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Combinatorial aspects of monomial ideals

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Abstract. To every squarefree monomial ideal one can associate a hypergraph. In this dissertation we will study some algebraic properties of monomial ideals via the combinatorial properties of the associated hypergraphs. In the first part of this thesis, we show that the Hilbert series of a monomial ideal can be obtained from the so called edge induced polynomial of the associated hypergraph. In the second part we focus on the quadratic case and we provide explicit formulas for some graded Betti numbers of these ideals in terms of combinatorial data of the associated hypergraphs.

Introduction

A hypergraph $\mathcal{H}$ on a finite set V is a family $\{\varepsilon_1, \dots, \varepsilon_m\}$ of nonempty distinct subsets of V with no proper containment relation (i.e., if $\varepsilon_i \subseteq \varepsilon_j$, then $i = j$). The elements $x_1, \dots, x_n$ of V are called vertices, and $\varepsilon_1, \dots, \varepsilon_m$ are the edges of the hypergraph. A graph is a hypergraph each of whose edges has cardinality 2.
An edge induced sub-hypergraph of $\mathcal{H}$ is a hypergraph $\mathcal{L} = \{\varepsilon_{l_1}, \dots, \varepsilon_{l_t}\}$ on the vertex set $V_{\mathcal{L}} := \bigcup_j \varepsilon_{l_j}$. The edge induced polynomial of a hypergraph $\mathcal{H}$ is $S_{\mathcal{H}}(x, y) = \sum_{i,j} \gamma_{ij} x^i y^j$, where γ_{ij} is the number of edge induced sub-hypergraphs of $\mathcal{H}$ with i vertices and j edges.

Suppose that I is a squarefree monomial ideal in $R = \mathbb{K}[x_1, \dots, x_n]$, where $\mathbb{K}$ is a field. One can associate a hypergraph $\mathcal{H}(I)$ on the vertex set $\{x_1, \dots, x_n\}$ to I, simply by considering the unique set $\mathcal{G}(I)$ of minimal monomial generators of I as edges of $\mathcal{H}(I)$. Note that $\mathcal{H}(I)$ is a graph if and only if I is quadratic. Considering this connection, it is natural to ask which algebraic properties of I that can be read from the combinatorial properties of $\mathcal{H}(I)$.

If I is a monomial ideal, then the quotient ring R/I can be written as a direct sum $R/I = \bigoplus_{i \geq 0} M_i$ of $\mathbb{K}$-vector spaces satisfying $M_i.M_j \subseteq M_{i+j}$. The Hilbert series of R/I, $\mathrm{Hilb}(R/I; t) = \sum_{i \geq 0} \dim_{\mathbb{K}}(M_i) t^i$, is an interesting invariant which contains much information about R/I.

Paul Renteln (2002) proved that if I is quadratic, then the Hilbert series of R/I can be obtained from the so called edge induced polynomial of $\mathcal{H}(I)$. Later in 2005, Ferrarello and Fröberg, by a careful use of the inclusion–exclusion principle, gave a short and easy proof of this fact.

In the first part of this thesis by using topological methods from combinatorics (that can be considered as natural generalizations of the inclusion–exclusion principle), we generalize this result by showing that the same result holds for any squarefree monomial ideal.

The second part deals with quadratic monomial ideals. These ideals has been studied extensively, since the pioneering works by Fröberg (1988) and by Simis, Vasconcelos, and Villarreal (1994). One of the most interesting problems in this direction is to provide connections between the resolution of the ideal and combinatorial properties of the associated graph.

Recall that, associated to I, there exists a minimal graded free resolution of the form

$$0 \leftarrow I \leftarrow \bigoplus_j R(-j)^{b_{0,j}} \leftarrow \cdots \leftarrow \bigoplus_j R(-j)^{b_{p,j}} \leftarrow 0$$

where $p \leq n$ and $R(-j)$ is the free R-module obtained by shifting the degrees of R by j. The number $b_{i,j}$ is called ij-th graded Betti number of I.

