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av

Isabelle Shankar

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MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

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Isabelle Shankar

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Abstract

In this paper, we present the theory for Fenchel-Lagrange duality and then use this to look at some nonconvex optimization problems. Specifically, we consider an optimization problem with DC objective functions and DC inequality constraints, a few fractional programming problems and a DC programming problem containing a composition with a linear continuous operator. The various primal problems considered are convexified and given Fenchel-Lagrange type dual problems as well as constraint qualifications for strong duality. Later, these results are reformulated into Farkas-type theorems to give a concise presentation of the relationship of each primal problem to its dual problem.

1 Introduction

In recent years, many new optimization methods and techniques have arisen from the need to consider various real-world problems that cannot be solved through convex programming alone. As a part of this trend, many authors have begun expanding beyond convex optimization problems to DC programming. These problems, which will be elaborated on in Section 5, have functions which are written as difference of convex, or DC, functions. The many advantages of DC functions allow for a wider range of application. Being nonconvex, DC optimization problems cover many types of real-world problems. In fact, the set of DC functions defined on a compact convex set X of $\mathbb{R}^n$ is dense in the set of continuous functions on X. So in theory, every continuous function can be closely approximated by a DC function. Furthermore, the special structure of having a positive and a negative convex function allows us to use many tools of convex analysis when studying DC programming.

The focus of this paper is the use of Fenchel-Lagrange duality to find dual problems to some DC programming problems and, through the results for the DC optimization problems, fractional programming problems as well. The framework for Fenchel-Lagrange duality is given in Section 4, in the context of convex optimization. Fenchel-Lagrange duality, a theory combining the Lagrange dual with the Fenchel dual, was developed by Boţ, Grad and Wanka in [7] as a response to Geometric Duality. Using less convoluted methods, they generalized the results of Geometric duality in convex optimization. When applied to DC programming in [6], they look at the problem

$$(P_{DC}) \qquad \inf_{\substack{g_i(x)-h_i(x)\leq 0 \\ i=1,\ldots,m,\, x\in X}} \{g(x)-h(x)\}$$

where $g,h:\mathbb{R}^n\to\overline{\mathbb{R}}, g_i,h_i:\mathbb{R}^n\to\overline{\mathbb{R}}, i=1,\ldots,m$ are proper and convex functions, and X is a nonempty convex subset of $\mathbb{R}^n$. By convexifying this primal problem, they are able to take the Fenchel-Lagrange dual of a sub problem, leading to a dual problem for (P_{DC}). This method will be described in Section 6.

Other DC programming and fractional programming problems we are interested in will also be discussed in Section 6. These include problems done by various author as well as my own work. Specifically, my addition to the body of literature is an evaluation of the fractional programming problem

$$(P^0_{FP_0}) \qquad \inf_{\substack{g_i(x)-h_i(x)\leq 0 \\ i=1,\ldots,m,\, x\in X}} \left\{\frac{g(x)}{h(x)}\right\}$$

where $X\subseteq\mathbb{R}^n$ is nonempty and convex, $g:\mathbb{R}^n\to\overline{\mathbb{R}}$ is proper and convex, $h:\mathbb{R}^n\to\overline{\mathbb{R}}$ is concave such that $-h$ is proper and lower semicontinuous over the feasible set of the problem, and $g_i,h_i:\mathbb{R}^n\to\mathbb{R}$ for $i=1,\ldots,m$, are proper and convex functions. This is an extension of the work done in [5]. Furthermore, the problems of part 6.4 are independently developed in this paper via the methods of [6]. To the primal problem

$$(P_A) \qquad \inf_{\substack{\phi_i(x)-\psi_i\leq 0 \\ i=1,\ldots,m,\, x\in X}} \{g_1(x)-g_2(x)+h_1(Ax)-h_2(Ax)\}$$

where $g_1,g_2,h_1,h_2,\phi_i,\psi_i:\mathbb{R}^n\to\overline{\mathbb{R}}$, for $i=1,\ldots,m$, are proper convex functions and $A\in\mathbb{R}^{n\times n}$ is a linear continuous operator we find the dual problem

$$(D_A) \qquad \inf_{\substack{x^*\in\mathrm{dom}(g_2^*) \\ y^*\in\mathrm{dom}(h_2^*) \\ z^*\in\prod_{i=1}^m \mathrm{dom}(\psi_i^*)}} \sup_{\substack{p\in\mathbb{R}^n \\ q\geq 0}} \{-(g_1+h_1\circ A)^*(p+x^*+A^Ty^*)+g_2^*(x^*)+h_2^*(y^*)$$
$$-\left(\sum_{i=1}^m q_i\phi_i\right)^*_X\left(\sum_{i=1}^m q_iz_i^*-p\right)+\sum_{i=1}^m q_i\psi_i^*(z_i^*)\}$$

and also give conditions for strong duality. The case where $\psi_i \equiv 0$ is also considered.

As mentioned, we will evaluate these problems using the theory of Fenchel-Lagrange duality. After finding duals to the problems, the remainder of the paper will look at some Farkas-type results in regards to Fenchel-Lagrange duality in general and how this may be applied to the problems of Section 6.

Before diving into the DC programming problems, we must present some preliminary information. Here we give some basic definitions, notation and concepts well known in the field of optimization and convex analysis. They are given out of necessity in order to develop the work in this text clearly. Beginning with some notation, throughout this paper the *interior* and *relative interior* of a set X will be denoted by $\text{int}(X)$ and $\text{ri}(X)$ respectively. Given two vectors in $\mathbb{R}^n$, $x = (x_1, \ldots, x_n)^T$ and $y = (y_1, \ldots, y_n)^T$, the usual inner product is denoted by

$$x^T y = \sum_{i=1}^{n} x_i y_i$$

For a function f, the *epigraph* of f is denoted $\text{epi}(f)$. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a given function, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is called the *extended real line.* Supposing a function f is convex, the *effective domain* of f will be denoted $\text{dom}(f) = \{x \in \mathbb{R}^n \,|\, f(x) < +\infty\}$. Furthermore, a convex function f is called *proper* if the effective domain is nonempty and if for all $x \in \mathbb{R}^n$, $f(x) > -\infty$. If f is concave, then the effective domain is $\text{dom}(f) = \{x \in \mathbb{R}^n \,|\, f(x) > -\infty\}$ and it is called proper if $-f$ is proper as a convex function.

Some more basic optimization theory must be given before working directly with the main problems. The next two sections will therefore deal with two well know optimization dual theories: Lagrange duality and Fenchel duality. With these dual problems, we can define the Fenchel-Lagrange dual which will be used to tackle various DC programming problems in Section 6. We begin with Lagrange duality.

2 Lagrange Duality

A well known method in optimization is to analyze a given problem, called the primal problem, via an associated dual problem. One such dual is the Lagrange dual problem, for which the framework is briefly discussed in this section. To this end, consider a convex optimization problem,

$$(P_C) \qquad \inf_{\substack{g_i(x)\leq 0 \\ i=1,\ldots,m \\ Ax=b,\, x\in X}} \{f(x)\}$$

where A is an $l \times n$ matrix, $b \in \mathbb{R}^l$, X is convex, $f : \mathbb{R}^n \to \mathbb{R}$ is convex, and $g_i : \mathbb{R}^n \to \mathbb{R}$ are convex functions for $i = 1, \ldots, m$. From (P_C) we can define the following equation, with $\theta : \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}$,

$$\theta(q, p) = \inf_{x\in X} \{f(x) + \sum_{i=1}^{m} q_i g_i(x) + \sum_{i=1}^{l} p_i h_i(x)\}$$

called the Lagrangian dual function, where $q \in \mathbb{R}^m$, $p \in \mathbb{R}^l$, and $h_i(x)$ is the ith component of $Ax - b$. The Lagrange dual problem is defined as

$$(D_{C_L}) \qquad \sup_{q\geq 0} \inf_{x\in X} \{f(x) + \sum_{i=1}^{m} q_i g_i(x) + \sum_{i=1}^{l} p_i h_i(x)\}$$

or simply by

$$(D_{C_L}) \qquad \sup_{q\geq 0}\{\theta(q, p)\}$$

where by $q \geq 0$ we mean $q = (q_1, \ldots, q_m)$ and $q_i \geq 0$ for $i = 1, \ldots, m$. This notation will be used throughout the paper.

