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Elements of homological algebra and Hochschild complexes

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Abstract

This essay is aimed to be an introduction to homological algebra and Hochschild complexes for those who are familiar with only basic abstract algebra.

The first part deals with the fundamentals in homological algebra, which is a branch with lot of applications in algebraic geometry, algebraic topology, mathematical physics and many other branches.

The second part introduces Hochschild complexes and Hochschild (co)homology, named after Gerhard Paul Hochschild, who invented Hochschild cohomology in 1945. This part deals with homology theory for associative algebras over rings, and ends up with computing the Hochschild (co)homology of $\mathbb{R}[x]/\langle x^2\rangle$.

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Contents

1 Introduction to Homological Algebra

1.1 Definitions

Definition 1.1.1. - Modules. Let R be a unital ring and V an abelian group. If there exists a map

$$R \times V \to V$$

$$(\lambda, v) \mapsto \lambda v$$

satisfying

(i) $\lambda(\mu \mathbf{v}) = (\lambda\mu)\mathbf{v}, \quad \forall\ \lambda, \mu \in$R and $\forall\ \mathbf{v} \in V$

(ii) $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}, \quad \forall\ \lambda, \mu \in$R and $\forall\ \mathbf{v} \in V$

(iii) $\lambda(\mathbf{v} + \mathbf{w}) = \lambda\mathbf{v} + \lambda\mathbf{w}, \quad \forall\ \lambda \in$R and $\forall\ \mathbf{v}, \mathbf{w} \in V$

(iv) $1v = v, \quad \forall\ \mathbf{v} \in V$ (where 1 denotes the multiplicative identity in R)

then V is called a R-module.

In the special case when R is a field and V admits a basis, we say that V is a vector space over R.

Definition 1.1.2. - Direct product of modules. Let $\{V_i\}_{i\in I}$ be a family of R-modules. The direct product of these R-modules is the set all sequences $(x_i)_{i\in I}$ where $x_i \in V_i$, equipped with an addition operation

$(x_i)_{i\in I} + (y_i)_{i\in I} = (x_i + y_i)_{i\in I}$,

and a mulitplication operation

$\lambda(x_i)_{i\in I} = (\lambda x_i)_{i\in I}$, where $\lambda \in R$.

The direct product of $\{V_i\}_{i\in I}$ is denoted by $\prod\limits_{i\in I} V_i$.

Definition 1.1.3. - Direct sum of modules. The direct sum of a family of R-modules $\{V_i\}_{i\in I}$, denoted by $\bigoplus\limits_{i\in I} V_i$, is the subset of $\prod\limits_{i\in I} V_i$ consisting of all sequences $(x_i)_{i\in I}$ with only a finite number of non-zero elements. Notice that when I is finite, $\bigoplus\limits_{i\in I} V_i$ and $\prod\limits_{i\in I} V_i$ coincides.

Definition 1.1.4. - Homomorphisms of modules. Homomorphisms of R-modules, $f : V \to W$, has to satisfy all conditions for homomorphisms of groups with following extra condition:

$$f(\lambda v) = \lambda f(v) \quad \forall \lambda \in R \text{ and } \forall v \in V.$$

The set of all homomorphisms from V to W is denoted by $\mathrm{Hom}(V, W)$.

Definition 1.1.5. - $\mathbb{Z}$-graded modules. A direct sum of a family of modules $\{V_i\}_{i\in I}$ is called $\mathbb{Z}$-graded if the modules are indexed by the elements of $\mathbb{Z}$ (i.e. $I = \mathbb{Z}$) and is denoted by $\bigoplus\limits_{i\in \mathbb{Z}} V_i$.

Definition 1.1.6. - Homogenous homomorphism. A homomorphism $f : V \to W$, where V and W are $\mathbb{Z}$-graded modules, is called homogenous of degree n if $f(V_i) \subset W_{i+n}, \ \forall i \in \mathbb{Z}$.
For a given homogenous homomorphism f, $|f|$ denotes the degree of homogenety of f.

Definition 1.1.7. - Chain and cochain complexes. A chain complex (V, d) is a $\mathbb{Z}$-graded module, V, equipped with a homogeneous endomorphism d of degree -1 (i.e. $d(V_i) \subseteq V_{i-1}$), such that $d^2(V_i) = 0, \ \forall i \in \mathbb{Z}$.

$$\cdots \longleftarrow V_i \xleftarrow{d} V_{i+1} \xleftarrow{d} V_{i+2} \longleftarrow \cdots$$

An illustrative diagram of a chain complex

A cochain complex (V, d) is defined analogously with the difference that d is of degree 1.

$$\cdots \longrightarrow V_i \xrightarrow{d} V_{i+1} \xrightarrow{d} V_{i+2} \longrightarrow \cdots$$

The endomorphism d described above is called the differential of the complex (whether it is a chain or cochain complex).

Definition 1.1.8. - Homology and cohomology. Let (V, d) be a chain complex. Then the sets
$\quad \mathrm{Ker}\ d := \{v \in V \mid dv = 0\}$
$\quad \mathrm{Im}\ d := \{v \in V \mid v = dv', \text{ for some } v' \in V\}$
are two submodules of V.

Both Ker d and Im d can be regarded as $\mathbb{Z}$-graded modules with:
$\quad \mathrm{Ker}_i\ d := \mathrm{Ker}\ d \cap V_i \quad (= \mathrm{Ker}(d : V_i \to V_{i-1}))$
$\quad \mathrm{Im}_i\ d := \mathrm{Im}\ d \cap V_i \quad (= \mathrm{Im}(d : V_{i+1} \to V_i))$

It is obvious that $\mathrm{Im}\ d \subseteq \mathrm{Ker}\ d$ since for any $v \in \mathrm{Im}\ d$ we have according to definition that $v = dv'$ for some $v' \in V$. Hence $dv = ddv' = 0 \Rightarrow v \in \mathrm{Ker}\ d$.

Now since we have shown that Im d is a submodule of Ker d we can define the following quotient:

$$H(V) := \frac{\mathrm{Ker}\ d}{\mathrm{Im}\ d}$$

which also can be viewed as a $\mathbb{Z}$-graded module with

$$H_i(V) := \frac{\mathrm{Ker}_i\ d}{\mathrm{Im}_i\ d}.$$

$H(V)$ is called the homology of the chain complex (V, d).
Elements of Ker d is called cycles, and elements of Im d is called boundaries. Observe that every boundary is a cycle, but the reverse is not necesserly true.

In an analogously way, we define cohomologies, cocyles and coboundaries for cochain complexes.

Definition 1.1.9. - Exact sequences. A complex (V, d) is called an exact sequence if Ker $d =$ Im d, i.e. that every (co)cycle is a (co)boundary.

Observe that V is exact is equivalent with that the homology $H(V)$ is the set of the trivial complex $(..., 0, 0, 0, ...)$.

A short exact sequence is an exact sequence (V, d) with at most three non-trivial modules (i.e. not sets consisting of just 0).

$$0 \longrightarrow P \longrightarrow Q \longrightarrow R \longrightarrow 0$$

An illustrative diagram of a short exact sequence

Definition 1.1.10. - Homomorphism of complexes. Let (V, d) and (W, d') be two complexes. Then $f : V \to W$ is a homomorphism of complexes if it satisfy all conditions of homomorphisms of modules with the additional property that f commutes the differentials, i.e. $f \circ d = d' \circ f$.

1.2 Exact Triangles

Propostion 1.2.1. *For any homomorphism of complexes $f : (V, d) \to (W, d')$, we have that f induces canonically a homomorphism $H(f)$ of the associated (co)homology groups $H(f) : H(V) \to H(W)$.*

Proof. We start with proving following two implications:

$$\begin{aligned} v \in \text{Ker } d \Rightarrow f(v) \in \text{Ker } d', \\ v \in \text{Im } d \Rightarrow f(v) \in \text{Im } d'. \end{aligned} \tag{1}$$

This follows from the commutative diagram below (commutativity follows from the *Definition 1.1.10.*):

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \downarrow d & & \downarrow d' \\ dV & \xrightarrow{f} & d'W \end{array}$$

From the diagram we get that if $v \in$ Ker d, then $f(v) \in$ Ker d' since $d'f(v) = f(dv) = f(0) = 0$.

We have also that if $v \in \text{Im } d = dV$ then $f(v) \in d'W = \text{Im } d'$. Hence the implications in (1) is proven.

Now we define $H(f) : H(V) \to H(W)$ to be the map that takes the equivalence class $[v] \in H(V)$ to the equivalence class $[f(v)] \in H(W)$ (the equivalence class $[f(v)]$ exists since we have shown that $f(v) \in \text{Ker } d'$ if $v \in \text{Ker } d$). Now we have to show that this map is well defined map. We take another representative of the equivalence class $[v]$, say $v' = v + z$ where $z \in \text{Im } d$ and check that $H(f)$ maps $[v']$ and $[v]$ to the same element in $H(W)$.

$$[v'] \mapsto [f(v')] = [f(v+z)] = [f(v) + \underbrace{f(z)}_{\in \text{Im } d'}] = [f(v)] \leftarrow\!\shortmid [v].$$

$\square$

Lemma 1.2.2. Let

$$0 \xrightarrow{b} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} 0$$

be a short exact sequence. Then f is injective and g is surjective.

Proof. We have

$$\text{Ker } f = \text{Im } b = \{0\}$$

which yields that f is injective (f is injective iff its kernel contains only the zero element). We have also that

$$\text{Im } g = \text{Ker } h = C$$

implying that g is surjective. $\square$

Theorem 1.2.3. (Exact triangle)

Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence of complexes, i.e. A, B and C are cochain complexes, and f and g be homomorphisms homogeneous of degree 0. The associated sequence of cohomology groups

$$\cdots \longrightarrow H_n(A) \xrightarrow{H_n(f)} H_n(B) \xrightarrow{H_n(g)} H_n(C) \xrightarrow{\delta_n} H_{n+1}(A) \longrightarrow \cdots$$

is an exact sequence, where δ_n is the induced map from $H_n(C)$ to $H_{n+1}(A)$ (i.e δ is a homomorphism $H(C) \to H(A)$ homogeneous of degree 1).

