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Introduction to Differential Manifolds and their Cohomology

av

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Abstract

This exposition starts with basic algebraic definitions such as *module*, *tensor product*, *tensor algebra*, as well as *symmetric* and *skew-symmetric algebras.* Next, we define *n-dimensional manifolds* as topological spaces locally homeomorphic to balls in $\mathbb{R}^n$. In the smooth case we define *differential forms* of arbitrary orders. Then we will prove the fundamental *Stokes theorem* for differential forms, which, in particular, explain how a surface integral of a vector field over an oriented surface is related to the volume integral of its divergence over the body bounded by the surface. We will investigate Stokes theorem for cuboids, simplices and general manifolds. Finally, we define the notion of *de Rham cohomology* of a smooth manifold using the famous *Poincaré lemma.* De Rham cohomology is a analytical way of approaching the algebraic topology of a manifold. De Rham theorem claims that de Rham cohomology group of a manifold is isomorphic with its singular homology group. For purposes of illustration, we provide a connection between the vector analytic notions such as *gradient, divergence* and *curl* in $\mathbb{R}^3$ and the singular homology of the corresponding objects. At the end we will take a look at *Morse inequalities. Morse theory* gives a direct way of analyzing the topology of a manifold by studying smooth real-valued functions on it.

Sammanfattning

Detta examensarbete utgår ifrån grundläggande definitioner såsom *modul*, *tensorprodukt*, *tensoralgebra*, både *symmetriska* och *asymmetriska algebra.* Därefter definierar vi *n-dimensionella mångfalder* som topologiska rum lokal homeomorfa till bollar i $\mathbb{R}^n$. Sedan bevisar vi den fundamentala satsen, *Stokes sats*, för differentialformer, som framför allt förklarar hur en ytintegral av ett vektorfält över en orienterad yta hänger ihop med volymintegralen över dess divergens över kroppen avgränsas av ytan. Vi betraktar satsen för både block, simplex och generella mångfalder. Vi definierar begreppet *de Rham kohomologin* under användandet av den välkända *Poincaré lemma.* De Rham kohomologin är ett analytiskt sätt att närma sig algebraiska topologin av en mångfald. De Rham teorem påstår att de Rham kohomologi gruppen av en mångfald är isomorfisk med dess singulär homologi grupp. Vi etablerar, för illustrationsändamål, sambandet mellan vektor analytiska begrepp såsom *gradient, divergens* och *rotation* i $\mathbb{R}^3$ och den singulära homologin med motsvarande objekter. Mot slutet av arbetet betraktar vi *Morse olikheter. Morse teorin* ger ett direkt sätt för att analysera topologin av mångfalder genom att studera differentialfunktioner på dem.

Contents

1 Elements of Linear Algebra with focus on Tensor products

1.1 Modules

The notion of a vector space can easily be generalized by permitting instead of a field a ring R as the set of scalars. Doing so we introduce a so called *R-module*. For example, every abelian group is a $\mathbb{Z}$-module. Besides modules we will define in this section the *tensor product* of modules and finally the *tensor algebra*.

I refer mainly to [KER], [STU] and [BOS] in this section.

1.1.1 Left and Right modules

Let R be a commutative ring.

Definition 1. 1. We call an operation $R \times M \to M$, $(r, m) \mapsto rm$ where R acts on an abelian group M by scalar multiplication a *R-module* or *left* R-module (notation $R - mod$) if for all $r_1, r_2, r \in R$ and $m_1, m_2, m \in M$ the following holds:

$$\begin{aligned} r \cdot (m_1 + m_2) &= rm_1 + rm_2 \\ (r_1 + r_2)m &= r_1 m + r_2 m \qquad\qquad (*) \\ r_1 \cdot (r_2 m) &= (r_1 r_2) \cdot m \\ 1 \cdot m &= m \end{aligned}$$

2. We call an operation $M \times R \to M$, $(m, r) \mapsto mr$ where the abelian group M acts on R by scalar multiplication a *right* R-module (notation $mod - R$) if properties analogous to $(*)$ hold.

1.1.2 Examples of R-modules

1. R itself is an R-module (with multiplication in R);
2. Every K-vector space, where $R = K$ is a field;
3. Every abelian group G is a $\mathbb{Z}$-module: all n terms of $nx = x + ... + x$ lie in G and therefore $(-n)x = -(nx)$ lies in G for every $x \in G$ and $n \in \mathbb{N}$.

1.1.3 R-module homomorphisms

Definition 2. A mapping $\varphi : M \to M'$ with an R-module M, M' is called *R-module homomorphism* or *R-linear*, if for $m, m' \in M, r \in R$ the following holds:

$$\varphi(m + m') = \varphi(m) + \varphi(m'),$$
$$\varphi(rm) = r\varphi(m).$$

In the same way we can generalize the notion of a *K-algebra* and obtain a *R-algebra*. For example, the set of all endomorphisms of a R-module $End_R M := \{\varphi : M \to M | \varphi \text{ is } R\text{-linear}\}$ is an R-algebra with the properties:

$$(\varphi + \psi)(m) := \varphi(m) + \psi(m),$$
$$(\varphi \circ \psi)(m) := \varphi(\psi(m)),$$
$$(r\varphi)(m) := r\varphi(m),$$

for all $m \in M, r \in R, \varphi, \psi \in End_R M$.

Definition 3. An additive subgroup N of an R-module is called its *submodule*, if $rn \in N$ for all $n \in N, r \in R$.

1.2 Tensor product

Let R be a ring, M be a right-R-module, and N be a left-R-module. Let U be the submodule of $\mathbb{Z}^{M \times N}$; i.e. U is generated by the elements of the following form:

$$(m + m', n) - (m, n) - (m', n),$$
$$(m, n + n') - (m, n) - (m, n'),$$
$$(mr, n) - (m, rn),$$

where $r \in R, m, m' \in M$ and $n, n' \in N$.

Definition 4. Define the *tensor product of M and N over R* by $\mathbb{Z}^{M\times N}/U =: M \otimes_R N$. In other words, for some elements $m \in M$ and $n \in N$ we define $m \otimes n$ as the residue class $(m, n) + U$ in $M \otimes_R N$.

For all $r \in R, m, m' \in M, n, n' \in N$ the following properties hold:

- $$\begin{aligned}(m + m') \otimes n &= m \otimes n + m' \otimes n,\\ m \otimes (n + n') &= m \otimes n + m \otimes n',\end{aligned}$$ $$mr \otimes n = m \otimes rn;$$

- every element $z \in M \otimes_R N$ has a representation of the form $$z = \sum_{i=1}^{j} m_i \otimes n_i \text{ where } m_i \in M, n_i \in N, j \in \mathbb{N}.$$ This decomposition is, in general, non-unique.

- If R is commutative, then $M \otimes_R N$ has the property: $$r(m \otimes n) = mr \otimes n = m \otimes rn.$$

Example 1. For $M = \mathbb{R}^2$ and $N = \mathbb{R}^3$ the tensor product $T = M \otimes N$ is given by 2×3 real valued matrices. Take $x = (x_1, x_2)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. The tensor product combines x and y in the *rank* 1 matrix xy^T:

$$\begin{aligned} x \otimes y = xy^T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} = \\ = \begin{pmatrix} x_1y_1 & x_1y_2 & x_1y_3 \\ x_2y_1 & x_2y_2 & x_2y_3 \end{pmatrix} = \sum_{i=1}^{2}\sum_{j=1}^{3} x_i y_j E_{i,j}, \end{aligned}$$

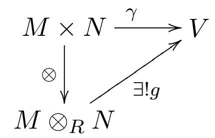
where $E_{i,j}$ is the 2×3 matrix with 1 at the (i, j)th entry and 0 else. The combinations of *rank* 1 matrices give all 2 times 3 matrices and thus the dimension of $T = \mathbb{R}^2 \otimes \mathbb{R}^3$ is 6.

1.2.1 Universality of tensor product

Theorem 1. *Let V be a $\mathbb{Z}-$module. Then every bilinear mapping $\gamma : M \times N \to V$ induces a unique homomorphism*

$$g : M \otimes_R N \to V \textit{ satisfying } g(m \otimes n) = \gamma(m, n).$$

In other words, the following diagram is commutative:



Proof. By bilinearity of γ we have

$$\gamma(mr, n) = \gamma(m, rn) \text{ for all } m \in M, n \in N, r \in R. \tag{1}$$

Define

$$\begin{aligned} g : (M \otimes N = \mathbb{Z}^{M \times N}/U) &\to V \\ ((m \otimes n) = U + (m, n)) &\mapsto (m, n), \end{aligned}$$

where U is the submodule of $\mathbb{Z}^{M \times N}$ with $U \subset \ker(g)$ defined as above.

- well-definedness: indeed, consider $U + (m, n) = U + (m', n')$ for $(m, n), (m', n') \in M \times N$, then there exists $u \in U$ with $(m, n) = u + (m', n')$. It follows $g((m, n)) = \gamma(u + (m', n')) = \gamma(u) + \gamma((m', n'))$. Since $u \in U \subset \ker(g)$ we get $g((m, n)) = g((m', n'))$.

- g is a homomorphism:

  (i) for $(m, n), (m', n') \in M \times N$ we have:

$$\begin{aligned} &g((U + (m, n)) + (U + (m', n'))) \\ &\quad = g(U + ((m, n) + (m', n'))) = \\ &\quad = \gamma((m, n) + (m', n')) = \\ &\quad = \gamma((m, n)) + \gamma((m', n')) = \\ &\quad = g(U + (m, n)) + g(U + (m', n')); \end{aligned}$$

  (ii) for $(m, n) \in M \times N, r \in R$ we have:

$$\begin{aligned} &g((U + (m, n))r) = g(U + (m, n)r) = \\ &\quad = \gamma((m, n)r) = \gamma(m, n)r = \\ &\quad = g(U + (m, n))r. \end{aligned}$$

- commutativity: $g \circ \otimes = \gamma$ holds because by construction we have for all $(m, n) \in M \times N$:

$$g \circ \otimes(m, n) = g(m \otimes n) = g(U + (m, n)) = \gamma(m, n).$$

- uniqueness: let h be a homomorphism with $h \circ \otimes = \gamma$, then for each $(m,n) \in M \times N$ we have:

$$\begin{aligned} h(U+(m,n)) &= h \circ \otimes(m,n) = \\ &= \gamma(m,n) = g \circ \otimes(m,n) = \\ &= g(U+(m,n)) \end{aligned}$$

$$\Rightarrow h = g.$$

□

Theorem 2. *Tensor product induces two canonical $\mathbb{Z}$-isomorphisms:*

$$M \otimes_R R \to M, m \otimes r \mapsto mr,$$
$$R \otimes_R N \to N, r \otimes n \mapsto rn.$$

Proof. Take the bilinear mapping $\gamma : M \times R \to M, (m,r) \mapsto mr$, which satisfies equation (1). By Theorem 1 (universality of tensor product) there exists exactly one $\mathbb{Z}$-linear mapping $g : M \otimes_R R \to M$ with $g(m \otimes r) = mr$ for all $m \in M, r \in R$. This mapping is bijective and its invertion is given by $M \to M \otimes_R R, m \mapsto m \otimes 1$. □

Lemma 1. *(Tensor product of vector spaces)*
Let $\mathbb{K}$ be a field and $M = V$ and $N = W$ be vector spaces over $\mathbb{K}$. Then $V \otimes_{\mathbb{K}} W$ is a $\mathbb{K}$-vector space with elements of the form $v \otimes w = \sum_{i \in I, j \in J} \lambda_i \mu_j (e_i \otimes f_j)$ where $E = \{e_i | i \in I\}$ a basis to V (i.e. $v = \sum_{i \in I} \lambda_i e_i$), $F = \{f_j | j \in J\}$ a basis to W (i.e. $w = \sum_{j \in J} \mu_j f_j$). Furthermore, $\dim(V \otimes W) = \dim V \cdot \dim W$.

Proof. We can interprete tensor product of finite-dimensional vector spaces as the space of matrices, where we determine the rows with index $I = \{1, ..., n\}$ and the columns with index $J = \{1, ..., m\}$. Then the entries of the columns are the multiplicities of $v = \sum_{i \in I} \lambda_i e_i$ and the rows multiplicities of $w = \sum_{j \in J} \mu_j f_j$. Tensor multiplication $v \otimes w$ has the following properties:

$$\begin{aligned} (v' + v'') \otimes w &= v' \otimes w + v'' \otimes w, \\ v \otimes (w' + w'') &= v \otimes w' + v \otimes w'', \\ (\lambda v) \otimes w &= \lambda(v \otimes w) = v \otimes (\lambda w), \end{aligned}$$

where $v, v', v'' \in V, w, w', w'' \in W, \lambda \in \mathbb{K}$.

In general, commutativity does not hold, because for $v \in V, w \in W$ the vectors $v \otimes w \in V \otimes W$ and $w \otimes v \in W \otimes V$ lie in the same space only in case $V \equiv W$. Even then the equality $v \otimes w = w \otimes v$ is not necessarily true. Constructing unique ordered pairs $E \times F = \{(e_i, f_j)|i \in I, j \in J\}$ from the initial two bases $E = \{e_i|i \in I\}$ and $F = \{f_j|j \in J\}$ as cartesian products we get that the dimension of $V \otimes W$ equals the product of the dimensions of V and W.

$\square$

1.3 Tensor algebra

Definition 5. Let R be a commutative ring with an unit element and let M, N be R-modules. By definition $M \otimes_R N$ is a R-module.
We define the *i-fold tensor product* by $M^{\otimes i} = \underbrace{M \otimes_R M \otimes_R \ldots \otimes_R M}_{i \text{ times}}$.
Note: for $i = 0$ we get $M^{\otimes 0} = R$ and for $i = 1 M^{\otimes 1} = M$.
Now we are able to define the *tensor algebra*

$T(M) = \bigoplus_{i \geq 0} M^{\otimes i} = R \oplus M \oplus (M \otimes M) \oplus (M \otimes M \otimes M) \oplus \ldots,$

where $\oplus$ denotes the direct sum.

Example 2. Considering for $R = \mathbb{R}$ and $M = \mathbb{R}^2$ we get: $T(\mathbb{R}^2) = \bigoplus_{i=0}^{n} T^i(\mathbb{R}^2) = \bigoplus_{i=0}^{n} (\mathbb{R}^2)^{\otimes i} = \mathbb{R} \oplus \mathbb{R}^2 \oplus (\mathbb{R}^2 \otimes \mathbb{R}^2) \oplus \ldots.$ Expansion of the first three summands we can obtain bases: $\mathbb{R} = \langle x \rangle$, $\mathbb{R}^2 = \langle x, y \rangle$, $\mathbb{R}^2 \otimes \mathbb{R}^2 = \langle xx, xy, yx, yy \rangle$, since $\begin{pmatrix} x \\ y \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xx & xy \\ yx & yy \end{pmatrix}$.

Lemma 2. 1. *Let R be a commutative ring and M_1, M_2 be R-modules. Then*

$$T(M_1 \oplus M_2) \twoheadrightarrow T(M_1) \otimes T(M_2)$$

defines an isomorphism, which we call canonical.

2. *Let V be a vector space and let $V^* = Hom(V, K)$, the set of homomorphisms from V to K, be its dual space. Then there is an isomorphism:*

$$\begin{aligned} T(V) \otimes T(V^*) &\cong (\bigoplus_{r \geq 0} V^{\otimes r}) \otimes (\bigoplus_{s \geq 0} (V^*)^{\otimes s}) \\ &= \bigoplus_{r,s \geq 0} (V^{\otimes r} \otimes (V^*)^{\otimes s}) \end{aligned}$$

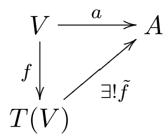
In other words, an arbitrary element x in $T(V) \otimes T(V^*)$ is represented by

$$x = \sum_{r,s \geq 0} x_{r,s} \ ; \ \text{where } x_{r,s} \in V^{\otimes r} \otimes (V^*)^{\otimes s}.$$

Note: $x_{1,0} \in V \otimes K$ is a vector and $x_{0,1} \in K \otimes V^*$ is a linear form.

Proof. 1. Follows similiarly as in theorem 2 in section 1.2.1.

2. Consider the commutative diagram:



$$\tilde{f} \circ a = f \ \ : \ \ \tilde{f}(v_1 \otimes v_2 \otimes ..) = f(v_1) \cdot f(v_2) \cdot ...$$

where f is a linear map, a the direct sum of the i-fold tensor product and A is a $\mathbb{K}$-algebra.

□

1.3.1 Symmetric algebra

Definition 6. Let $M^{\otimes n}$ be a n-fold tensor product of a R-module M and let $T(M)$ be the corresponding tensor algebra. Consider the ideal X in $T(M)$ generated by

$$[m_1, m_2] = m_1 \otimes m_2 - m_2 \otimes m_1,$$

where $m_1, m_2 \in M$. We define the *symmetric algebra* of M as

$$S(M) = T(M)/X = \bigoplus_{n \geq 0} S^n(M) = R \oplus M \oplus S^2(M) \oplus ...,$$

where $S^n(M) = T^n(M)/X$.

Example 3. Considering for $R = \mathbb{R}$ and $M = \mathbb{R}^2$ we get: $S(\mathbb{R}^2) = \bigoplus_{i=0}^n S^i(\mathbb{R}^2) = \bigoplus_{i=0}^n (\mathbb{R}^2)^{\otimes i} = \mathbb{R} \oplus \mathbb{R}^2 \oplus (\mathbb{R}^2 \otimes \mathbb{R}^2) \oplus ....$ Expansion of the first three summands we can obtain bases: $\mathbb{R} = \langle x \rangle$, $\mathbb{R}^2 = \langle x, y \rangle$, $\mathbb{R}^2 \otimes \mathbb{R}^2 = \langle xx, xy, yy \rangle$, since $\begin{pmatrix} x \\ y \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xx & xy \\ yx & yy \end{pmatrix}$ and $xy = yx$.

1.3.2 Skew-symmetric algebra

Definition 7. Let again $M^{\otimes n}$ be a n-fold tensor product of a R-module M and let $T(M)$ be the corresponding tensor algebra. Consider the ideal Y in $T(M)$ generated by

$$[m, m] = m \otimes m,$$

where $m \in M$. We define the *skew-symmmetric* (or Grassmann algebra) of M as

$$\Lambda(M) = T(M)/Y = \bigoplus_{n \geq 0} \Lambda^n(M) = R \oplus M \oplus \Lambda^2(M) \oplus ...,$$

where $\Lambda^n(M) = T^n(M)/Y$ is the so called *n-th exterior power* of M.

Definition 8. In the above notations the exterior product $\wedge$ of two elements $m_1, m_2 \in \Lambda(M)$ is defined as

$$m_1 \wedge m_2 = m_1 \otimes m_2 (modY).$$

Properties of the exterior power, $m_i \in M$ for $i = 1, ..., n$:

1. $m_1 \wedge m_2 \wedge ... \wedge m_n = \overline{(m_1 \otimes m_2 \otimes ... \otimes m_n)} =$
   $= \overline{(m_1 \otimes m_2 \otimes ... \otimes m_k)} \otimes \overline{(m_{k+1} \otimes m_{k+2} \otimes ... \otimes m_n)},$
2. $m_i = m_j \;\Rightarrow\; m_i \wedge m_j = 0,$
3. $m_i \wedge m_j = -m_j \wedge m_i.$

Example 4. Skew-symmetric algebra on vector spaces.
In particular, we can define the skew-symmetric algebra of a vector space V over a field $\mathbb{K}$. Consider the index set $I = \{i_1, i_2, ..., i_n | i_1 \leq ... \leq i_n\}$ and the generating system, $e_I = e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_n}$. Then:

$$\Lambda(V) = \bigoplus_{I \in \{1,...,n\}} \mathbb{K} e_I.$$

For two vectors $v = (v_1, v_2)^T = v_1 e_1 + v_2 e_2$ and $w = (w_1, w_2)^T = w_1 e_1 + w_2 e_2$ in $\Lambda(V)$ the exterior product is given as follows:

$$\begin{aligned} v \wedge w &= (v_1 e_1 + v_2 e_2) \wedge (w_1 e_1 + w_2 e_2) \\ &= v_1 w_1 e_1 \wedge e_1 + v_1 w_2 e_1 \wedge e_2 + \\ &\quad + v_2 w_1 e_2 \wedge e_1 + v_2 w_2 e_2 \wedge e_2 \\ &= (v_1 w_2 - v_2 w_1) e_1 \wedge e_2. \end{aligned}$$

2 Differentiable Manifolds

Differentiable manifolds are higher dimensional analogues of surfaces. We should, nevertheless, not think of a manifold as an object, which always sits inside an Euclidian space, but rather as an abstract object. After the definition of a differentiable manifold we want to describe points locally by n real numbers, local coordinates.