It is a well-known fact that Hilbert series can be computed by using the graded Betti numbers, so the minimal free resolution is a finer invariant than Hilbert series. On the other hand unlike the case of Hilbert series, the graded Betti numbers depend on the characteristic of the ground field $\mathbb{K}$. However, even if we restrict our study to the those cases that are independent of the characteristic, the subgraph polynomial would not be a good candidate. For instance, the squarefree monomial ideals with 2-linear resolution (i.e. those monomial ideals I such that $b_{i,j}(I) = 0$ if $j \neq i + 1$) are corresponded to complement of chordal graphs, by a Theorem of Fröberg (1988). Then the fact that there is no restriction on the difference between the number of edges and vertices of a subgraph of those graphs, shows that to find a combinatorial picture of graded Betti numbers, the subgraph polynomial would not be a good candidate.

In the second part of this thesis we will provide connections between some small graded Betti numbers of a quadratic monomial ideal and the number of induced subgraphs of its associated graph.

ON THE HILBERT SERIES OF MONOMIAL IDEALS

AFSHIN GOODARZI

Abstract. To every squarefree monomial ideal one can associate a hypergraph. In this paper we show that the Hilbert series of a squarefree monomial ideal, can be obtained from the so called edge induced polynomial of the associated hypergraph.

1. Introduction

Suppose that I is a squarefree monomial ideal in $R = \mathbb{K}[x_1, \ldots, x_n]$, where $\mathbb{K}$ is a field. One can associate a hypergraph $\mathcal{H}(I)$ on the vertex set $\{x_1, \ldots, x_n\}$ to I, simply by considering the unique set $\mathcal{G}(I)$ of minimal monomial generators of I as edges of $\mathcal{H}(I)$.

The *edge induced (sub-hypergraph) polynomial* of a hypergraph $\mathcal{H}$ is $S_{\mathcal{H}}(x, y) = \sum_{i,j} \gamma_{ij} x^i y^j$, where γ_{ij} is the number of edge induced sub-hypergraphs (see Section 2) of $\mathcal{H}$ with i vertices and j edges. It was shown by Renteln [9] (see also [4]) that if I is quadratic, then the Hilbert series of the quotient R/I can be computed from the edge induced polynomial of $\mathcal{H}(I)$. Note that in this case $\mathcal{H}(I)$ will be a graph.

The aim of this note is to generalize this result by showing that the same result holds for any squarefree monomial ideal. More precisely we will prove the following result.

Theorem 1.1. *Let $I \subset R = \mathbb{K}[x_1, \ldots, x_n]$ be a squarefree monomial ideal and $\mathcal{H} = \mathcal{H}(I)$ its associated hypergraph. Then*

$$\mathrm{Hilb(R/I, t)} = \frac{\mathrm{S}_{\mathcal{H}}(\mathrm{t}, -1)}{(1-\mathrm{t})^{\mathrm{n}}}.$$

The structure of the paper is as follows. Section 2 reviews basic concepts and terminology. In Section 3 we discuss the foundation for our proof. Finally the main result is proved in Section 4.

2. Basic Concepts

In this section we recall some basic concepts. We refer to the books by Munkres [8], Berge [1] and Miller and Sturmfels [7] for more details and unexplained terminology.

2.1. **Topological Preliminaries.** Let Δ be a simplicial complex on the vertex set $V = \{x_1, \dots, x_n\}$ and S a commutative ring with unity. $\widetilde{H}_{.}(\Delta; S)$ (resp. $\widetilde{H}^{.}(\Delta; S)$) stands for the reduced simplicial homology (resp. cohomology) of Δ over S. Instead of $\widetilde{H}_{.}(\Delta; \mathbb{K})$, we will use the notation $\widetilde{H}_{.}(\Delta)$. Also $\widetilde{\beta}_i(\Delta) = \dim_{\mathbb{K}} \widetilde{H}_i(\Delta)$ is the i-th *reduced Betti number* of Δ over $\mathbb{K}$.