Before proving this theorem we have to prove that the induced map δ exists.

Proposition 1.2.4. *Let A, B, C, f, g be as in Theorem 1, then there exists a degree 1 homomorphism, $\delta : H(C) \to H(A)$.*

Proof. (i) For any $[c_n] \in H_n(C)$ we have that $d_C c_n = 0$, since c_n is a cocycle. (ii) Since g is surjective (according to Lemma 1) there exists an element $b_n \in B_n$ such that $g(b_n) = c_n$.

(i) and (ii) with the commutative property in *Definition 1.1.10.* we get the following diagram

$$\begin{tikzcd}
b_n \arrow[r, mapsto, "g"] \arrow[d, mapsto, "d_B"] & c_n \arrow[d, mapsto, "d_C"] \\
b_{n+1} \arrow[r, mapsto, "g"] & 0
\end{tikzcd}$$

We see that $b_{n+1} \in \text{Ker } g$, implying $b_{n+1} \in \text{Im } f$ (due to the exactness), i.e. there exists an element $a_{n+1} \in A_{n+1}$ such that $f(a_{n+1}) = b_{n+1}$.

$$\begin{tikzcd}
 & b_n \arrow[r, mapsto, "g"] \arrow[d, mapsto, "d_B"] & c_n \arrow[d, mapsto, "d_C"] \\
a_{n+1} \arrow[r, mapsto, "f"] & b_{n+1} \arrow[r, mapsto, "g"] & 0
\end{tikzcd}$$

We have that $d_B b_{n+1} = d_B^2 b_n = 0$ yielding following diagram

$$\begin{tikzcd}
 & b_n \arrow[r, mapsto, "g"] \arrow[d, mapsto, "d_B"] & c_n \arrow[d, mapsto, "d_C"] \\
a_{n+1} \arrow[r, mapsto, "f"] \arrow[d, mapsto, "d_A"] & b_{n+1} \arrow[r, mapsto, "g"] \arrow[d, mapsto, "d_B"] & 0 \arrow[d, mapsto, "d_C"] \\
a_{n+2} \arrow[r, mapsto, "f"] & 0 \arrow[r, mapsto, "g"] & 0
\end{tikzcd}$$

Since f is injective according to Lemma 1 we have that $a_{n+2} = 0$, implying a_{n+1} is a cocycle. Hence the equivalence class $[a_{n+1}] \in H_{n+1}(A)$ exists.

We define δ to be the map that maps $[c_n]$ to $[a_{n+1}]$. We show that this map is well defined by choosing another representant of $[c_n]$, say $c_n + \tilde{c}_n$, where $\tilde{c}_n \in \text{Im } d_C$ and look where it is mapped under δ.

Since $\tilde{c}_n \in \text{Im } d_C$, $\exists\ \tilde{c}_{n-1} \in C_{n-1}$ such that $d_C \tilde{c}_{n-1} = \tilde{c}_n$, and since g is surjective $\exists\ \tilde{b}_{n-1}$ such that $g(\tilde{b}_{n-1}) = \tilde{c}_{n-1}$

This yields following diagram

$$\begin{tikzcd}
 & \tilde{b}_{n-1} \arrow[r, mapsto, "g"] \arrow[d, mapsto, "d_B"] & \tilde{c}_{n-1} \arrow[d, mapsto, "d_C"] \\
 & \tilde{b}_n \arrow[r, mapsto, "g"] \arrow[d, mapsto, "d_B"] & \tilde{c}_n \arrow[d, mapsto, "d_C"] \\
\tilde{a}_{n+1} \arrow[r, mapsto, "f"] & 0 \arrow[r, mapsto, "g"] & 0
\end{tikzcd}$$

Since f is injective, $\tilde{a}_{n+1} = 0$. This yields that $[\tilde{c}_n]$ is mapped to $[\tilde{a}_{n+1} = 0]$ under δ.

It is obvious that δ preserves addition, since it depends on homomorphisms that do preserve addition.
Hence $\delta_n[c_n + \tilde{c}_n] = \delta_n[c_n] + \delta_n[\tilde{c}_n] = [a_{n+1}] + [0] = [a_{n+1}] = \delta_n[c_n]$.

Hence δ is well defined, and the proposition is proven. $\square$

Proof of main theorem *(Exact Triangle).* We have to prove three equalities to obtain a proof.

Equality 1: Im $H_n(f) =$ Ker $H_n(g)$**.**
Since $g \circ f = 0$, it follows that $H_n(g) \circ H_n(f) = 0$, implying

$$\text{Im } H_n(f) \subseteq \text{Ker } H_n(g). \tag{2}$$

For any $[b_n] \in$ Ker $H_n(g)$ we have that $[g(b_n)] = [0]$, i.e. $g(b_n) \in$ Im d_C. Hence $\exists\ \tilde{c}_{n-1} \in C_{n-1}$ such that $d_C \tilde{c}_{n-1} = g(b_n)$.

The surjectivity of g and the commutativity described in *Definition 1.1.10.* yields following diagram.

$$\begin{tikzcd}
\widetilde{b}_{n-1} \arrow[rr, mapsto, "g"] \arrow[d, mapsto, "d_B"] & & \widetilde{c}_{n-1} \arrow[d, mapsto, "d_C"] \\
\widetilde{b}_n \arrow[rr, mapsto, "g"] & & g(b_n) \\
 & b_n \arrow[ur, mapsto, "g"'] &
\end{tikzcd}$$

Obviously $b_n - \widetilde{b}_n \in \text{Ker } g \overset{\text{exactness}}{\Longleftrightarrow} b_n - \widetilde{b}_n \in \text{Im } f \quad \Longleftrightarrow$
$\Longleftrightarrow \exists\ a_n \in A_n$ such that $f(a_n) = b_n - \widetilde{b}_n$.

Hence $H_n(f)([a_n]) = [f(a_n)] = [b_n - \underbrace{\widetilde{b}_n}_{\in \text{Im } d_B}] = [b_n] \in \text{Im } H_n(f)$.

We have shown that $[b_n] \in \text{Ker } H_n(g) \Longrightarrow [b_n] \in \text{Im } H_n(f)$, implying that Ker $H_n(g) \subseteq$ Im $H_n(f)$. This together with (2) yields a proof of *Equality 1.*

Equality 2: Im $H_n(g) = \text{Ker } \delta_n$.
For any $[c_n] \in \text{Im } H_n(g)$ there exists an equivalence class $[b_n] \in H_n(B)$ such that $H_n(g)([b_n]) = [c_n]$. Now according to *Proposition 1.2.4* , δ_n is the function taking $[c_n]$ to $[a_{n+1}] \in H_{n+1}(A)$ where $f(a_{n+1}) = d_B b_n = 0$ (b_n is a cocycle). Since f is injective, $a_{n+1} = 0$, so $\delta_n([c_n]) = [a_{n+1}] = [0]$. Hence $[c_n] \in \text{Ker } \delta_n$, implying Im $H_n(g) \subseteq \text{Ker } \delta_n$.

For any $[c_n] \in \text{Ker } \delta_n$ we have following diagram of elements (non-unique):

$$\begin{tikzcd}
a_n \arrow[d, mapsto, "d_A"] & b_n \arrow[d, mapsto, "d_B"] \arrow[r, mapsto, "g"] & c_n \arrow[d, mapsto, "d_C"] \\
a_{n+1} \arrow[r, mapsto, "f"] & b_{n+1} \arrow[r, mapsto, "g"] & 0
\end{tikzcd}$$

The existence of a_n follows from that $[a_{n+1}] = \delta([c_n])$. But $[c_n]$ is assumed to be in Ker δ_n, which means that a_{n+1} is a coboundary.
Obviously $b_n - f(a_n)$ is a cocycle of d_B (follows from the commutativity of f with the differentials). Hence

$$H_n(g)[b_n - f(a)] = [g(b_n) - \underbrace{g(f(a_n))}_{=0}] = [g(b_n)] = [c_n].$$

Hence $[c_n] \in \text{Im } H_n(g)$, implying that Ker $\delta_n \subseteq \text{Im } H_n(g)$. Since we have shown that Ker δ_n and Im $H_n(g)$ are subsets of each other, *Equality 2* is proven.

Equality 3: Im $\delta_n = \text{Ker } H_{n+1}(f)$.

For any $[a_{n+1}] \in \text{Im } \delta_n$, there exists a $[c_n] \in H_n(C)$ such that
$\delta_n([c_n]) = [a_{n+1}]$. We have according to Proposition 2 that $f(a_{n+1}) = d_B b_n$, for some $b_n \in B_n$ where $g(b_n) = c_n$.

Hence $H_{n+1}(f)([a_{n+1}]) = [f(a_{n+1})] = [\ \underbrace{d_B b_n}_{\text{coboundary}}\ ] = [0]$. This shows that $[a_{n+1}] \in \text{Ker } H_{n+1}(f)$, implying that Im $\delta_n \subseteq \text{Ker } H_{n+1}(f)$.

For any $[a_{n+1}] \in \text{Ker } H_n(f)$ we have that $f(a_{n+1}) = d_B b_n$ for some $b_n \in B_n$. Let $c_n := g(b_n)$. Obviously c_n is a cocycle since

$$d_C c_n = d_C g(b_n) = g(d_B b_n) = g(f(a_{n+1})) = 0.$$

Hence $[c_n] \in H_n(C)$ exists and is mapped to $[a_{n+1}]$ by δ_n, so $[a_{n+1}] \in \text{Im } \delta_n$. This proves that Ker $H_{n+1}(f) \subseteq \text{Im } \delta_n$.