I refer mainly to [STU], [WAR] and [HOF] in this section.

2.1 Topological n- dimensional manifolds

We begin with some notions of topology. For our considerations it is enough to refer to [HIS].

Definition 9. Let X be a set and $P(X)$ its power set. A *topology* T is a family of sets, open subset of $P(X)$, with the following properties:

- $\emptyset, X$ are open sets,
- the intersection of finitely many open sets is open,
- any union of open sets is open.

Then one calls the pair (X, T) a *topological space.*

Definition 10. A system B of subsets of (X, T) is called *basis of topology*, if

- every open set of B is open with respect to T,
- every open set of (X, T) is representable as a union of sets of B. Here one understands the empty union as the empty set $\emptyset$.

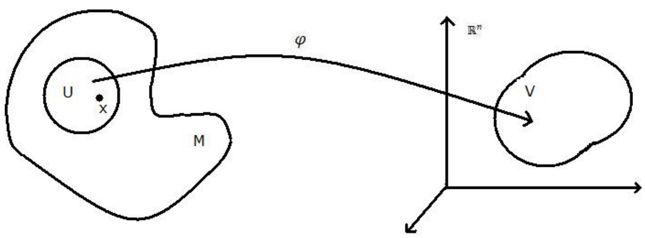
Definition 11. A topological space (X, T) satisfies the *second axiom of countability*, if and only if X has a countable basis consisting of open sets.

Now we can define the term of topological n-dimensional manifold.

Definition 12. We call a set M a *topological n-dimensional manifold*, if M is a topological space such that for each point $x \in M$ there exists an open neighborhood $U(x)$ and a locally bijective and continuous map φ:

$$U \xrightarrow{\varphi} V \subset \mathbb{R}^n$$

Remark 1. A connected 1-dimensional topological manifold is homeomorphic to a segment of a straight line.



PICTURE 1: TOPOLOGICAL N-DIMENSIONAL MANIFOLD

Remark 2. *Submanifolds* are subsets of manifolds, which are self manifolds. Generally speaking, a submanifold of a manifold is what a subspace is of a vector space; only that submanifolds and manifolds are of the same dimension. Suppose, for example, the Earth's surface as a manifold, then a meridian is a submanifold.

2.2 Local coordinates on topological manifolds

Let M be a manifold.

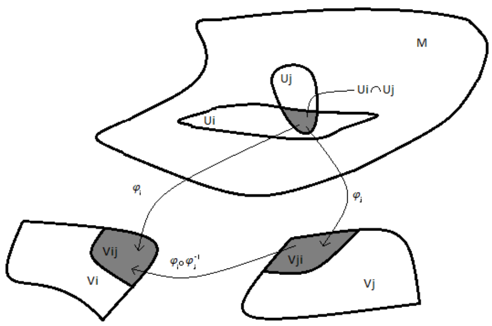
Definition 13. Given an open covering $M = \bigcup_{i \in I} U_i$ and open sets $V_i \in \mathbb{R}^n$, we call the homeomorphisms $\varphi_i : U_i \to V_i$ *local charts* on M. The φ_i determine the change of local coordinates on U_i to local coordinates on V_i.

To be able to describe properties of a manifold M, which is as mentioned earlier, somehow a object without reference system, we want to transfer M to coordinates we can work better with, namely coordintes in $\mathbb{R}^n$. Not to distort everything this change of local coordinates has to happen smooth with help of local charts.

Definition 14. We call the following smooth composition of two maps *change of local coordinates*:

$$\varphi_i \circ \varphi_j^{-1} : V_{ji}(= \varphi_j(U_i \cap U_j)) \to V_{ij}(= \varphi_i(U_i \cap U_j))$$

where both maps act on the subset $U_i \cap U_j \subset M$.



Picture 2: change of local coordinates

2.2.1 Differentiable Manifolds

Definition 15. A topological manifold is said to be C^k-*differentiable*, respectively C^∞-*differentiable* or *smooth*, if and only if

(i) M satisfies the 2^{nd} axiom of countability and

(ii) the local change of coordinates on M is C^k, C^∞, respectively.

2.3 Atlas

As for world atlases, we need several charts to depict the world and the more charts we take, the better is the representation of reality. Nevertheless, we need to assure that the charts are compatible with each other; that we can smoothly stick the charts together to one big atlas.

Definition 16. An *atlas* is the set of $U_i \subset M$ with the local charts $\varphi_i : U_i \to V_i$ and $i \in I$ an index set: $\mathfrak{A} = \{(U_i; \varphi_i)|i \in I\}$.

Definition 17. Let $U, U_i, U_i' \subset M$ and $V, V_i, V_i' \subset \mathbb{R}^n$ be open subsets.

- A map $\varphi : U \to V$ is called *compatible with a given atlas* $\mathfrak{A}$ *on a smooth manifold* M if and only if the operation $\varphi \circ \varphi_i^{-1}$: $\varphi_i(U \cap U_i) \to \varphi(U \cap U_i)$ is a diffeomorphism (i.e. $\varphi \circ \varphi_i^{-1}$ and its inverse are smooth).
- Two atlases $\mathfrak{A} = (U_i; \varphi_i : U_i \to V_i)$, $\mathfrak{A}' = (U_i'; \varphi_i' : U_i' \to V_i')$ are called *compatible* on M if and only if
  - (i) every chart of (U_i, φ_i) is compatible with $\mathfrak{A}'$ and
  - (ii) every chart of (U_i', φ_i') is compatible with $\mathfrak{A}$.

Definition 18. We call the union of all maps which are compatible with a given atlas $\mathfrak{A}$ the *maximal Atlas* $\mathcal{A} = \bigcup_{\varphi_i \in \mathfrak{A}} \mathfrak{A}$.

Remark 3. It follows from the defintion that every atlas $\mathfrak{A}$ is contained in the unique maximal atlas $\mathcal{A}$.

Remark 4. (Refinement of open covers)
Take an open cover $\mathcal{U} = (U_i|i \in I)$ of a manifold M. Let the open sets $\mathcal{V} = (V_j|j \in I)$ be a refinement of $\mathcal{U}$. This means, $\mathcal{V}$ is a new cover of M such that every set in $\mathcal{V}$ is contained in some set in $\mathcal{U}$. We can select the indices by the function $\alpha : J \to I$; $j \mapsto i$: then $V_j \subset U_{\alpha(j)}$.

This is motivated by the *Heine-Borel Theorem* claiming that: if the set M is covered by the union of open covers $\bigcup_{i \in I} U_i \supseteq M$ then there exists a subindex set $J \subset I$, so is M even representable by the union over the new indexed covers $\bigcup_{j \in J} U_j \supseteq M$. In other words, every open cover of a compact subset of $\mathbb{R}^n$ has a finite subcover. See for further considerations [RUD].

Theorem 3. *(Heine - Borel)*
For a subset S of $\mathbb{R}^n$ the following statements are equivalent:

(a) S is closed and bounded;

(b) every open cover of S has a finite subcover (i.e. S is compact).

Proof. (a) $\Rightarrow$ (b) (closed and bounded implies compact): Assume that S is bounded. We can cover $S \subset \mathbb{R}^n$ by the Cartesian product of n intervals $[a_i, b_i]$, $a_i < b_i$, $a_i, b_i \in \mathbb{R}$, $i = 1, ..., n$: $S \subseteq T_0 = [a_1, b_1] \times [a_s, b_2] \times ... \times [a_n, b_n]$. We can divide each side of T_0 into halves. Doing so we get 2^n smaller cells. To get a contradiction we assume that T_0 is not compact, i.e. T_0 is covered by open sets $\{G_\alpha\}$ which have no finite subcover. Now take the 2^n smaller cells into account: at least one of those cannot be covered by a open subcellection of $\{G_\alpha\}$. Otherwise the whole T_0 would be covered in this way. Call this small cell T_1. Considering T_1 we divide its sides in halves again and pick out the next cell T_2, which is not covered by a open subcollection of $\{G_\alpha\}$. Continuing this process we construct a sequence of cells with the following properties:

(i) $T_0 \supset T_1 \supset T_2 \supset ... \supset T_k \supset ...$;

(ii) T_k is not covered by any finite subcollection of $\{G_\alpha\}$;

(iii) the length of all sides T_k tends to zero when k goes to ∞, i.e.: $\lim_{k \to \infty} \frac{b_k - a_k}{2^k} = 0$

By Cantor's lemma we get $\bigcap_{k=1}^{\infty} T_k \neq \emptyset$, which means that there is a point p in the intersection of all enclosed cells, $p \in T_k$ for all $k \in \mathbb{N}$. Because G_α is open there exists an open ball $\mathfrak{B}(p)$ around p. Since p sits in all T_k this ball $\mathfrak{B}(p)$ works as the finite subcover. Since we covered S by the n-cell T_0 we can make a selection of open balls covering S and we get a contradiction.

(b) $\Rightarrow$ (a) (compact implies closed and bounded): (1) Compact implies closed: Taken a point y in the complement to S: $y \in S^C$. For all $x \in S$ there exist nonintersecting neighborhoods $\mathcal{B}_x$ containing x and $\mathcal{B}_y^x$ containing y. The union of all $\mathcal{B}_x$ builds an open cover of S, $S \subset \bigcup_{x \in S} \mathcal{B}_x$. Since S is assumed to be compact, there exists a open, finite subcover $\mathcal{B}_{x_1}, ..., \mathcal{B}_{x_n}$ of S. Consider the neighborhood of y, which lies outside S: $\bigcap_{k=1}^{n} \mathcal{B}_y^{x_k}$. This means that y (any point outside S) can´t be a limit point of S and therefore all limit points must already lie inside S.

(2) Compact implies bounded: Consider an open ball centered at a common point of S of any desired radius. Because all points have the

same distance to the ball´s boundary, it can cover any set. Since all balls in the subcover are contained in the largest open ball entirely lying in that subcover, they must be bounded. Therefore the set S which is covered now by the bounded smaller balls, must be bounded as well. $\square$

3 Tangent and Cotangent vectors and Differential forms

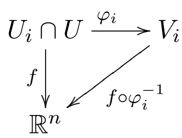
In this section M is a differentiable manifold and let $\mathfrak{A} = \{(U_i; \varphi_i) | i \in I\}$ be a smooth atlas on M with local charts $\varphi_i : U_i \to V_i$, where $U_i \subseteq M$ and $V_i \subseteq \mathbb{R}^n$.

I refer mainly to [FRI],[FOR],[GSI] and [HIT] in this section.

3.1 Smooth maps

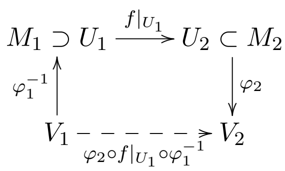
- Let $U \subset M$ be an open subset and let p be a point in U. We say that the map $f : U \to \mathbb{R}^n$ is *smooth at p* if and only if for any chart φ_i the composition $f \circ \varphi_i^{-1}$ is smooth at $\varphi_i(p)$, i.e.

  $f : U \to \mathbb{R}^n$ is smooth at $p \in U$
  $\Leftrightarrow$ for each chart φ_i the composition $f \circ \varphi_i^{-1}$ is $\varphi(U_i \cap U) \to \mathbb{R}^n$ is smooth in $\varphi_i(p)$.



- Let M_1 and M_2 be manifolds, $U_i \subset M_i$ and $V_i \subset \mathbb{R}^n$ be open subsets for $i = 1, 2$. We say that the map $f : M_1 \to M_2$ is *smooth at p* if and only if the composition of f and two charts $\varphi_i : U_i \to V_i$, $f(U_1) \subset U_2$, $\varphi_2 \circ f|_{U_1} \circ \varphi_1^{-1} : V_1 \to V_2$ is smooth; i.e.

  $f : M_1 \to M_2$ is smooth
  $\Leftrightarrow$ for each two charts $\varphi_i : U_i \to V_i$, $U_i \subset M_i$, $i = 1, 2$, $f(U_1) \subset U_2$, the restriction $\varphi_2 \circ f|_{U_1} \circ \varphi_1^{-1} : V_1 \to V_2$ is smooth.



Definition 19. We call $f : M_1 \to M_2$ a *diffeomorphism*, if

(i) f is a homeomorphism: f is bijective, f, f^{-1} are continuous,

(ii) f, f^{-1} are C^∞-differentiable.

Even if we define manifolds with the help of local charts, there is no chart marked as 'more preferably' than another. According to this, the choice of charts chould later be rather unimportant. We define a smooth function, which maps every point p on a manifold M to a real number. Then we get, collecting all these functions, a set of all smooth functions from M to $\mathbb{R}$.

Definition 20. We define the *ring of smooth functions* by $\mathcal{C}^\infty(U) = \{f : M \to \mathbb{R} | f \text{ is smooth for all } p \in M\}$.

This definition is given without local charts and this is meaningsfull, because it is possible to define a map from any set to the real numbers. This means, in other words, that the value of a function is independent of the choice of local coordinates.

Remark 5.

- For any $p \in U$ consider the set $\mathfrak{M}_p = \{f \in \mathcal{C}^\infty(U) | f(p) = 0\}$. This is a maximal ideal in $\mathcal{C}^\infty(U)$.
- $\mathcal{C}^\infty(U)$ is a $\mathbb{R}$-vector space.

Proof.

- By definition a *maximal ideal* of a ring R is an ideal $\mathfrak{M}$ satisfying:

  $$\text{for each ideal } \mathfrak{a} \in R \text{ with } \mathfrak{M} \subset \mathfrak{a} \subset R: \text{ either } \mathfrak{a} = \mathfrak{M} \text{ or } \mathfrak{a} = R$$

  Denote by ev_0 the mapping which evaluates every function at the origin. The image under ev_0 is $\mathbb{R}$ and its kernel, the set of all $f(p) = 0$, is a maximal ideal.
- The axioms of a vector space are easily verified.

□

These functions have many applications, for example, the electrical potential can be understood as such a scalar field. But since the electrical and the magnetic fields are attracting each other, we need to introduce a reference system, which respects both forces and specifies both, a scalar and a direction at a point p on a manifold M. We need later on a vector field.

3.2 Tangent vector

Because we are not longer in a Euclidean space but in a curved space, we can´t define vectors globally. Each point p of a manifold M has its own vector space.

Definition 21. Let M be a differentiable manifold and $p \in M$ be a point on M. Define the *vector* X on M at p as the map from $\mathcal{C}^{\infty}_{M,p}$ to $\mathbb{R}^n$, mapping every smooth function $f \in C^{\infty}_{M,p}$ on a differentiable manifold M to a real number, which satisfies the following properties:

- X is a $\mathbb{R}$-linear mapping: $(\lambda X_1 + \nu X_2)f = \lambda(X_1 f) + \nu(X_2 f)$
- X respects the *Leibnitz* rule:
  $X(f_1 + f_2) = X(f_1)f_2(p) + f_1(p)X(f_2)$

We can consider vectors as directional derivatives. They specify the rate of change of a scalar field in the direction they point.

Definition 22. For a function $f : M \to \mathbb{R}^n$ and the on M in local coordinates represented points $p = (p_1, p_2, ..., p_n), h = (h_1, h_2, ..., h_n)$ the *directional derivative* is defined as:

$$D_h(f) = \lim_{t \to 0} \frac{f(p+th) - f(p)}{t} = \sum_{i=1}^{n} h_i \frac{\partial}{\partial x_i}|_p(f).$$

In the special case: $h_i = e_i$ the unit vectors: $D_{e_i}(f) = \frac{\partial}{\partial x_i}|_p(f)$; $i = 1, ..., n$, we get the usual partial derivative.

The direction a vector points can be determined by calculating the derivative in direction of a curve: let $k : \mathbb{R} \to M$ be a smooth curve and let $p \in M$be a point on M with $p = k(t_p)$ for $t_p \in \mathbb{R}$. The map which at a given point p assigns to each function $f \in C^{\infty}_{M,p}$ the real number

$$\frac{df(k(t))}{dt}|_{t=t_p}$$

is called tangent vector. We note that this coincides due to the above defintion 21 of a vector with the general derivative and we define finally

$$X : \mathcal{C}^{\infty}_{M,p} \longrightarrow \mathbb{R}$$
$$f \longmapsto \frac{df(k(t))}{dt}|_{t=t_p}$$

as the *tangent vector* at $p = k(t_p)$ of a smooth curve $k : \mathbb{R} \to M$, $t_p \in \mathbb{R}$.

Given a tangent vector X at p, we define the *set of tangent vectors* to M at p as $T_{M,p} = \{X | X \text{ tangent vector in } p\}$.

Note:

- $T_{\mathbb{R}^n,p} = \mathbb{R}\frac{\partial}{\partial x_1}|_p + \mathbb{R}\frac{\partial}{\partial x_2}|_p + ... + \mathbb{R}\frac{\partial}{\partial x_n}|_p$
- $T_{S^{n-1},p} = \{\sum_{i=1}^n \lambda_i \frac{\partial}{\partial x_i} | \sum_{i=1}^n \lambda_i p_i = 0\} \subseteq T_{\mathbb{R}^n,p}$

Here we denote the $(n-1)$-dimensional unit sphere:

$$S^{n-1} = \{x \in \mathbb{R}^n | \, \|x\| = 1\}$$

For $f(x) = x_1^2 + ... + x_n^2 - 1$ is

$$S^{n-1} = \{x \in \mathbb{R}^n | f(x) = 0\}$$

and further is $\text{grad} f(x) = (2x_1, ..., 2x_n)$ implying that $\text{grad} f(x) \neq 0$ for all $x \in S^{n-1}$.

3.3 Tangent bundle

Definition 23. Given a smooth manifold M we define its *tangent bundle* $T(M)$ as the union of tangential vectors over all points p on the manifold M:

$$T(M) = \bigcup_{p \in M} T_{M,p}$$

Although we need later charts with local coordinates on manifolds M to define differentiablility of functions, which are acting on M, we have defined the term of vector independent of coordinates. Further, our notion of a vector got the vivid idea of a direction, which might be defined by a curve.

3.4 Vector field

Until now we have no possibility to compare elements of the tangent space at one point with the tangent space at another point. Before we get rid of this problem we first define a vector field.

Definition 24. We define a *vector field* for each point p on a smooth manifold M as the mapping from M to the tangent bundle $T(M)$. A point $p \in M$ is mapped to its vector $V(p) \in T_{M,p}$:

$$V : U \longrightarrow T(M)$$
$$p \longmapsto \sum_{i=1}^{n} \lambda_i(p) \frac{\partial}{\partial x_i}|_p$$

Note: We call V smooth if and only if the generating functions λ_i are smooth for $i = 1, ..., n$

Theorem 4. *Consider a k-dimensional manifold M and a point $p \in M$, then:*

(a) $T_{M,p}$ is a k-dimensional subspace in $\mathbb{R}^n$.

(b) Consider a map $\varphi : M \to V$, where $V \subset \mathbb{R}^k$ open and a point $p \in U$. The vectors $\frac{\partial \varphi}{\partial t_1}(p), ..., \frac{\partial \varphi}{\partial t_k}(p)$ form a basis for $T_{M,p}$.