If $f_i(\Delta)$ denotes the number of i-faces (faces of cardinality $i+1$) of Δ, then the Euler-Poincaré formula says that the *reduced Euler characteristics* of Δ is

$$\sum_{i \geq -1} (-1)^i f_i(\Delta) = \sum_{i \geq 0} (-1)^i \widetilde{\beta}_i(\Delta). \tag{1}$$

The *combinatorial Alexander dual* of a simplicial complex Δ is the simplicial complex on the same ground set defined by

$$\Delta^* = \{F \subset V | V \setminus F \notin \Delta\}.$$

There exists a close connection between the homology of a simplicial complex and cohomology of its Alexander dual:

$$\widetilde{H}_i(\Delta) \cong \widetilde{H}^{n-i-3}(\Delta^*). \tag{2}$$

2.2. **Combinatorial Preliminaries.** Let $V = \{x_1, \dots, x_n\}$ be a finite set. A *hypergraph* on V is a family $\mathcal{H} = \{\varepsilon_1, \dots, \varepsilon_m\}$ of nonempty distinct subsets of V with no proper containment relation (i.e., if $\varepsilon_i \subseteq \varepsilon_j$, then $i = j$).

The elements $x_1, \dots, x_n$ of V are called *vertices*, and $\varepsilon_1, \dots, \varepsilon_m$ are the *edges* of the hypergraph. A graph is a hypergraph each of whose edges has cardinality 2.

Let $\mathcal{H} = \{\varepsilon_1, \dots, \varepsilon_m\}$ be a hypergraph on the vertex set V and $\mathcal{L} = \{\varepsilon_{l_1}, \dots, \varepsilon_{l_t}\} \subset \{\varepsilon_1, \dots, \varepsilon_m\}$. We say that $\mathcal{L}$ is an *edge induced sub-hypergraph* of $\mathcal{H}$ on the vertex set $V_{\mathcal{L}} := \bigcup_j \varepsilon_{l_j}$. A *vertex induced sub-hypergraph* of $\mathcal{H}$ induced by $W \subseteq V$ is $\mathcal{H}_W = \{\varepsilon \in \mathcal{H} | \varepsilon \subseteq W\}$.

An *independent set* in a hypergraph $\mathcal{H} = \{\varepsilon_1, \dots, \varepsilon_m\}$ is a subset W of vertices of $\mathcal{H}$ such that $\varepsilon_j \nsubseteq W$ for all j. The collection $\bar{\Delta}(\mathcal{H})$ of all independent set of $\mathcal{H}$ forms a simplicial complex that is called the *independence complex* of $\mathcal{H}$.

2.3. **Algebraic Preliminaries.** Let $\mathbb{K}$ be a field and $R = \mathbb{K}[x_1, \dots, x_n]$ a polynomial ring. Assume that $M = R/I$ is a monomial quotient. Then $M = \bigoplus_{i \geq 0} M_i$, where M_i is the vector space of the homogeneous elements of M of degree i. The *Hilbert series* of M is

$$\mathrm{Hilb}(M; t) = \sum_{i \geq 0} \dim_{\mathbb{K}}(M_i) t^i.$$

The Hilbert series of every monomial quotient $M = R/I$ can be expressed as a rational function

$$\operatorname{Hilb}(M;t) = \frac{\mathcal{K}(M;t)}{(1-t)^n}.$$

The numerator of this expression, $\mathcal{K}(M;t)$, is called the *K-polynomial* of M.