Since we have shown that the two sets are subsets of each other, *Equality 3* is proven. $\square$

Remark 1. The theorem is named *Exact triangle* since the triangular diagram below illustrates the theorem:

$$\begin{array}{ccc} H(A) & \xrightarrow{H(f)} & H(B) \\ \delta \nwarrow & & \swarrow H(g) \\ & H(C) & \end{array}$$

Remark 2. Corresponding theorems where A, B and C are chain complexes, or when f and g are homogeneous of degree other than 1, can be proven in a similar way.

1.3 Homotopy

Definition 1.3.1. - Homotopic maps. Two homomorphisms of chain (respectively cochain) complexes $f, g : V \to W$ is said to be homotopic if there exists a homomorphism $s : W \to V$ homogeneous of degree -1 (respectively 1) such that

$$f - g = d_W \circ s + s \circ d_V.$$

Theorem 1.3.2. *Two homotopic homomorphisms $f, g : V \to W$ induce the same map between the (co)homologies. I.e.*

$$H(f) = H(g) : H(V) \to H(W).$$

(Definition and proof of existence of $H(f), H(g)$ is found in *Proposition 1.2.1.*)

Proof. For any $[v] \in H(v)$ (v has to be a (co)cycle) we have that

$$H(f)([v]) - H(g)([v]) = [d_W(s(v)) - s(\underbrace{d_V(v)}_{=0})] = [\ \underbrace{d_W(s(v))}_{\text{(co)boundary}}\ ] = [0]$$

Hence

$$H(f)([v]) = H(g)([v])$$

□

1.4 Resolutions

Definition 1.4.1. - Free modules. An R-module M is free if it admits a basis i.e. there exists a sequence $\{x_i\}_{i \in I}$ of elements in M, such that $\{x_i\}_{i \in I}$

is a linear independent set which spans M.

Lemma 1.4.2. *An R-module M is free iff $M \cong \bigoplus_{i\in I} R$.*

Proof. It is obvious that $\bigoplus_{i\in I} R$ is free with basis consisting of the elements $\{b_i\}_{i\in I}$ where b_i is 1 in position i and 0 in all other positions.

If M admits a basis $\{m_i\}_{i\in I}$ then we define a homomorphism $M \to \bigoplus_{i\in I} R$ such that $\sum \lambda_i m_i \mapsto \sum \lambda_i b_i$ (b_i is 1 in position i and 0 in all other positions). This is obviously an isomorphism, proving that $M \cong \bigoplus_{i\in I} R$. $\square$

Lemma 1.4.3 *For any R-module M there exists a free R-module F that surjects on M under some homomorphism $F \to M$.*

Proof. According to Lemma 1.4.2. we have that $F = \bigoplus\limits_{m\in M} R$ is a free module. Define $f : \bigoplus\limits_{m\in M} R \to M$, where $\sum \lambda r_m \mapsto \sum \lambda m$, where r_m is 1 in position m and zero in all other positions. This is obviously a surjective homomorphism. $\square$.

Definition 1.4.4. - Projective modules. An R-module M is projective if for any homomorphism $f : M \to N$ and any surjection $g : N' \to N$, there exists a homomorphism $\widetilde{f} : M \to N'$ such that the diagram below commutes

$$\begin{tikzcd} & M \arrow[dl, "\widetilde{f}"'] \arrow[d, "f"] \\ N' \arrow[r, "g"'] & N \end{tikzcd}$$

Lemma 1.4.5. *Free modules are projective.*

Proof. Let M be a free R-module with the basis $\{m_i\}_{i\in I}$. Let $f : M \to N$ be a homomorphism and $g : N' \to N$ be a surjection.

Let $\{n'_i\}_{i\in I}$ be a sequence of elements in N' such that $g(n'_i) = f(m_i)$ (such n'_i exists due to surjectivity). Define $\widetilde{f} : M \to N'$, $m_i \mapsto n'_i$.

Since M is free, any element $m \in M$ is on the form $m = \sum \lambda_i m_i$. This implies

$$g\left(\widetilde{f}(m)\right) = g\left(\widetilde{f}\left(\sum \lambda_i m_i\right)\right) = g\left(\sum \lambda_i \widetilde{f}(m_i)\right) =$$

$$= \sum \lambda_i g(n'_i) = \sum \lambda_i f(m_i) = f\left(\sum \lambda_i m_i\right) = f(m),$$

proving the commutativity. $\square$

Definition 1.4.6. - Resolutions. Let M be an R-module and (F, d) be a non-negative complex, i.e. $F_n = 0$ for $n < 0$. Assume that the sequence

$$\cdots \longrightarrow F_n \xrightarrow{d} F_{n-1} \xrightarrow{d} \cdots \xrightarrow{d} F_0 \xrightarrow{d_0} M \longrightarrow 0$$

is exact. Then the complex (F, d) is called a resolution of M.

Observe that we make a distinction between d and d_0. d_0 is not a part of the data in the resolution (F, d).

Defintion 1.4.7. - Free and Projective Resolutions. A resolution F is called free (or projective) if F_n is free (or projective) for all $n \geq 0$.

Proposition 1.4.8. *Any module admits a free resolution.*

Proof. Let M be an arbitrary R-module. We have according to *Lemma 1.4.3.* that there exists a free module F_0 that surjects on M under some homomorphism d_0. Using again *Lemma 1.4.3.* there exists a free module F_1 that surjects on the kernel $K_0 = \mathrm{Ker}(F_0 \to M)$ and is restricted to it. Continuing this way, we're getting a chain of free modules F_n that obviously form a free resolution of M.

Proposition 1.4.9. *Any module admits a projective resolution.*

Proof. It follows from *Lemma 1.4.5.* that free resolutions are projective resolutions as well, and hence the assertion follows from *Proposition 1.4.8.* $\square$

Theorem 1.4.10. *For any resolutions P and Q of a module M, with P projective, there exists a homomorphism of complexes $f : P \to Q$, homogeneous of degree 0.*

Proof. We prove the theorem by induction.

Claim 1 (base case): *There exists a homomorphism $f_0 : P_0 \to Q_0$ that commutes with the maps $d_0^P : P_0 \to M$ and $d_0^Q : Q_0 \to M$.*

Since the map $Q_0 \to M$ is surjective (due to exactness) and P_0 is projective, there exists a map $f_0 : P_0 \to Q_0$ that commutes with d_0^P and d_0^Q.

Claim 2: *Let $B(Q_n) = \mathrm{Im}_n\ d_Q$. Then $f_n \circ d_P(P_{n+1}) \subseteq B(Q_n)$.*

For any element $p_{n+1} \in P_{n+1}$ we have following diagram of elements:

$$\begin{array}{ccccccc}
p_{n+1} & \overset{d_P}{\longmapsto} & p_n & \overset{d_P}{\longmapsto} & 0 & \overset{d_P}{\longmapsto} & \cdots \\
 & & \Big\downarrow{\scriptstyle f_n} & & \Big\downarrow{\scriptstyle f_{n-1}} & & \\
 & & q_n & \overset{d_Q}{\longmapsto} & 0 & \overset{d_Q}{\longmapsto} & \cdots
\end{array}$$

We see that any element in P_{n+1} is mapped to a cycle under $f \circ d_P$. But the exactness implies that this cycle is a boundary as well, proving the claim.

Claim 3 (inductive step): *Assume that f restricted to P_k, $0 \leq k \leq n$, given by the homomorphisms $f_k : P_k \to Q_k$, gives rice to a commutative diagram:*

$$\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{d_P} & P_{n+1} & \xrightarrow{d_P} & P_n & \xrightarrow{d_P} & P_{n-1} & \xrightarrow{d_P} & \cdots & \xrightarrow{d_P} & P_0 & \searrow & \\
 & & & & \downarrow f_n & & \downarrow f_{n-1} & & & & \downarrow f_0 & & M \\
\cdots & \xrightarrow{d_Q} & Q_{n+1} & \xrightarrow{d_Q} & Q_n & \xrightarrow{d_Q} & Q_{n-1} & \xrightarrow{d_Q} & \cdots & \xrightarrow{d_Q} & Q_0 & \nearrow &
\end{array}$$

We want to show that we can construct f in a way so that if $f_{n+1} : P_{n+1} \to Q_{n+1}$ is included in the diagram above, the diagram would still be commutative.

We have shown in *Claim 2* that $f_n \circ d_P : P_{n+1} \to B(Q_n)$ and that Q_{n+1} surjects on $B(Q_n)$, so by the projectivity of P_{n+1} there is a function $f_{n+1} : P_{n+1} \to Q_{n+1}$, that fulfills $d_Q \circ f_{n+1} = f_n \circ d_P$.

Hence the existence of f has been proven inductively. $\square$

Theorem 1.4.11. *Let P and Q be two resolutions of a module M, where P is projective. Let $f, g : P \to Q$ be two homomorphisms of complexes homogeneous of degree 0. Then f and g are homotopic.*

Proof. Let s_i denote some homomorphism $P_i \to Q_{i+1}$. We prove the theorem in some steps and by induction.

Claim 1: $\mathrm{Im}(f_0 - g_0) \subseteq B(Q_0)$ *where $B(Q_n)$ is the image of Q_{n+1} under d_Q.*

For any element $p_0 \in P_0$ we have

$$d_0^Q[f(p_0) - g(p_0)] = d_0^Q[f(p_0)] - d_0^Q[g(p_0)] \overset{\text{commutativity}}{=} d_0^P(p_0) - d_0^P(p_0) = 0$$

Hence we have that $f(p_0) - g(p_0) \in \mathrm{Ker}\ d_0^Q$, implying $f(p_0) - g(p_0) \in B(Q_0)$ due to exactness.