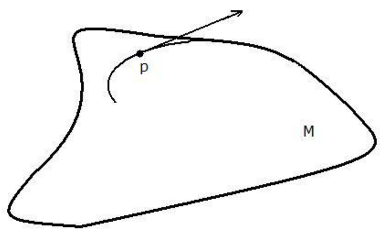
(c) Let $N \subset \mathbb{R}^n$ be an open neighborhood of p and $f_1, ..., f_{n-k} : N \to \mathbb{R}$ smooth functions with

$$M \cap N = \{x \in N : f_1(x) = ... = f_{n-k}(x) = 0\} \textit{ and}$$

$$rank \frac{\partial(f_1, ..., f_{n-k})}{\partial(x_1, ..., x_n)}(p) = n - k$$

Then:

$$T_{(m,p)} = \{v \in \mathbb{R}^n | < v, gradf_j(p) >= 0 \textit{ for } j = 1, ..., n-k\}$$



Picture 3: tangent vector to M at p

Proof. Consider T_1 as a vector space spanned by $\frac{\partial \varphi}{\partial t_1}(p), ..., \frac{\partial \varphi}{\partial t_1}(p)$ and $T_2 = \{v \in \mathbb{R}^n | < v, \text{grad} f_j(p) >= 0 \text{ for } j = 1, ..., n-k\}$. We want to show: $T_1 \subset T_{M,p} \subset T_2$. Since T_1 and T_2 are both k-dimensional subspaces of $\mathbb{R}^n$ we have shown then, that necessarily $T_1 = T_{M,p} = T_2$ and therefore the theorem holds.

(i) (Inclusion $T_1 \subset T_{M,p}$) Set

$$v = \lambda_1 \frac{\partial \varphi}{\partial t_1}(p) + ... + \lambda_k \frac{\partial \varphi}{\partial t_k}(p)$$

to denote any vector of T_1. We define a curve

$$\psi :]-\epsilon, \epsilon[\to M \subset \mathbb{R}^n \text{ by } \psi(\tau) : \varphi(p_1 + \lambda_1 \tau, ..., p_k + \lambda_k \tau)$$

We have $\psi(0) = \psi(p)$ and by chain rule $\psi'(0) = \lambda_1 \frac{\partial \varphi}{\partial t_1}(p) + ... + \lambda_k \frac{\partial \varphi}{\partial t_k}(p) = v$; this means $v \in T_{M,p}$

(ii) (Inclusion $T_{M,p} \subset T_2$) Consider $v \in T_{M,p}$, i.e. $v \in \psi'(0)$ for a smooth curve $\psi :]-\epsilon, \epsilon[\to M \subset \mathbb{R}^n$ with $\psi(0) = p$. Since this curve proceeds in M it is for $j = 1, ..., n-k$:

$$f_j(\psi(\tau)) = 0 \text{ for } |\tau| < \epsilon_1 \text{ , } (0 < \epsilon_1 \leq \epsilon)$$

After differentiation we get

$$0 = \sum_{i=0}^{n} \frac{\partial f_j}{\partial x_i}(\psi(0)) \frac{d\psi_i}{d\tau}(0) =$$
$$= < \text{grad} f_j(p), \psi'(0) > = < v, \text{grad} f_j(p) >$$

which means $v \in T_2$.

□

Definition 25. Let M be a k-dimensional manifold with scalar product $\langle ., . \rangle$ of $\mathbb{R}^n$ and $p \in M$. We call a vector $v \in \mathbb{R}^n$ a *normal vector* on M in p, if v is perpendicular to $T_{M,p}$:

$$\langle v, w \rangle = 0 \text{ for all } w \in T_{M,p}$$

Normal vectors to M at p build the $(n-k)$- dimensional normal bundle $N_{M,p} \subset \mathbb{R}^n$. By the above theorem $\text{grad} f_1(p), ..., \text{grad} f_{n-k}(p)$ form a basis of $N_{M,p}$.

3.5 Cotangent space to M at p

In the linear algebra there is the concept of dual vector space. Given a vector space V the dual space V^* consists of all linear maps from V to $\mathbb{R}$. We apply this concept on our tangent spaces at every point p on a manifold M and get the corresponding cotangent spaces.

Definition 26. We call $T^*_{M,p} = \mathrm{Hom}(T_{M,p}, \mathbb{R})$, i.e. the dual space to $T_{M,p}$, *cotangent space* to M at p. The map

$$\varphi^* : T^*_{N,q} \to T^*_{M,p}$$

from the cotangent space to N at q to the cotangent space to M at p is called *cotangent vector*.

Remark 6. Let φ^* be as above $\varphi^* : T^*_{N,q} \to T^*_{M,p}$ and let φ the map between the set of tangent vectors given by $\varphi : T_{M,p} \to T_{N,q}$. Then

$$\varphi^*(\omega)(p) = \omega(\varphi)(p)$$

holds for $p \in T_{M,p}$ and $\omega \in T^*_{N,q}$.

3.6 Differential forms

We want to investigate the elements of the just defined spaces. For example, the elements of cotangent space are called 1-forms. After we have considered this most basical case we go on with higher order differential forms.

3.6.1 1-forms

Definition 27. (a) A *differential form of order 1* (or a *pfaffian form*) on an open set $U \in \mathbb{R}^n$ is given by the mapping:

$$\omega : U \to \bigcup_{p \in U} T^*_p(U)$$

where $\omega(p) \in T^*_p(U)$ for all $p \in U$. In other words, this differential form maps every point $p \in U$ to a cotangent vector $\omega(p) \in T^*_p(U)$. We write the value of $\omega(p)$ on the tangent vector $v \in T_{U,p}$ as $< \omega(p), v >$.

(b) Given a smooth function $f : U \to \mathbb{R}$ we define its *total differential df* as the differential 1-form given by: for $p \in U$ and $v \in T_{U,p}$:

$$< df(p), v > := < \mathrm{grad} f(p), v > = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p) v_i.$$

The definition works even for a smooth manifold M. Then: $\omega : M \to \bigcup_{p \in M} T^*_{M,p}$. Then we just need to use instead of the standard coordinates in $\mathbb{R}^n$ the local coordinates of the manifold M.

Representation of 1-forms in coordinates

Likewise we constructed by $\left(\frac{\partial}{\partial x_1}\right)_p, ..., \left(\frac{\partial}{\partial x_n}\right)_p$ a basis for the tangent space we want to construct a dual basis for the cotangent space. For this we use the standard coordinate system $(x_1, ..., x_n)$ of $\mathbb{R}^n$. The process gets the same, if one understands this system to be the local coordinate system of a manifold M.

Consider the differentials $dx_1, dx_2, ..., dx_n$ of a coordinate system $(x_1, x_2, ..., x_n)$ of $\mathbb{R}^n$. The i-th coordinate function is given by

$$x_i : \mathbb{R}^n \to \mathbb{R}; (p_1, p_2, ..., p_n) \mapsto p_i$$

Let $e_j = (0, ...0, 1, 0, ...0)$ be the j-th basis vector in $\mathbb{R}^n$, where the 1 stands at the j-th place. By definition we have:

$$< dx_i(p), e_j >= \frac{d}{dt} x_i(p + te_j)|_{t=0} = \frac{d}{dt}(p_i + t\delta_{ij})|_{t=0} = \delta_{ij}$$

where δ_{ij} is the Kronecker delta. Therefore we see that the cotangent vectors $dx_1(p), ..., dx_n(p)$ form a basis of $T^*_{\mathbb{R}^n,p}$, which is dual to $e_1, e_2, ..., e_n$. Every cotangent vector $\varphi \in T^*_{\mathbb{R}^n,p}$ can be written as

$$\varphi = \sum_{i=0}^{n} c_i dx_i(p)$$

with uniquely determined coefficients $c_i \in \mathbb{R}$. We can conclude that every 1-form ω of an open set $U \subset \mathbb{R}^n$ can be uniquely written as

$$\omega = \sum_{i=0}^{n} f_i dx_i \text{ resp. for all } p \in U : \omega(p) = \sum_{i=0}^{n} f_i(p) dx_i(p)$$

with functions $f_i : U \to \mathbb{R}$.

Lemma 3. *Let $f : U \to \mathbb{R}$ be a smooth function on open set $U \subset \mathbb{R}^n$. Then:*

$$df = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} dx_i$$

Proof. We have to show that at every point $p \in U$ the right-hand side and the left-hand side give the same value on every tangent vector $v \in T_{U,p}$. Since $< dx_i(p), v >= v_i$ we have:

$$< \sum \frac{\partial f}{\partial x_i}(p) dx_i(p), v >= \sum \frac{\partial f}{\partial x_i} =< df(p), v >$$

□

3.6.2 Differential q-forms on M

Definition 28. We call the mapping ω on a manifold M given by

$$\omega : M \to \bigcup_{p \in M} \Lambda^q(T^*_{M,p})$$

a *skew-symmetric q-form*. The set of all skew-symmetric q-forms at p on M forms a vector space $\Lambda^q T^*_{M,p}$. Note: $\Lambda^0 T^*_{M,p} := \mathbb{R}, \Lambda^1 T^*_{M,p} = T^*_{M,p}$.

Representation of a q-form in local coordinates

Again we consider the coordinate system $(x_1, ..., x_n)$ of $\mathbb{R}^n$ and again one can understand those as local coordinates of the manifold M. In canonical coordinate functions $x_1, x_2, ..., x_n$ of $\mathbb{R}^n$ a basis of $\Lambda^q T^*_{U,p}$ is given by the elements:

$$dx_{i_1}(p) \wedge ... \wedge dx_{i_q}(p), \text{ for } 1 \leq i_1 < i_2 < ... < i_q \leq n.$$

Every differential q-form is representable as:

$$\omega : U \to \bigcup_{p \in U} \Lambda^q(T^*_{M,p}),$$

$$\omega = \omega_1 \wedge \omega_2 \wedge ... \wedge \omega_q = \sum_I \alpha_I(x_1, x_2, ..., x_q) dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_q},$$

where each w_k, $(k = 1, ..., q)$ is a 1-form, I is an index set and α_I are unique smooth functions. For all $p \in U$ this means:

$$\begin{aligned} \omega(p) &= \omega_1(p) \wedge \omega_2(p) \wedge ... \wedge \omega_q(p) \\ &= \sum_I \alpha_I(p)(x_1(p), x_2(p), ..., x_q(p)) dx_{i_1}(p) \wedge dx_{i_2}(p) \wedge ... \wedge dx_{i_q}(p). \end{aligned}$$

In the following we will derive the theorems and defintions on an open subset $U \subset \mathbb{R}^n$ with the standard coordinate system $(x_1, ..., x_n)$. On a smooth manifold M with local coordinates $(x_1, ..., x_n)$ all this looks similar, because we provide local charts sending each point of M to $\mathbb{R}^n$ and which can be glued smoothly together.

Exterior derivative of a differential form

For $U \subset \mathbb{R}^n$ and the q-form $\omega = \sum_{i_1<...<i_q} f_{i_1...i_q} dx_{i_1} \wedge ... \wedge dx_{i_q}$ define the $(q+1)$-form $d\omega$ by

$$d\omega := \sum_{i_1<...<i_q} df_{i_1...i_q} \wedge dx_{i_1} \wedge ... \wedge dx_{i_q}.$$

Example 5. Take the smooth 1-form $\omega = \sum_{i=1}^n f_i dx_i$. Since $df_i = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} dx_j$ we have:

$$d\omega = \sum_{i,j=1}^n \frac{\partial f_i}{\partial x_j} dx_i \wedge dx_j.$$

Further, $dx_i \wedge dx_i = 0$ and $dx_i \wedge dx_j = -dx_j \wedge dx_i$ and finally:

$$d\omega = \sum_{i<j}^n \left(\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j} \right) dx_i \wedge dx_j.$$

Theorem 5. *Let $U \subset \mathbb{R}^n$ be an open set and $\omega, \omega_1, \omega_2$ be smooth q-forms in U and σ be a smooth r-form, $\lambda, \mu \in \mathbb{R}$:*

(i) $d(\lambda\omega_1 + \mu\omega_2) = \lambda d\omega_1 + \mu d\omega_2$

(ii) $d(\omega \wedge \sigma) = (d\omega) \wedge \sigma + (-1)^q \omega \wedge (d\sigma)$

(iii) $d(d\omega) = 0$.

Proof. (i) Follows directly from general rules of differentiation.

(ii) Consider first the case $q = r = 0$, i.e. two smooth functions $f, g : U \to \mathbb{R}$. By the chain rule, $\frac{\partial}{\partial x_i}(fg) = \frac{\partial}{\partial x_i} \cdot g + f \cdot \frac{\partial}{\partial x_i}$, we get

$$d(fg) = gdf + fdg = df \wedge g + f \wedge dg.$$

Now in the general case we have

$$\omega = \sum_{|I|=q} f_I dx_I, \sigma = \sum_{|J|=r} g_J dx_J : \ \omega \wedge \sigma = \sum_{I,J} f_I g_J dx_I \wedge dx_J.$$

We get:

$$\begin{aligned} d(\omega \wedge \sigma) &= \sum_{I,J} (g_J df_I + f_I dg_J) \wedge dx_I \wedge dx_J \\ &= \sum_{I,J} (g_J df_I \wedge dx_I \wedge dx_J + (-1)^q f_I dx_I \wedge dg_J \wedge dx_j) \\ &= (d\omega) \wedge \sigma + (-1)^q \omega \wedge (d\sigma). \end{aligned}$$

(iii) Consider first the case $q = 0$. For a smooth function $f : U \to \mathbb{R}$ we have: $df = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} dx_i$. By the above example we have:

$$d(df) = \sum_{i<j} \left\{ \frac{\partial}{\partial x_i} \left(\frac{\partial f}{\partial x_j} \right) - \frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_i} \right) \right\} = 0$$

Consider now a twice continuous differentiable q-form

$$\omega = \sum_{|I|=q} f_I dx_I : \; d\omega = \sum_{|I|=q} df_I \wedge dx_I.$$

Since $d(dx_I) = d(1 \wedge dx_I) = 0$ follows together with (ii):

$$d(d\omega) = \sum_{I} \{ d(df_I) \wedge dx_I - df_I \wedge d(dx_I) \} = 0.$$

□

Definition 29. Let $U \in \mathbb{R}^n$ be an open set.

(i) A smooth q-form ω in U is called *closed*, if $d\omega = 0$.

(ii) For $q \geq 1$ we call a smooth q-form ω in U *exact*, if there exists a smooth $(q-1)$-form η in U such that $\omega = d\eta$.

Note: Every exact form is closed, since $d \circ d = 0$.

3.6.3 Differential $n-1$-forms on M

We can define differential forms of order $n-1$ on the open n-dimensional manifold M. We use the following elements as basis

$$e_i(p) = (-1)^{i-1}(dx_1 \wedge ... \wedge \widehat{dx_i} \wedge ... \wedge dx_n)(p) \text{ for } 1 \leq i \leq n.$$

The 'hat' above dx_i means that this factor has to be left out. This allows us to write a $(n-1)$-dimensional smooth form of the vector space $\Lambda^{n-1} T^*_{M,p}$, as

$$\omega = \sum_{i=1}^{n} (-1)^{i-1} f_i (dx_1 \wedge ... \wedge \widehat{dx_i} \wedge ... \wedge dx_n) = \sum_{I=1}^{n} f_i(p) e_i$$

with smooth coefficients functions $f_i : M \to \mathbb{R}$. Since

$$(-1)^{i-1} \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_i} dx_j \right) \wedge dx_1 \wedge ... \wedge \widehat{dx_i} \wedge ... \wedge dx_n =$$
$$= \frac{\partial f_i}{\partial x_i} dx_1 \wedge ... \wedge dx_i \wedge ..., \wedge dx_n$$

we get

$$d\omega = \left(\sum_{i=1}^{n} \frac{\partial f_i}{\partial dx_i} \right) dx_1 \wedge ... \wedge dx_n.$$

We collect the functions f_i in a vector field $f = (f_1, ..., f_n) : M \to \mathbb{R}^n$ and define the following $n-$tuple of $(n-1)$-forms:

$$d\vec{S} := (dS_1, ..., dS_n),\ dS_i := (-1)^{i-1} dx_1 \wedge ... \wedge \widehat{dx_i} \wedge ... \wedge dx_n.$$

We will study $d\vec{S}$ closer when we consider integration on manifolds.

We can now determine the $n-1$-form ω as the scalar product between f and $d\vec{S}$:

$$\omega = f \cdot d\vec{S} := \sum_{i=1}^{n} f_i \cdot dS_i.$$

This leads us to the definition:

$$d\omega = d(f \cdot d\vec{S}) = div(f) dx_1 \wedge ... \wedge dx_n,$$

where

$$div(f) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$$

is the divergence of the vector field f.

Example 6. (Differential 2-forms on $\mathbb{R}^2$)
We parametrize the standard coordinates of $\mathbb{R}^2$ with the polar coordinates: $x = r\cos\varphi$, $y = r\sin\varphi$.

$$\begin{aligned} dx \wedge dy &= \left(\frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \varphi} d\varphi \right) \wedge \left(\frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \varphi} d\varphi \right) = \\ &= (\cos\varphi dr - r\sin\varphi d\varphi) \wedge (\sin\varphi dr + r\cos\varphi d\varphi) = \\ &= \cos\varphi \sin\varphi dr \wedge dr + r\cos^2\varphi dr \wedge d\varphi + \\ &\quad - r\sin^2\varphi d\varphi \wedge dr - r^2 \sin\varphi d\varphi \wedge d\varphi = \\ &= r\cos^2\varphi dr \wedge d\varphi + r\sin^2 dr \wedge d\varphi = \\ &= r(\cos^2\varphi + \sin^2\varphi) dr \wedge d\varphi = \\ &= r dr \wedge d\varphi. \end{aligned}$$

3.7 $*$ - operation

Problems can often be solved by transition to other coordinates, which are better adjusted to the problem. Change of coordinates from the standard coordinates in $\mathbb{R}^n$ to polar coordinates (see the example above) or change of local coordinates from a manifold M to $\mathbb{R}^n$ as we have already seen provide good examples. In particular, this means that one maps a point x lying in an arbitrary set U with coordinates $(x_1, ..., x_n)$ uniquely to another point $f(x) = y$ with the coordinates $(y_1, ..., y_n)$. Then the set U has under this transformation f the form $f(U) = V$. Functions g on V can then be pulled back to U, by composition $g \circ f$. Now we want to investigate how differential forms can be pulled back.

3.7.1 Pullback of Differential forms

Let M be a smooth manifold and $U \subset M$ be a q-dimensional submanifold. Let $V \subset \mathbb{R}^m$ be an open subset with a q-form $\omega = \sum_{i_1<...<i_q} f_{i_1...i_q} dx_{i_1} \wedge ... \wedge dx_{i_q}$. Furthermore consider a smooth map φ:

$$\varphi = (\varphi_1, ..., \varphi_m) : U \to V.$$

Constructing the pullback of differential forms we should demand that it doesn't matter if we first perfom differentiation and after that pullback or the other way round. This means, we are looking for a pull back operation φ^*, which commutes with differentiation.