The K-polynomial of M can be computed from its finite free resolution. Recall that associated to M is a *minimal graded free resolution* of the form

$$0 \leftarrow M \leftarrow \bigoplus_j R(-j)^{b_{0,j}} \leftarrow \cdots \leftarrow \bigoplus_j R(-j)^{b_{p,j}} \leftarrow 0$$

where $p \leq n$ and $R(-j)$ is the free R-module obtained by shifting the degrees of R by j. The number $b_{i,j}$ is called *ij-th graded Betti number* of M. One can compute the K-polynomial of M using this graded Betti numbers

$$\mathcal{K}(M;t) \quad = \quad \sum_{i=0}^{n} \sum_{j \in \mathbb{Z}} (-1)^i b_{i,j}(M) t^j. \tag{3}$$

The Hochster's formula (see [7, Corollary 5.12]) expresses the graded Betti numbers of the Stanley-Reisner ring of a simplicial complex in terms of the reduced Betti numbers of some subcomplexes. The following equivalent form of this formula shall be more useful for our purpose

$$b_{i,j}(R/I) = \sum_{W=j} \widetilde{\beta}_{j-i-1}(\bar{\Delta}(\mathcal{H}(I)_W)). \tag{4}$$

3. Edge Cover Complex

In this section we discuss the foundation of our proof. Let $\mathcal{H} = \{\varepsilon_1, \ldots, \varepsilon_m\}$ be a hypergraph on the vertex set V. An *edge cover* of $\mathcal{H}$ is a subset $\{\varepsilon_{l_1}, \ldots, \varepsilon_{l_t}\}$ of the edges of $\mathcal{H}$ such that $\bigcup_j \varepsilon_{l_j} = V$. An edge cover of cardinality t will be called a t-cover.

Clearly if E is an edge cover, then every subset F of $\{\varepsilon_{l_1}, \ldots, \varepsilon_{l_t}\}$ which contains E is also an edge cover. So, by considering the collection of complements of edge covers, we have a simplicial complex $\Lambda(\mathcal{H})$ on $\{\varepsilon_{l_1}, \ldots, \varepsilon_{l_t}\}$. We will call this complex the *edge cover complex* of $\mathcal{H}$. In the case when $\mathcal{H}$ is a graph, the edge cover complex has been studied in [6] and [5].

Denote by $\Gamma(\mathcal{H})$ the simplicial complex on the vertex set $\{\varepsilon_1, \ldots, \varepsilon_m\}$ as those subsets of $\{\varepsilon_1, \ldots, \varepsilon_m\}$ whose union is not all of V. This complex appeared in [3], where the authors [3, Theorem 2] showed that

$$\widetilde{H}_i(\bar{\Delta}(\mathcal{H}); S) \cong \widetilde{H}^{|V|-3-i}(\Gamma(\mathcal{H}); S).$$

It is easy to see that $\Gamma(\mathcal{H})$ is the collection of all non-edge covers of $\mathcal{H}$, hence a subset E of $\{\varepsilon_1, \dots, \varepsilon_m\}$ is not in $\Gamma(\mathcal{H})$ (E is an edge cover) if and only if the complement of E is in $\Lambda(\mathcal{H})$. In other words

$$\Gamma(\mathcal{H}) = \Lambda(\mathcal{H})^*.$$

Now using combinatorial Alexander duality, one can deduce the following result which the special case when $\mathcal{H}$ is a graph has been proved in [5].

Proposition 3.1. *Let $\mathcal{H} = \{\varepsilon_1, \dots, \varepsilon_m\}$ be a hypergraph on the vertex set $V = \{x_1, \dots, x_n\}$. Then*

$$\widetilde{H}_i(\bar{\Delta}(\mathcal{H}); S) \cong \widetilde{H}_{m-n+i}(\Lambda(\mathcal{H}); S).$$

Remark 3.2. An immediate but useful consequence of Proposition 3.1 is the following formula which relates the reduced Euler characteristics of independence complex and edge cover complex of a hypergraph with n vertices and m edges.

$$\widetilde{\chi}(\Lambda(\mathcal{H})) = (-1)^{m-n}\widetilde{\chi}(\bar{\Delta}(\mathcal{H})) \tag{5}$$

4. PROOF OF THEOREM 1.1

In order to prove Theorem 1.1 we will show the validity of the following two claims.