It is natural at this point to wonder at the nature of the relationship between (P_C) and (D_{C_L}). First, for a problem (P), the optimal value is denoted by $v(P)$. Therefore, $v(P_C)$ and $v(D_{C_L})$ are the optimal values for (P_C) and (D_{C_L}) respectively. That is,

$$v(P_C) = \inf_{\substack{g_i(x)\leq 0 \\ i=1\ldots,m \\ Ax=b, x\in X}} \{f(x)\} \qquad \text{and} \qquad v(D_{C_L}) = \sup_{q\geq 0}\{\theta(q, p)\}$$

Returning to the relationship between the dual and primal problems, it is always true that $v(P_C) \geq v(D_{C_L})$. This is know as *weak duality.* A weak duality theorem for Lagrange duality can be found in most (if not all) optimization books, such as [1], [2], and [8].

While this information is useful, the next step is to determine when this inequality becomes an equality, i.e. when $v(P_C) = v(D_{C_L})$. This is know as *strong duality.* To attain strong duality between the primal problem $v(P_C)$ and its Lagrange dual problem $v(D_{C_L})$, we define a constraint qualification (CQ) known as Slater's condition: there exists an $x' \in \mathrm{ri}(\mathcal{D})$, where

$$\mathcal{D} = \bigcap_{i=1}^{m} (\mathrm{dom}(g_i)) \cap \mathrm{dom}(f) \cap X$$

such that $g_i(x') < 0$, for $i = 1, \ldots, m$, and $Ax' = b$.

Slater's condition can be refined by distinguishing the constraint functions g_i which are affine. Define sets $L := \{i \in \{1, \ldots, m\} \,|\, g_i : \mathbb{R}^n \to \mathbb{R} \text{ is an affine function}\}$ and $N := \{1, \ldots, m\} \setminus L$. Then the refined Slater's condition is that $\exists x' \in \mathrm{ri}(\mathcal{D})$ such that $Ax' = b$, $g_i(x') \leq 0$ for $i \in L$, and $g_i(x') < 0$ for $i \in N$. Notice that the only difference is that the affine functions g_i are no longer strictly less than 0 at x'.

Theorem 2.2 states that both Slater's condition and the refined CQ imply strong duality between the primal optimization problem and its Lagrange dual. Before this can be proven, however, we need what is known as the separating hyperplane theorem:

Theorem 2.1. Let A and B be two nonempty convex sets in $\mathbb{R}^n$ such that $A \cap B = \emptyset$. Then there exist $\alpha \in \mathbb{R}$ and $u \neq 0$ in $\mathbb{R}^n$ such that

$$u^T x \leq \alpha, \forall x \in A \qquad \text{and} \qquad u^T x \geq \alpha, \forall x \in B$$

In other words, there exists a hyperplane $H = \{x \,|\, u^T x = \alpha\}$ that separates sets A and B.

Now we present a strong duality theorem for the Lagrange dual problem. The proof will be for the refinement, as that is what will be used later in the paper. For the original version of the proof, which only proves strong duality under the unrefined Slater's condition, see [8].

Theorem 2.2. Suppose Slater's condition (or its refinement) holds. Then there is strong duality between (P_C) and its Lagrange dual problem (D_{C_L}). Furthermore, the dual optimal value is attained when $v(D_{C_L}) > -\infty$.

Proof. Suppose, by Slater's condition, that there exists $x' \in \text{ri}(\mathcal{D})$ such that $g_i(x') < 0$, for $i = N$, $g_i(x') \leq 0$ for $i \in L$, and $Ax' = b$, where $\mathcal{D}$, L and N are defined above. Consider the functions $g_i(x)$ for $i = 1, \dots, m$. We order these, so that we create two categories, one group where the function is zero at x' and the rest in the other group. Thus, we have $g_i(x') = 0$ for $i = 1, \dots, k$ where $k \leq m$, which are only affine functions, and $g_i(x') < 0$ for $i = k+1, \dots, m$, which may include some affine functions. Since $g_i(x), i = 1, \dots, k$ are affine, we can lump them into the matrix A without changing the set of feasible points. Thus the equality constraints look like this:

$$\begin{bmatrix} & & \\ & A & \\ & & \\ a_{11} & \dots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1k} & \dots & a_{nk} \end{bmatrix} x = \begin{bmatrix} b_1 \\ \vdots \\ b_l \\ a_{01} \\ \vdots \\ a_{0k} \end{bmatrix}$$

Call this new matrix $\hat{A}$ and the vector on the right hand side $\hat{b}$. For simplicity, we also assume that the matrix $\hat{A}$ has rank $l + k$ and that $\mathcal{D}$ has a nonempty interior so that $\text{int}(\mathcal{D}) = \text{ri}(\mathcal{D})$. By Slater's condition, $v(P_C) < \infty$, since there is a feasible point in $\text{dom}(f)$. Furthermore, if $v(P_C) = -\infty$, then by weak duality $v(D_{C_L}) = -\infty$ and hence strong duality holds. Therefore, we consider the case where $v(P_C)$ is finite.

Define two disjoint sets, S_1 and S_2. First,

$$S_1 = \{(u, v, t) \in \mathbb{R}^{m-k} \times \mathbb{R}^{l+k} \times \mathbb{R} \,|\, \exists x \in \mathcal{D} \text{ for which } \hat{g}(x) \leq u, \hat{h}(x) = v, f(x) \leq t\}$$

where by $\hat{g}(x) \leq u$ we mean that $g_i(x)$, $i = k+1, \dots, m$, is less than or equal to the components of $u = (u_1, \dots, u_{m-k})$, and $\hat{h}(x) = v$ means that the ith component on $\hat{A}$ is equal to v_i of $v = (v_1, \dots, v_{l+k})$. Second,

$$S_2 = \{(0, 0, s) \in \mathbb{R}^{m-k} \times \mathbb{R}^{l+k} \times \mathbb{R} \,|\, s < v(P_C)\}$$

To use the separating hyperplane theorem we must show that S_1 and S_2 are convex and do not intersect. Starting with convexity, consider two points $(u_1, v_1, t_1), (u_2, v_2, t_2) \in S_1$. We want to show that the line segment $\lambda(u_1, v_1, t_1) + (1-\lambda)(u_2, v_2, t_2)$ is contained in S_1 for $\lambda \in [0, 1]$. It follows from how the set is defined, that for the two points in S_1, there exist $x_1, x_2 \in \mathcal{D}$ such that $\hat{g}(x_1) \leq u_1, \hat{h}(x_1) = v_1, f(x_1) \leq t_1$ and $\hat{g}(x_2) \leq u_2, \hat{h}(x_2) = v_2, f(x_2) \leq t_2$. By the convexity of g_i for $i = k+1, \dots, m$,

$$\hat{g}(\lambda x_1 + (1-\lambda)x_2) \leq \lambda \hat{g}(x_1) + (1-\lambda)\hat{g}(x_2) \leq \lambda u_1 + (1-\lambda)u_2$$

for $\lambda \in [0, 1]$. Likewise, since f is convex,

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda t_1 + (1-\lambda)t_2$$

Finally,

$$\hat{A}(\lambda x_1 + (1-\lambda)x_2) - \hat{b} = \lambda(\hat{A}x_1) + (1-\lambda)(\hat{A}x_2) - \hat{b} = \lambda v_1 + (1-\lambda)v_2 - \hat{b}$$

and $\hat{h}(\lambda x_1 + (1-\lambda)x_2) = \lambda v_1 + (1-\lambda)v_2$. Hence, S_1 is convex. Similarly, for S_2, suppose that $(0,0,s_1),(0,0,s_2) \in S_2$. Then $s_1 < v(P_C)$ and $s_2 < v(P_C)$. We want to show that for $\lambda \in [0,1]$, $\lambda s_1 + (1-\lambda)s_2 < v(P_C)$. This is obviously true at the endpoints since for $\lambda = 0$ we have $\lambda s_1 + (1-\lambda)s_2 = s_2$ and for $\lambda = 1$, $\lambda s_1 + (1-\lambda)s_2 = s_1$. So we consider $\lambda \in (0,1)$. Then,

$$\lambda s_1 < \lambda v(P_C) \qquad \text{and} \qquad (1-\lambda)s_2 < (1-\lambda)v(P_C)$$

which implies,

$$\lambda s_1 + (1-\lambda)s_2 < \lambda v(P_C) + (1-\lambda)v(P_C) = v(P_C)$$

proving that S_2 is convex.