Definition 30. The *pullback* $\varphi^*\omega$ *of a differential form* ω is defined by:

$$\varphi^*\omega := \sum_{i_1<...<i_q} (f_{i_1...i_q} \circ \varphi) d\varphi_{i_1} \wedge ... \wedge d\varphi_{i_q}.$$

Let $t_1, ..., t_m$ denote the canonical coordinate functions in $\mathbb{R}^m$. Then define:

$$d\varphi_i = \sum_{j=1}^{m} \frac{\partial \varphi_i}{\partial t_j} dt_j.$$

Example 7. The case $q = 1$: $\omega = \sum_{i=0}^{n} f_i dx_i$. Then

$$\varphi^*\omega = \sum_{i=1}^{n}\sum_{j=1}^{m} (f_i \circ \varphi)\frac{\partial \varphi_i}{\partial t_j} dt_j, \text{ which means}$$

$$\varphi^*\omega = \sum_{j=1}^{m} g_j dt_j \text{ with } g_j = \sum_{i=1}^{n} (f_i \circ \varphi)\frac{\partial \varphi_i}{\partial t_j}.$$

This can be represented in the matrix form as follows. Let

$$D\varphi = \begin{pmatrix} \frac{\partial \varphi_1}{\partial t_1} & \cdots & \frac{\partial \varphi_1}{\partial t_m} \\ \vdots & & \vdots \\ \frac{\partial \varphi_n}{\partial t_1} & \cdots & \frac{\partial \varphi_n}{\partial t_m} \end{pmatrix}$$

be the Jacobian matrix of φ and both $f = (f_1, ..., f_n)$ and $g = (g_1, ..., g_m)$ abbreviated to row vectors, then: $g = (f \circ \varphi)D\varphi$.

Theorem 6. *Let M be a smooth manifold, $U \subset M$ be a submanifold, $V \subset \mathbb{R}^m$ be an open subset and $\varphi : U \to V$ be a smooth map. Further, let $\omega, \omega_1, \omega_2$ be q-forms and σ be a r-form in V and $\lambda, \mu \in \mathbb{R}$. Then:*

(i) $\varphi^*(\lambda\omega_1 + \mu\omega_2) = \lambda\varphi^*\omega_1 + \mu\varphi^*\omega_2$;

(ii) $\varphi^*(\omega \wedge \sigma) = (\varphi^*\omega) \wedge (\varphi^*\sigma)$;

(iii) $d(\varphi^*\omega) = \varphi^*(d\omega)$;

(iv) $\psi : W \to U$ *another smooth function on an open set* $W \subset \mathbb{R}^p$, *then:* $\psi^*(\varphi^*\omega) = (\varphi \circ \psi)^*\omega$.

Proof. (i) Follows from the linearity of derivatives.

(ii) Set $\omega = \sum_I f_I dx_I$, which yields to $\varphi^*\omega = \sum_I (f_I \circ \varphi) d\varphi_I$ and $\sigma = \sum_J g_J dx_J$, which yields to $\varphi^*\sigma = \sum_J (g_J \circ \varphi) d\varphi_J$. We get the equalities:

$$\begin{aligned} \varphi^*(\omega \wedge \sigma) &= \varphi^*\left((\sum_I f_I dx_I) \wedge (\sum_J g_J dx_J) \right) \\ &= \varphi^*\left(\sum_{I,J} f_I g_J dx_I \wedge dx_J \right) \\ &= \sum_{I,J} (f_I g_J \circ \varphi) dx_I \wedge dx_J \\ &= \left(\sum_I (f_I \circ \varphi) d\varphi_I \right) \wedge \left(\sum_J (g_J \circ \varphi) d\varphi_J \right) \\ &= (\varphi^*\omega) \wedge (\varphi^*\sigma). \end{aligned}$$

(iii) Consider first the case $q = 0$; i.e. we have a smooth function $f : U \to \mathbb{R}$. Using the chain rule we get:

$$
\begin{aligned}
d(\varphi^* f) = d(f \circ \varphi) &= \sum_{j=1}^{m} \frac{\partial (f \circ \varphi)}{\partial t_j} dt_j = \\
&= \sum_{j=1}^{m} \sum_{i=1}^{n} \left(\frac{\partial f}{\partial dx_i} \circ \varphi \right) \frac{\partial \varphi_i}{\partial dt_j} dt_j = \\
&= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \circ \varphi \right) d\}\varphi_i = \\
&= \varphi^* \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \right) = \varphi^*(df).
\end{aligned}
$$

For any smooth q-form $\omega = \sum_I f_I dx_I$ we get

$$
\varphi^* \omega = \sum_I (f_I \circ \varphi) d\varphi_I,
$$

where we abbreviated $d\varphi_I := d\varphi_{i_1} \wedge ... \wedge d\varphi_{i_q}$ with $I = (i_1, ..., i_q)$. Since the function f_I is smooth and φ_i is smooth too, follows that the differential form $\varphi^* f$ is smooth as well. Using (ii) we get:

$$
\begin{aligned}
d(\varphi^* \omega) &= \sum d(f_I \circ \varphi) \wedge d\varphi_I = \sum \varphi^*(df_I) \wedge \varphi^*(dx_I) = \\
&= \varphi^* \left(\sum df_I \wedge dx_I \right) = \varphi^*(d\omega).
\end{aligned}
$$

This proves the most important property: commutativity of differentiation and pullback.

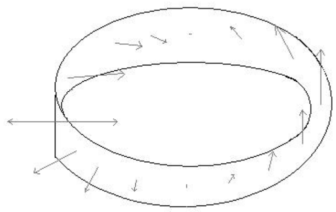
(iv) Set $\Psi := \varphi \circ \psi$. For $1 \leq i \leq n$ we get $\Psi_i = \varphi_i \circ \psi_i$ and with (iii) $d\Psi_i = d(\varphi_i \circ \psi) = \psi^*(d\varphi_i)$. Further we get:

$$
\begin{aligned}
\psi^*(\varphi^* \omega) = \psi^* \left(\sum_I (f_I \circ \varphi) d\varphi_I \right) &= \\
&= \sum (f_I \circ \varphi \circ \psi) \psi^*(d\varphi_I) = \\
&= \sum (f_I \circ \Psi) d\Psi_I = \\
&= \Psi^* \left(\sum f_I dx_I \right) = (\varphi \circ \psi)^* \omega.
\end{aligned}
$$

□

4 Integration of Differential forms and Stokes Theorem

Differential q-forms can be integrated over q-dimensional submanifolds. Important in this construction is the orientation of the submanifold or at least the possibility to orientate it. Informally, for a hypersurface in $\mathbb{R}^n$ 'to be orientable' means that it has 'two sides'. An example of a non-orientable manifold is the *Möbius strip*, see below. We can colour the whole strip by just one continued brush stroke, which gives the clue to what it means to be non-orientable.



PICTURE 4: MÖBIUS STRIP

More mathematically, orientability means, that there exists a continuous normal vector.

I refer mainly to [FOR],[FRI] and [HIT] in this section.

4.1 Orientation of submanifolds

In order to integrate over smooth submanifolds we need to explain the meaning of orientation of a submanifold. For this we recall the notions of *local charts* and *atlases*: Let $\mathfrak{A} = \{(U_i; \varphi_i)|i \in I\}$ and $\mathfrak{A}' = \{(U_i'; \varphi_i')|i' \in I'\}$ be two given atlases. In other words, there exist two homeomorphic maps $\varphi_i : U_i \to V_i$ and $\varphi_i' : U_i' \to V_i'$ acting on open covers $\mathcal{U} = (U_i|i \in I)$, $\mathcal{U}' = (U_i'|i \in I)$, respectively, of a submanifold M and two open subsets $V_i, V_i' \subset \mathbb{R}^n$.

Definition 31. Let $U, V \subset \mathbb{R}^q$ be open sets and $\varphi : U \to V$ be a diffeomorphism, which is called

(i) *orientation-preserving*, if $det\,(D\varphi(x)) > 0$ for all $x \in U$,

(ii) *orientation-reversing*, if $det\,(D\varphi(x)) < 0$ for all $x \in U$,

where $D\varphi$ is the Jacobian of φ.

Definition 32. (i) We call two maps φ_i, φ_i' *equally oriented*, if a coordinate change between those maps is orientation-preserving;

(ii) An atlas $\mathfrak{A}$ is called *oriented*, if two maps on $\mathfrak{A}$ are equally oriented;

(iii) An orientation of a submanifold is determined by the behaviour of its atlases:
We define the residue-class of oriented atlases on M as:
$\mathfrak{A} \sim \mathfrak{A}' \Leftrightarrow$ two maps $(\varphi_i : U_i \to V_i)$ on $\mathfrak{A}$ and
$(\varphi_i' : U_i' \to V_i')$ on $\mathfrak{A}'$ are equally oriented;

(iv) A submanifold M is said to be *orientable*, if M has an oriented atlas. A *oriented manifold* is indicated by the pair (M, σ) where σ is an orientation of M;

(v) We call a map $(\varphi : U \to V)$ *positive orientable with respect to* σ, if all maps $\varphi_i : U_i \to V_i$ of an atlas of the oriented manifold (M, σ) are equally oriented to φ. In other words, all maps on an atlas, which belongs to the manifold M with orientation σ, are equally oriented to the map φ in question.

Example 8. (circle)
Consider $S^1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$, which is a 1-dimensional submanifold of $\mathbb{R}^2$. Its atlas is formed by two maps:

$$\varphi_1 : T_1 := \,]-\pi, \pi[\, \to S^1 \setminus (-1, 0)$$

$$\varphi_2 : T_2 := \,]0, 2\pi[\, \to S^2 \setminus (1, 0)$$

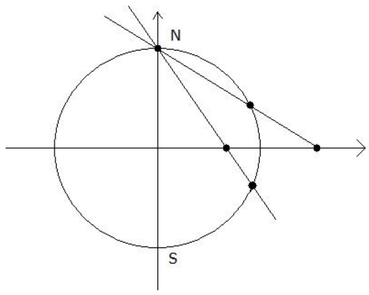
where $\varphi_j(t) := (\cos t, \sin t)$ is the standard parametrization of S^1 for $t \in T_j$, $j = 1, 2$. To change the parameters we consider a transformation:

$$\tau : T_1 \setminus \{0\} \to T_2 \setminus \{\pi\}$$

given by

$$\tau(t) = \begin{cases} t + 2\pi & \text{for } -\pi < t < 0 \\ t & \text{for } 0 < t < \pi \end{cases}$$

and $\varphi_1|_{T_1 \setminus \{0\}} = \varphi_2 \circ \tau$. Because $\tau'(t) > 0$ for all $t \in T_1 \setminus \{0\}$, then $\{\varphi_1, \varphi_2\}$ is an oriented atlas.



Picture 5: Stereographic projection of the circle

Let us collect our results corncerning orientation:

Let $(x_1, ..., x_n)$ be a basis in $\mathbb{R}^n$. This basis is called *positively oriented*, if $det(x_1, ..., x_n) > 0$, where x_i are understood as column vectors for each $i = 1, ..., n$.

Let (M, σ) be a k-dimensional submanifold and $\varphi : U \to V \subset \mathbb{R}^k$ a positively oriented chart on $U \subset M$. Then is $(\frac{\partial \varphi}{\partial t_1}(p), ..., \frac{\partial \varphi}{\partial t_k}(p))$ a basis for tangent space $T(M) = \bigcup_{p \in M} T_{M,p}$. Thus we can claim that $det(\frac{\partial \varphi}{\partial t_1}(p), ..., \frac{\partial \varphi}{\partial t_k}(p)) > 0$ indicates a positively oriented chart φ.

Especially, for a hypersurface $M \subset \mathbb{R}^n$, i.e. $n-1$-dimensional manifold M, its chart $\varphi : M \to \mathbb{R}^n$ is positively oriented if

$$det \left(\nu(p), \frac{\partial \varphi}{\partial t_1}(p), ..., \frac{\partial \varphi}{\partial t_{n-1}}(p) \right) > 0,$$

where $\nu(p)$ is the normal vector on M at p. This is since $(\nu(p), \frac{\partial \varphi}{\partial t_1}(p), ..., \frac{\partial \varphi}{\partial t_{n-1}}(p))$ is a basis of $\mathbb{R}^n$.

Definition 33. The *normal vector space* on a $n-1$-dimensional manifold M is a smooth map

$$\nu : M \to \mathbb{R}^n$$

such that for each $p \in M$ the vector $v(p)$ is a normal vector on M at p. This means $v(p)$ is perpendicular to $T_{M,p}$ and has length 1.

Example 9. (2-sphere)
Consider the orientated 2-sphere $S^2 = \{x \in \mathbb{R}^3 | \, \|x\| = 1\}$ and the map Ψ on the rectangle $\Omega :=]0, \pi[\times]\alpha, \alpha + 2\pi[\subset \mathbb{R}^2$:

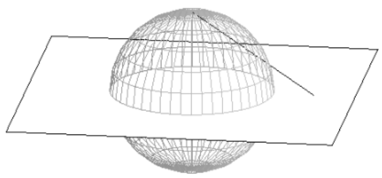
$$\begin{aligned} \Psi : \Omega &\to \Psi(\Omega) \subset S^2 \\ (\vartheta, \varphi) &\mapsto (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta) \end{aligned}.$$

Claim: The map Ψ is positively oriented.
Proof: For $x \in S^2$ is the outer normal vector $\nu(x) = x$. We have to show that for all $(\vartheta, \varphi) \in \Omega$:

$$det\left(\Psi(\vartheta, \varphi), \frac{\partial\Psi}{\partial\vartheta}(\vartheta, \varphi), \frac{\partial\Psi}{\partial\vartheta}(\vartheta, \varphi)\right) > 0.$$

Indeed, we get since $\vartheta \in]0, \pi[$

$$det\begin{pmatrix} \sin\vartheta\cos\varphi & \cos\vartheta\cos\varphi & -\sin\vartheta\sin\varphi \\ \sin\vartheta\sin\varphi & \cos\vartheta\sin\varphi & \sin\vartheta\cos\varphi \\ \cos\vartheta & -\sin\vartheta & 0 \end{pmatrix} = \sin\vartheta > 0.$$



PICTURE 6: PROJECTION OF A 2-SPHERE

4.2 Integration of a q-form

Let M be a smooth q-dimensional manifold and $U \subset M$ be a smooth submanifold. Let $\omega = \sum_I f_I dx_I$ be a continuous q-form.

Definition 34. Let $\omega = f dx_1 \wedge dx_2 \wedge ... \wedge dx_q$ be a q-form on an open set $U \subset \mathbb{R}^q$. We call ω *integrable* over a subset $A \subset U$, if $f|_A$ is integrable with respect to the usual Lebesgue measure and we write

$$\int_A \omega := \int_A f(x)dx.$$

The following theorem concerns orientation-preserving, orientation-reversing respectively of integration of differential forms.

Theorem 7. *Let $U, V \subset \mathbb{R}^q$ and $\varphi : U \to V$ be smooth. Further let ω be a smooth q-form on V and $A \subset U$ be a compact subset. Then*

$$\int_{\varphi A} \omega = \int_A \varphi^* \omega \text{ if } \varphi \text{ orientation-preserving;}$$
$$\int_{\varphi A} \omega = -\int_A \varphi^* \omega \text{ if } \varphi \text{ orientation-reversing.}$$

Proof. Set $\omega = f dx_1 \wedge ... \wedge dx_q$. Looking at Example 4 and extending it from the 1-dimensional to q-dimensional case, we have instead of the single-index matrix $D\varphi = \left(\frac{\partial \varphi_i}{\partial x_j}\right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$ a multi-index matrix $D\varphi = \left(\frac{\partial \varphi_{i_\nu}}{\partial x_\mu}\right)_{1 \leq \nu, \mu \leq q}$. Further by definition of the $\wedge$-product and its skew-symmetric property we get:

$$d\varphi_{i_1} \wedge ... \wedge d\varphi_{i_q} = det D\varphi \cdot dx_1 \wedge ... \wedge dx_q.$$

This leads us to

$$\varphi^* \omega = (f \circ \varphi) det D\varphi \cdot dx_1 \wedge ... \wedge dx_q.$$

By coordination change we get the desired result:

$$\int_{\varphi A} \omega = \int_{\varphi A} f(x) d^n(x) = \int_A f(\varphi(x)) |det D\varphi(x)| d^n x.$$

□

Now let in addition to the above notations M be an manifold with orientation σ. We want to define the integral of ω over (A, σ), where $A \subset U \subset M$ is a submanifold with induced orientation σ: $\int_{(A,\sigma)} \omega$.

(a) Let us first assume that there exists a positively oriented map $\varphi : U \to V$ on (M, σ), such that $A \subset U \subset M$ and $V \subset \mathbb{R}^q$. Set

$$\int_{(A,\sigma)} \omega := \int_{\varphi^{-1}(A)} \varphi^* \omega = \int_{\varphi^{-1}(A)} \sum_I (f_I \circ \varphi) d\varphi_I =$$
$$= \sum_I \int_{\varphi^{-1}(A)} (f_I \circ \varphi) d\varphi_I,$$

where the last equality holds due to smoothness. We shall show now, that this definition is independent of the choosen chart. For this, take another positively oriented map $\varphi_1 : U_1 \to V_1$ with $A \subset U_1$. There exists an orientation-preserving transformation

$$\tau : U \to U_1 \text{ with } \varphi = \varphi_1 \circ \tau.$$

By theorem 7 the independence of choice of coordinates for the above definition of the integral holds and we get:

$$\int_{\varphi^{-1}(A)} \varphi^*\omega = \int_{\varphi^{-1}(A)} (\varphi_1 \circ \tau)\omega = \int_{\tau^{-1}(\varphi_1^{-1}(A))} \tau^*(\varphi_1^*\omega) =$$

$$= \int_{\varphi_1^{-1}(A)} \varphi_1^*\omega.$$

(b) We reduce the case, where A is not contained in one of the preimages of φ, by using the technique of *partition of unity* to the previous case:

Partition of unity:

The partion of unity is a finite family of continuous functions $(f_i, i \in I)$ with $f_i \geq 0$ for all $i \in I$ such that:

(i) $\sum_{i \in I} f_i(x) = 1$ for all $x \in A$;

(ii) $\{supp(f_i)\}_{i \in I}$ is locally finite (a system of subsets is called *locally finite* if a neighborhood of all elements in the set intersects with finitely many subsets);

(iii) For every i there exists a positively oriented map: $\varphi_i : U_i \to V_i$ with $A_i := A \cap supp(f_i) \subset V_i$.

Now we define

$$\int_{(A,\sigma)} \omega := \sum_{i \in I} \int_{\varphi_i^{-1}(A_i)} (\alpha_i \circ \varphi_i)\varphi_i^*\omega.$$

Again this formula is independent of the choice of a map and a partition of unity.

Remark 7. 1. We have to be carefull with the orientation of the manifold M when we integrate over a subset $A \subset M$:

$$\int_{(A,-\sigma)} \omega = -\int_{(A,\sigma)} \omega.$$

2. For a 1-dimensional oriented submanifold $M \subset U$ we can associate the above defined integration formula to the fundamental theorem of calculus: Namely, let $\omega = \sum_{i=1}^n f_i dx_i$ a continuous 1-form (differential form of order 1) on U and $\varphi : I \to V$ a positive oriented map of M with an open interval $I \subset \mathbb{R}$, i.e. a smooth curve. Let $[a,b] \subset I$ a compact subinterval and $A := \varphi([a,b,])$. For ω we have the pullback $\varphi^*\omega = \sum_{i=1}^n (f_i \circ \varphi)\varphi_i' dt$. Then

$$\int_{\varphi|_{[a,b]}} \omega = \int_{[a,b]} \varphi^*\omega = \sum_{i=1}^n \int_a^b (f_i \circ \varphi)\varphi_i' dt,$$

which coincides with the earlier defined integral

$$\int_A \omega = \int_a^b \varphi^* \omega.$$

4.3 Stokes theorem

In the following we use the expressions as introduced in section 3.6.3 differential $n-1$-forms on M.