- **Claim 1.** $S_{\mathcal{H}}(t, -1) = \sum_{j\in\mathbb{Z}} \sum_{|W|=j} (-1)^{|E_W|-1}\widetilde{\chi}(\Lambda(\mathcal{H}_W))t^j$.
- **Claim 2.** $\mathcal{K}(R/I; t) = \sum_{j\in\mathbb{Z}} \sum_{|W|=j} (-1)^{|E_W|-1}\widetilde{\chi}(\Lambda(\mathcal{H}_W))t^j$.

Proof of Claim 1. If we fix a subset W of $V = \{x_1, \dots, x_n\}$ and consider all edge induced sub-hypergraphs on the vertex set W with possible edges and then sum over all choices of $W \subset V$, we obtain that

$$S_{\mathcal{H}}(x, y) = \sum_j \sum_{|W|=j} \left(\sum_i \gamma_i(W) y^i \right) x^j$$

where $\gamma_i(W)$ is the number of edge induced sub-hypergraphs $\mathcal{L}$ of $\mathcal{H}$ with $V_{\mathcal{L}} = W$. Note that $\gamma_i(W)$ equals to the number of i-covers of $\mathcal{H}_W$. Now if we denote by E_W the set of edges of $\mathcal{H}_W$. The complementation map $c : E_W \to E_W$ induces a one-one correspondence between the set of all k-covers of $\mathcal{H}_W$ and the set of $(|E_W| - k - 1)$-faces of $\Lambda(\mathcal{H}_W)$, for all k. So we have

$$S_{\mathcal{H}}(t, -1) = \sum_j \sum_{|W|=j} \left(\sum_i (-1)^i f_{|E_W|-i-1}(\Lambda(\mathcal{H}_W)) \right) t^j$$

therefore the Euler-Poincaré formula yields that

$$S_{\mathcal{H}}(t,-1)=\sum_{j\in\mathbb{Z}}\sum_{|W|=j}(-1)^{|E_W|-1}\widetilde{\chi}(\Lambda(\mathcal{H}_W))t^j.$$

□

Proof of Claim 2.

$$\begin{aligned}
\mathcal{K}(R/I;t) &= \sum_{i=0}^{n}\sum_{j\in\mathbb{Z}}(-1)^i b_{i,j}t^j \quad \text{(by 3)}\\
&= \sum_{i=0}^{n}\sum_{j\in\mathbb{Z}}(-1)^i\left(\sum_{W=j}\widetilde{\beta}_{j-i-1}(\bar{\Delta}(\mathcal{H}_W))\right)t^j \quad \text{(by 4)}\\
&= \sum_{j\in\mathbb{Z}}\sum_{|W|=j}\left(\sum_{i=0}^{n}(-1)^i\widetilde{\beta}_{j-i-1}(\bar{\Delta}(\mathcal{H}_W))\right)t^j\\
&= \sum_{j\in\mathbb{Z}}\sum_{|W|=j}(-1)^{j-1}\widetilde{\chi}(\bar{\Delta}(\mathcal{H}_W))t^j\\
&= \sum_{j\in\mathbb{Z}}\sum_{|W|=j}(-1)^{|E_W|-1}\widetilde{\chi}(\Lambda(\mathcal{H}_W))t^j. \quad \text{(by 5)}
\end{aligned}$$

□

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A STUDY OF GRADED BETTI NUMBERS OF QUADRATIC MONOMIAL IDEALS

AFSHIN GOODARZI

Abstract. To every squarefree quadratic monomial ideal one can associate a simple graph. In this note we provide explicit formulas for some graded Betti numbers of these ideals in terms of the combinatorial data of the associated graph.

1. Introduction

Let $\mathbb{K}$ be a field and let $G = (V, E)$ be a simple graph (i.e. a graph without loops or multiple edges) on the vertex set $V = \{x_1, \ldots, x_n\}$. The edge ideal of G, $I(G)$, is the ideal of $R := \mathbb{K}[x_1, \ldots, x_n]$ generated by the set of monomials $x_i x_j$ for all $\{x_i, x_j\} \in E$. The quotient ring $M(G) = R/I(G)$ is called the edge ring of G. Every squarefree quadratic monomial ideal $I \subset R$ is indeed an edge ideal $I = I(G)$, for some graph G.