Next we must show that $S_1 \cap S_2 = \emptyset$. For a contradiction, suppose there is a $(u,v,t) \in S_1 \cap S_2$. Since $(u,v,t) \in S_2$, $u = v = 0$ and $t < v(P_C)$. Moreover, since $(u,v,t) = (0,0,t) \in S_1$, there must exist an $x \in \mathcal{D}$ such that $\hat{g}(x) \leq 0, \hat{h}(x) = 0, f(x) \leq t < v(P_C)$, and hence that $g_i(x) \leq 0, i = 1,\ldots,m, Ax = b, f(x) \leq t < v(P_C)$. This is impossible since $v(P_C)$ is the optimal value of the primal problem. Hence S_1 and S_2 do not intersect. By Theorem 2.1, there exist $\alpha \in \mathbb{R}$ and $(\mu,\nu,\tau) \neq 0$ such that

$$\mu^T u + \nu^T v + \tau t \geq \alpha, \forall (u,v,t) \in S_1 \tag{1}$$

and

$$\mu^T u + \nu^T v + \tau t \leq \alpha, \forall (u,v,t) \in S_2 \tag{2}$$

Equation (1) implies that $\mu \geq 0$ and $\tau \geq 0$, since otherwise $\mu^T u + \tau t$ would be unbounded from below, contradicting (1). Equation (2) states that $\tau t \leq \alpha$ for all $t < v(P_C)$ which implies that $\tau v(P_C) \leq \alpha$. Thus from (1) and (2), we have that for any $x \in \mathcal{D}$,

$$\mu^T \hat{g}(x) + \nu^T(\hat{A}x - \hat{b}) + \tau f(x) \geq \alpha \geq \tau v(P_C) \tag{3}$$

Now we consider two cases; $\tau > 0$ and $\tau = 0$. First, consider the case where $\tau = 0$. Then (3) becomes,

$$\mu^T \hat{g}(x) + \nu^T(\hat{A}x - \hat{b}) \geq 0$$

for all $x \in \mathcal{D}$. From Slater's condition,

$$\mu^T \hat{g}(x') \geq 0$$

However, since $\mu \geq 0$ and $\hat{g}(x') < 0$, we find that $\mu = 0$. Furthermore, the fact that $(\mu,\nu,\tau) \neq 0$ and $\mu = \tau = 0$ implies that $\nu \neq 0$. From (3) we now have

$$\nu^T(\hat{A}x - \hat{b}) \geq 0$$

for all $x \in \mathcal{D}$. However, from Slater's condition, there exists $x' \in \operatorname{int}(D)$ such that $\nu^T(\hat{A}x' - \hat{b}) = 0$ which implies that there are points in $\mathcal{D}$ satisfying $\nu^T(\hat{A}x - \hat{b}) < 0$ unless $\hat{A}^T\nu = 0$. This contradicts the assumption that the rank of $\hat{A}$ is $l + k$. By contradiction, we have shown that $\tau \neq 0$.

Let $\tau > 0$. Dividing (3) by τ gives,

$$\frac{1}{\tau}\mu^T \hat{g}(x) + \frac{1}{\tau}\nu^T(\hat{A}x - \hat{b}) + f(x) \geq v(P_C)$$

for all $x \in \mathcal{D}$. We can rewrite this by redistributing the affine functions which we added to the equality constraints. If $\mu = (\mu_1,\ldots,\mu_{m-k})$, $\nu = (\nu_1,\ldots,\nu_l,\nu_{l+1},\ldots,\nu_{l+k})$, then define vectors $q := \frac{1}{\tau}(\mu_1,\ldots,\mu_{m-k},\nu_{l+1},\ldots,\nu_{l+k})$ and $p := \frac{1}{\tau}(\nu_1,\ldots,\nu_l)$. The equation above becomes

$$q^T g(x) + p^T(Ax - b) + f(x) \geq v(P_C)$$

By taking the infimum over x it follows that $v(D_C) \geq v(P_{C_L})$. From weak equality then we have that $v(D_C) = v(P_{C_L})$, so that strong duality holds and the optimal value of the dual problem is attained at (p,q). $\square$

We will use Theorem 2.2 later in the paper to prove strong duality between a primal problem and its Fenchel-Lagrange dual. This paper will deals with optimization problems that have only inequality constraints, such as,

$$(P) \qquad \inf_{\substack{g_i(x)\leq 0\\ i=1,\ldots,m\\ x\in X}} \{f(x)\}$$

where the functions, f, $g_i, i = 1, \ldots, m$ are convex as usual. In this case, the Lagrange dual problem of (P) is

$$(D_L) \qquad \sup_{q\geq 0} \inf_{x\in X} \{f(x) + \sum_{i=1}^{m} q_i g_i(x)\}$$

where $q \in \mathbb{R}^m$. Indeed, weak duality still holds, as does strong duality under both Slater's condition and its refinement.

Next, we present another optimization theory known as Fenchel duality, sometimes called conjugate duality.

3 Conjugate Duality

In order to discuss the theory of Fenchel Duality, we must first define the convex conjugate function. Therefore the following section will give a brief introduction to conjugate and biconjugate functions before introducing the Fenchel dual problem.

3.1 Conjugate Functions

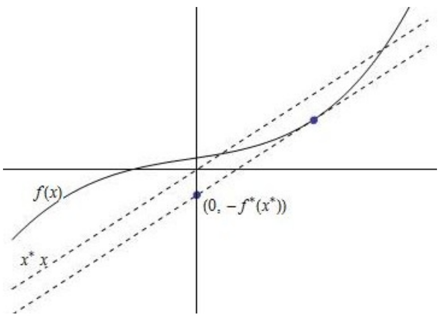
Definition 3.1. Let $X \subseteq \mathbb{R}^n$ be nonempty. The conjugate relative to the set X of a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, denoted $f_X^* : \mathbb{R}^n \to \overline{\mathbb{R}}$, is defined by

$$f_X^*(x^*) = \sup_{x \in X}\{x^{*T}x - f(x)\} = -\inf_{x \in X}\{f(x) - x^{*T}x\}$$

If $X = R^n$, then this becomes the classical conjugate of f, $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$,

$$f^*(x^*) = \sup\{x^{*T}x - f(x)\} = -\inf\{f(x) - x^{*T}x\}$$

The definition is illustrated in the following figure:



Given a function f, for each value $x^* \in \mathbb{R}^n$, the conjugate function of f, $f^*(x^*)$ is the (signed) point where the hyperplane, that has normal $(-x^*, 1)$ and supports the epigraph of f, intercepts the vertical axis. In other words, it is the maximum gap between the linear function $x^{*T}x$ and $f(x)$. To further understand conjugates, consider the following examples. First, take an easy example where $f : \mathbb{R} \to \mathbb{R}$ is an affine function, $f(x) = \alpha x + \beta$, where α, β are real scalars. The conjugate is

$$f^*(x^*) = \sup\{x^*x - \alpha x - \beta\} = \sup\{(x^* - \alpha)x\} - \beta$$

The supremum is unbounded except at $x^* = \alpha$. Hence

$$f^*(x^*) = \begin{cases} -\beta & x^* = \alpha \\ +\infty & \text{otherwise} \end{cases}$$