Claim 2 (base case): *There exists homomorphisms s_0, s_{-1} such that $f_0 - g_0 = d_Q \circ s_0 + s_{-1} \circ d_P$.*

We have proven in *Claim 1* that $f_0 - g_0 : P_0 \to B(Q_0)$. Since Q_1 surjects on $B(Q_0)$ and P_0 is projective, there exists a homomorphism $s_0 : P_0 \to Q_1$ that commutes with the differentials, i.e.

$$f_0 - g_0 = d_Q \circ s_0$$

We have that s_{-1} is the zero map since it is defined on $P_{-1} = 0$ (distinct between d_P and d_0^P since d_P maps P_0 to $P_{-1} = 0$, while d_0^P maps P_0 to M), so $s_{-1} \circ d_P = 0$, which completes the proof of the claim.

Claim 3 (inductive step): *Assume that $f_n - g_n = d_Q \circ s_n + s_{n-1} \circ d_P$, then there exists a homomorphism s_{n+1} such that $f_{n+1} - g_{n+1} = d_Q \circ s_{n+1} + s_n \circ d_P$.*

Obviously $f_{n+1} - g_{n+1} - s_n \circ d_P : P_{n+1} \to Q_{n+1}$. We want to show that the image of this function is restricted to $B(Q_{n+1})$ (which coincides with the kernel of $d_Q : Q_{n+1} \to Q_n$), by applying d_Q on it. For any element $p_{n+1} \in P_{n+1}$ we have that

$$d_Q[(f_{n+1} - g_{n+1} - s_n \circ d_P)(p_{n+1})] =$$

$$= d_Q[f_{n+1}(p_{n+1})] - d_Q[g_{n+1}(p_{n+1})] - d_Q[s_n(d_P(p_{n+1}))] =$$

$$= \big[\text{We can add } (-s_{n-1}(d_P(d_P(p_{n+1})))) = 0\big] =$$

$$= d_Q[f_{n+1}(p_{n+1})] - d_Q[g_{n+1}(p_{n+1})] \underbrace{- d_Q[s_n(d_P(p_{n+1}))] - s_{n-1}[d_P(d_P(p_{n+1}))]}_{= -(d_Q \circ s_n + s_{n-1} \circ d_P)(d_P(p_{n+1}))}$$

$$= \underbrace{d_Q[f_{n+1}(p_{n+1})] - d_Q[g_{n+1}(p_{n+1})]}_{\substack{= f_n[d_P(p_{n+1}) - g_n[d_P(p_{n+1})] \\ \text{acc. to commutativity property}}} - \underbrace{(d_Q \circ s_n + s_{n-1} \circ d_P)(d_P(p_{n+1}))}_{\substack{= f_n[d_P(p_{n+1})] - g_n[d_P(p_{n+1})] \\ \text{acc. to assumption}}} = 0$$

Hence $\mathrm{Im}(f_{n+1} - g_{n+1} - s_n \circ d_P) \subseteq \mathrm{Ker}_{n+1}\, d_Q = B(Q_{n+1})$. We see that we can restrict the range of $f_{n+1} - g_{n+1} - s_n \circ d_P$ to $B(Q_{n+1})$. Obviously Q_{n+2} surjects on $B(Q_{n+1})$ under d_Q, so there exists $s_{n+1} : P_{n+1} \to Q_{n+2}$ such that $d_Q \circ s_{n+1} = f_{n+1} - g_{n+1} - s_n \circ d_P$ (since P_{n+1} is projective). This is equivalent to have $f_{n+1} - g_{n+1} = d_Q \circ s_{n+1} + s_n \circ d_P$.

Now we define s to be s_n when s is restricted to P_n. Then we have that

$$f - g = d_Q \circ s + s \circ d_P.$$

$\square$

1.5 Tensor Products and Algebras

Definition 1.5.1. - Tensor product. Let V and W be two R-modules and define

$$T[V \times W] := \left\{ \sum_{k=1}^{n} \lambda_k (v, w) \mid \lambda_k \in R,\ (v, w) \in V \times W \right\}.$$

Let $S \subseteq T[V \times W]$ be a subset containing all possible combination of elements of the form

$$(v + v', w) - (v, w) - (v', w), \quad (v, w + w') - (v, w) - (v, w')$$

$$\lambda(v, w) - (\lambda v, w), \quad \lambda(v, w) - (v, \lambda w)$$

where $v, v' \in V,\ w, w' \in W,\ \lambda \in R$

The tensor product of V and W is defined as the quotient space $T[V \times W]/S$ and denoted by $V \otimes_R W$ or $V \otimes W$ if it is clear from the context that V and W are R-modules.
The image of an element of $(v, w) \in T[V \times W]$ under the projection from $T[V \times W]$ to $V \otimes W$, is denoted by $v \otimes w$.

Definition 1.5.2. If M and N are two R-modules and $f : M \to M'$ and $g : N \to N'$ are homomorphisms of modules, then $f \otimes g : M \otimes N \to M' \otimes N'$ is defined by the map $m \otimes n \mapsto f(m) \otimes g(n)$. It is easy to check that this map is a well-defined homomorphism.

Definition 1.5.3. - Associative algebras. A module, A, equipped with homomorphism μ, with following properties

$$\mu : A \otimes A \to A$$

$$a_1 \otimes a_2 \mapsto a_1 a_2$$

$$\text{where } (a_1 a_2) a_3 = a_1 (a_2 a_3),$$

is called an associative algebra.

Remark. If we let μ define a multiplication on A then A becomes a ring.

Definition 1.5.4. - Tor. Let P be a projective resolution of an R-module M. Then for any R-module N we define following associated complex

$$\cdots \stackrel{id_N \otimes d_P}{\longrightarrow} N \otimes_R P_n \stackrel{id_N \otimes d_P}{\longrightarrow} \cdots \stackrel{id_N \otimes d_P}{\longrightarrow} N \otimes_R P_0 \longrightarrow 0$$

which we denote by $N \otimes_R P$.

$Tor^R(M,N)$ is defined to be the homology of the complex above, i.e. $Tor^R_n(M,N) = H_n(N \otimes_R P)$.

Remark. Observe that the resolution P is not specified in the definition of Tor, but it is still well defined due to next proposition.

Proposition 1.5.5. *$Tor^R_n(M,N)$ is independent of the choice of projective resolution of M.*

Proof. Let P and Q be two different projective resolutions of M. We have according to Theorem 1.4.10. that there exists homomorphisms $f : P \to Q$ and $g : Q \to P$ with $|f| = |g| = 0$.

According to Theorem 1.4.11. $g \circ f : P \to P$ is homotopic to id_P (the identity function on P).

This means that there exists a homomorphism $s_P : P_n \to P_{n+1}$ (i.e. $|s| = 1$) such that

$$id_p - g \circ f = d_P \circ s_P + s_P \circ d_P \tag{3}$$

Obviously $id_N \otimes id_P$ and $id_N \otimes (g \circ f)$ are homomorphisms homogeneous of degree 0 on $N \otimes P$. Now tensoring equation (3) with id_N from left yields

$$id_{N\otimes P} - id_N \otimes (g \circ f) = d_{N\otimes P} \circ (id_N \otimes s_P) + (id_N \otimes s_P) \circ d_{N\otimes P} \tag{4}$$

where $d_{N\otimes P}$ is the the differential $id_N \otimes d_P$ of $N \otimes P$.
From (4) we get that $id_{N\otimes P}$ and $id_N \otimes (g \circ f)$ are homotopic.

This means that

$$H(id_N \otimes (g \circ f)) \cong H(id_{N\otimes P}) \tag{5}$$

according to *Theorem 1.3.2.*.

The equalities below (which follows easily from defintion of tensor product and Proposition 1.2.1.)

$$H(id_N \otimes (g \circ f)) = H(id_N \otimes g) \circ H(id_N \otimes f)$$

and

$$H(id_{N\otimes P}) = id_{H(N\otimes P)}$$

implies together with (5) that

$$H(id_N \otimes g) \circ H(id_N \otimes f) \cong id_{H(N\otimes P)}. \tag{6}$$

Now since $f \circ g : Q \to Q$ (distinct between $g \circ f$ and $f \circ g$) we can in an analogous way as above derive

$$H(id_N \otimes f) \circ H(id_N \otimes g) \cong id_{H(N\otimes Q)}. \tag{7}$$

From (6) and (7) we can conclude that $H(id_N \otimes f)$ and $H(id_N \otimes g)$ are each others inverses, meaning that $H(id_N \otimes f)$ (and $H(id_N \otimes g)$) is injective and surjective. Hence $H(f \otimes id_N)$ is an ismomorphism $H(P \otimes N) \to H(Q \otimes N)$ implying $H(P \otimes N) \cong H(Q \otimes N)$. □

1.6 Ext

Lemma 1.6.1. *For any R-modules M and N, $\operatorname{Hom}(M, N)$ is also an R-module ($\operatorname{Hom}(M, N)$ defined in Definition 1.1.4.)*

Proof. It is obvious that $(\operatorname{Hom}(M, N), +)$ satisfies all group axioms (closure, associativity, existence of an (additive) identity, existence of (additive) inverses). The R-module structure of $\operatorname{Hom}(M, N)$ is given by the map

$$R \times \operatorname{Hom}(M, N) \to \operatorname{Hom}(M, N)$$

$$(\lambda, \phi) \mapsto \lambda\phi.$$

Obviously this structure satisfies all module axioms. □

Definition 1.6.2. For any R-module N and any homomorphism of R-modules $f : M \to M'$ induces a homomorphism $f^* : \operatorname{Hom}(M', N) \to \operatorname{Hom}(M, N)$ given by the map $\phi \mapsto \phi \circ f$.