In recent years there have been a flurry of work investigating connections between properties of an edge ideal and its associated graph. We refer the reader to the survey articles [5] and [6] and references there, for more details and information.

The aim of this note is to provide connections between some small graded Betti numbers of the edge ring of a graph and the number of its induced subgraphs (see Theorem 4.1).

The structure of this note is as follows. First in Section 2, we will brifely recall some basic concepts and terminology. In Section 3 we will set up our foundation for the proof. Finally the main result is proved in Section 4.

2. Basic Concepts

In this section we will recall some basic notions. We refer to the books by Bruns and Herzog [1] and Diestel [2] for undefined terminology and more details.

2.1. **Hilbert Series and K-Polynomial.** Let $\mathbb{K}$ be a field and $R = \mathbb{K}[x_1, \ldots, x_n]$ a polynomial ring. Assume that $M = R/I$ is a monomial quotient. Then $M = \bigoplus_{i \geq 0} M_i$, where M_i is the vector space of the homogeneous elements of M of degree i. The *Hilbert series* of M is

$$\mathrm{Hilb}(M;t) = \sum_{i \geq 0} \dim_{\mathbb{K}}(M_i) t^i.$$

The Hilbert series of every monomial quotient $M = R/I$ can be expressed as a rational function

$$\mathrm{Hilb}(M;t) = \frac{\mathcal{K}(M;t)}{(1-t)^n}.$$

The numerator of this expression, $\mathcal{K}(M;t)$, is called the *K-polynomial* of M.

2.2. **Minimal Graded Resolution.** Associated to M, there exists a *minimal graded free resolution* of the form

$$0 \leftarrow M \leftarrow \bigoplus_j R(-j)^{b_{0,j}} \leftarrow \cdots \leftarrow \bigoplus_j R(-j)^{b_{p,j}} \leftarrow 0$$

where $p \leq n$ and $R(-j)$ is the free R-module obtained by shifting the degrees of R by j. The number $b_{i,j}$ is called *ij-th graded Betti number* of M.

From the fact that Hilbert Series is additive relatively to exact sequences, one can compute the K-polynomial of M using this graded Betti numbers

$$\mathcal{K}(M;t) = \sum_{i=0}^{n} \sum_{j \in \mathbb{Z}} (-1)^i b_{i,j}(M) t^j. \tag{1}$$

2.3. **Hochster's Formula.** The Hochster's formula expresses the graded Betti numbers of the Stanley-Reisner ring of a simplicial complex in terms of the reduced Betti numbers of some subcomplexes. The following form of this formula shall be more useful for our purpose

$$b_{i,j}(R/I(G)) = \sum_{|W|=j} \widetilde{\beta}_{j-i-1}(\bar{\Delta}(G_W)), \tag{2}$$

where G_W is the induced subgraph of G on the vertex set W. Recall that edges of G_W are those edges of G which have both endpoints in W.

3. K-Polynomial of the Edge Ring

In this section we provide a combinatorial formula to compute the coefficients of the K-polynomial of an edge ring. A more general connection between K-polynomial of monomial quotient rings and associated combinatorial objects can be found in [4].

The number of induced subgraphs of G, isomorphic to a given graph H will be denoted by $\#_G(H)$. We will also denote by $\mathcal{G}^j$, the set of all non-isomorphic graphs on j vertices and without any isolated vertex.