The next example will help in evaluating problems from 6.4. Let h be a convex function on $\mathbb{R}^n$ and define $f(x) = h(A(x - \alpha)) + x^T\alpha^* + \beta$ where A is a one-to-one linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$, $\alpha, \alpha^* \in \mathbb{R}^n$, and $\beta \in \mathbb{R}$. Then, letting $y = A(x - \alpha)$, the conjugate is

$$\begin{aligned} f^*(x^*) &= \sup_x\{x^Tx^* - h(A(x-\alpha)) - x^T\alpha^* - \beta\} \\ &= \sup_y\{(A^{-1}y + \alpha)^Tx^* - h(y) - (A^{-1}y + \alpha)^T\alpha^* - \beta\} \\ &= \sup_y\{(A^{-1}y)^T(x^* - \alpha^*) - h(y)\} + \alpha^T(x^* - \alpha^*) - \beta \\ &= \sup_y\{y^TA^{*-1}(x^* - \alpha^*) - h(y)\} + \alpha^T(x^* - \alpha^*) - \beta \\ &= h^*(A^{*-1}(x^* - \alpha^*)) + \alpha^T(x^* - \alpha^*) - \beta \end{aligned}$$

where A^* is the adjoint of A.

For the last example, consider the following definition:

Definition 3.2. Given $X \subseteq \mathbb{R}^n$, its indicator function, denoted $\delta_X : \mathbb{R}^n \to \overline{\mathbb{R}}$ is defined by,

$$\delta_X(x) = \begin{cases} 0 & x \in X \\ +\infty & \text{otherwise} \end{cases}$$

The conjugate is easily calculated to be the function $\sigma_X : \mathbb{R}^n \to \overline{\mathbb{R}}$,

$$\sigma_X(u) = \sup_{x \in X} u^T x$$

This is known as the *support function* of the set X.

Note: The indicator function is a very important tool in optimization and will be used throughout the paper. For instance, when used with conjugates, it becomes possible to switch between a classical conjugate and a conjugate relative to a set. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a function and let X be a nonempty subset of $\mathbb{R}^n$:

$$f_X^*(x^*) = \sup_{x \in X} \{x^{*T} x - f(x)\} = \sup_{x \in \mathbb{R}} \{x^{*T} x - f(x) - \delta_X(x)\} = (f + \delta_X)^*(x^*)$$

A notable property of the conjugate is that it is always convex, whether or not the original function itself is convex. This is due to f^* being the pointwise supremum of a family of convex functions. Because of this property, we sometimes distinguish between two conjugates, the *convex conjugate* and the *concave conjugate*, denoted f_* for a concave function f. The convex conjugate is what we defined in Definition 3.1. The distinction then is simply that if f_* is the concave conjugate of f, $-f_*$ is the convex conjugate.

Another property of conjugate functions is known as the *Young-Fenchel Inequality.* For all $x, x^* \in \mathbb{R}^n$, it holds that

$$f(x) + f^*(x^*) \geq x^{*T} x$$

This inequality has many important consequences in optimization. A chief concern is how to attain equality. To achieve this, we present an important definition that will be used later in the paper.

Definition 3.3. Let f be a convex function. For an arbitrary $x \in \mathbb{R}^n$ such that $f(x) \in \mathbb{R}$, the subdifferential of the function f at x is the set

$$\partial f(x) = \{x^* \in \mathbb{R}^n \mid f(y) - f(x) \geq (y - x)^T x^*, \forall y \in \mathbb{R}^n\}$$

Furthermore, the function f is subdifferentiable at $x \in \mathbb{R}^n$ with $f(x) \in \mathbb{R}$ if $\partial f(x) \neq \emptyset$.

Applying Definition 3.3 to the Young-Fenchel Inequality gives an if and only if statement for equality which will be referred to later in the paper (see Section 6). That is, if $f(x) \in \mathbb{R}$, then

$$f(x) + f^*(x^*) = x^{*T} x \Leftrightarrow x^* \in \partial f(x) \tag{4}$$

One final property of conjugate functions is presented in the following lemma:

Lemma 3.1. Let $f_1, \ldots, f_m : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper and convex functions such that $\bigcap_{i=1}^{m} \mathrm{ri}(\mathrm{dom}(f_i))$ is not empty. Then,

$$\left(\sum_{i=1}^{m} f_i\right)^*(x^*) = \inf\{\sum_{i=1}^{m} f_i^*(x_i^*); x^* = \sum_{i=1}^{m} x_i^*\}$$

and for each $x^* \in \mathbb{R}^n$ the infimum is attained.

This is a very useful lemma and will be needed later in the paper.

When discussing conjugate functions, a natural next step is to consider the conjugate of a conjugate function, $(f^*)^*$. This is know as the biconjugate and is denoted simply by f^{**}. Questions arise regarding what it looks like in comparison to the original function, whether it is ever equal to f. The remainder of this section gives a brief introduction to the biconjugate, starting with the definition:

Definition 3.4. Given a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, the biconjugate of f is

$$f^{**}(x^{**}) = \sup\{x^{**T}x^* - f^*(x^*)\} = -\inf\{f^*(x^*) - x^{**T}x^*\}$$

In general the biconjugate does not equal f. Instead it is always true that $f^{**}(x) \leq f(x)$ for any function f. Equality can hold given certain circumstances, seen in Lemma 3.2. Before presenting this lemma, however, we need the following definition:

Definition 3.5. Let X be a topological space and consider the function $f : X \to \overline{\mathbb{R}}$. If the set

$$f^{-1}((\alpha, \infty]) = \{x \in X \mid f(x) > \alpha\}$$

is open in X for all $\alpha \in \mathbb{R}$, then f is said to be lower semicontinuous.

Using this definition, we have the following lemma:

Lemma 3.2. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper function. Assume that f is also lower semicontinuous and convex. Then $f^{**}(x) = f(x)$.

In fact, by the Fenchel Moreau theorem, $f = f^{**}$ if and only if the above assumptions hold, or if either $f \equiv +\infty$ or $f \equiv -\infty$. However, we will only be concerned with the case presented in Lemma 3.2.

3.2 Fenchel Duality

As with Lagrange duality, Fenchel duality is about assigning a dual problem, called the Fenchel or conjugate dual problem, to a primal problem. In this case we work in the context of the particular problem:

$$(P_F) \qquad \inf_{x \in \mathbb{R}^n} \{f(x) - g(x)\}$$

where $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a proper and convex function and $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a proper and concave function (so that $-g$ is convex). Notice that this is still a convex optimization problem, since the sum of two convex functions is itself convex. The Fenchel dual problem to (P_F) is

$$(D_F) \qquad \sup_{x^* \in \mathbb{R}^n} \{g_*(x^*) - f^*(x^*)\}$$

where g_* is the concave conjugate of g and f^* is the convex conjugate of f. Thus the objective function of (D_F) is

$$g_*(x^*) - f^*(x^*) = \inf\{x^{*T}x - g(x)\} - \sup\{x^{*T}x - f(x)\}$$

Given these two problems, do weak and strong duality hold? Weak duality is, in fact, always true, i.e. $v(P_F) \geq v(D_F)$. It follows directly from the Young-Fenchel Inequality. That is, since

$$f(x) + f^*(x^*) \geq x^{*T}x \geq g(x) + g_*(x^*)$$

we get that for all $x, x^* \in \mathbb{R}^n$,

$$f(x) - g(x) \geq g_*(x^*) - f^*(x^*)$$

The main theorem of this section, called Fenchel's Duality Theorem, gives conditions for strong duality between (P_F) and (D_F). It is presented in this paper as it is found in [12, p. 47]. To prove it, however, requires to following results, found in [12]:

Lemma 3.3. For every convex function f, $\mathrm{ri}(\mathrm{epi}(f)) = \{(x,\mu) \,|\, x \in \mathrm{ri}(\mathrm{dom}(f)), f(x) < \mu < \infty\}$.