Let M be a smooth q-dimensional manifold and ω be a smooth $(q-1)$-form in $U \subset M$. For a smooth vector field $f = (f_1, ..., f_q) : U \to \mathbb{R}^q$ and

$$d\vec{S} = (dS_1, ..., dS_q), \; dS_i = (-1)^{i-1} dx_1 \wedge ... \wedge \widehat{dx_i} \wedge ... \wedge dx_q$$

the differential form can be represented as the scalar product between f and $d\vec{S}$:

$$\omega = f \cdot d\vec{S} = \sum_{i=1}^{q} f_i \cdot dS_i.$$

4.3.1 Stokes Theorem

Stokes theorem, which is generalizing the Gauss theorem of classical calculus, relates a surface integral over a oriented surface and the volume integral of a scalar field. We will return to the classical Stokes theorem in the next section when looking closer on homology and integration over cuboids and simplices. Below we present it in a more general form.

Theorem 8. *(Stokes) Let M be a q-dimensional manifold and ω be a smooth $(q-1)$-form in $U \subset M$. For a compact set $A \subset U$ with a smooth boundary ∂A which is oriented by an outer normal field one has:*

$$\int_A d\omega = \int_{\partial A} \omega.$$

Remark 8. For one dimension ($q = 1$) Theorem 8 coincides with the Fundamental theorem of calculus in one variable:

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Proof. Set $\omega = \sum(-1)^{i-1} f_i dx_1 \wedge ... \wedge \widehat{dx_i} \wedge ... \wedge dx_q$. Then $\omega = f \cdot d\vec{S}$ as above with the smooth vector field $f = (f_1, ..., f_q) : U \to \mathbb{R}^q$. Let $\nu : \partial A \to R^q$ be the unit outer normal field. Then:

$$\int_{\partial A} \omega = \int_{\partial A} < f(x), \nu(x) > dS(x).$$

Moreover, ω is given by:

$$d\omega = div\ f dx_1 \wedge ... \wedge dx_n.$$

Therefore (by the Gauss theorem):

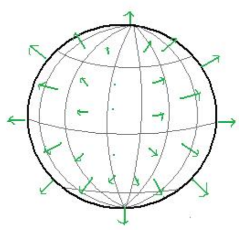
$$\int_A d\omega = \int_A div f(x) d^q x = \int_{\partial A} < f(x), \nu(x) > dS(x).$$

□

Remark 9. (*Gauss theorem*) In the same notation as above:

$$\int_A div f(x) d^q x = \int_{\partial A} < f(x), \nu(x) > dS(x).$$

This theorem is a special case of the Stokes Theorem and explains the relation between the volume integral over a body (with its normal vectors) and the surface integral over the surface area element.



PICTURE 7: NORMAL VECTORS TO A 2-SPHERE IN $\mathbb{R}^3$

Example 10. Consider a differential form

$$\omega = \sum_{i=1}^{n} (-1)^{i-1} x_i dx_1 \wedge ... \wedge \widehat{dx_i} \wedge ... \wedge dx_n$$

in $\mathbb{R}^n$ and its exterior derivate $d\omega = n dx_1 \wedge ... \wedge dx_n$. For this example we need to look first at the map $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(x) = x$. The divergence of f is then given by $div f(x) = \sum_{k=1}^{n} \frac{\partial x_k}{\partial x_k} = n$. The n-dimensional volume is defined as the volume integral over the n-dimensional element: $Vol_n(A) = \int_A \frac{1}{n} div f(x) d^n x$, where A is a compact subset of $\mathbb{R}^n$ with smooth boundary. Using this and Stokes Theorem we get:

$$\begin{aligned} Vol_n(A) &= \frac{1}{n}\int_A d\omega = \frac{1}{n}\int_{\partial A} \omega = \\ &= \frac{1}{n}\int_{\partial A} \sum_{i=1}^{n} (-1)^{i-1} x_i dx_1 \wedge ... \wedge \widehat{dx_i} \wedge ... \wedge dx_n. \end{aligned}$$

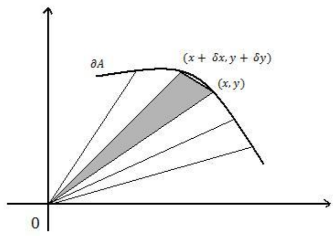
In case of $\mathbb{R}^2$ with coordinates (x, y) we get the classical formula for the area of a compact subset $A \subset \mathbb{R}^2$ with a smooth boundary. Namely:

$$Area_2(A) = \frac{1}{2}\int_{\partial A} (x dy - y dx).$$

Using this to express the oriented area of a triangle with vertices $(0,0), (x, y), (x + \delta x, y + \delta y)$, where $\delta x, \delta y > 0$, we get

$$Area_2(\triangle) = \frac{1}{2} det \begin{pmatrix} x & x + \delta x \\ y & y + \delta y \end{pmatrix} = \frac{1}{2}(x\delta y - y\delta x)$$

The whole A can be approximatively composed of such small triangles.



PICTURE 8: APPROXIMATING AN AREA BY SMALL TRIANGLES

4.3.2 Using holomorphic functions

Definition 35. A complex-valued function $f : U \to \mathbb{C}$ defined on an open subset $U \subset \mathbb{C} \cong \mathbb{R}^2$ with $z = x + iy \in \mathbb{C}$ is called *holomorphic*, if f has continuous partial derivatives and

$$\frac{\partial f}{\partial \bar{z}} = 0 \text{ where } \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Take, for example, $g : U \to \mathbb{C}$, $z \mapsto \frac{1}{z-a}$. Since $\frac{\partial g}{\partial \bar{z}} = \frac{1}{2}\left(-\frac{1}{(x+iy-a)^2} - i\frac{i}{(x+iy-a)^2}\right) = 0$ then is g holomorphic in $U\backslash\{a\}$. If $f : U \to \mathbb{C}$ is another holomorphic function, then is even $h(z) = \frac{f(z)}{z-a}$ a holomorphic function in $U\backslash\{a\}$:

$$
\begin{aligned}
\frac{\partial h(z)}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial h(z)}{\partial x} + i\frac{\partial h(z)}{\partial y}\right) \\
&= \frac{1}{2}\left(\frac{\partial \frac{f(z)}{z-a}}{\partial x} + i\frac{\partial \frac{f(z)}{z-a}}{\partial y}\right) \\
&= \frac{1}{2}\left(\frac{\partial f(x)}{\partial x}\frac{1}{z-a} - \frac{f(z)}{(x+iy-a)^2} + i\frac{\partial f(x)}{\partial y}\frac{1}{z-a} - i\frac{if(z)}{(x+iy-a)^2}\right) \\
&= \frac{1}{2}\left[\underbrace{\left(\frac{f(z)}{\partial x} + i\frac{f(z)}{\partial y}\right)}_{=\frac{\partial f(z)}{\partial \bar{z}}}\frac{1}{z-a}\right] \\
&= \frac{1}{2}\underbrace{\frac{\partial f(z)}{\partial \bar{z}}}_{=0 \text{ holomorphic}}\frac{1}{z-a} = 0
\end{aligned}
$$

The corresponding differential form is given by $\omega = \frac{f(z)}{z-a}dz$.

Theorem 9. *(Cauchy integral formula)*
Consider an open subset $U \subset \mathbb{C}$, a holomorphic function $f : U \to \mathbb{C}$ and a compact subset $A \subset U$ with smooth boundary. Then for every point a inside A one gets:

$$f(a) = \frac{1}{2\pi i}\int_{\partial A}\frac{f(z)}{z-a}dz.$$

Proof. To prove the integral formula we assume that $D_{epsilon} \subset A$ is a disk around any point $a \in A$ having a sufficiently small radius $\epsilon > 0$: $|z-a| = \epsilon$. Using $(\bar{z}-\bar{a})(z-a) = |z-a|^2 = \epsilon^2$ we get:

$$\int_{\partial D_{epsilon}}\frac{f(z)}{z-a}dz = \int_{|z-a|=\epsilon}\frac{f(z)}{z-a}dz = \frac{1}{\epsilon^2}\int_{|z-a|=\epsilon}(\bar{z}-\bar{a})f(z)dz.$$

Since the form $\omega = \frac{f(z)}{z-a}dz = \frac{1}{\epsilon^2}(\bar{z}-\bar{a})f(z)dz$ is closed (i.e. $d\omega = 0$, because $\frac{f(z)}{z-a}$ is holomophic), we can compute:

$$d((\bar{z}-\bar{a})f(z)dz) = d(\bar{z}-\bar{a}) \wedge (f(z)dz) = f(z)d\bar{z} \wedge dz.$$

Further, we have

$$d\bar{z} \wedge dz = (dx - idy) \wedge (dx + idy) = 2idx \wedge dy.$$

Thus we get:

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial D_\epsilon}\frac{f(z)}{z-a}dz &= \frac{1}{\pi\epsilon^2}\int_{|z-a|\leq\epsilon} f(z)dx \wedge dy = \\ &= \frac{1}{\pi}\int_{|\zeta|\equiv 1} f(a+\epsilon\zeta)d\xi \wedge d\eta,\end{aligned}$$

where we use the substitution: $\zeta = \xi + i\eta = \frac{z-a}{\epsilon}$. Since f is a holomorphic function the right-hand side converges to $f(a)$ as $\epsilon \to 0$. The left-hand side is independent of ϵ which means that we get the equality:

$$f(a) = \frac{1}{2\pi i}\int_{\partial D_\epsilon}\frac{f(z)}{z-a}dz.$$

To complete the proof we claim $\int_{\partial A}\omega = \int_{\partial B}\omega$ for $A, B = D_\epsilon \subset U \subset \mathbb{R}^n$ open sets and ω a smooth $(n-1)$-form on $U\backslash\{a\}$ for $a \in \mathring{A} \cap \mathring{B}$.

Choose $\delta > 0$ arbitrary such that

$$K_\delta := \{x \in \mathbb{R}^n \mid \|x-a\| \leq \delta\} \subset \mathring{A} \cap \mathring{B}.$$

Define $A_\delta := A\backslash\mathring{K}_\delta$ and $B_\delta := B\backslash\mathring{K}_\delta$. Both, A_δ and B_δ, are compact sets in $U\backslash\{a\}$. Since $d\omega = 0$ we get

$$\int_{\partial A_\delta}\omega = \int_{\partial B_\delta}\omega = 0.$$

The boundary ∂A_δ consists of ∂A and the negativly oriented ∂K_δ; likewise ∂B_δ consists of ∂B and the negativly oriented ∂K_δ. Thus

$$\int_{\partial A}\omega - \int_{\partial K_\delta}\omega = 0 = \int_{\partial B}\omega - \int_{\partial K_\delta}\omega.$$

This general case provides the case we need above and we get finally:

$$f(a) = \frac{1}{2\pi i}\int_{\partial A}\frac{f(z)}{z-a}dz = \frac{1}{2\pi i}\int_{\partial D_\delta}\frac{f(z)}{z-a}dz.$$

□

Example 11. Choosing: $f \equiv 1$ and $a = 0$ in the above setting we get:

$$f(0) = 1 = \frac{1}{2\pi i}\int_{\partial A}\frac{dz}{z}.$$

Since $\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{x-iy}{x^2+y^2}$ the integrand is given by:

$$\frac{dz}{z} = \frac{x-iy}{x^2+y^2}(dx+idy) = \frac{xdx+ydy}{x^2+y^2} + i\frac{xdy-ydx}{x^2+y^2} = M + iN.$$

Comparing the imaginary and the real part with the above integral

$$1 = \frac{1}{2\pi i}\int_{\partial A}\frac{1}{z}dz = \int_{\partial A}\frac{M}{2\pi i} + \int_{\partial A}\frac{N}{2\pi} = -i\int_{\partial A}\frac{M}{2\pi} + \int_{\partial A}\frac{N}{2\pi}$$

we get:

$$\int_{\partial A} M = \int_{\partial A}\frac{xdx+ydy}{x^2+y^2} = 0 \text{ and } \int_{\partial A} N = \int_{\partial A}\frac{xdy-ydx}{x^2+y^2} = 2\pi.$$

5 De Rham Cohomology and its relation to vector calculus

In this section we present a short introduction of the so-called *De Rham cohomology* of manifolds. Its construction uses sequences of vector spaces of differential forms and their exterior derivatives. Boundary operators on cuboids and simplices define *simplicial homology.* The *De Rham theorem* provides an isomorphism between those two concepts. First we need the *Poincaré lemma*, which shows that closed forms represent classes in de Rham cohomology groups.

I refer mainly to [FOR], [FRI] and [UTO] in this section.

Before we begin with the new ideas, we revise the property 'closed' especially for 1-forms to draw consequences on the existence of primitives. A necessary condition for the existence of a primitive of a 1-form is given by the following property.

Definition 36. Let M be a n-dimensional manifold and $\omega = \sum_{i=1}^{n} f_i dx_i$ be a smooth 1-form. Then ω is called *closed*, if

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \text{ for all } i, j = 1, ..., n.$$

This definition of closedness coincides with our earlier definition ($d\omega = 0$).

$$\begin{aligned} d\omega &= \sum_{i,j=1}^{n} \frac{\partial f_i}{\partial x_j} dx_i \wedge dx_j = \\ &= \sum_{i<j}^{n} \underbrace{\left(\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j} \right)_{=0 \text{ by defintion 33}}} dx_i \wedge dx_j = \\ &= 0 \end{aligned}$$

If ω has a primitive F, ω is necessarily closed. Since $dF = \omega$ and $f_i = \partial F/\partial x_i$ we have

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{F}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{F}{\partial x_j}\right) = \frac{\partial f_j}{\partial x_i}.$$

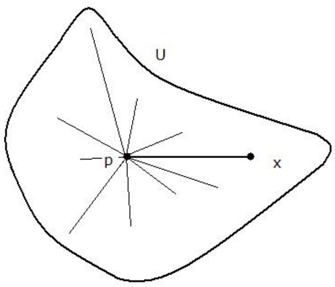
This is, however, not yet sufficient for the opposite direction as the following example shows.

Example 12. Define $\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ on $U = \mathbb{R}^2\backslash\{0\}$. This 1-form has no primitive but satisfies the above condition:

$$\frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right).$$

we need thus an additional definition, which assures the existence of a primitve.

Definition 37. A subset $U \subset \mathbb{R}^q$ is called *contractible* with respect to a point $p \in U$, if for any other point $x \in U$ the connecting segments $\{(1-t)p + tx | 0 \le t \le 1\}$ lie totally in U.



PICTURE 9: CONTRACTIBLE SET

Now we return to general k-forms and establish a connection between closed and exact forms.

Lemma 4. *Let $U \subset \mathbb{R}^n$ be a contractible set with respect to $p \in U$ and ω be a smooth closed 1-form on U. Then ω has a primitive $F : U \to \mathbb{R}$.*

Proof. We can assume that U is contractible with respect to the origin. Define the primitive F for $\omega = \sum_{i=1}^{n} f_i dx_i$ by

$$F(x) := \int_0^1 \left(\sum_{i=1}^{n} f_i(tx)x_i \right) dt$$

for $x \in U$. The integral is defined, because the whole line tx lies in U for $0 \leq t \leq 1$ by contractibility. Using $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ we get: $\partial F/\partial x_i = 0$. □

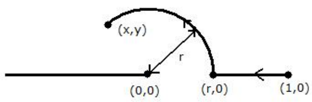
Example 13. We return to the counterexample and modify the set $U = \mathbb{R}^2 \backslash \{0\}$ such that we get a contractible set. Namely, define $V = \mathbb{R}^2 \backslash \{(x,0)|x \leq 0\}$. Especially V is contractible with respect to the point $(1,0) \in V$. Since the form $\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ is closed it has a primitive F. We can calculate the integral by

$$F(x,y) = \int_{(1,0)}^{(x,y)} \omega,$$

where we integrate over an arbitrary piecewise smooth curve in V from $(1,0)$ to (x,y). Set $r := \sqrt{x^2+y^2}$, then:

$$F(x,y) = \int_{(1,0)}^{(r,0)} \omega + \int_{(r,0)}^{(x,y)} \omega.$$

Choose the curve as shown in picture 10.



Picture 10: example 13

Then

$$\int_{(1,0)}^{(r,0)} \omega = 0 \text{ and } \int_{(r,0)}^{(x,y)} \omega = \varphi.$$

where φ is given by the conditions $x = r\cos\varphi$, $y = r\sin\varphi$ for $-\pi < \varphi < \pi$. Thus

$$\varphi = F(x,y)) = \begin{cases} \tan^{-1}\frac{x}{y}, & \text{for } x > 0 \\ \pi/2 - \tan^{-1}\frac{x}{y}, & \text{for } y > 0 \\ -\pi/2 - \tan^{-1}\frac{x}{y}, & \text{for } y < 0 \end{cases}$$

One varifies $dF = \omega$.

To prove the Poincaré lemma we need the following statement.

Lemma 5. *Let $U \subset \mathbb{R}, V \subset \mathbb{R} \times \mathbb{R}^n$ be two open subsets such as $[0,1] \times U \subset V$. Define two mappings*

$$\Psi_0, \Psi_1 : U \to V \text{ by } \Psi_0(x) := (0,x), \Psi_1(x) := (1,x).$$

For a smooth closed k-form σ on V $(k \geq 1)$, there exists a smooth $(k-1)$-form η on U with

$$\Psi_1^*\sigma - \Psi_0^*\sigma = d\eta.$$

Proof. We denote coordinates on $V \subset \mathbb{R} \times \mathbb{R}^n$ by $(t, x_1, x_2, ..., x_n)$ and we represent σ as

$$\sigma \sum_{|I|=k} f_I dx_I + \sum_{|J|=k-1} g_J dt \wedge dx_J.$$

The pullbacks of Ψ_0 and Ψ_1 are given by

$$\Psi_0^*\sigma = \sum_I f_I(0,x)dx_I \text{ and } \Psi_1^*\sigma = \sum_I f_I(1,x)dx_I.$$

Calculating the exterior derivative of σ, we get:

$$d\sigma = \sum_I \frac{\partial f_I}{\partial t} dt \wedge dx_I + \sum_I \sum_{i=1}^n \frac{\partial f_I}{\partial x_i} dx_i \wedge dx_I - \sum_J \sum_{i=1}^n \frac{\partial g_J}{\partial x_i} dt \wedge dx_i \wedge dx_J.$$

Since $d\sigma = 0$ (σ is closed), the middle term vanishes and factoring out dt we have:

$$\sum_I \frac{\partial f_I}{\partial t} dx_I = \sum_J \sum_{i=1}^n \frac{\partial g_J}{\partial x_i} dx_i \wedge dx_J.$$

Integrating both sides with respect to t from 0 to 1 we get:

$$\int_0^1 \frac{\partial f_I}{\partial t}(t,x)dt = f_I(1,x) - f_I(0,x),$$

$$\int_0^1 \frac{\partial g_J}{\partial x_i}(t,x)dt = \frac{\partial}{\partial x_i} \int_0^1 g_J(t,x)dt.$$

Substituting this in $\Psi_1^*\sigma - \Psi_0^*\sigma$ we get:

$$\begin{aligned}\Psi_1^*\sigma - \Psi_0^*\sigma &= \sum_I f_I(1,x)dx_I - \sum_I f_I(0,x)dx_I = \\ &= \sum_I (f_I(1,x) - f_I(0,x))dx_I = \\ &= \sum_J \left(\frac{\partial}{\partial x_i} \int_0^1 g_J(t,x)dt \right) dx_I \wedge dx_J = d\eta.\end{aligned}$$

for an $\eta := \sum_{|J|=k-1} \left(\int_0^1 g_J(t,x)dt \right) dx_J$. □

Theorem 10. *(The Poincaré lemma)*
Let $U \subset \mathbb{R}^n$ be a contractible set and ω be a smooth closed k-form $(k \geq 1)$ in U. Then ω is exact.