Propotion 1.6.3 *If (C, d) is a chain complex with R-module structure, and M is any R-module, then*

$$\cdots \xrightarrow{d^*} \operatorname{Hom}(C_{n-1}, N) \xrightarrow{d^*} \operatorname{Hom}(C_n, N) \xrightarrow{d^*} \operatorname{Hom}(C_{n+1}, N) \xrightarrow{d^*} \cdots$$

is a cochain complex.

Proof. We want to prove that $(d^*)^2 = 0$. Pick $\phi \in \operatorname{Hom}(C, N)$ arbitrary. Then

$$(d^*)^2(\phi) = (d^* \circ d^*)(\phi) = (\phi \circ d) \circ d = \phi \circ (d \circ d) = 0.$$

□

Definition 1.6.4. - *Ext.* Let M be an R-module and (P, d) be a projective resolution of M. For any R-module N we have following associated complex

$$0 \longrightarrow \operatorname{Hom}(P_0, N) \xrightarrow{d^*} \operatorname{Hom}(P_1, N) \xrightarrow{d^*} \cdots$$

with $\operatorname{Hom}(P_0, N)$ in position 0. This complex is denoted by $\operatorname{Hom}(P, N)$. $Ext^R(M, N)$ is defined to be the homology of the complex above, i.e.
$Ext^R_n(M, N) = H_n(Hom(P, N))$.

Proposition 1.6.5. *$Ext(M, N)$ is independent of choice of projective resolution of M.*

Proof. The proof is almost identical to the proof of Proposition 1.5.4. The main difference is that instead of tensoring

$$id_p - g \circ f = d_P \circ s_P + s_P \circ d_P$$

with id_N, we take their induced maps defined in Definition 1.6.2 and get

$$id_p^* - f^* \circ g^* = s_P^* \circ d_P^* + d_P^* \circ s_P^*$$

(observe that $(g \circ f)^*$ is given by $f^* \circ g^*$). Continuing as in the proof of Proposition 1.5.4. we get that $H(f^*)$ and $H(g^*)$ are each others inverses implying that

$$H(f^*) : H(\mathrm{Hom}(Q, N)) \to H(\mathrm{Hom}(P, N))$$

is an isomorphism, proving that $H(\mathrm{Hom}(Q, N)) \cong H(\mathrm{Hom}(P, N))$. $\square$

2 Hochschild Homology and Cohomology

In this section we will study Hochschild (co)chain complexes and their (co)homology. To distincit between Hochschild homolohy and cohomology, the Hochschild chain complexes and homology is denoted by $C_\bullet$ respective $H_\bullet$ and are indexed in the lower right side, while the Hochschild cochain complexes and cohomology is denoted by $C^\bullet$ respective $H^\bullet$ and are indexed in the upper right side.

We will during all this section consider non-negative complexes, i.e. complexes with the trivial module, denoted by 0, in all negative positions.

2.1 Hochschild Homology

Definition 2.1.1. - A^{op}, the opposite algebra of A. If A is an algebra with μ defining its multiplication, then A^{op} contains the same elements as A but equipped with a multiplication given by $a_1 *_{\mathrm{op}} a_2 = \mu^{\mathrm{op}}(a_1, a_2) = \mu(a_2, a_1) = a_2 a_1$.

Definition 2.1.2 - Bimodules over algebras. If A is an algebra and M is an A-module from both right and left side, satisfying $a(ma') = (am)a'$ where $a, a' \in A,\ m \in M$, then M is called a bimodule over A.

A bimodule over A can also be considered as a right module over
$A^{\mathrm{e}} = A \otimes A^{\mathrm{op}}$, via $m(a \otimes a') = a'ma$ (where $a \otimes a' \in A^{\mathrm{e}},\ m \in M$).

Definition 2.1.3 - $C_\bullet(A, M)$ and the Hochschild Boundary. Let A be an algebra and M a bimodule over A. Consider the non-negatively $\mathbb{Z}$-graded module $C_\bullet(A, M) = \bigoplus_{n=0}^{\infty}(M \otimes A^{\otimes n})$, where $M \otimes A^{\otimes 0}$ is defined to be M. The Hochschild boundary $b : C_\bullet(A, M) \to C_\bullet(A, M)$ is a homomorphism homogeneous of degree -1, with b_n, the Hochschild boundary restricted to $C_n(A, b)$, defined as follows:

$$b_n : C_n(A, M) \to C_{n-1}(A, M), \quad \forall\ n \geq 1$$

$$m \otimes a_1 \otimes \cdots \otimes a_n \mapsto (ma_1 \otimes a_2 \otimes \cdots \otimes a_n) + \sum_{i=1}^{n-1} (-1)^i (m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n)$$
$$+ (-1)^n (a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1}).$$

We can express b_n in a more compressed way if we define the maps
$d_{n_i} : C_n(A, M) \to C_{n-1}(A, M),\ i = 0, 1, 2, ..., n$, where

$$d_{n_0}(m \otimes a_1 \otimes \cdots \otimes a_n) := ma_1 \otimes a_2 \otimes \cdots \otimes a_n$$
$$d_{n_i}(m \otimes a_1 \otimes \cdots \otimes a_n) := m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \quad \text{for } 1 \leq i \leq n-1$$

$$d_{n_n}(m \otimes a_1 \otimes \cdots \otimes a_n) := a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1}$$

If we let $b'_n = \sum_{i=0}^{n-1}(-1)^i d_{n_i}$ (the sum goes only up to $n-1$), obviously $b_n = b'_n + (-1)^n d_{n_n}$.

Proposition 2.1.4. *The maps b' and b defined in Definition 2.1.3. are differentials, i.e. $b' \circ b' = b \circ b = 0$.*

Proof. For b we have that

$$b_{n-1} \circ b_n = \sum_{\substack{0 \le i \le n \\ 0 \le j \le n-1}} (-1)^{i+j} d_{(n-1)_j} \circ d_{n_i}$$

We divide this sum into two sums, $S_1 + S_2$ where

$$S_1 := \sum_{0 \le j < i \le n} d_{(n-1)_j} \circ d_{n_i}$$

$$S_2 := \sum_{0 \le i \le j \le n-1} d_{(n-1)_{i-1}} \circ d_{n_j}$$

Now for any summand $(-1)^{i+j} d_{(n-1)_j} \circ d_{n_i}$ in S_1 cancels out with the summand $(-1)^{i+j-1} d_{(n-1)_{i-1}} \circ d_{n_j}$ in S_2. The correspondence between the summands in S_1 and their outcancelling summands in S_2 is $1-1$, and since both sums have the same number of elements ($1+2+...+n = (n+1)n/2$ elements), every summand in S_2 has also a corresponding outcancelling summand in S_1. This proves that $b_{n-1} \circ b_n = 0$ or more generally $b \circ b = 0$ (since n was choosen arbitrary).

The proof of $b' \circ b' = 0$ can be constructed in a similar way. $\square$

Corollary/Definition 2.1.5. - Hochschild Chain Complex and Homology. $(C_\bullet(A, M), b)$ is a chain complex (corollary of *Proposition 2.1.4.*), called the Hochschild chain complex of A with coefficients in M.

$$\cdots \xrightarrow{b} M \otimes A^{\otimes n} \xrightarrow{b} M \otimes A^{\otimes n-1} \xrightarrow{b} \cdots \xrightarrow{b} M \otimes A \xrightarrow{b} M \longrightarrow 0$$

An illustrative diagram of $C_\bullet(A, M)$.

The homology of a Hoschschild chain complex is called Hochschild homology denoted by $H_\bullet(A, M)$.

Definition 2.1.6. - The bar complex C^{bar}. Let A be an algebra and b' is defined as in Definition 2.1.3. Then the bar complex $C^{\text{bar}}(A)$ is given by

$$C^{\text{bar}}(A) : \quad \cdots \xrightarrow{b'} A^{\otimes n} \xrightarrow{b'} \cdots \xrightarrow{b'} A^{\otimes 2}$$

Proposition 2.1.7. *Let C be an arbitrary complex with the differential d. If there exists a homomorphism $s : C \to C$ such that*

$$d \circ s + s \circ d = id_C \tag{8}$$

then the C is exact.

Proof. It is obvious that Im $d \subseteq$ Ker d since $d \circ d = 0$ (one of the conditions of the differential). We want to show that (8) implies Ker $d \subseteq$ Im d .

Pick $c \in$ Ker d arbitrary. Then

$$d(s(c)) + s(\underbrace{d(c)}_{=0}) = id_C(c) = c \quad \Longleftrightarrow \quad d(s(c)) = c$$

$$\Longrightarrow \quad c \in \text{Im } d.$$

□

Proposition 2.1.8. *If A is unital, then $C^{\text{bar}}(A)$ is a resolution of A (the map from $A^{\otimes 2}$ to A is also given by b').*

Proof. We want to prove that

$$\cdots \xrightarrow{b'} A^{\otimes n} \xrightarrow{b'} \cdots \xrightarrow{b'} A^{\otimes 2} \xrightarrow{b'} A \longrightarrow 0$$

is an exact sequence.
In order to do that we have to prove that $A^{\otimes 2}$ surjects on A under b' (since all of A is mapped to 0), and then prove that Im $b' =$ Ker b'.

b' restricted to $A^{\otimes 2}$ is obviosly given by the map $a \otimes a' \mapsto aa'$. For any element $a \in A$ we have that $b'(1 \otimes a) = 1a = a$ proving the that $A^{\otimes 2}$ surjects on A under b'.

Consider $s_n : C_n^{\text{bar}}(A) \to C_{n+1}^{\text{bar}}(A)$, where $s_n(a_1 \otimes ... \otimes a_n) = 1 \otimes a_1 \otimes \cdots \otimes a_n$.