Theorem 3.1. Let A and B be non-empty convex sets in $\mathbb{R}^n$. There exists a hyperplane H separating A and B properly, i.e. not both A and B are contained in the hyperplane H, if and only if $\mathrm{ri}(A) \cap \mathrm{ri}(B) = \emptyset$.

Now we are ready to present the theorem for strong duality. It will be needed in the next section for proving strong duality between the primal problem (P) and the dual problem (D_{FL}), called the Fenchel-Lagrange dual.

Theorem 3.2 (Fenchel's Duality Theorem)**.** Let f be a proper and convex function on $\mathbb{R}^n$ and g be a proper and concave function on $\mathbb{R}^n$. Then

$$\inf_x \{f(x) - g(x)\} = \sup_{x^*} \{g_*(x^*) - f^*(x^*)\}$$

if one of the following conditions holds:

(a) $\mathrm{ri}(\mathrm{dom} f) \cap \mathrm{ri}(\mathrm{dom} g) \neq \emptyset$

(b) f and g are closed and $\mathrm{ri}(\mathrm{dom} g_*) \cap \mathrm{ri}(\mathrm{dom} f^*) \neq \emptyset$

Under (a) the supremum is attained at some x^*. Under (b) the infimum is attained at some x. If both conditions are satisfied, then the infimum and supremum are necessarily finite.

Proof. We saw above that weak duality holds, that is $v(P_F) \geq v(D_F)$.

If the infimum is $-\infty$, then by weak duality the supremum is also $-\infty$. Thus suppose $v(P_F)$ is not $-\infty$. Assume (a) holds. This implies that $v(P_F)$ is finite. To show that $v(P_F) = v(D_F)$ and the supremum is attained, we only need to show that there exists a vector x^* such that $g_*(x^*) - f^*(x^*) \geq v(P_F)$. To this end, let $v(P_F) = \alpha$ and consider the epigraphs

$$C = \{(x,\mu) \,|\, x \in \mathbb{R}^n, \mu \in \mathbb{R}, \mu \geq f(x)\} \qquad \text{and} \qquad D = \{(x,\mu) \,|\, x \in \mathbb{R}^n, \mu \in \mathbb{R}, \mu \leq g(x) + \alpha\}$$

These are convex sets in $\mathbb{R}^{n+1}$. By Lemma 3.3,

$$\mathrm{ri}(C) = \{(x,\mu) \,|\, x \in \mathrm{ri}(\mathrm{dom}(f)), f(x) < \mu < \infty\}$$

Since $f(x) - g(x) \geq v(P_F)$ implies that $f(x) \geq g(x) + \alpha$, we know that $\mathrm{ri}(C) \cap D = \emptyset$. Thus, by Theorem 3.1, there exists a hyperplane H in $\mathbb{R}^{n+1}$ which separates C and D properly.

Suppose that H is vertical. Then its projection on $\mathbb{R}^n$ would be a hyperplane separating the projections of C and D properly. The projections of C and D are $\mathrm{dom}(f)$ and $\mathrm{dom}(g)$ respectively. By the assumption (a), however, these cannot be separated properly. Thus by contradiction H is not vertical. This implies that H is the graph of an affine function $h(x) = x^T x^* - \beta$. From this we have that

$$f(x) \geq x^T x^* - \beta \geq g(x) + \alpha$$

for all $x \in \mathbb{R}^n$. The left hand side implies that $\beta \geq x^T x^* - f(x)$. Taking the supremum over x gives

$$\beta \geq \sup\{x^T x^* - f(x)\} = f^*(x^*)$$

Likewise, the right hand side gives us

$$\beta + \alpha \leq \inf\{x^T x^* - g(x)\} = g_*(x^*)$$

It follows that $g_*(x^*) - f^*(x^*) \geq \alpha = v(P_F)$. Thus, under assumption (a), strong duality holds and the supremum is attained at x^*.

Assume, now, that (b) holds. Then f and g are closed which implies that they are lower semicontinuous. Thus, by Lemma 3.2, $f = f^{**}$ and $g = g^{**}$ and the same argument given for (a) can be used for strong duality. □

With the two duality theories explained, we move on to the main duality theory of the paper, Fenchel-Lagrange duality.

4 Fenchel-Lagrange Duality

4.1 Framework

Assume that X is a nonempty subset of $\mathbb{R}^n$, $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a convex and proper function, and that $g = (g_1, \ldots, g_m)^T : \mathbb{R}^n \to \mathbb{R}^m$ is a vector-valued function such that g_i is convex for $i = 1, \ldots, m$. We consider the convex optimization problem,

$$(P) \qquad \inf_{\substack{g(x) \leq 0 \\ x \in X}} \{f(x)\}$$

Note that by $g(x) \leq 0$ we mean that $g_i(x) \leq 0$ for $i = 1, \ldots, m$.

In [3], Boţ uses perturbation functions to derive dual problems to a given primal problem. Using this method he computes two well-known dual problems, the Lagrange dual and the Fenchel dual. Moreover, he uses a third perturbation function to determine the Fenchel-Lagrange dual problem. The theory of duality regarding the Fenchel-Lagrange dual is thoroughly discussed in [3], [4], [6],[7] . To start, we introduce the perturbation function $\Phi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$,

$$\Phi(x, y, z) = \begin{cases} f(x + y) & x \in X, g(x) \leq z \\ +\infty & \text{otherwise} \end{cases}$$

The next step is to calculate its conjugate, $\Phi^* : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$,

$$\begin{aligned} \Phi^*(x^*, p, q) &= \sup_{\substack{x, y \in \mathbb{R}^n, \\ z \in \mathbb{R}^m}} \{x^{*T} x + p^T y + q^T z - \Phi(x, y, z)\} \\ &= \sup_{\substack{x \in X, y \in \mathbb{R}^n, \\ g(x) \leq z}} \{x^{*T} x + p^T y + q^T z - f(x + y)\} \end{aligned}$$

To make further calculations, we introduce two new variables, $r := x + y$ and $s := z - g(x)$ to get rid of y and z. Inserting this into the above function gives the following,

$$\begin{aligned} \Phi^*(x^*, p, q) &= \sup_{\substack{x \in X \\ r \in \mathbb{R}^n \\ s \geq 0}} \{x^{*T} x + p^T (r - x) + q^T (s + g(x)) - f(r)\} \\ &= \sup_{s \geq 0} \{q^T s\} + \sup_{r \in \mathbb{R}^n} \{p^T r - f(r)\} + \sup_{x \in X} \{(x^* - p)^T x + q^T g(x)\} \\ &= \begin{cases} f^*(p) - \inf_{x \in X} \{(p - x^*)^T x - q^T g(x)\} & q \leq 0, q \in \mathbb{R}^m \\ +\infty & \text{otherwise} \end{cases} \end{aligned}$$

All the information needed for the dual problem is now available. According to [3], given a perturbation function, the dual problem is defined as,

$$(D) \qquad \sup_{\substack{p \in \mathbb{R}^n \\ q \in \mathbb{R}^m}} \{-\Phi^*(0, p, q)\}$$

which in the case of Fenchel-Lagrange duality becomes

$$(D_{FL}) \qquad \sup_{\substack{p \in \mathbb{R}^n \\ q \geq 0}} \{-f^*(p) + \inf_{x \in X} \{p^T x + q^T g(x)\}\}$$

Note that the sign of q was changed. Also, $\inf_{x \in X} \{p^T x + q^T g(x)\} = \inf_{x \in X} \{q^T g(x) - (-p)^T x\} = -(q^T g)^*_X(-p)$ so that the dual problem can be equivalently written as,

$$(D_{FL}) \qquad \sup_{\substack{p \in \mathbb{R}^n \\ q \geq 0}} \{-f^*(p) - (q^T g)^*_X(-p)\}$$

4.2 Weak and Strong Duality

As in the above sections on duality, this section will elaborate on weak and strong duality for the Fenchel-Lagrange dual problem.

