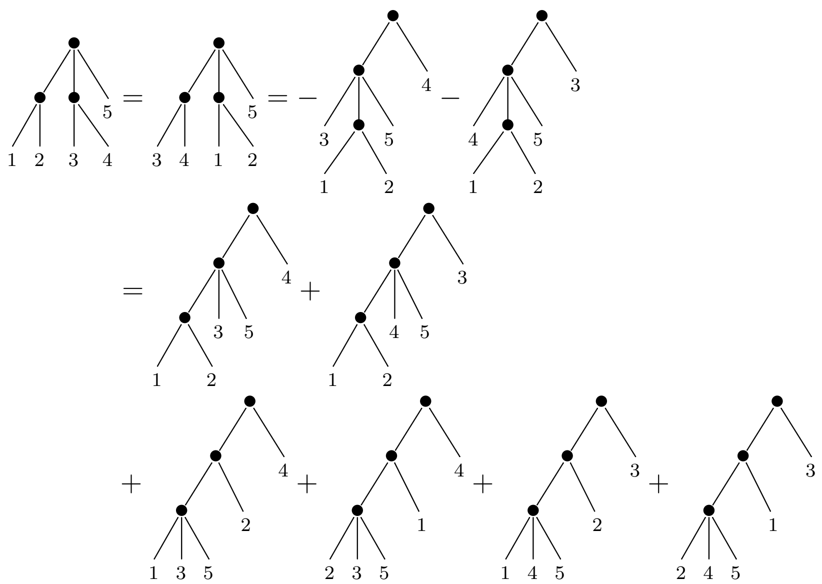


There are two ways of rewriting this tree, either we use the rewriting rule on the first or use the symmetric action and apply the rewriting rule on the second argument. We have to check that the results are equivalent in $\mathcal{C}om \circ 2\text{-}\mathcal{L}ie$.

First argument first:

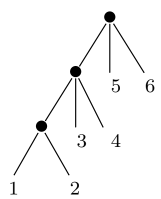


Second argument first:



We see that the two expressions are already equivalent in $\mathcal{C}om \circ 2\text{-}\mathcal{L}ie$, thus we do not introduce any new relations in this case.

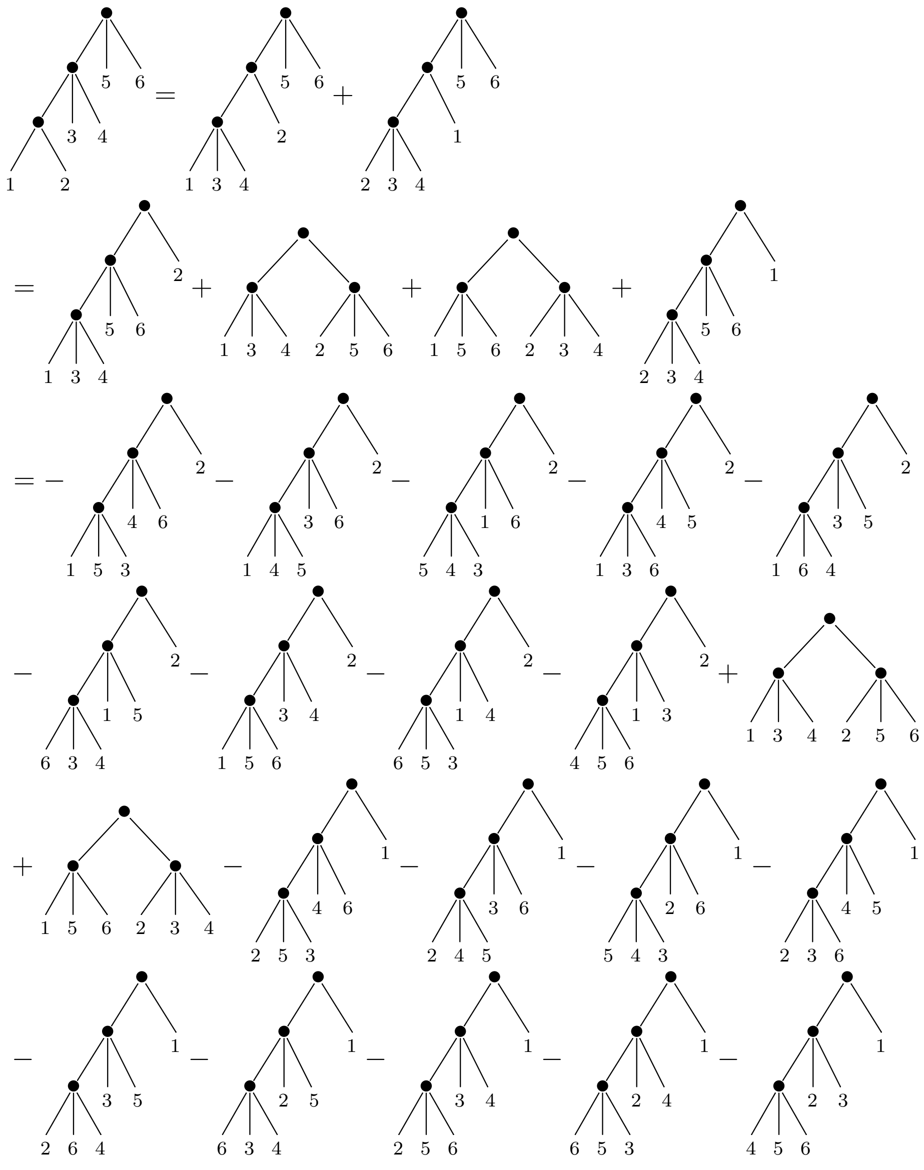
Case 3:



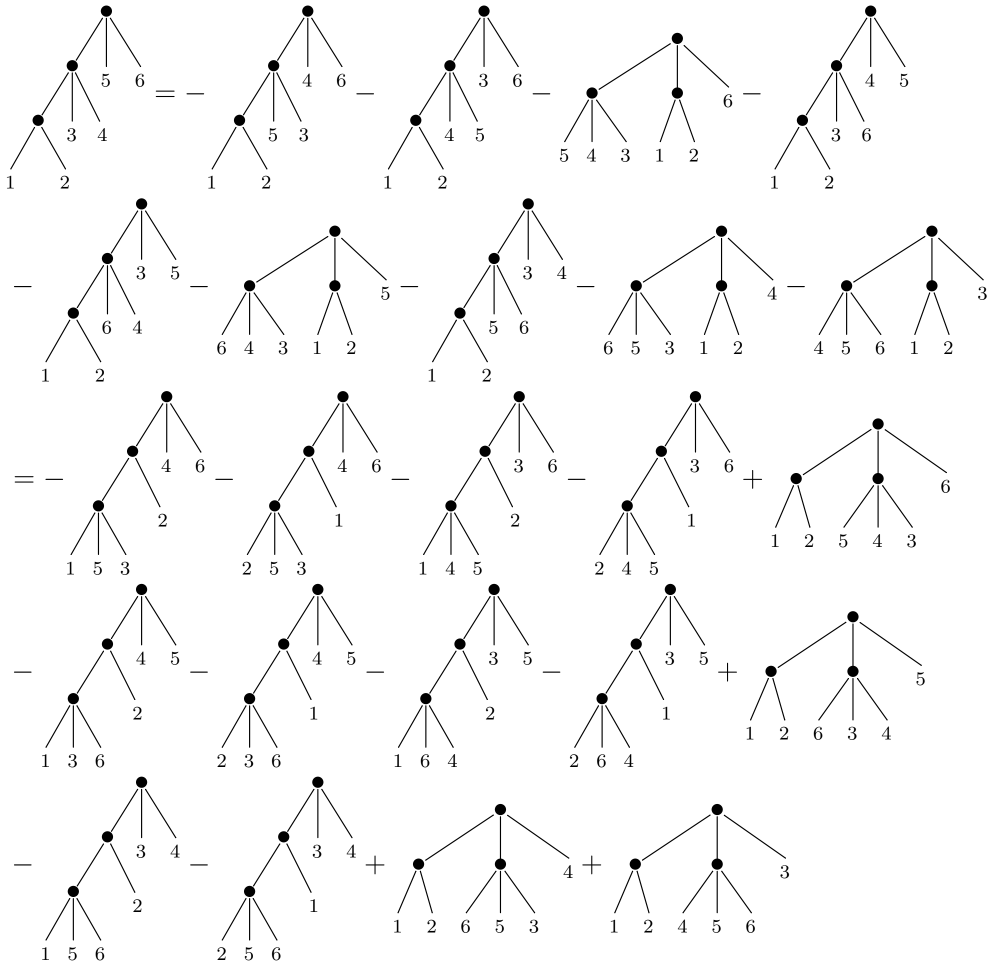
There are two ways of rewriting this tree, either we use the rewriting rule first or the rules coming from $R_{2\text{-}\mathcal{L}ie}$. We have to check that the results are equivalent in $\mathcal{C}om \circ 2\text{-}\mathcal{L}ie$. We choose a basis for the relation in $2\text{-}\mathcal{L}ie$ where the permutations are as follows:

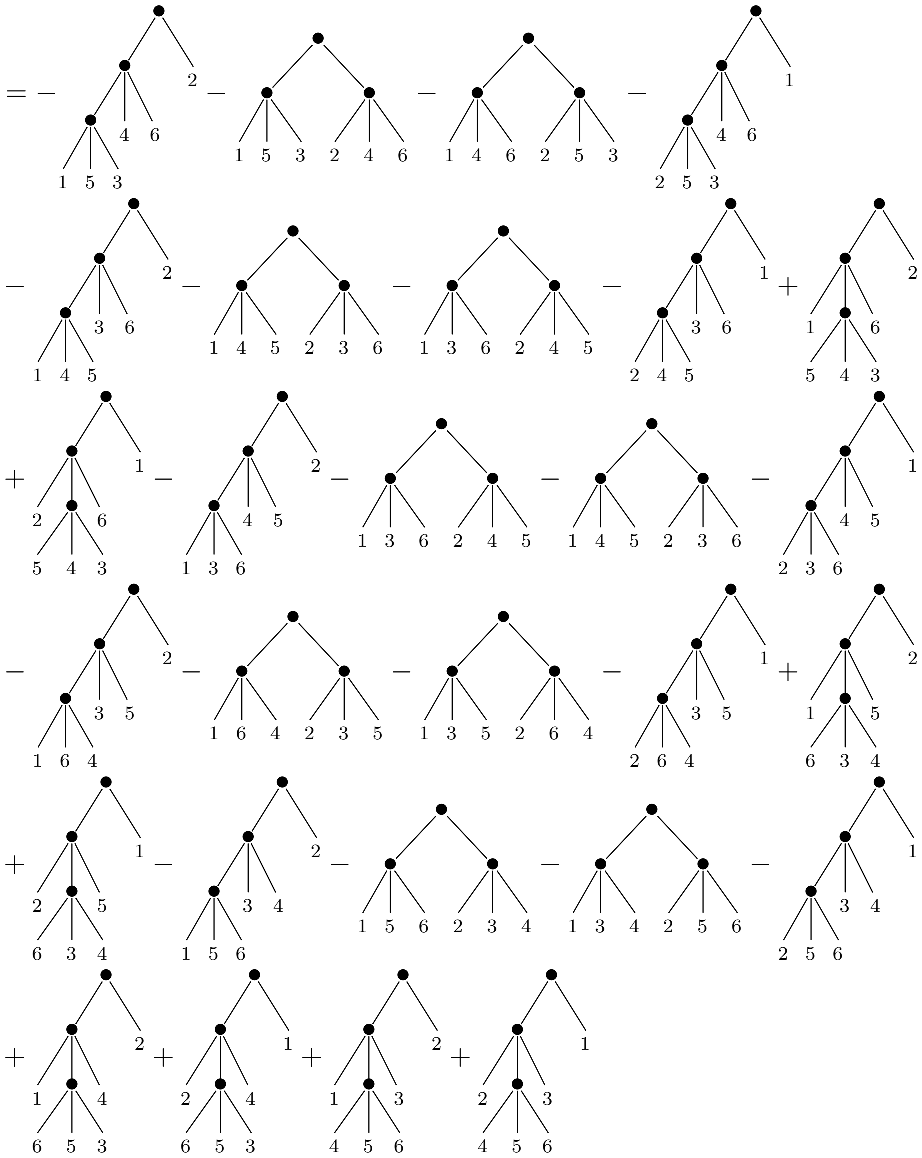
$$\{12345, 14235, 13425, 43215, 12534, 15324, 52314, 14523, 54213, 34512\}.$$

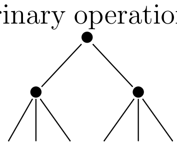
Rewriting law first:



10 term rule first:





We see that the two expressions are already equivalent in $\mathcal{C}om \circ 2\text{-}\mathcal{L}ie$ since firstly the trinary operation is alternating causing minus signs. Secondly, all terms of the form  cancel because the trinary operation is of odd degree causing a minus sign to appear from the Koszul sign rule. Thus we do not introduce any new relations in this case either. Since we do not introduce any new relations p_λ is a monomorphism, thus inducing a distributive law. □

Corollary 7.3.1. *The operad* 2-$\mathcal{G}erst$ *is Koszul.*

Proof. By above theorem, theorem 7.1.1, theorem 7.2.1 and theorem 6.8.3. □

7.4 Further Directions

Knowing that the operad 2-$\mathcal{G}erst$ is Koszul we can consider the cobar construction on the quadratic dual to obtain the dg operad 2-$\mathcal{G}erst_\infty$. What structure does it have? How does the corresponding cohomology theory for 2-Gerstenhaber algebras look. Is there any direct correspondence between the cochains of the operad $\mathcal{M}$ and 2-$\mathcal{G}erst_\infty$?

There is a theory of k-Lie algebras following [HW]; how would the corresponding operad k-$\mathcal{G}erst$ look? Are they Koszul? We could also consider combinations of several different operations of higher arities. Are any of them self-dual as the operad of Poisson algebras is?

8 Acknowledgements

Firstly, I would like to thank my advisor Professor Sergei Merkulov for introducing me to the wonderful theory of operads and for suggesting an interesting project to work on. Secondly, I would like to thank Professor Ralf Fröberg for reading the manuscript carefully and for his valuable comments. Thirdly, I would like to thank my dad for helping me out with numerous questions about LaTeX and for proof-reading.

References

[Bec] J. Beck, *Distributive laws*, Sem. on Triples and Categorical Homology Theory (ETH, Zürich, 1966/67), Lect. Notes Math. 80, Springer-Verlag, Berlin, 1969, 119–140.

[Coh] F. Cohen, *The homology of C_{n+1}-spaces, The homology of iterated loop spaces (Cohen, Frederick R. and Lada, Thomas J. and May, J. Peter)*, Lecture Notes in Mathematics, Vol. 533, Springer-Verlag, Berlin, 1976.

[DM] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. **36** 1969, 75–109.

[Dev] S. L. Devadoss, *Tessellations of moduli spaces and the mosaic operad*, Homotopy invariant algebraic structures (Baltimore, MD, 1998), Contemp. Math., vol. 239, Amer. Math. Soc., Providence, RI, 1999, 91–114.

[DK] V. Dotsenko and A. Khoroshkin, *Gröbner bases for operads*, Duke Math. J. **153**, no. 2 (2010), 363–396.

[EHKR] P. Etingof, A. Henriques, J. Kamnitzer and E. M. Rains, *The cohomology ring of the real locus of the moduli space of stable curves of genus* 0 *with marked points*, Ann. of Math. (2) **171** (2010), no. 2, 731–777.

[GK] V. Ginzburg and M. Kapranov, *Koszul duality for operads*, Duke Math. J. **76** (1994), no. 1, 203–272.

[HW] P. Hanlon and M. Wachs, *On Lie k-algebras*, Adv. Math., **113** (1995), no. 2, 206–236.

[Lei] T. Leinster, *Higher operads, higher categories*, London Mathematical Society Lecture Note Series, vol. 298, Cambridge University Press, Cambridge, 2004.

[LV] J-L. Loday and B. Valette, *Algebraic operads v 0.99*,

`http://www-irma.u-strasbg.fr/∼loday/PAPERS/LodayVallette.pdf` (19.01.2012).

[Mac] S. Mac Lane, *Categories for the working mathematician*, 2nd ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998.

[Mar] M. Markl, *Distributive laws and Koszulness*, Ann. Inst. Fourier (Grenoble) **46** (1996), no. 2, 307–323.

[MSS] M. Markl, S. Shnider and J. Stasheff, *Operads in algebra, topology and physics*, Mathematical Surveys and Monographs, vol. 96, American Mathematical Society, Providence, RI, 2002.

[Mas] W. S. Massey, *Singular homology theory*, Graduate Texts in Mathematics, vol. 70, Springer-Verlag, New York, 1980.

[May] P. May, *The geometry of iterated loop spaces*, Springer-Verlag, Berlin, 1972. Lecture Notes in Mathematics, Vol. 271.