Now consider $s \circ b' + b' \circ s$ restricted to $C_n^{\text{bar}}(A)$. This is

$$s_{n-1} \circ b'_n + b'_{n+1} \circ s_n = \sum_{i=0}^{n-1} (-1)^i s_{n-1} \circ d_{n_i} + \sum_{i=0}^{n} (-1)^i d_{(n+1)_i} \circ s_n.$$

It is easy to check that the k'th summand in the first sum cancels out with the $(k+1)$'th summand in the second sum. The only summand that does not cancels out is $d_{(n+1)_0} \circ s_n$ in the second sum.

Obviously $d_{(n+1)_0} \circ s_n = id_{C_n^{\text{bar}}(A)}$. Hence $s \circ b + b \circ s = id_{C^{\text{bar}}(A)}$ implying that Im $b' =$ Ker b' according to Proposition 2.1.7. $\square$

Proposition 2.1.9. *Let M be a bimodule over a unital algebra A (or equivalently, M is a right A^{e}-module) and consider $A^{\otimes n}$ as a left-side A^{e}-module, given by $(a \otimes a')(a_1 \otimes \cdots \otimes a_n) = aa_1 \otimes a_2 \otimes \cdots \otimes a_{n-1} \otimes a_n a'$. Then we have follwing isomorphism:*

$$M \otimes_{A^{\mathrm{e}}} A^{\otimes n+2} \cong M \otimes_R A^{\otimes n}$$

Proof. Every nonzero element $m \otimes_{A^{\mathrm{e}}} (a_1 \otimes \cdots \otimes a_{n+1}) \in M \otimes_{A^{\mathrm{e}}} A^{\otimes n+2}$ can be decomposed in following way:

$$m \otimes_{A^{\mathrm{e}}} (a_1 \otimes \cdots \otimes a_{n+1}) = m \otimes_{A^{\mathrm{e}}} [(a_1 \otimes a_{n+2})(1 \otimes a_2 \otimes \cdots \otimes a_{n+1} \otimes 1)] =$$

$$= m(a_1 \otimes a_{n+2}) \otimes_{A^{\mathrm{e}}} [1 \otimes a_2 \otimes \cdots \otimes a_{n+1} \otimes 1] =$$
$$= a_{n+2} m a_1 \otimes_{A^{\mathrm{e}}} [1 \otimes a_2 \otimes \cdots \otimes a_{n+1} \otimes 1]$$

Now we define the map $f : M \otimes_{A^{\mathrm{e}}} A^{\otimes n+2} \to M \otimes_R A^{\otimes n}$ by

$$a_{n+2} m a_1 \otimes_{A^{\mathrm{e}}} [1 \otimes a_2 \otimes \cdots \otimes a_{n+1} \otimes 1] \mapsto a_{n+2} m a_1 \otimes a_2 \otimes \cdots \otimes a_{n+1}$$

This map is an isomorphism since it is obviously surjective ($m \otimes b_1 \otimes \cdots \otimes b_n$ is the image of $m \otimes_{A^{\mathrm{e}}} (1 \otimes b_1 \otimes \cdots \otimes b_n \otimes 1)$) and injective (the kernel contains just the zero). $\square$

Theorem 2.1.10. *If A is a unital algebra and projective as an R-module, then for any A^{e}-module M we have that*

$$H(A, M) = Tor^{A^{\mathrm{e}}}(A, M)$$

where $H(A, M)$ is the homology of the Hochschild complex $C(A, M)$, called the Hochschild homology.

Proof. If A is R-projective then $A^{\otimes n}$ is R-projective. From that it follows that $A^{\otimes n+2}$ is A^{e}-projective, since for any homomorphism of A^{e}-modules $\phi : A^{\otimes n+2} \to M$, we have that

$$\phi(a_1 \otimes \cdots \otimes a_{n+2}) = a_1 \cdot \phi(1 \otimes a_2 \otimes \cdots \otimes a_{n+1} \otimes 1) \cdot a_{n+2}.$$

But $\phi(1 \otimes b_1 \otimes \cdots \otimes b_n \otimes 1)$ where the b_i:s run over all possible values is given by a homomorphism of R-modules $f : A^{\otimes n} \to M$, and from this the projectivity follows.

So from above we conclude that $C^{\text{bar}}(A)$ is a projective resolution of A

as an A^{e}-module since A is assumed to be R-projective. Tensoring this projective resolution with M results in following complex:

$$\cdots \xrightarrow{id_M \otimes b'} M \otimes_{A^{\mathrm{e}}} A^{\otimes n} \xrightarrow{id_M \otimes b'} \cdots \xrightarrow{id_M \otimes b'} M \otimes_{A^{\mathrm{e}}} A^{\otimes 2} \longrightarrow 0 \tag{9}$$

According to *Proposition 2.1.9.* we have that $M \otimes_{A^{\mathrm{e}}} A^{\otimes n+2} \cong M \otimes_R A^{\otimes n}$. Hence, due to isomorphism, the map

$$id_M \otimes b'_{n+2} : M \otimes_{A^{\mathrm{e}}} A^{\otimes n+2} \to M \otimes_{A^{\mathrm{e}}} A^{\otimes n+1}$$

induces a map $M \otimes_R A^{\otimes n} \to M \otimes_R A^{\otimes n-1}$, given by $f_{n+1} \circ (id_M \otimes b'_{n+2}) \circ f_n^{-1}$ where f is the isomorphism defined in *Proposition 2.1.9.* (the diagram below may explain how to derive this map).

$$\begin{tikzcd}
M \otimes_{A^{\mathrm{e}}} A^{\otimes n+2} \arrow[r, "id_M \otimes b'"] & M \otimes_{A^{\mathrm{e}}} A^{\otimes n+1} \arrow[d, "f_{n+1}"] \\
M \otimes_R A^{\otimes n} \arrow[u, "f_n^{-1}"] & M \otimes_R A^{\otimes n-1}
\end{tikzcd}$$

We have

$$f_n^{-1}\big(m \otimes a_1 \otimes \cdots \otimes a_n\big) = m \otimes_{A^{\mathrm{e}}} (1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1).$$

Hence

$$(id_M \otimes b'_{n+2}) \circ f_n^{-1}\big(m \otimes a_1 \otimes \cdots \otimes a_n\big) = (id_M \otimes b'_{n+2})\big(m \otimes_{A^{\mathrm{e}}} (1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1)\big) =$$

$$= m \otimes_{A^{\mathrm{e}}} \big(a_1 \otimes \cdots \otimes a_n \otimes 1\big) + \sum_{i=1}^{n-1} (-1)^i m \otimes_{A^{\mathrm{e}}} (1 \otimes a_1 \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \otimes 1) +$$
$$+ (-1)^n m \otimes_{A^{\mathrm{e}}} (1 \otimes a_1 \cdots \otimes a_n).$$

Hence

$$f_{n+1} \circ (id_M \otimes b'_{n+2}) \circ f_n^{-1}\big(m \otimes a_1 \otimes \cdots \otimes a_n\big) = (m a_1 \otimes a_2 \otimes \cdots \otimes a_n) +$$

$$+ \sum_{i=1}^{n-1} (-1)^i (m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) + (-1)^n (a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1})$$
$$= b_n\big(m \otimes a_1 \otimes \cdots \otimes a_n\big).$$

From the calculations above we see that $id_M \otimes b'$ induces the Hochschild boundary b. Thus the complex in (9) and the Hochschild complex (see Corollary/Definition 2.1.5.) are isomorphic, and hence also their homology. But the homology of the complex in (9) is exactly $Tor^{A^{\mathrm{e}}}(A, M)$, and the theorem follows. $\square$

2.2 Hochschild Cochain Cohomology

Definition 2.2.1. - Hochschild Coboundary. For any R-algebra A and any A-bimodule M, consider the $\mathbb{Z}$-graded module

$$C^\bullet(A,B) = \bigoplus_{n=0}^{\infty} \mathrm{Hom}(A^{\otimes n}, M),$$

where $C^0(A,M) = \mathrm{Hom}(A^{\otimes 0}, M)$ is defined to be M. The Hochschild coboundary $\beta : C^\bullet(A,M) \to C^\bullet(A,M)$ is a homomorphism of degree 1 with β_n, the Hochschild coboundary restricted to $C_n^\bullet(A,M)$, defined as follows:

$$\beta_n : \mathrm{Hom}(A^{\otimes n}, M) \to \mathrm{Hom}(A^{\otimes n+1}, M)$$

$$\phi \mapsto \beta_n \phi$$

with

$$(\beta_n\phi)(a_1\otimes\cdots\otimes a_{n+1}) := a_1\phi(a_2\otimes\cdots\otimes a_{n+1}) + \sum_{i=1}^{n}(-1)^i\phi(a_1\otimes\cdots\otimes a_i a_{i+1}\otimes\cdots\otimes a_{n+1})$$

$$+(-1)^{n+1}\phi(a_1\otimes\cdots\otimes a_n)a_{n+1}$$

Proposition 2.2.2. *β is a differential.*

Proof. The proof is very similar to the proof of Proposition 2.1.4. □

Definition 2.2.3. - Hochschild Cochain Complex and Cohomology. Since the Hochschild coboundary β is a differential (*Proposition 2.2.2.*), we have that $(C^\bullet(A,M), \beta)$ is a cochain complex called the Hochschild cochain complex of A with coefficients in M, and its cohomology is called Hochschild cohomology denoted by $H^\bullet(A,M)$.

$$0 \longrightarrow M \xrightarrow{\beta} \mathrm{Hom}(A,M) \xrightarrow{\beta} \mathrm{Hom}(A^{\otimes 2}, M) \xrightarrow{\beta} \cdots$$

An illustrative diagram of $C^\bullet(A,M)$.