Theorem 4.1. Weak duality holds between the primal problem (P) and the Fenchel-Lagrange dual problem (D_{FL}), i.e. $v(P) \geq v(D_{FL})$.

Unlike weak duality, strong duality does not always hold. That is, $v(P) = v(D_{FL})$ is not true in general. In order for there to be no duality gap, we need an optimality condition. First, define sets $L := \{i \in \{1, \ldots, m\} \,|\, g_i : \mathbb{R}^n \to \mathbb{R} \text{ is an affine function}\}$ and $N := \{1, \ldots, m\} \setminus L$. Then we have the following constraint qualification:

$$(CQ) \qquad \exists x' \in \bigcap_{i=1}^{m} \mathrm{ri}(\mathrm{dom}(g_i)) \cap \mathrm{ri}(\mathrm{dom}(f)) \cap \mathrm{ri}(X) : \begin{cases} g_i(x') \leq 0 & i \in L \\ g_i(x') < 0 & i \in N \end{cases}$$

Recall the refinement of Slater's condition from Section 2,

$$\exists x' \in \mathrm{ri}(\bigcap_{i=1}^{m} (\mathrm{dom}(g_i)) \cap \mathrm{dom}(f) \cap X)$$

such that $g_i(x') \leq 0$ for $i \in L$ and $g_i(x') < 0$ for $i \in N$. It is easy to see how similar this condition is to (CQ). In fact, to take advantage of the similarity in the proof for strong duality, we first need the following theorem.

Theorem 4.2. Let I is a finite index set and let C_i be a convex set in $\mathbb{R}^n$ for $i \in I$. Suppose that the sets $\mathrm{ri}(C_i)$ have at least one point in common, then

$$\mathrm{ri}(\bigcap_{i\in I} C_i) = \bigcap_{i \in I} \mathrm{ri}(C_i)$$

Now we a prepared to present the theorem for strong duality between (P) and (D_{FL}).

Theorem 4.3. Assume that $v(P) < -\infty$. If (CQ) is fulfilled, then there is strong duality between the primal problem (P) and the Fenchel-Lagrange dual problem (D_{FL}), i.e. $v(P) = v(D_{FL})$ and there exists an optimal solution to (D_{FL}).

Proof. By Theorem 4.2, the (CQ) gives that

$$\exists x' \in \bigcap_{i=1}^{m} \mathrm{ri}(\mathrm{dom}(g_i)) \cap \mathrm{ri}(X) \cap \mathrm{ri}(\mathrm{dom}(f)) = \mathrm{ri}(\bigcap_{i=1}^{m} (\mathrm{dom}(g_i)) \cap X \cap \mathrm{dom}(f))$$

Thus we can use the refined Slater's condition and by Theorem 2.2 there exists a $\bar{q} \geq 0$ such that

$$v(P) = \sup_{q \geq 0} \inf_{x \in X} \{f(x) + q^T g(x)\} = \inf_{x \in X} \{f(x) + \bar{q}^T g(x)\}$$

By defining a function $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ as

$$h(x) = \begin{cases} \bar{q}^T g(x), & \text{if } x \in X \\ +\infty, & \text{if } x \notin X \end{cases}$$

the last term can be written as

$$v(P) = \inf_{x \in \mathbb{R}^n} \{f(x) + h(x)\}$$

Since $\mathrm{ri}(\mathrm{dom}(f)) \cap \mathrm{ri}(\mathrm{dom}(h)) = \mathrm{ri}(\mathrm{dom}(f)) \cap \mathrm{ri}(X) \neq \emptyset$, by Theorem 3.2 there exists a $\bar{p} \in \mathbb{R}^n$ such that

$$\begin{aligned}
v(P) = \inf_{x\in\mathbb{R}^n}\{f(x) + h(x)\} &= \sup_{p\in\mathbb{R}^n}\{-f^*(p) - h^*(-p)\} \\
&= -f^*(\bar{p}) - h^*(-\bar{p}) \\
&= -f^*(\bar{p}) - \sup_{x\in\mathbb{R}^n}\{-\bar{p}^T x - h(x)\} \\
&= -f^*(\bar{p}) - \sup_{x\in X}\{-\bar{p}^T x - \bar{q}^T g(x)\} \\
&= -f^*(\bar{p}) - (\bar{q}^T g)^*_X(-\bar{p})
\end{aligned}$$

This is the objective function of the Fenchel-Lagrange dual problem at $(\bar{p}, \bar{q})$. By Theorem 4.1, the supremum is attained at $(\bar{p}, \bar{q})$ hence this is the optimal solution of (D_{FL}). □

Notice in the proof that we first take the Lagrange dual of the primal problem and then we take the Fenchel dual of the Lagrange dual problem. Both steps rely on the (CQ) to give strong duality. Thus it is clear why (D_{FL}) is given its name.

5 DC Programming

This section will give an overview of DC (difference of convex) functions and DC programming problems. Parts 1 and 2 below come from [11].

5.1 DC Functions

Definition 5.1. Let X be nonempty and convex subset of $\mathbb{R}^n$. A real-valued function $f : X \to \mathbb{R}$ is called DC on X if there exist two convex functions $g, h : X \to \mathbb{R}$ such that f can be written as $f(x) = g(x) - h(x)$. Each representation of this form is said to be a DC decomposition of f. If $X = \mathbb{R}^n$ then f is just called a DC function.

The following propositions give some insight into the usefulness of DC functions.

Propostion 5.1. Let f and f_i for $i = 1, \ldots, n$ be DC functions. Then the following are also DC functions:

(i) $\sum_{i=1}^n \lambda_i f_i(x)$ for some $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, n$

(ii) Both $\max_{i=1,\ldots,n}\{f_i(x)\}$ and $\min_{i=1,\ldots,n}\{f_i(x)\}$

(iii) $|f(x)|$, $\max\{0, f(x)\}$, and $\min\{0, f(x)\}$

(iv) $\prod_{i=1}^n f_i(x)$

Propostion 5.2. Every function $f : \mathbb{R}^n \to \mathbb{R}$ whose second partial derivatives are continuous everywhere is DC.

Propostion 5.3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a DC function and let $g : \mathbb{R} \to \mathbb{R}$ be convex. Then their composition $(g \circ f)(x) = g(f(x))$ is DC.

5.2 DC Programming Problems

DC programming problems are optimization problems that involve DC functions. That is the objective function can be DC, DC functions can be found among the constraints, or a combination of this. For now, consider the problem

$$(P_{DC}) \qquad \min_{\substack{f_i(x) \leq 0 \\ i=1,\ldots m, \\ x \in X,}} \{f(x)\}$$

where $f, f_i, i = 1, \ldots, m$ are DC functions and X is a closed convex subset of $\mathbb{R}^n$.

It is worth noting that this problem can be transformed into another well-known form, which Horst and Thoai do on pages 4 and 5 of [11]:

$$\min_{\substack{\Psi_i(x) \leq 0 \\ i=1,\ldots,m \\ x \in X}} \{c(x)\}$$

where c is a linear function, X is still a closed convex subset of $\mathbb{R}^n$, and Ψ is concave. This is called a *canonical DC program*. More generally, if c is convex then this is called a *generalized canonical DC program*. Thus we see that the canonical DC program is in a class of reverse convex problems.

Now that the preliminaries have been covered, we come to the main work of the paper. The next Section will cover different DC and fractional programming problems, finding for each primal problem its dual problem.

6 Fenchel Lagrange Duality applied to some DC Programs

As mentioned in the Introduction, the dual problems to each primal problem will be defined via Fenchel-Lagrange duality, discussed in Section 4. Then we will give a constraint qualification for each pair of problems, which is need for strong duality. In order to outline the method, we start with the problem presented originally by Boţ, Hodrea, and Wanka. We will use the process and the results of this first problem in the subsequent problems of the section.