Proof. We can assume that U is contractible with respect to the origin. Set

$$\varphi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, (t, x) \mapsto tx,$$

where $V := \varphi^{-1}(U)$ is the preimage, then: $]0,1[\, \times U \subset V$. Define Ψ_0 and Ψ_1 as in the additional lemma above. By Theorem 6(iii) of section 3.8 $\sigma = \varphi^*\omega$ is closed $(d\sigma = d(\varphi^*\omega) = \varphi^* \underbrace{(d\omega)}_{=0} = 0)$. We can use the above lemma to obtain a $(k-1)$-form η on U satisfying

$$\Psi_1^*\sigma - \Psi_0^*\sigma = d\eta.$$

Since $\varphi \circ \Psi_1 = id_U$ $[\varphi(\Psi_1(x)) = \varphi(1, x) = 1 \cdot x = x]$ and $\varphi \circ \Psi_0 = 0$ $[\varphi(\Psi_0(x)) = \varphi(0, x) = 0 \cdot x = 0]$ is the constant mapping to the origin, we get:

$$\Psi_1^*\sigma = \Psi_1^*(\varphi^*\omega) = (\varphi \circ \Psi_1)^*\omega = id_U^*\omega = \omega \text{ and}$$

$$\Psi_0^*\sigma = \Psi_0^*(\varphi^*\omega) = (\varphi \circ \Psi_0)^*\omega = 0 \circ \omega = 0.$$

This implies that $\omega = d\eta$ by $\Psi_1^*\sigma - \Psi_0^*\sigma = \omega - 0 = \omega = d\eta$. ☐

5.1 De Rham cohomology complex and cohomology groups

We can now reformulate the *Poincaré lemma* to introduce the concept of De Rham cohomology for a contractible set $U \subset \mathbb{R}^n$ and the vector space $\Omega^k(U)$ of smooth $k-$forms in U. Namely, if $\omega \in \Omega^k(U)$, then $d\omega$ is smooth, i.e. $d\omega \in \Omega^{k-1}(U)$ and it is closed, i.e. $d(d\omega) = 0$. If $d\omega = 0$, then by the Poincaré lemma we can find a $(k-1)$-form η, such as $d\eta = \omega$. One can check that the constructed form η is smooth as well. This allows us to define a sequence:

$$0 \to \mathbb{R} \to \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \dots \xrightarrow{d} \Omega^{n-1}(U) \xrightarrow{d} \Omega^n(U) \xrightarrow{d} 0, \quad (2)$$

of vector spaces and maps which is *exact* for a contractible U. The latter notion means that for any k $d(\Omega^{k-1}(U))$ coincides with the kernel of $d : \Omega^k \to \Omega^{k+1}$, i.e.:

$$Im(\Omega^{k-1}(U) \xrightarrow{d} \Omega^k(U)) = \ker(\Omega^k(U) \xrightarrow{d} \Omega^{k+1}(U))$$
$$\mathbb{R} = \ker(\Omega^0(U) \xrightarrow{d} \Omega^1(U)).$$

If U is any arbitrary open set then the sequence (2) is not necessarily exact. But since the image of $d^2 = d \circ d$ at least belongs to the kernel $Im \subset \ker$ we can build the quotient spaces:

$$H^k_{de\ R}(U) := \frac{\ker(\Omega^k(U) \xrightarrow{d} \Omega^{k+1}(U))}{Im(\Omega^{k-1}(U) \xrightarrow{d} \Omega^k(U))}, (k \geq 0);$$

where we set $\Omega^{-1}(U) = 0$. We call $H^k_{de\ R}(U)$ the *k-th de Rham cohomology of U*. For U contractible the Poincaré lemma says that:

$$H^k_{de\ R}(U) = \begin{cases} 0, & \text{for } k \geq 1 \\ \mathbb{R}, & \text{for } k = 0. \end{cases}$$

Definition 38. Given an arbitrary smooth manifold M we define the *k-th de Rham cohomology of M* as the quotient space of the space of closed k-forms modulo the subspace of exact k-forms:

$$H^k_{de\ R}(M) = \{\text{ closed } k\text{-forms}\}/\{\text{ exact } k\text{-forms}\}.$$

The main purpose of the de Rham cohomology is to compare *closed* and *exact* differential forms on a manifold, namely: we say that two closed forms α and β in the set of k-forms $\Omega^k(M)$ on a manifold M are *cohomologous*, if their difference $\alpha - \beta$ is an exact form:

$$\begin{aligned} &\alpha \sim \beta \text{ where } d\alpha = d\beta = 0 \text{ for } k-\text{forms } \alpha, \beta \\ &\quad\Leftrightarrow (\alpha - \beta) = d\eta \text{ for some } (k-1)\text{-form } \eta. \end{aligned}$$

Example 14. Consider $M = \mathbb{R}$. We get the sequence

$$0 \longrightarrow \Omega^0(\mathbb{R}) \xrightarrow{d} \Omega^1(\mathbb{R}) \longrightarrow 0.$$

Then $H^0_{de\ R}(\mathbb{R}) = \ker(d) \simeq \mathbb{R}$, the constant functions on $\mathbb{R}$. Further $H^1_{de\ R}(\mathbb{R}) = \Omega^1(\mathbb{R})/im(d) = \{0\}$, the 2-forms on $\mathbb{R}$ are 0. Overall, $H^k_{de\ R}(\mathbb{R}) = 0$ for $k \geq 1$.

Example 15. Consider $M = S^1$,the circle. There are no non-zero k-forms on S^1 except possibly for $k = 0, 1$. The cohomology $H^k_{de\ R}(S^1)$ is zero for $k > 1$. There are no exact 0-forms (i.e. $d\omega \neq 0$) and the closed 0-form is a constant function, so that we have

$$H^0_{de\ R}(S^1) \cong \mathbb{R}.$$

Further, the angle θ is not everywhere well-defined, because the polar coordinates functions are 2π- periodic. But its differential $d\theta$ is a well-defined and nonvanishing 1-form on S^1. Nevertheless $d\theta$ is not exact, because if it were, then integrating it over S^1 we would get 0, since $d\theta = 0$, but the integral has value 2π. We claim now that all closed $1-$forms on S^1 are proportional to $d\theta$, i.e. for any closed ω then there is a constant c, such that $\omega - cd\theta$ is exact. Set $\omega = f(\theta)d\theta$ and

$$c = \frac{1}{2\pi}\int_{S^1}\omega$$

and

$$g(\theta) = \int_0^\theta (f(\theta) - c)d\theta.$$

Since θ is 2π-periodic then $g(\theta+2\pi n) = g(\theta)$ for every $n \in \mathbb{N}$. Further, g is a well-defined C^∞-function in S^1. Furthermore is $dg = (f(\theta) - c)d\theta = \omega - cd\theta$ and this means that every 1-form in S^1 differs from a real multiple of $d\theta$ by an exact form and therefore:

$$H^1_{de\ R}(S^1) \cong \mathbb{R}.$$

Later we will calculate the de Rham cohomology in other cases, such as $\mathbb{R}^n$ and the n-sphere S^n. Now we study some general properties of de Rham cohomology on a n-dimensional smooth manifold M.

5.2 Properties of de Rham cohomology

Lemma 6. *The de Rham cohomology of a n-dimensional manifold M, has the following properties:*

- $H^p_{de\ R}(M) = 0$ *if* $p > n$;
- *for* $a \in H^p_{de\ R}(M), b \in H^q_{de\ R}(M)$ *there is a bilinear product* $ab \in H^{p+q}_{de\ R}(M)$ *which satisfies*

$$ab = (-1)^{pq}ba;$$

- *for any smooth mapping* $f : M \to N$ *the pullback*

$$f^* : H^p_{de\ R}(N) \to H^p_{de\ R}(M)$$

commutes with the $\wedge$*-product.*

Proof. • Clear since $\Lambda^p T^* = 0$ for $p > n$.

- Recall the skew-symmetric property of $\wedge$-product. Consider two elements a and b, which define a cohomology class $[\alpha]$, $[\beta]$, respectivley, by the above defined quotient space. If $a = \alpha$ and $b = \beta$, then define a representative of the $\wedge$-product of a and b as:

$$ab = [\alpha \wedge \beta].$$

We need to check that by this we really define a cohomology class (the difference of two closed forms should be exact):

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta = 0 \text{ since } \alpha, \beta \text{ are closed.}$$

Now choose a different representative $\alpha' = \alpha + d\gamma$ for a. This gives:

$$\alpha' \wedge \beta = (\alpha + d\gamma) \wedge \beta = \alpha \wedge \beta + d(\gamma \wedge \beta).$$

Since $d\beta = 0$ the last summand is $d(\gamma \wedge \beta) = d\gamma \wedge \beta$. Then:

$$\alpha' \wedge \beta - \alpha \wedge \beta = d\gamma \wedge \beta,$$

which means that two elements $\alpha' \wedge \beta$ and $\alpha \wedge \beta$ differ by an exact form. Therefore they define the same cohomology class. Similarly for the element β.

- Consider the pull-back operation on forms. Since:

$$df^*\alpha = f^*d\alpha$$

f^* defines a map of de Rham cohomology. The product is preserved by linearity of pullback operation, i.e.

$$f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$$

and one verifies:

(i) $d\omega = 0 \Rightarrow d(f^*\omega) = 0$ for $\omega \in H^p_{de\ R}(N)$,

(ii) $[\omega_1] = [\omega_2] \Rightarrow \omega_2 - \omega_1 = d\alpha \Rightarrow f^*\omega_2 - f^*\omega_1 = f^*(\omega_2 - \omega_1) = f^*d\alpha = df^*\alpha \Rightarrow [f^*\omega_2] = [f^*\omega_1]$ for $\omega_{1,2} \in H^p_{de\ R}(N)$.

□

The latter property of de Rham cohomology is independent of the chosen map f. Thus we can show that for a smooth family of smooth maps f_t the result of their action on the de Rham cohomology is independent of t. Consider now a smooth map

$$f : M \times [a, b] \to N.$$

We say that f is smooth if it is the restriction of a smooth map defined on a slightly bigger open interval $M \times (a - \epsilon, b + \epsilon)$.

This is the so called *homotopy invariance* of de Rham cohomology.
We define:

Definition 39. Let M and N be two topological spaces and $f, g : M \to N$ two smooth maps. We say f and g are *homotope*, if there exists a smooth map

$$F : M \times [0,1] \to N$$

such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for some $x \in M$.

Example 16. It might be most vivid to think of the second parameter as the time. Then F describes the continuous deformation of f into g: We start at time 0 with map f and end at time 1 with map g. Let M be an arbitrary set and $N = \mathbb{R}^n$. Define $f, g : M \to \mathbb{R}^n$ by $F(x,t) = (1-t) \cdot f(x) + t \cdot g(x)$, then f and g are homotope.

Theorem 11. *Take the smooth map $f : M \times [0,1]$ and set $f_t(x) = f(x,t)$. Consider the induced map $f_t^* : H^p_{de\ R}(N) \to H^p_{de\ R}(M)$ on de Rham comomology. Then:*

$$f_1^* = f_0^*.$$

Proof. Let $a \in H^p_{de\ R}(N)$ be represented by a closed p-form α and the pullback form $f^*\alpha$ on $M \times [0,1]$. Recall that for a $p-$form $\alpha = \sum_I g_I dx_I$ its pullback is given by $f^*\alpha = \sum_I (g_I \circ f) df_I$. We can decompose the pullback in the form

$$f^*\alpha = \beta + dt \wedge \gamma,$$

where β is a p-form on M, depending on t, and γ is a $(p-1)$-form an M depending on t. In other words, β is $f_t^*\alpha$. We can think of γ as the mapping $(x,s) \mapsto (x, s+t)$ in the group of diffeomorphisms of $M \times (a,b)$ generated by a vector field $X = \partial/\partial t$. Then $\gamma = i_X f^*\alpha$, where i_X means the interior product, see remark below. We know that α is closed, so for the exterior derivative on M we get:

$$d_M f^*\alpha = f^*(d_M\alpha) = 0 = d_M\beta + dt \wedge \frac{\partial \beta}{\partial t} - dt \wedge d_M\gamma.$$

We get

$$\frac{\partial \beta}{\partial t} = d_M\gamma.$$

Using

$$\frac{\partial}{\partial t} f_t^*\alpha = \frac{\partial(\beta + dt \wedge \gamma)}{\partial t} = \frac{\partial \beta}{\partial t}$$

we get by integration over t

$$f_1^*\alpha - f_0^*\alpha = \int_0^1 \frac{\partial}{\partial t} f_t^* \alpha dt = d\int_0^1 \gamma dt.$$

The latter formula means that the difference of the closed forms $f_1^*\alpha$ and $f_0^*\alpha$ is an exact form and

$$f_0^*\alpha = f_1^*\alpha.$$

□

Remark 10. Given a vector field X on a manifold M we define the interior product i_X as a linear map

$$i_X : \Omega^p(M) \to \Omega^{p-1}(M)$$

such that

- $i_X df = X(f)$;
- $i_X(\alpha \wedge \beta) = i_X\alpha \wedge \beta + (-1)^p \alpha \wedge i_X\beta$ if $\alpha \in \Omega^p(M)$.

If the vector field is given by

$$X = \sum_i a_i \frac{\partial}{\partial x_i}$$

and $\alpha = dx_1 \wedge dx_2 \wedge ... \wedge dx_p$ is a p-form, then

$$i_X\alpha = a_1 dx_2 \wedge ... \wedge dx_p - a_2 dx_1 \wedge dx_3 \wedge ... \wedge dx_p + ...$$

In particular

$$i_X(i_X\alpha) = a_1a_2dx_3 \wedge ... \wedge dx_p - a_2a_1dx_3 \wedge ... \wedge dx_p + ... = 0.$$

Example 17.

$$\alpha = dx \wedge dy, \qquad X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

then

$$i_X\alpha = xdy - ydx.$$

Lemma 7.

$$H^p_{de\ R}(\mathbb{R}^n) = 0 \textit{ for all } p > 0.$$

Proof. We want to use again homotopy invariance and define therefore similiarly to the above situation: $f : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ by

$$f(x,t) = tx.$$

Denote the identity map by f_1, i.e. $f_1(x) = x$. It induces the pullback:

$$f_1^* : H^p_{de\ R}(\mathbb{R}^n) \to H^p_{de\ R}(\mathbb{R}^n),$$

which itself is the identity. On the other hand, for $t = 0$ we get $f_0(x) = 0$ is the constant mapping to zero. In particular, the derivative vanishes, which means that the pullback of any p-form of degree greater than zero is the zero map. So:

$$f_0^* : H^p_{de\ R}(\mathbb{R}^n) \to H^p_{de\ R}(\mathbb{R}^n), \text{ for } p > 0$$

vanishes. Since we have shown that de Rham cohomology is independent of the paramenter t is $f_0^* = f_1^*$ which means that $H^p_{de\ R}(\mathbb{R}^n)$ vanishes for $p > 0$. □

We can now apply the Poincaré lemma again. First we show that for $p > 0$

$$H^p_{de\ R}(U) = 0 \text{ for any contractible set } U \subset \mathbb{R}^n.$$

Lemma 8. *For any open, contractible set $U \subset \mathbb{R}^n$ the de Rham cohomology vanishes for all $p > 0$.*

Proof. Denoting by $a \in U$ the point such as for any $x \in U$ the straight segment $\overline{ax}$ lies entirely in U we can use the same argument as above by using the map: $f_t : M \times \mathbb{R}^n \to M \times \mathbb{R}^n$ given by $f_t(a,tx) = (a,tx)$ to show that $H^p_{de\ R}(M \times \mathbb{R}^n) \cong H^p_{de\ R}(M)$. Consider $\mathbb{R}/\mathbb{Z}$ (which is isomorphic to S^1). Define the two sets $U_0 = p(0,1), U_1 = p(-1/2,1/2)$ where $p : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ the quotient homomorphism. Since any two elements in the subset $p^{-1}(a)$ differ by an integer, p restricted to $(0,1)$ or $(-1/2,1/2)$ is injective and so we have coordinate charts

$$\varphi_0 = p^{-1} : U_0 \to (0,1) \text{ and } \varphi_1 = p^{-1} : U_1 \to (-1/2,1/2).$$

Then U_0 and U_1 cover $\mathbb{R}/\mathbb{Z}$ because $0 \in U_1$. We verify:

$$\varphi_0(U_0 \cap U_1) = (0,1/2) \cup (1/2,1), \quad \varphi_1(U_0 \cap U_1) = (-1/2,0) \cup (0,1/2)$$

are open sets. If $x \in (0,1/2)$ then $\varphi_1\varphi_0^{-1}(x) = x$ and if $x \in (1/2,1)$ the atlas map is $\varphi_1\varphi_0^{-1}(x) = x - 1$. We got a smooth altas. Thus the 1-form $dx = d(x-1)$ is well-defined and nowhere vanishing. Further, since $\mathbb{R}/\mathbb{Z}$ is compact and every function attains its minimum where $df = 0$, we see that

$$H^1_{de\ R}(\mathbb{R}/\mathbb{Z}) \neq 0.$$

Take any closed 1-form $\alpha = g(x)dx$, where the function $g(x)$ is a periodic function with period 1 ($g(x) = g(x+1)$). To solve the equation $df = \alpha$ is the same as solving $f'(x) = g(x)$ which we can do in $\mathbb{R}$:

$$f(x) = \int_0^x g(s)ds.$$

We can attain $f(x+1) = f(x)$ if and only if

$$\int_0^1 g(x)dx = 0.$$

This means that:

$$\alpha = g(x)dx = \left(\int_0^1 g(s)ds\right) dx + df$$

and thus $H^1_{de\ R}(\mathbb{R}/\mathbb{Z}) \cong \mathbb{R}$. □

We can now formulate a more general result about de Rham cohomology of the spheres.

Theorem 12. *For n-sphere S^n $(n > 0)$ the de Rham cohomology is:*

$$H^p_{de\ R}(\mathbb{R}) \cong \begin{cases} \mathbb{R} & \text{if } p = 0 \text{ or } p = n \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have settled the case $n = 1$, so assume $n > 1$.

If $p > n$, the group vanishes. If $p = 0$, we have $H^0_{de\ R}(S^n) \cong \mathbb{R}$ for all n.

Consider now $n > 1$ and first assume $1 < p$: Decompose S^n into two open sets U and V, where those are supposed to be the complements of the closed balls around the north and the south pole respectively. Stereographic projection gives a diffeomorphic mapping to open balls in $\mathbb{R}^n$. If α is a closed p- form for $1 < p < n$, then by the Poincaré lemma there exist $(p-1)$-forms u, v such that $\alpha = du$ and $\alpha = dv$. On the intersection $U \cap V$ we have

$$d(u - v) = \alpha - \alpha = 0,$$

which means that $(u - v)$ is closed. But by construction

$$U \cap V \cong S^{n-1} \times \mathbb{R}.$$

Thus

$$H^{p-1}_{de\ R}(U \cap V) \cong H^{p-1}_{de\ R}(S^{n-1}),$$

which vanishes by induction. Therefore $u - v = d\omega$ on $U \cap V$.

Consider $U \cap V$ as a product of S^{n-1} with an open interval, e.g. $U \cap V = S^{n-1} \times (-2, 2)$. Let $\varphi(s)$ be a continuous piecewise defined function equal to 1 for $s \in (-1, 1)$ and supported in $(-2, 2)$. Take arbitrary smaller sets $U' \subset U$ and $V' \subset V$ such that $U' \cap V' = S^{n-1} \times (-1, 1)$. Then define a form on S^n by extending $\varphi\omega$ by zero. We have u on U' and $v + d(\varphi\omega)$ on V' with $u = v + d(\varphi\omega)$ on $U' \cap V'$. This defines a $(p-1)$-form β on S^n such that $\beta = u$ on U' and $v + d(\varphi\omega)$ on V' and $\alpha = d\beta$ on U' and V'. Therefore $\alpha = d\beta$ globally, which means that the cohomology of α is zero and thus $H^p_{de\ R}(S^n) = 0$ for $1 < p < n$.