Example 2.2.4 We have that

$$H^0(A,M)) = \frac{\mathrm{Ker}_0\ \beta}{\mathrm{Im}_{-1}\ \beta} = \frac{\{m \in M \mid am - ma = 0,\ \forall a \in A\}}{\{0\}} =$$

$$= \{m \in M \mid am - ma = 0,\ \forall a \in A\}.$$

We see that $H^0(C^\bullet(A,M))$ is the set of all elements in M that commutes with all elements in A.

We have also

$$H^1(A,M)) = \frac{\mathrm{Ker}_1\ \beta}{\mathrm{Im}_0\ \beta} =$$

$$= \frac{\{\phi \in \mathrm{Hom}(A,M) \mid a_1\phi(a_2) - \phi(a_1a_2) + \phi(a_1)a_2 = 0,\ \forall a_1, a_2 \in A\}}{\{am - ma \mid a \in A, m \in M\}}.$$

Theorem 2.2.5 *If A is a unital and projective algebra then*

$$H^\bullet(A,M) = Ext^{A^{\mathrm{e}}}(A,M).$$

Proof. Every A^{e}-homomorphism $\phi : A^{\otimes n+2} \to M$ is uniquely determined by an R-homomorphism $f : A^{\otimes n} \to M$ where f is given by the equality $\phi(a_1 \otimes \cdots \otimes a_{n+2}) = a_1 f(a_2 \otimes \cdots \otimes a_{n+1})a_{n+2}$. By the same equality we can also show the converse, namely that every R-homomorphism $f : A^{\otimes n} \to M$ is uniquely determined by some A^{e}-homomorphism $\phi : A^{\otimes n+2} \to M$. This proves an isomorphism

$$\mathrm{Hom}_{A^{\mathrm{e}}}(A^{\otimes n+2}, M) \cong \mathrm{Hom}_R(A^{\otimes n}, M) \tag{10}$$

Moreover, if A is unital, we have that $f(b_1 \otimes ... \otimes b_n) = \phi(1 \otimes b_1 \otimes \cdots \otimes b_n \otimes 1)$, implying that R-homomorphisms can always be expressed in terms of A^{e}-homomorphisms and vice versa.

The Hochschild coboundary β induces due to (10) a corresponding map
$\gamma : \mathrm{Hom}_{A^{\mathrm{e}}}(A^{\otimes n+2}, M) \to \mathrm{Hom}_{A^{\mathrm{e}}}(A^{\otimes n+3}, M)$.

Now pick $\phi \in \mathrm{Hom}_{A^{\mathrm{e}}}(A^{\otimes n+2}, M)$ arbitrary and let $\psi := \gamma(\phi)$. According to above ϕ and ψ are given by R-homomorphisms f and g respectively, where $g = \beta(f)$. Hence

$$\psi(a_1 \otimes ... \otimes a_{n+3}) = a_1 g(a_2 \otimes ... \otimes a_{n+2})a_{n+3} = a_1[\beta(f)(a_2 \otimes ... \otimes a_{n+2})]a_{n+3}$$

$$= a_1\Big[a_2 f(a_3 \otimes \cdots \otimes a_{n+2}) + \sum_{i=2}^{n+2}(-1)^{i-1}(a_2 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+2}) +$$

$$+(-1)^{n+1} f(a_2 \otimes \cdots \otimes a_{n+1})a_{n+2}\Big]a_{n+3} =$$

$$= \phi\Bigg(a_1a_2 \otimes a_3 \otimes \cdots \otimes a_{n+3} + \sum_{i=2}^{n+2}(-1)^{i-1}(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+3})$$

$$+f(a_1 \otimes \cdots \otimes a_{n+1} \otimes a_{n+2}a_{n+3})\Bigg) = \phi \circ b'(a_1 \otimes \cdots \otimes a_{n+3}) = b'^*(\phi)(a_1 \otimes \cdots \otimes a_{n+3}).$$

We see that the induced map γ is the map b'^* where b' is the map defined in Definition 2.1.3. and b'^* is the induced map between Hom-groups defined in Definition 1.6.2..
Hence $C^\bullet(A, M) \cong \mathrm{Hom}(C^{\mathrm{bar}}(A), M)$. If, moreover, A is projective $C^{\mathrm{bar}}(A)$ is projective resolution, implying

$$H^\bullet(A, M) = H(\mathrm{Hom}(C^{\mathrm{bar}}(A), M)) \cong Ext^{A^{\mathrm{e}}}(A, M).$$

□

2.3 Hochschild (co)homology of $\mathbb{R}[x]/\langle x^2 \rangle$.

Let $A := \mathbb{R}[x]/\langle x^2 \rangle$ be an algebra over $\mathbb{R}$ with multiplication defined in the natural way. We want to find the Hochschild homology $H_\bullet(A, A)$, and cohomology $H^\bullet(A, A)$.
To do that we want to find a projective A^{e}-resolution of A so we can apply *Theorem 2.1.10.* and *Theorem 2.2.5.* on it.

We start with considering the diagram

$$\cdots \xrightarrow{g_3} P_2 \xrightarrow{g_2} P_1 \xrightarrow{g_1} P_0 \xrightarrow{f} A \longrightarrow 0 \tag{11}$$

where all $P_i = A \otimes A$ and

$$f[(a + bx) \otimes (c + dx)] := (a + bx)(c + dx) = ac + (ad + bc)x$$

and

$$g_i[(a+bx)\otimes(c+dx)] := \begin{cases} ax \otimes cx & \text{if } i \text{ odd} \\ ax \otimes cx + (a + bx) \otimes cx + ax \otimes (c + dx) & \text{if } i \text{ even} \end{cases}$$

It is obvious that f is surjective. It is also clear that Ker f = Im g_1 and that Ker g_i = Im g_{i+1} for all $i \geq 1$.

Hence the diagram in (11) is exact, so (P, g) is a resolution of A as an A^{e}-module. To show that this resolution is projective we have to show that $P_i = A \otimes A$ is a projective A^{e}-module. We have that $A \otimes A^{\mathrm{op}} = A^{\mathrm{e}}$ is a free A^{e}-module according to *Lemma 1.4.2.*. Since multiplication in A is commutative, $A \otimes A = A \otimes A^{\mathrm{e}}$, so $A \otimes A$ is also a free A^{e}-module and hence also a projective A^{e}-module (according to *Lemma 1.4.5.*).

$$\cdots \xrightarrow{g_3} P_2 \xrightarrow{g_2} P_1 \xrightarrow{g_1} P_0 \tag{12}$$

An illustrative diagram of the projective resolution of A.

$H_\bullet(A, A)$.

According to *Theorem 2.1.10.* we can compute $H_\bullet(A, A)$ by computing $Tor^{A^e}(A, A)$. In order to that, we're considering the associated chain complex of the projective resolution in (12):

$$\cdots \xrightarrow{id_A \otimes g_3} A \otimes_{A^e} P_2 \xrightarrow{id_A \otimes g_2} A \otimes_{A^e} P_1 \xrightarrow{id_A \otimes g_1} A \otimes_{A^e} P_0 \longrightarrow 0$$

We have according to *Proposition 2.1.9.* that $A \otimes_{A^e} P_i = A \otimes_{A^e} (A \otimes A)$ as an A^e-module is isomorphic to A as an R-module.

Hence the associated chain complex above is isomorphic to

$$\cdots \xrightarrow{\widetilde{g}_3} A \xrightarrow{\widetilde{g}_2} A \xrightarrow{\widetilde{g}_1} A \longrightarrow 0 \tag{13}$$

where $\widetilde{g}_i : A \to A$ is the map induced by $id_A \otimes g_i$, given by
$f \circ (id_A \otimes g_i) \circ f^{-1}$ where f is the function defined in *Proposition 2.1.9.* (for motivation see proof of *Theorem 2.1.10.*).

Hence

$$\widetilde{g}_i(a + bx) = f \circ (id_A \otimes g_i) \circ f^{-1}(a + bx) = \begin{cases} 0 & \text{if } i \text{ odd} \\ 2ax & \text{if } i \text{ even.} \end{cases}$$

This results in following equalities

$$\begin{array}{c} \text{Ker } \widetilde{g}_{2n-1} = A \quad \text{and} \quad \text{Im } \widetilde{g}_{2n-1} = 0, \quad \forall n \in \mathbb{Z}_{\geq 1} \\ \text{Ker } \widetilde{g}_{2n} = \{bx \in A\} \quad \text{and} \quad \text{Im } \widetilde{g}_{2n} = \{2ax \in A\}, \quad \forall n \in \mathbb{Z}_{\geq 1}. \end{array} \tag{14}$$

Now we are ready to calculate $H_\bullet(A, A)$ by calculating the homology of (13).

$$H_0(A, A) = \frac{A}{\text{Im } \widetilde{g}_1} \overset{(14)}{=} \frac{A}{\{0\}} = A$$

$$H_{2n-1} = \frac{\text{Ker } \widetilde{g}_{2n-1}}{\text{Im } \widetilde{g}_{2n-2}} \overset{(14)}{=} \frac{A}{\{2ax \in A\}} = \{a \in \mathbb{R}\} = \mathbb{R}, \quad \text{for all } n \geq 1$$

$$H_{2n} = \frac{\text{Ker } \widetilde{g}_{2n}}{\text{Im } \widetilde{g}_{2n-1}} \overset{(14)}{=} \frac{\{bx \in A\}}{\{0\}} = \{bx \in A\} \cong \{b \in \mathbb{R}\} = \mathbb{R}, \quad \text{for all } n \geq 1$$

Hence we have found $H_\bullet(A, A)$ completely.

$H^\bullet(A, A)$.