6.1 DC objective function and inequality constraints

Consider the problem from [6],

$$(P_{DC}) \qquad \inf_{\substack{g_i(x)-h_i(x)\leq 0 \\ i=1,\ldots,m,\, x\in X}} \{g(x)-h(x)\}$$

where $g, h : \mathbb{R}^n \to \overline{\mathbb{R}}, g_i, h_i : \mathbb{R}^n \to \overline{\mathbb{R}}, i = 1, \ldots, m$ are proper and convex functions, and X is a nonempty convex subset of $\mathbb{R}^n$. Also suppose that

$$\bigcap_{i=1}^{m} \mathrm{ri}(\mathrm{dom}(g_i)) \cap \mathrm{ri}(\mathrm{dom}(g)) \cap \mathrm{ri}(X) \neq \emptyset \tag{5}$$

Consider the feasible set, denoted $\mathcal{F}(P_{DC}) = \{x \in X \,|\, g_i(x) - h_i(x) \leq 0, i = 1, \ldots, m\}$, of (P_{DC}). We suppose that $\mathcal{F}(P_{DC}) \neq \emptyset$. Furthermore, assume that h is lower semicontinuous on $\mathcal{F}(P_{DC})$ and that h_i is subdifferentiable on $\mathcal{F}(P_{DC})$ for $i = 1, \ldots, m$. Then we have the following lemma:

Lemma 6.1. Given the assumptions presented so far, the following is true:

$$\mathcal{F}(P_{DC}) = \bigcup_{\substack{y_i^*\in\mathrm{dom}(h_i^*) \\ i=1,\ldots,m}} \{x \in X \,|\, g_i(x) - x^T y_i^* + h_i^*(y_i^*) \leq 0, i = 1, \ldots, m\}$$

Proof. Let $x \in \mathcal{F}(P_{DC})$, then $x \in \bigcap_{i=1}^{m} \mathrm{dom}(h_i)$. Since $h_i, i = 1, \ldots, m$, is subdifferentiable, there exists a $y_i^* \in \partial h_i(x)$ for $i = 1, \ldots, m$. Thus by equation (4) above, for $i = 1, \ldots, m$,

$$h_i(x) + h_i^*(y^*) = y^{*T} x$$
$$-y^{*T} x + h_i^*(y^*) = -h_i(x)$$
$$g_i(x) - y^{*T} x + h_i^*(y^*) = g_i(x) - h_i(x) \leq 0$$

Therefore, x is in the union above and we have one inclusion.

Next, we prove the opposite inclusion, $\supseteq$. Let $x \in X$ such that $g_i(x) - x^T y_i^* + h_i^*(y_i^*) \leq 0, i = 1, \ldots, m$. Then $g_i(x) < +\infty$ for $i = 1, \ldots, m$. Also, let $y^* = (y_1^*, \ldots, y_m^*) \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)$. By the Young-Fenchel inequality we have that $h_i(x) + h_i^*(y^*) \geq y^{*T} x$. Since $g_i(x) < +\infty$ for $i = 1, \ldots, m$, we get from the inequality

$$g_i(x) - h_i(x) \leq g_i(x) - y^{*T} x + h_i^*(y^*) \leq 0$$

for $i = 1, \ldots, m$. Thus $x \in \mathcal{F}(P_{DC})$ and therefore the sets are in fact equal. $\square$

We now derive another form of (P_{DC}). First, since h is proper, convex and semicontinuous on $\mathcal{F}(P_{DC})$, then $h(x) = h^{**}(x) = \sup_{x^*\in\mathrm{dom}(h^*)} \{x^{*T} x - h^*(x^*)\}$. Hence,

$$\begin{aligned} v(P_{DC}) = \inf_{x\in\mathcal{F}(P_{DC})} \{g(x) - h(x)\} &= \inf_{x\in\mathcal{F}(P_{DC})} \{g(x) - \sup_{x^*\in\mathrm{dom}(h^*)} \{x^{*T} x - h^*(x^*)\}\} \\ &= \inf_{x\in\mathcal{F}(P_{DC})} \{g(x) + \inf_{x^*\in\mathrm{dom}(h^*)} \{-x^{*T} x + h^*(x^*)\}\} \\ &= \inf_{x^*\in\mathrm{dom}(h^*)} \inf_{x\in\mathcal{F}(P_{DC})} \{g(x) - x^{*T} x + h^*(x^*)\} \end{aligned}$$

Using Lemma 6.1 gives the final form of (P_{DC}):

$$(P_{DC}) \qquad \inf_{\substack{x^* \in \mathrm{dom}(h^*) \\ y^* \in \prod_{i=1}^m \mathrm{dom}(h_i^*)}} \quad \inf_{\substack{g_i(x) - {y^*}^T x + h_i^*(y^*) \leq 0 \\ i=1,\ldots,m,\, x \in X}} \{g(x) - {x^*}^T x + h^*(x^*)\}$$

This is the form for which we will find a dual problem. To do so, notice that the inner infimum is a convex optimization problem. It therefore it will be treated as a separate problem. We will find a dual to the inner infimum and then "reattach" the outer infimum to this to get (D_{DC}). Hence, consider for some fixed $x^* \in \mathrm{dom}(h^*)$ and $y^* \in \prod_{i=1}^m \mathrm{dom}(h_i^*)$ the following convex optimization problem,

$$(P_{x^*,y^*}) \qquad \inf_{\substack{g_i(x) - {y^*}^T x + h_i^*(y^*) \leq 0 \\ i=1,\ldots,m,\, x \in X}} \{g(x) - {x^*}^T x + h^*(x^*)\}$$

To simplify the problem, let $f(x) = g(x) - {x^*}^T x + h^*(x^*)$ and $f_i(x) = g_i(x) - {y^*}^T x + h_i^*(y^*)$ for $i = 1, \ldots, m$. Then the problem becomes

$$(P_{x^*,y^*}) \qquad \inf_{\substack{f_i(x) \leq 0 \\ i=1,\ldots,m, \\ x \in X}} \{f(x)\}$$

where $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is convex and proper and $f_i : \mathbb{R}^n \to \overline{\mathbb{R}}$ are proper and convex for $i = 1, \ldots, m$. Taking the Lagrange dual gives,

$$(D_{x^*,y^*}) \qquad \sup_{q \geq 0} \inf_{x \in X} \{f(x) + \sum_{i=1}^m q_i f_i(x)\}$$

where $q = (q_1, \ldots, q_m)^T \in \mathbb{R}^n$. By the definition of conjugates,

$$\begin{aligned} \inf_{x \in X} \{f(x) + \sum_{i=1}^m q_i f_i(x)\} &= -(- \inf_{x \in X} \{f(x) + \sum_{i=1}^m q_i f_i(x) - 0^T x\}) \\ &= -\big(f + \sum_{i=1}^m q_i f_i\big)_X^*(0) \end{aligned}$$

Recall that we assumed (5), which implies that $\bigcap_{i=1}^m \mathrm{ri}(\mathrm{dom}(f_i)) \cap \mathrm{ri}(\mathrm{dom}(f)) \neq \emptyset$, and that functions f, f_i, $i = 1, \ldots, m$, are proper and convex. Hence we can apply Lemma 3.1,

$$\begin{aligned} -\big(f + \sum_{i=1}^m q_i f_i\big)_X^*(0) &= -\big(f + \sum_{i=1}^m q_i f_i + \delta_X\big)^*(0) \\ &= - \inf_{p \in \mathbb{R}^n} \{f^*(p) + \left(\sum_{i=1}^m q_i f_i + \delta_X\right)^*(-p)\} \\ &= - \inf_{p \in \mathbb{R}^n} \{f^*(p) + \left(\sum_{i=1}^m q_i f_i\right)_X^*(-p)\} \\ &= \sup_{p \in \mathbb{R}^n} \{-f^*(p) - \left(\sum_{i=1}^m q_i f_i\right)_X^*(-p)\} \end{aligned}$$

Returning to the dual problem, we can use the equation above to write it in the equivalent form,

$$(D_{x^*,y^*}) \qquad \sup_{\substack{p \in \mathbb{R}^n \\ q \geq 0}} \{-f^*(p) - \left(\sum_{i=1}^m q_i f_i\right)_X^*(-p)\}$$

It should be noted that this is exactly the Fenchel-Lagrange dual of the convex optimization problem (P_{x^*,y^*}) with objective and constraint functions f and f_i, $i = 1,\dots,m$. Note that the process involves first taking the Lagrange dual and then using conjugates to essentially reformulate it via the Fenchel dual.