Take now the case $p = 1$. The difference $u - v$ is a function on $U \cap V$ and since $d(u - v) = 0$, the function itself must be a constant, say $c = u - v$. Then $d(v + c) = \alpha$ and the pair of function u on U and $v + c = u$ on V coincide on the untersection $U \cap V$. Therefore we can define a function f such that $df = \alpha$. Again the cohomology vanishes.

Finally consider the last case $p = n$: The form $u - v$ defines a class in $H^{n-1}_{de\ R}(U \cap V) \cong H^{n-1}_{de\ R}(S^{n-1}) \cong \mathbb{R}$. Let ω be an $(n-1)$-form an S^{n-1}. We pull this form back to $S^{n-1} \times (-2, 2)$ by projection on the first factor. Then $H^{n-1}_{de\ R}(S^{n-1} \times (-2, 2))$ is spanned by ω and we get

$$u - v = \lambda\omega + d\omega$$

for some $\lambda \in \mathbb{R}$. $H^n_{de\ R}(S^n)$ is at most 1-dimensional.
We now need to find a cohomology class for $H^n_{de\ R}(S^n)$ for which $\lambda \neq 0$. If $\lambda = 0$ $u - v = d\omega$, i.e. u and v differ by an exact form. Then we can apply the above case $p = 1$. Consider

$$\varphi dt \wedge \omega,$$

which is extended by zero outside the intersection $U \cap V$. Then

$$\left(\int_{-2}^{t} \varphi(s) ds \right) \omega$$

vanishes for $t < -2$ and extended by zero, it defines a form u in U such that $du = \alpha$. For $t > 2$ the integral is not vanishing but we can change the integral into

$$v = \left(\int_{-2}^{t} \varphi(s) ds \right) \omega - \left(\int_{-2}^{2} \varphi(s) ds \right) \omega.$$

The expression extends by zero to V and satisfies $dv = \alpha$. This means, that the above λ equals

$$\lambda = \int_{-2}^{2} \varphi(s)ds > 0.$$

The latter is equivalent to the statement that $H^n_{de\ R}(S^n) \cong \mathbb{R}$.

□

5.3 Simplicial homology

An obvious disadvantage of de Rham cohomology is that $H^k_{de\ R}(M)$ is defined for smooth manifolds M only. We now define first n-dimensional cuboids and simplices. Then we develop the concept of simplicial homology of a smooth manifold in order to formulate the de Rham Theorem. We will mainly concentrate on $\mathbb{R}^3$, because in this case we are able to relate our result to the vector-analytical concepts such as gradient, divergence and curl. The de Rham theorem claims that there is a duality between de Rham cohomology and simplicial homology. This gives us a method to avoid the restriction of de Rham cohomology.

Definition 40. Let $Q = [a_1, b_1] \times ... \times [a_n, b_n] \subset \mathbb{R}^n$ be a compact cuboid, $U = U(Q) \subset \mathbb{R}^n$ an open neighborhood and $\sigma : U \to \mathbb{R}^m$ a smooth map. Then one calls $\sigma(Q)$ a *n-dimensional singular cuboid* in $\mathbb{R}^m$ and $|\sigma| := \sigma(Q)$ the *support* of σ.

If it is possible to decompose a manifold into such pieces, one hopes to get information on the properties of the space by investigating the decomposition. If we demand the differentiability of the map from simplices onto the manifold, those are the perfect objects for the integration of differential forms.
We investigate first the behavior of cuboids and develop the analogue properties for simplicies. We will reformulate the already in section 4 stated Stokes theorem again - this time for cuboids and simplices.

Definition 41. A *n-dimensional chain* in $\mathbb{R}^m$ is a linear combination, denoted by Γ, which maps every n-dimensional cuboid σ in $\mathbb{R}^m$ to an integer n_σ for all σ. If $\Gamma(\sigma_\kappa) = n_\kappa$ for $\kappa = 1, ..., k$ and $\Gamma(\sigma) = 0$ for all other simplices, then one writes

$$\Gamma = n_1\sigma_1 + ... + n_k\sigma_k.$$

The set $|\Gamma| := |\sigma_1| \cup ... \cup |\sigma_k|$ is called the *support* of Γ.

If $\Gamma(\sigma) = 0$ for all σ, then on writes $\Gamma = 0$ and $|\Gamma| = \emptyset$.

Let $\Gamma = \sum_{\kappa=1}^{k} n_\kappa \sigma_\kappa$ be a chain and ω be a n-form on an neighborhood of $|\Gamma|$, then we can write the integral

$$\int_\Gamma \omega = n_1 \int_{\sigma_1} \omega + ... + n_k \int_{\sigma_k} \omega.$$

If $\Gamma = 0$, we define $\int_\Gamma \omega := 0$ for every n-form ω.

We now recall the boundary operators. Let
$Q = [a_1, b_1] \times ... \times [a_n, b_n] \subset \mathbb{R}^n$ be a compact cubiod, then we define

$$\partial_i^u Q := \{(x_1, ..., x_n) \in Q : x_i = a_i\},$$
$$\partial_i^o Q := \{(x_1, ..., x_n) \in Q : x_i = b_i\}.$$

Obviously

$$\partial Q = \bigcup_{i=1}^{n} (\partial_i^u Q \cup \partial_i^o Q).$$

We define by

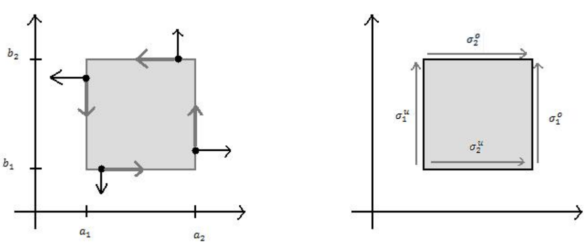
$$\sigma_i^u(x_1, ..., \widehat{x_i}, ..., x_n) := (x_1, ..., x_{i-1}, a_i, x_i, ..., x_n),$$
$$\sigma_i^o(x_1, ..., \widehat{x_i}, ..., x_n) := (x_1, ..., x_{i-1}, b_i, x_i, ..., x_n)$$

$(n-1)$- dimensional cuboids

$$\sigma_i^u, \sigma_i^o : [a_1, b_1] \times ... \times \widehat{[a_i, b_i]} \times ... \times [a_n, b_n] \to \mathbb{R}^n;$$

this corresponds the ith lower side and the ith upper side of Q.

This parametrization also specifies the orientation. For $n = 2$ the bottom and top will be passed from left to right, the sides from the bottom up.



Picture 11: Orientation,parametrization of the boundary ($n = 2$)

For the integration of an $(n-1)$-form over the boundary of Q we want to orientate the boundary such that the normal vector always points outward.

The parametrization $\sigma : [a, b] \to \mathbb{R}^2$ of a boundary piece of Q should always be chosen such that $\{n_x, \sigma'(t)\}$ forms at $\sigma(t) = x$ a positive oriented basis of the tangent space $T_x(\mathbb{R}^2)$, where n_x is the outer normal vector at x. For σ_2^u and σ_1^o is this automatically satisfied, whereas the orientation of σ_2^o and σ_1^u must be transversed. Therefore, the chain $\partial_+ Q = \sigma_2^u + \sigma_1^o - \sigma_2^o - \sigma_1^u$ describes a positive oriented boundary of Q, which has q always on its left side. Analogously we call an orthonormal basis $\{a_1, ..., a_{n-1}\}$ of a tangent space positive oriented, if $\{n, a_1, ..., a_{n-1}\}$ is a positive basis of $\mathbb{R}^n$, where n the normal vector.

Thus, in the n-dimensional case one calls the chain

$$\partial_+ Q := \sum_{i=1}^{n} (-1)^{i-1} (\sigma_i^o - \sigma_i^u)$$

the *(positively) oriented boundary* of Q. Clearly: $|\partial_+ Q| = \partial Q$.

We can now formulate the Stokes theorem for cuboids.

Theorem 13. *(Stokes theorem for cuboids)*
Let $Q \subset \mathbb{R}^n$ be a compact cuboid and ω be a $(n-1)$-form on an open neighborhood of Q. Then:

$$\int_Q d\omega = \int_{\partial_+ Q} \omega.$$

Proof. Let ω be given by

$$\omega = f dx_1 \wedge ... \wedge \widehat{dx_j} \wedge ... \wedge dx_n.$$

Then

$$d\omega = \frac{\partial f}{\partial x_j} (-1)^{j-1} dx_1 \wedge ... \wedge dx_n.$$

Denote the parametrizations of the sides of Q again by σ_i^u and σ_i^o.

Then it follows:

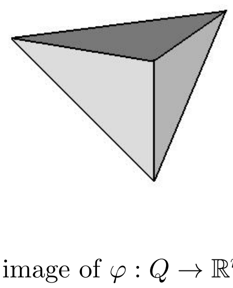
$$\int_Q d\omega = \int_Q \frac{\partial f}{\partial x_j}(x_1, ..., x_n)(-1)^{j-1} dx_1 ... dx_n =$$
$$= (-1)^{j-1} \cdot \int_{a_n}^{b_n} ... \int_{a_j}^{b_j} ... \int_{a_1}^{b_1} \frac{\partial f}{\partial x_j}(x_1, ..., x_j, ..., x_n) dx_1 ... dx_j ... dx_n =$$
$$= (-1)^{j-1} \cdot \int_{a_n}^{b_n} ... \widehat{\int_{a_j}^{b_j}} ... \int_{a_1}^{b_1}$$
$$[f \circ \sigma_j^o(x_1, ..., \widehat{x_j}, ..., x_n) - f \circ \sigma_j^u(x_1, ..., \widehat{x_j}, ..., x_n)] dx_1 ... \widehat{dx_j} ... dx_n =$$
$$= (-1)^{j-1} [\int_{\sigma_j^o} \omega - \int_{\sigma_j^u} \omega] =$$
$$= \sum_{i=1}^n (-1)^{i-1} [\int_{\sigma_i^o} \omega - \int_{\sigma_i^u} \omega] =$$
$$= \int_{\partial_+ Q} \omega,$$

since $(\sigma_i^u)^* \omega = 0$ and $(\sigma_i^o)^* \omega = 0$ for $i \neq j$. □

Analogously for simplices

Now we perform the analogous considerations for simplices.

A simplex is the simplest polytope. Each of its vertices expands a simplex into the next dimension, i.e. a n-dimensional simplex is created by adding a vertex to a $n-1$-dimensional simplex in the next higher dimension. Thus, a n-dimensional simplex has $n+1$ vertices. For example, a point, an interval, a triangle and the tetrahedron are simplices in dimensions 0 to 3.



PICTURE 12: TETRAHEDRON

Let the image of $\varphi : Q \to \mathbb{R}^n$ be a p- dimensional singular simplex. Q is the Cartesian product of p intervals $[a, b]$, $a, b \in \mathbb{R}$; e.g. for

$p = 1$: $Q = [0, 1]$, the image of $\varphi : [0, 1] \to \mathbb{R}^n$ the 1-dimensional singular simplex. Then one calls the $(p-1)$-dimensional chain

$$\partial_+ \varphi := \sum_{i=1}^{n} (-1)^{i-1} (\varphi \circ \sigma_i^o - \varphi \circ \sigma_i^u)$$

the *posivitely oriented boundary* of φ. For $S := \varphi(Q)$ so denote $bS := |\partial_+ \varphi|$. If $p = n$, then is $bS = \partial S$, the topological boundary of S. This is no longer true for $p < n$.

For $\Gamma = \sum_{\kappa=1}^{k} n_\kappa \sigma_\kappa$, one writes $\partial_+ \Gamma := \sum_{\kappa=1}^{k} n_\kappa \partial_+ \sigma_\kappa$. One can prove, that $\partial_+ \partial_+ \sigma = 0$ for every simplex σ.

Theorem 14. *(Stokes Theorem for p-dimensional simplices)*
Let $B \subset \mathbb{R}^n$ be an open subset, the image of $\varphi : Q \to B$ a p-dimensional simplex and let ω be a $(p-1)$-form on B, then:

$$\int_{\partial_+ \varphi} \omega = \int_{\varphi(Q)} d\omega.$$

Proof.

$$\begin{aligned}
\int_{\varphi(Q)} d\omega = \int_Q \varphi^*(d\omega) = \int_Q d(\varphi^* \omega) = \int_{\partial Q} \varphi^* \omega = \\
= \sum_{i=1}^{p} (-1)^{i-1} [\int_{\sigma_i^o} \varphi^* \omega - \int_{\sigma_i^u} \varphi^* \omega] = \\
= \sum_{i=1}^{p} (-1)^{i-1} [\int_{\varphi \circ \sigma_i^o} \omega - \int_{\varphi \circ \sigma_i^u} \omega] = \\
= \int_{\partial_+ \varphi} \omega.
\end{aligned}$$

□

Let us now consider the special cases $n = 2$ and $n = 3$ to establish connections to the classical vector calculus. For this, we need to recall the notions of gradient, divergence and curl on the differentiable vector field $A \subset \mathbb{R}^3$ and a differentiable function $f : \mathbb{R}^3 \to \mathbb{R}$.

Definition 42. For a differentiable vector field $A \subset \mathbb{R}^3$ and a differentiable function $f : \mathbb{R}^3 \to \mathbb{R}$:

$$\begin{aligned}
\text{Nabla operator: } \nabla &:= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), \\
\text{Gradient: } \nabla \bullet f &:= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right), \\
\text{Divergence: } \nabla \bullet A &:= \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right), \\
\text{Curl: } \nabla \times A &:= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right), \\
\text{Laplace operator: } \Delta &:= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.
\end{aligned}$$

Case $n = 2$:

Every 1-form is representable by $\omega = \omega_F = fdx + gdy$ with $F := (f, g)$ and it is

$$d\omega = (g_x - f_y)dx \wedge dy = (rotF)dx \wedge dy.$$

For $(x, y) = \Psi(u, v) = (\Psi_1(u, v), \psi_2(u, v))$, then is

$$\Psi^*\omega = (f\circ\Psi)d\Psi_1 + (g\circ\Psi)d\Psi_2 = ((F \circ \Psi) \bullet \Psi_u)\, du + ((F \circ \Psi) \bullet \Psi_v)\, dv.$$

Since $\Psi^*(hdx \wedge dy) = (h \circ \Psi) \cdot det(J_\Psi)du \wedge dv$, the equation $\Psi^*(d\omega) = d(\Psi^*\omega)$ we get

$$((rotF) \circ \Psi) \cdot det(J_\Psi)du \wedge dv = rot\left((F \circ \Psi) \cdot J_\Psi\right) du \wedge dv.$$

Case $n = 3$:

Denote by $ds := (dx_1, dx_2, dx_3)$ a 1-form, by $dS := (dx_2 \wedge dx_3, dx_3 \wedge dx_1, dx_1 \wedge dx_2)$ a 2-form and further, we write $dV := dx_1 \wedge dx_2 \wedge dx_3$ for a 3-form in $\mathbb{R}^3$. Let A and B be smooth vector fields. Then we can write the differential forms of orders $0, 1, 2, 3$ on A and for a differentiable function f as follows:

$$\begin{aligned}
&\omega_0 := f, \\
&\omega_1 := \omega_A := A_1dx_1 + A_2dx_2 + A_3dx_3) = A \cdot ds, \\
&\omega_2 := \Omega_A := A_1dx_2 \wedge dx_3 + A_2dx_3 \wedge dx_1 + A_3dx_1 \wedge dx_2 = A \cdot dS, \\
&\omega_3 := BdV.
\end{aligned}$$

Lemma 9. *The following relations hold for a smooth function f and smooth vector fields A, B:*

$$\begin{aligned} df &= \omega_{gradf}, \\ d(\omega_A) &= \Omega_{rotA}, \\ d(\Omega_B) &= (divB)dV. \end{aligned}$$

Proof. The first relation follows from the identity $df = \sum \frac{\partial f}{\partial x_i} dx_i$. The second can be derived as:

$$\begin{aligned} d(\omega_A) = {} & dA_1 \wedge dx_1 + dA_2 \wedge dx_2 + dA_3 \wedge dx_3 = \\ & = (A_1)_{x_2} dx_2 \wedge dx_1 + (A_1)_{x_3} dx_3 \wedge dx_1 + \\ & \quad + (A_2)_{x_1} dx_1 \wedge dx_2 + (A_2)_{x_3} dx_3 \wedge dx_2 + \\ & \quad + (A_3)_{x_1} dx_1 \wedge dx_3 + (A_3)_{x_2} dx_2 \wedge dx_3 = \\ & = ((A_2)_{x_1} - (A_1)_{x_2})\, dx_1 \wedge dx_2 + \\ & \quad + ((A_1)_{x_3} - (A_3)_{x_1})\, dx_3 \wedge dx_1 + \\ & \quad + ((A_3)_{x_2} - (A_2)_{x_3})\, dx_2 \wedge dx_3 = \\ & = \Omega_{rotA}. \end{aligned}$$

where we apply the definitions of $rotA$ and Ω_C (set $C = rotA$) from above. The third relation is verified as:

$$\begin{aligned} d(\Omega_B) &= dB_1 \wedge dx_2 \wedge dx_3 + dB_2 \wedge dx_3 \wedge dx_1 + dB_3 \wedge dx_1 \wedge dx_2 = \\ &= ((B_1)_{x_1} + (B_2)_{x_2} + (B_3)_{x_3})\, dx_1 \wedge dx_2 \wedge dx_3 = \\ &= (divB)dV. \end{aligned}$$

□

It is possible now to deduce relations we know from vector calculus to the Stokes theorem. We will consider two most important equations and additionally two relations between 1- and 2- forms.

Theorem 15. *Let f be a smooth function and let A be a smooth vector field. Then:*

(i) $rot\ gradf = 0$,

(ii) $div\ rotA = 0$.

Proof. We use that $d \circ d = 0$ for 0- and 1- forms:

(i) $0 = ddf = d(\omega_{grad\ f}) = \Omega_{rot\ gradf}$, which means $rot\ gradf = 0$,

(ii) $0 = dd\omega_A = d(\Omega_{rot\ A}) = (div\ rotA)dV$, which means $div\ rotA = 0$.

□

Theorem 16. *Let A, B be smooth vector fields, let ω_A, ω_B be 1- forms and $\Omega_B, \Omega_{A\times B}$ be 2-forms. Then:*

(i) $\omega_A \wedge \omega_B = \Omega_{A\times B}$,

(ii) $\omega_A \wedge \Omega_B = (A\ B)dV$.

Proof. (i)

$$\begin{aligned}
&\omega_A \wedge \omega_B = \\
&= (A_1 dx_1 + A_2 dx_2 + A_3 dx_3) \wedge (B_1 dx_1 + B_2 dx_2 + B_3 dx_3) = \\
&= (A_2B_3 - A_3B_2)dx_2 \wedge dx_3 + (A_3B_1 - A_1B_3)dx_3 \wedge dx_1 + \\
&+ (A_1B_2 - A_2B_1)dx_1 \wedge dx_2 = \\
&= \Omega_{A\times B}.
\end{aligned}$$

(ii) We denote with $< i, j, k >$ the triple $(1, 2, 3)$ or its cyclic permutation. Then is:

$$\begin{aligned}
\omega_A \wedge \Omega_B &= \left(\sum_{\nu=1}^{3} A_\nu dx_\nu\right) \wedge \left(\sum_{<i,j,k>} B_i dx_j \wedge dx_k\right) = \\
&= \sum_{<i,j,k>} A_i B_i dx_i \wedge dx_j \wedge dx_k = \\
&= \left(\sum_{\nu=1} A_\nu B_\nu\right) dx_1 \wedge dx_2 \wedge dx_3 = \\
&= (A\ B)dV.
\end{aligned}$$

□

Collecting the previous facts together we get the following diagram for $0, 1, 2, 3$- forms.

$$\begin{array}{ccccccc}
\{\text{0-forms}\} & \xrightarrow{d} & \{\text{1-forms}\} & \xrightarrow{d} & \{\text{2-forms}\} & \xrightarrow{d} & \{\text{3-forms}\} \\
\| & & \uparrow & & \uparrow & & \uparrow {\scriptstyle \cdot dV} \\
\{\text{functions}\} & \xrightarrow{grad} & \{\text{vector fields}\} & \xrightarrow{rot} & \{\text{vector fields}\} & \xrightarrow{div} & \{\text{functions}\}
\end{array}$$

Generalization of this diagram to arbitrary dimensions is the so called de Rham Chain complex, see below.