According to *Theorem 2.2.5.* we can compute $H^\bullet(A, A)$ by computing $Ext^{A^e}(A, A)$. In order to that, we're considering the associated cochain complex of the projective resolution in (12):

$$0 \longrightarrow \text{Hom}_{A^e}(P_0, A) \xrightarrow{g_1^*} \text{Hom}_{A^e}(P_1, A) \xrightarrow{g_2^*} \text{Hom}_{A^e}(P_2, A) \xrightarrow{g_3^*} \cdots$$

We have according to (10) (in proof of *Theorem 2.2.5*) that

$$\mathrm{Hom}_{A^e}(P_i, A) = \mathrm{Hom}_{A^e}(A \otimes A, A) \cong \mathrm{Hom}_R(A^{\otimes 0}, A) = A.$$

Hence the associated cochain complex above is isomorphic to

$$0 \longrightarrow A \xrightarrow{\widetilde{g}_1^*} A \xrightarrow{\widetilde{g}_2^*} A \xrightarrow{\widetilde{g}_3^*} \cdots \tag{15}$$

where $\widetilde{g}_i^*$ is the map induced by g_i^*.

Now due to the isomorphism in (10), any element $a + bx \in A$ has a corresponding element $\phi \in \mathrm{Hom}_{A^e}(A^{\otimes 2}, A)$ where $\phi(1 \otimes 1) = a + bx$. Obviously $g_i^*(\phi) = \phi \circ g_i \in \mathrm{Hom}_{A^e}(A^{\otimes 2}, A)$ is the corresponding element of $\widetilde{g}_i^*(a + bx) \in A$. Hence for all $n \in \mathbb{Z}_{\geq 1}$ we have

$$\widetilde{g}_{2n-1}^*(a+bx) = (\phi \circ g_{2n-1})(1 \otimes 1) = \phi(x \otimes x) = x\phi(1 \otimes 1)x = x(a+bx)x = 0$$

and

$$\widetilde{g}_{2n}^*(a + bx) = (\phi \circ g_{2n})(1 \otimes 1) = x(a + bx)x + (a + bx)x + x(a + bx) = 2ax.$$

This results in

$$\begin{array}{c} \mathrm{Ker}\ \widetilde{g}_{2n-1}^* = A \quad \text{and} \quad \mathrm{Im}\ \widetilde{g}_{2n-1}^* = 0, \quad \forall n \in \mathbb{Z}_{\geq 1} \\ \mathrm{Ker}\ \widetilde{g}_{2n}^* = \{bx \in A\} \quad \text{and} \quad \mathrm{Im}\ \widetilde{g}_{2n}^* = \{2ax \in A\}, \quad \forall n \in \mathbb{Z}_{\geq 1}. \end{array} \tag{16}$$

Now we are ready to calculate $H^\bullet(A, A)$ by calculating the homology of (15).

$$H^0(A, A) = \frac{\mathrm{Ker}\ \widetilde{g}_1^*}{\{0\}} \overset{(16)}{=} \frac{A}{\{0\}} = A$$

$$H^{2n-1} = \frac{\mathrm{Ker}\ \widetilde{g}_{2n}^*}{\mathrm{Im}\ \widetilde{g}_{2n-1}^*} \overset{(16)}{=} \frac{\{bx \in A\}}{\{0\}} = \{b \in \mathbb{R}\} = \mathbb{R}, \quad \text{for all } n \geq 1$$

$$H^{2n} = \frac{\mathrm{Ker}\ \widetilde{g}_{2n+1}^*}{\mathrm{Im}\ \widetilde{g}_{2n}^*} \overset{(16)}{=} \frac{A}{\{2ax \in A\}} = \{bx \in A\} \cong \{b \in \mathbb{R}\} = \mathbb{R}, \quad \text{for all } n \geq 1.$$

Hence we have found $H^\bullet(A, A)$ completely and also derived $H_\bullet(A, A) = H^\bullet(A, A)$.

Another attempt to calculate $H_\bullet(A, A)$.

Theorem 2.3.1. *For any unital R-algebra A, let $\bar{A} := A/(R \cdot 1)$ ($R \cdot 1$ is the set of all elements in A which can be expressed as $\lambda 1$ for some $\lambda \in R$). Then we have following equivalence of Hochschild homologies:*

$$H_\bullet(A, M) = H_\bullet(\bar{A}, M).$$

The proof of this theorem requires knowledge over the level of this essay, so we're omitting the proof. Yet, we will use it to find $H_\bullet(A,A)$ where $A = \mathbb{R}[x]/\langle x^2 \rangle$ by calculating $H_\bullet(\bar{A},A)$, i.e. the homology of

$$\cdots \xrightarrow{b} A \otimes \bar{A}^{\otimes 3} \xrightarrow{b} A \otimes \bar{A}^{\otimes 2} \xrightarrow{b} A \otimes \bar{A} \xrightarrow{b} A \longrightarrow 0$$

where $\bar{A} = A/(R \cdot 1) = \{kx \in A \mid k \in \mathbb{R}\}$.

Now for any element in $A \otimes \bar{A}^{\otimes n}$ we have the following equality:

$$(m + kx) \otimes k_1 x \otimes k_2 x \otimes \cdots \otimes k_n x = \big(k_1 k_2 \cdots k_n (m + kx)\big) \otimes x \otimes \cdots \otimes x.$$

Hence any element in $A \otimes \bar{A}^{\otimes n}$ can be expressed as $(m + kx) \otimes x \otimes \cdots \otimes x$.

Now consider

$$b_n\big[(m+kx) \otimes x \otimes x \otimes \cdots \otimes x\big] = ((m+kx)x) \otimes x \otimes \cdots \otimes x +$$

$$+ \sum_{i=1}^{n-1} (-1)^i (m+kx) \otimes x \otimes \cdots \otimes x \cdot x \otimes \cdots \otimes x +$$

$$+ (-1)^n (x(m+kx)) \otimes x \otimes \cdots \otimes x.$$

It is obvious that it is only possible for the first and the last term to be non-zero in the expression above (the rest of terms are tensor products of elements including $x \cdot x = x^2 = 0$, making the whole element zero).
Hence

$$b_n\big[(m+kx) \otimes x \otimes x \otimes \cdots \otimes x\big] = mx \otimes x \otimes \cdots \otimes x + (-1)^n mx \otimes x \otimes \cdots \otimes x.$$

From this we can see that

$$\text{Im } b_{2n-1} = 0 \quad \text{and} \quad \text{Ker } b_{2n-1} = A \otimes \bar{A}^{\otimes 2n-1}, \quad \forall n \in \mathbb{Z}_{\geq 1}$$

$$\text{Im } b_{2n} = \bar{A}^{\otimes 2n} \quad \text{and} \quad \text{Ker } b_{2n} = \bar{A}^{\otimes 2n+1} \text{ (i.e. } m = 0\text{)}, \quad \forall n \in \mathbb{Z}_{\geq 1}$$

Hence

$$H_0(\bar{A}, A) = \frac{A}{\text{Im } b_1} = \frac{A}{\{0\}} = A$$

$$H_{2n-1}(\bar{A}, A) = \frac{\text{Ker } b_{2n-1}}{\text{Im } b_{2n-2}} = \frac{A \otimes \bar{A}^{\otimes 2n-1}}{\bar{A}^{\otimes 2n}}$$

For any equivalence class $[(m+kx) \otimes x \otimes \cdots \otimes x] \in \frac{A \otimes \bar{A}^{\otimes 2n-1}}{\bar{A}^{\otimes 2n}}$ we have that

$$[(m+kx) \otimes x \otimes \cdots \otimes x] = [m \otimes x \otimes \cdots \otimes x] + \underbrace{[kx \otimes x \otimes \cdots \otimes x]}_{\in \bar{A}^{\otimes 2n}, \text{ so } = 0} = [m \otimes x \otimes \cdots \otimes x]$$

Hence every equivalence class in $H_{2n-1}(\bar{A}, A)$ is given by a unique real number m, and vice versa. Thus $H_{2n-1}(\bar{A}, A) = \mathbb{R}$, for all positive integers n.

Now consider

$$H_{2n}(\bar{A}, A) = \frac{\text{Ker } b_{2n}}{\text{Im } b_{2n-1}} = \frac{\bar{A}^{\otimes 2n+1}}{\{0\}} = \bar{A}^{\otimes 2n+1}.$$

Any element in $\bar{A}^{\otimes 2n+1}$ is on the form

$$k_1 x \otimes \cdots \otimes k_{2n+1} x = k_1 \cdots k_{2n+1} x \otimes x \otimes \cdots \otimes x = mx \otimes x \otimes \cdots \otimes x.$$

Hence every element in $\bar{A}^{\otimes 2n+1}$ is given by a unique real number m and vice versa. Hence $H_{2n}(\bar{A}, A) = \mathbb{R}$, for all positive integers n.

We have thus determined $H_\bullet(\bar{A}, A)$ completely which equals $H_\bullet(A, A)$ according to *Theorem 2.3.1.*.

Remark. Note that if we would calculate the Hochschild (co)homology out of its definition, without using *Theorem 2.1.10* (or *Theorem 2.3.1.*) and *Theorem 2.2.5.*, the task would be much harder. Take for example

$$H_1(A, A) = \frac{\text{Ker}(b : A \otimes A \to A)}{\text{Im}(b : A \otimes A^{\otimes 2} \to A \otimes A)}$$

where

$$\text{Im}(b : A \otimes A^{\otimes 2} \to A \otimes A) = \{(ac + (ad + bc)x) \otimes (e + fx) - (a + bx) \otimes (ce + (de + cf)x) +$$

$$+ (ea + (fa + eb)x) \otimes (c + dx) \mid a, b, c, d, e, f \in \mathbb{R}\}.$$

This set is very hard to interpret, which make it hard to calculate $H_1(A, A)$.

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