Proposition 3.1. *Let G be a graph and M be its edge ring, then the coefficient of t^j in $\mathcal{K}(M;t)$ is*

$$[\mathcal{K}(M;t)]_{t^j} = (-1)^{j-1} \sum_{H \in \mathcal{G}^j} \left(\#(H).\widetilde{\chi}(\bar{\Delta}(H)) \right).$$

Proof. Combining Hochster's formula and 1 yield

$$\mathcal{K}(M;t) = \sum_{i=0}^{n} \sum_{j \in \mathbb{Z}} (-1)^i \left(\sum_{|W|=j} \widetilde{\beta}_{j-i-1}(\bar{\Delta}(G_W)) \right) t^j$$

changing the order of summation, we get

$$\mathcal{K}(M;t) = \sum_{j \in \mathbb{Z}} \sum_{|W|=j} (-1)^{j-1} \widetilde{\chi}(\bar{\Delta}(G_W)) t^j$$

so

$$[\mathcal{K}(M;t)]_{t^j} = (-1)^{j-1} \sum_{|W|=j} \widetilde{\chi}(\bar{\Delta}(G_W)).$$

Now gathering all isomorphic cases, we obtain the desirable result. $\square$

The following running example, will be needed in the next section.

Example 3.2. If we denote by $K_4 - e$ the graph obtained by removing an edge from the complete graph K_4 and denote by $S_4 + e$ the graph obtained by connecting two non-adjacent vertices of the star graph S_4, then it is easy to check that

$$\mathcal{G}^4 = \{K_4, K_4 - e, S_4, S_4 + e, C_4, P_4, K_2 \uplus K_2\}$$

where C_4, P_4, and $K_2 \uplus K_2$ denote the cycle on 4 vertices, the path on 4 vertices, and disjoint union of two copies of complete graph on 2 vertices, respectively.

Furthermore, using Proposition 3.1 we have

$$\begin{aligned}[\mathcal{K}(M;t)]_{t^4} &= -3\#(K_4) - 2\#(K_4 - e) - \#(S_4 + e) \\ &\quad -\#(S_4) - \#(C_4) + \#(K_2 \uplus K_2).\end{aligned}$$

4. Main Result

In this section, by using the tool provided in the previous section, we will prove our main result, which is the following theorem.

Theorem 4.1. *Let M be the edge ring of a graph G. Then*
(i) $b_{2,4}(M) = \#(K_2 \uplus K_2)$,
(ii) $b_{3,4}(M) = \sum_{v\in G} \binom{\deg v}{3} - \#(K_4) + \#(C_4)$.

Proof. Part *(i)*: Using Hochster's formula, we have

$$b_{2,4}(M) = \sum_{|W|=4} \widetilde{\beta}_1(\bar{\Delta}(G_W)),$$

on the other hand, it is easy to see that $\widetilde{\beta}_1(\bar{\Delta}(G_W)) = 1$ if $\bar{\Delta}(G_W) = C_4$ and otherwise $\widetilde{\beta}_1(\bar{\Delta}(G_W)) = 0$. Therefore $b_{2,4}(M)$ counts the number of induced 4-cycles in the complement of G, or equivalently the number of induced $K_2 \uplus K_2$'s in G.
Part *(ii)*: If we fix a vertex v in G and choose 3 neighbours of v, the induced subgraph of G on these four vertices will be isomorphic to one of the following graphs

$$K_4, K_4 - e, S_4 + e, S_4,$$

since at least one of the vertices has degree 3. Now if we sum over all possible choices of v and its neighbours (i.e. $\sum_{v\in G} \binom{\deg v}{3}$), we count the number of K_4's 4 times (for every vertex once), the number of $K_4 - e$'s twice ($K_4 - e$ has two vertices of degree 3) and the others once. So we have

$$\sum_{v\in G} \binom{\deg v}{3} = 4\#(K_4) + 2\#(K_4 - e) + \#(S_4 + e) + \#(S_4).$$

Now Example 3.2 and part *(i)* imply that

$$\sum_{v\in G} \binom{\deg v}{3} - \#(K_4) + \#(C_4) = b_{2,4}(M) - [\mathcal{K}(M;t)]_{t^4}$$

so the result follows, since the right hand side of the above equality is $b_{3,4}(M)$ (by 1).

□

Remark 4.2. The part *(ii)* of Theorem 4.1 was conjectured by Eliahou and Villarreal [3, Conjecture 2.4] and has been proved in a different way by Roth and Van Tuyl [7, Proposition 2.8].

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