5.4 Main result - de Rham Theorem

Definition 43. The *de Rham Chain complex* is defined as follows:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d_{k-2}} & \Omega^{k-1}(M) & \xrightarrow{d_{k-1}} & \Omega^{k}(M) & \xrightarrow{d_{k}} & \Omega^{k+1}(M) & \xrightarrow{d_{k+1}} & \cdots \\
 & & \downarrow{\scriptstyle Int^{k-1}} & & \downarrow{\scriptstyle Int^{k}} & & \downarrow{\scriptstyle Int^{k+1}} & & \\
\cdots & \xrightarrow{\partial_{k-2}} & \varphi^{k-1} & \xrightarrow{\partial_{k-1}} & \varphi^{k} & \xrightarrow{\partial_{k}} & \varphi^{k+1} & \xrightarrow{\partial_{k+1}} & \cdots
\end{array}$$

where $(Int^k(\omega))(Q) = \int_A \omega$. By Stokes theorem this diagram is commutative. Define the de Rham cohomology and homology groups as

$$H^k_{de\ R}(M) = kerd_k/\ imd_{k-1} \text{ and } H^k(\varphi) = ker\partial_k/im\partial_{k-1}.$$

We get the homomorphism:

$$Int^k : H^k_{de\ R}(M) \to H^k(\varphi).$$

To formulate de Rham theorem we need:

- the image of $\varphi : Q \to B \subset \mathbb{R}^n$ a n-simplex for $Q \subset \mathbb{R}^n$, where B an open subset in $\mathbb{R}^n$. Now we specify $\varphi^k(Q^k)$ the set of k-dimensional simplices for the k-dimensional subset Q^k.
- The positive oriented boundary of a k-simplex: $\partial_{+\varphi\ k} = \sum_{i=1}^{k} (-1)^{i-1}(\varphi \circ \sigma_i^o - \varphi \circ \sigma_i^u)$.
- The set of k-forms ω_k on a smooth submanifold $M \subset \mathbb{R}^k$ is denoted by $\Omega^k(M)$.
- $Int^k(\omega_k) = \int_{\varphi^k} d\omega_k = \int_{\partial\varphi^k} \omega_k$.

Theorem 17. *(de Rham theorem) In the above notation, the mapping*

$$Int^k : H^k_{de\ R}(M) \to H^k(\varphi)$$

is an isomorphism.

Proof. We show that Int^k is an isomorphims by proving its surjectivity and injectivity.

- surjectivity: we need to assure, that for every k-simplex in φ^k there exists a k-form ω_k such that $Int^k(\omega_k) = Q^k$: Let $Q_k \in$

$H^k(\varphi^k)$ be given. Then $Q_k = Int^k(\omega_k)$. And applying the boundary operator we get:

$$\begin{aligned}\partial_k Q_k &= \partial_k Int^k(\omega_k) = \partial_k \int_{\varphi_k} d\omega_k = \partial_k \int_{\partial_k \varphi^k} \omega^k = \\ &= \int_{\partial_k \varphi^k} d_k \omega^k = \int_{\underbrace{\partial_k \circ \partial_k}_{=0} \varphi^k} \omega_k = \\ &= 0.\end{aligned}$$

Thus, $\omega_k \in ker\partial_k$, which implies $\omega_k \in H^k_{de\ R}(M)$.

- injectivity: let ω_k and η_k be k-forms: $\omega_k, \eta_k \in ker(Int^k)$. Then is $Int^k(\omega_k) = Int^k(\eta_k) = 0$ and further

$$0 = Int^k(\omega_k) = \int_{\varphi^k} d\omega_k = \int_{\partial\varphi^k} \omega_k.$$

This implies $\omega_k = 0$. As it is

$$0 = Int^k(\eta_k) = \int_{\varphi^k} d\eta_k = \int_{\partial\varphi^k} \eta_k.$$

This implies $\eta_k = 0$. Which shows that if the function attains the same image at two points, these preimages are the same.

□

6 Introduction to Morse theory

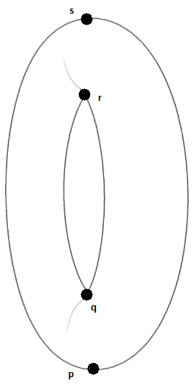
We will finally introduce the Morse theory (named after the US- mathematician Marston Morse), which gives a direct way to analyse the topology of a manifold by studying smooth functions on it. After explaining the basic concept we develop the theory formally. We will state and prove Morse lemma, the headstone in Morse theory. The last step is then to investigate the morse inequalities, which gives a upper bound for the number of critical points on manifolds.

I refer to mainly to [MIL] and [RBO] in this section.

6.1 Basic concept

Consider a hilly landscape M, which will later turns out to be a manifold. We define a function $f : M \to \mathbb{R}$, mapping to each point its elevation. The inverse is thus the contour line. Thus, $f^{-1}(a)$ for $a \in \mathbb{R}$ represents all points on M, which have the same elevation. Such a contour line is either a point, a closed curve or a closed curve with double points (saddel points). Even triple points or points of higher orders are possible, however, those are unstable. Now we flood this imaginary landscape with water. Our interest is focused on how the topology changes with increasing water level. Reaching a arbitrary elevation a the water covers all points below, i.e. the covered surface equals $f^{-1}(-\infty, a]$. The topology only changes if the continuously rising water lever reaches a critical point, e.g. a saddle or a valley point (maximum/ minimum). As we define later such points are those with gradient equal to zero. Each of these critical points one allocates a so called morse index (roughly speaking the number of independent directions around a critical point in which the function is decreasing: for minima 0, for saddles 1 and for maxima 2.

Example 18. We consider a torus M, which is standing tangentially on a horizontal plane E, with its 4 critical points p, q, r, s. Let p and s be the south-, respectively northpole of the torus and let q and r be the south-, respectively northpole af the hole.



PICTURE 13: TORUS

Further, let $f : M \to \mathbb{R}$ be the projection onto the vertical axis, mapping to each point of M its elevation above E. Then M^a is a subset of M given by $f^{-1}(a)$:

$$M^a := f^{-1}(-\infty, a] = \{x \in M | f(x) \leq a\} \subset M,$$

representing all points with an elevation less or equal to a. Now consider a coming from the negative infinity and running through the real numbers. During this process M^a attains exactly five types of topological equivalent objects:

(H1) if $a \in (-\infty, f(p) = 0)$, then $M^a = \emptyset$,
(H2) if $a \in (f(p), f(q))$, then M^a is a disk,
(H3) if $a \in (f(q), f(r))$, then M^a is a cylinder,
(H4) if $a \in (f(r), f(s))$, then M^a is a torus, where one has removed a disk,
(H5) if $a \in (f(s), \infty)$, then M^a is a equal to M.

As we notice, at the critical points p, q, r, s the type changes. We investigate the trasitions between these types:

(H1)→(H2) add a 0-cell to the empty set: putting a point into the void results in a one-point space, which is topologically equivalent to a disk (a disk is contractible to a point),
(H2)→(H3) add a 1-cell to the disk: we get a cylinder,
(H3)→(H4) add again a 1-cell to the cylinder: we get a torus with removed disk,
(H4)→(H5) add a disk (2-cell) to complete the torus.

Unformally, we get the idea of how the topology changes: is a passing a critical point of index γ one adds a γ-cell to M^a.

6.2 Formal preparation

Now we want wo make this considerations formally correct. Let M be a n dimensional manifold and $f : M \to \mathbb{R}$ be a smooth map.

Definition 44. A point $p \in M$ is called *critical point* of f, if the map $T_p f : T_{M,p} \to \mathbb{R}$ of tangent spaces is the zero-mapping. We denote the set of all critical points of f with $C_f \subset M$. The images of critical points $p \in M$ under f are called *critical values*: $f(p) = c$.

In other words, is $p \in M$ a critical point, so

$$\frac{\partial f}{\partial x_1}(p) = ... = \frac{\partial f}{\partial x_n}(p) = 0 \tag{3}$$

holds with respect to a local coordinates $(x_1, ..., x_n)$ of a chart (U_p, φ_p) on M.

We prove now the independence of coordinates of this defintion.

Lemma 10. *The above defintion of critical points is independent of the choice of local coordinates.*

Proof. Let $(x_1, ..., x_n)$ and $(y_1, ..., y_n)$ be two different local coordinate systems. Further, let p be a critical point of f with respect to $(x_1, ..., x_n)$. Changing coordinates we get

$$\frac{\partial f}{\partial y_i}(p) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\frac{\partial x_i}{\partial y_i}(p).$$

With equation (3) we get that even

$$\frac{\partial f}{\partial y_1}(p) = ... = \frac{\partial f}{\partial y_n}(p) = 0$$

holds. This means that p is also a critical point with respect to $(y_1, ..., y_n)$. □

We demand another property: a critical point should be nondegenerate to be somehow stable.

Definition 45. Let $p \in C_f$ be a critical point of f and let (U_p, φ_p) be a local chart with local coordinates $(x_1, ..., x_n)$. The Hessian $H_f(p)$ of f at p is thus given by

$$H_f(p) = \left(\frac{\partial^2 f}{\partial x_i \partial_j}(p) \right) = \begin{pmatrix} \frac{\partial^2 f}{(\partial x_1)^2}(p) & & \cdots & & \frac{\partial^2 f}{\partial x_1 \partial x_n}(p) \\ & \ddots & & & \\ \vdots & & \frac{\partial^2 f}{\partial x_i \partial x_j}(p) & & \vdots \\ & & & \ddots & \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(p) & & \cdots & & \frac{\partial^2 f}{(\partial x_1)^2}(p) \end{pmatrix}$$

A critical point p is called *nondegenerate* if $H_f(p)$ is nonsingular. In the singular case $det(H_f(p)) = 0$ we call p *generate*.

Also this defintion is independent of the choice of coordinates:

Lemma 11. *The nongeneracy of critical points is independent of the choice of the local coordinate system.*

Proof. Let $(x_1, ..., x_n)$ and $(y_1, ..., y_n)$ be two local coordinate systems of a nondegenerate critical point $p \in C_f$. Further, let $H_f^{(x_i)}(p)$ and $H_f^{(y_i)}(p)$ be the Hessian of f at p with respect to these coordinates. Then

$$H_f^{(x_i)}(p) = (J_\Phi(p))^t H_f^{(y_i)}(p) J_\Phi(p),$$

with

$$J_\Phi(p) = \left(\frac{\partial y_i}{\partial x_i}(p) \right)$$

the Jacobian of the coordinate change $\Phi : (x_1, ..., x_n) \to (y_1, ..., y_n)$, evaluated at p. Then

$$det\left(H_f^{(x_i)}(p)\right) = det\left((J_\Phi(p))^t\right) det\left(H_f^{(y_i)}(p)\right) det\left(J_\Phi(p)\right).$$

Since the Jacobian $J_\Phi(p)$ is nonsingular, is $det\left(H_f^{(x_i)}(p)\right) \neq 0$ if and only if $det\left(H_f^{(y_i)}(p)\right) \neq 0$. □

Now we define the index of a critical point $p \in C_f$.

Definition 46. Let $p \in C_f$ be a nondegenerate critical point and $H_f(p)$ the Hessian of f at p. The *index* $\lambda(p)$ of p is the number of negative eigenvalues of $H_f(p)$.

Note: $0 \leq \lambda(p) \leq dim(T_{M,p}) = dim(M) = n$.

6.3 Basics of Morse theory

Now we go into the Morse theory and begin with the defintion of a Morse function.

Definition 47. If the funtion $f : M \to \mathbb{R}$ has only nondegenerate critical points, we call it *Morse function.* The *Morse polynomial* of f expresses the behavior of its critical points p:

$$M_t(f) := \sum t^{\lambda(p)}.$$

Example 19. (Torus, revisited) As we have seen the torus has four critical point of index 0, 1, 1, 2 respectively. Thus, the Morse polynomial for the torus is given by:

$$M_t(f) = 1 + 2t + t^2.$$

It now follows the Morse lemma, which describes the local appearance of the function $f : M \to \mathbb{R}$ nearby a nondegenerate critial point $p \in C_f$. We will just give an idea of the proof, the whole proof can be found at [MIL]. The same with two immediatly following conclusions.

Lemma 12. *Let p be nondegenerate critical point of $f : M \to \mathbb{R}$ and let $\lambda(p)$ the index of p. Then there exists a local coordinate system $(y_1, ..., y_n)$ in a neighborhood U of p with $y_i(p) = 0$ for all $i = 1, ..., n$, such that f has the form*

$$f = f(p) - (y_1)^2 - ... - (y_\lambda)^2 + (y_{\lambda+1})^2 + ... + (y_n)^2$$

on U with respect to $(y_1, ..., y_n)$.

Proof. (Idea) Show first, that if such a presentation of f with respect to local coordinates of a local chart (U_p, φ_p) exists for a nongenerate critical point p, the λ indeed coincides with the index of f at p. Then, one need to show that such suitable coordinates exist. □

Lemma 13. *Nondegenerate critical points of f are isolated in C_f.*

Proof. (Idea) If $p \in C_f$ is nongenerate, says the Morse lemma, that f is locally in a neighborhood U_p of the form

$$f = f(p) - (y_1)^2 - ... - (y_\lambda)^2 + (y_{\lambda+1})^2 + ... + (y_n)^2.$$

Partial differentiation of f with respect to the local coordinates y_i ($i = 1, ..., n$) we can conclude that U_p has only p as critical point. Thus, the critical points are isolated. □

The Morse-lemma thus implies, that there exists a neighborhood $U \subset M$ of p (being a open subset of a smooth manifold M, U self is a smooth manifold), such that the restriction $f_{|U} : U \to \mathbb{R}$ is a Morse function.

Lemma 14. *Let M be a compact manifold and f be a Morse function on M. Then C_f is finite.*

Proof. First, a manifold M is called compact, if M as topological space is compact. We prove the lemma by contradiction: assume there is a Morse function $f : M \to \mathbb{R}$ on a compact manifold M with a infinite number of critical points p_m, $m \in \mathbb{N}$. We understand these critical points as a sequence $(p_m)_{m\in\mathbb{N}}$. Since M is compact $(p_{m_k})_{k\in\mathbb{N}}$ is a converging subsequence. Denote the limit with $p \in M$: $p := \lim_{k\to\infty} p_{m_k}$. Let $(x_1, ..., x_n)$ be a local coordinate system of a local chart (U_p, φ_p). We can by convergence assume that $(p_{m_k})_{k\in\mathbb{N}} \subset U_p$. Then

$$0 = \lim_{k\to\infty} \frac{\partial f}{\partial x_i}(p_{m_k}) = \frac{\partial f}{\partial x_i}(p) \text{ , i=1,...,n}$$

implies that also p is a critical point of f (change differentiation and taking the limit is allowed due to smoothness). But f was a Morse function and thus all critical points p are isolated, which is a contradiction to the converging sequence of critical points. □

6.4 Morse inequalities

Finally we state the weak Morse inequalities, principally saying that the number of critical numbers of index i is greater or equal to the dimension of the ith Betti number. The ith Betti number b_i of topological space X is the dimension of the ith singular homology group of X. In our case we only need to consider the Betti numbers b_0, b_1 and b_2, because only indices $0, 1, 2$ appear.

Definition 48. Let X be a topological space. The ith *Betti number* of X is given by

$$b_i(X) = dim_{\mathbb{Q}} H_i(X, \mathbb{Q}), \text{ for } I = 0, 1, 2, ...$$

where $H_i(X, \mathbb{Q})$ denotes the ith singular homology group with coefficients in $\mathbb{Q}$.

Informally one can think of the ith Betti number as the number of i-dimensional unconnected surfaces on X. Thus

- b_0 gives the number of connected components
- b_1 gives the number of 2-dimensional holes
- b_2 gives the number of 3-dimensional holes, resp. voids.

Example 20. (Torus, revisited) The torus has one connected component, the 'skin' ($b_0 = 1$), it has two 2-dimensional holes, one in the middle and one inside the torus ($b_1 = 2$) and further, it has one 3-dimensional void, the interior ($b_2 = 1$).

Lemma 15. *The number of critical points of index i is greater or equal to the ith Betti number.*

Before we proof the lemma we look a last time back at our torus.

Example 21. (Torus, revisited) The torus has one critical point of index 0: $1 \geq b_0 = 1$; two critical points of index 1: $2 \geq b_1 = 2$; and one critical point of index 2: $1 \geq b_2 = 1$.

Proof. Let M be a compact manifold and f a differentiable function on M with isolated, nongenerate critical points, i.e. f is a Morse function. Let $p_1 < ... < p_n$ be such points on M that M^{p_k} contains exactly k critical points and $M^{p_n} = M$. Denote with $H_*(M, M')$ the homology chain complex: the set of chains is the $\mathbb{Z}$-module generated by the critical points of f. The differential d of a complex sends a critical point p of index i to a sum of critical points p' with index $i-1$, whose coefficients correspond to the number of unparametrized lines from p to p'. Then:

$$\begin{aligned} H_*(M^{p_k}, M^{p_{k-1}}) &= H_*(M^{p_{k-1}} \smile e^{\lambda_k}, M^{p_{k-1}}) \\ &= H_*(e^{\lambda_k}, e'^{\lambda_k}) \\ &= \begin{cases} \text{coefficient group in dimension } \lambda_k \\ 0 \text{ otherwise} \end{cases} \end{aligned}$$

where λ_k the index of the critical point p_k and $\breve{} e^{\lambda_k}$ means to attach the λ_k-cell $= \{x \in \mathbb{R}^{\lambda_k} : \|x\| < 1\}$. One property (we will not show here) of Betti numbers is the subadditivity. Apply this to $\emptyset = M^{p_0} \subset M^{p_1} \subset ... \subset M^{p_n} = M$ and R_{λ_k} the kth Betti number we have:

$$R_i(M) < R_k(M^{p_k}, M^{p_{i-1}}) = k;$$

where k denotes the number of critical points of index λ_k. □

This was the final step in this thesis. Of course, there is a plenty of theorems and applications left to enlarge upon both differential topology and algebraic topology. This thesis had as it´s goal giving an overview over the basically concepts, such that one gets an impression and is curious to immerse oneself in one or the other subject.

Symbols

R	ring
mod- R	right - R - module
R -mod	left - R - module
$\otimes_R$	tensor product over R
$T(M)$	tensor algebra
$\oplus$	direct sum
$S(M)$	symmetric algebra
$\Lambda(M)$	skew- symmetric algebra
$(\varphi_i : U_i \to V_i)$	local coordinate change
$\mathfrak{A} = \{(U_i; \varphi_i) \| i \in I\}$	atlas
C^k (C^∞)	class of k-times differentiable (smooth) mappings
$T_{M,p}$	set of tanget vectors X in p on M
φ^*	cotangent vector
$T^*_{M,p}$	cotangent space
ω	differential form, k-form
$d\omega$	derviative of k-form
$\Omega^k(M)$	set of k-forms ω_k on a smooth submanifold $M \subset \mathbb{R}^k$
$\varphi^k(Q)$	k-dimensional simplex
$\partial_+\varphi_k$	positive oriented boundary of φ^k
$H^i_{de\ R}(M)$	i-th de Rham cohomology group of M
$H^i(\varphi)$	i-th homology group
$H_f(p)$	Hessian of function f evaluated at p
$J(p)$	Jacobian evaluated at p
$M_t(f)$	Morse polynomial of Morse function f
C_f	set of critical points of f
λ_p	index of point p
e^{λ_k}	λ_k-cell

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