In order to have (D_{x^*,y^*}) in terms of g, h, g_i, and h_i, for $i = 1,\dots,m$, we must calculate the conjugates found in the above form of the dual problem. Starting with the simpler of the two, $f(x) = g(x) - x^{*T} + h^*(x^*)$ has the following conjugate,

$$\begin{aligned} f^*(p) &= \sup\{p^T x - (g(x) - x^{*T}x + h^*(x^*))\} \\ &= \sup\{(p + x^*)^T x - g(x)\} - h^*(x^*) \\ &= g^*(p + x^*) - h^*(x^*) \end{aligned}$$

Next, given that $f_i(x) = g_i(x) - y_i^{*T}x + h_i^*(y_i^*)$,

$$\begin{aligned} \left(\sum_{i=1}^m q_i f_i\right)_X^*(-p) &= \sup_{x\in X}\left\{-p^T x - \left(\sum_{i=1}^m q_i(g_i(x) - y_i^{*T}x + h_i^*(y_i^*))\right)\right\} \\ &= \sup_{x\in X}\left\{\left(\sum_{i=1}^m q_i y_i^* - p\right)^T x - \sum_{i=1}^m q_i g_i(x)\right\} - \sum_{i=1}^m q_i h_i^*(y_i^*) \\ &= \left(\sum_{i=1}^m q_i g_i\right)_X^*\left(\sum_{i=1}^m q_i y_i^* - p\right) - \sum_{i=1}^m q_i h_i^*(y_i^*) \end{aligned}$$

Plugging these conjugates into the dual problem,

$$(D_{x^*,y^*}) \qquad \sup_{\substack{p\in\mathbb{R}^n \\ q\geq 0}}\left\{h^*(x^*) - g^*(p + x^*) + \sum_{i=1}^m q_i h_i^*(y_i^*) - \left(\sum_{i=1}^m q_i g_i\right)_X^*\left(\sum_{i=1}^m q_i y_i^* - p\right)\right\}$$

Since the dual problem (D_{x^*,y^*}) is the Fenchel-Lagrange dual of (P_{x^*,y^*}) by Theorem 4.1, weak duality holds. For strong duality, we must refer back to the constraint qualification in Section 4.2. In our case this becomes,

$$(CQ_{y^*}) \qquad \exists x' \in \bigcap_{i=1}^m \mathrm{ri}(\mathrm{dom}(g_i)) \cap \mathrm{ri}(\mathrm{dom}(g)) \cap \mathrm{ri}(X) : \begin{cases} g_i(x') - x'^T y_i^* + h_i^*(y_i^*) \leq 0 & i \in L \\ g_i(x') - x'^T y_i^* + h_i^*(y_i^*) < 0 & i \in N \end{cases}$$

where as before, $L = \{i \in \{1,\dots,m\} \,|\, g_i \text{ is affine}\}$ and $N = \{1,\dots,m\} \setminus L$. With this constraint qualification strong duality can be asserted.

Propostion 6.1. Assume $v(P_{x^*,y^*})$ is finite. If (CQ_{y^*}) is fulfilled, then strong duality holds between (P_{x^*,y^*}) and (D_{x^*,y^*}).

Proof. Evaluating the problem

$$(P_{x^*,y^*}) \qquad \inf_{\substack{f_i(x)\leq 0 \\ i=1,\dots,m, \\ x\in X}} \{f(x)\}$$

led to the Fenchel-Lagrange dual

$$(D_{x^*,y^*}) \qquad \sup_{\substack{p\in\mathbb{R}^n \\ q\geq 0}}\left\{-f^*(p) - \left(\sum_{i=1}^m q_i f_i\right)_X^*(-p)\right\}$$

Notice that, since x^* and y^* are fixed, $\mathrm{dom}(f) = \mathrm{dom}(g) \cap \mathrm{dom}(-x^{*T}x) = \mathrm{dom}(g) \cap \mathbb{R}^n = \mathrm{dom}(g)$ and likewise, $\mathrm{dom}(f_i) = \mathrm{dom}(g_i)$. Hence the constraint qualification (CQ_{y^*}) implies that

$$\exists x' \in \bigcap_{i=1}^{m} \mathrm{ri}(\mathrm{dom}(f_i)) \cap \mathrm{ri}(\mathrm{dom}(f)) \cap \mathrm{ri}(X) : \begin{cases} f_i(x') \leq 0 & \text{when } f_i \text{ is affine} \\ f_i(x') < 0 & \text{otherwise} \end{cases}$$

By Theorem 4.3, strong duality holds. That is, there exists a $\bar{q} = (\bar{q}_1, \ldots, \bar{q}_m) \in \mathbb{R}^n$, such that $\bar{q} \geq 0$, and a $\bar{p} \in \mathbb{R}^n$ such that

$$\begin{aligned} v(P_{x^*,y^*}) &= \sup_{\substack{p \in \mathbb{R}^n \\ q \geq 0}} \{-f^*(p) - \left(\sum_{i=1}^{m} q_i f_i\right)_X^* (-p)\} \\ &= -f^*(\bar{p}) - \left(\sum_{i=1}^{m} \bar{q}_i f_i\right)_X^* (-\bar{p}) \\ &= h^*(x^*) - g^*(\bar{p} + x^*) + \sum_{i=1}^{m} \bar{q}_i h_i^*(y_i^*) - \left(\sum_{i=1}^{m} \bar{q}_i g_i\right)_X^* \left(\sum_{i=1}^{m} \bar{q}_i y_i^* - \bar{p}\right) \end{aligned}$$

Hence $v(P_{x^*,y^*}) = v(D_{x^*,y^*})$ and an optimal solution for (D_{x^*,y^*}) is attained at $(\bar{p}, \bar{q})$. □

Given that there is strong duality between (P_{x^*,y^*}) and (D_{x^*,y^*}), it is natural to use this dual problem to define a dual to our original problem, (P_{DC}).

$$(D_{DC}) \qquad \inf_{\substack{x^* \in \mathrm{dom}(h^*) \\ y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)}} \sup_{\substack{p \in \mathbb{R}^n \\ q \geq 0}} \left\{ h^*(x^*) - g^*(p + x^*) + \sum_{i=1}^{m} q_i h_i^*(y_i^*) - \left(\sum_{i=1}^{m} q_i g_i\right)_X^* \left(\sum_{i=1}^{m} q_i y_i^* - p\right) \right\}$$

Because weak duality holds between the subproblem and its dual, weak duality also holds between (P_{DC}) and (D_{DC}). Furthermore, suppose that (CQ_{y^*}) is satisfied for all $y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)$. Then it is clear that $v(P_{DC}) = v(D_{DC})$.

Propostion 6.2. If (CQ_{y^*}) is fulfilled for all $y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)$. Then strong duality holds between (P_{DC}) and (D_{DC}).

Next we consider two possible cases of this problem: the first is when $h_i \equiv 0$, for $i = 1, \ldots, m$ and the second is the case where $h \equiv 0$. If it is true that $h_i \equiv 0$ as well as $h \equiv 0$ then the primal problem becomes the standard convex optimization problem and the Fenchel-Lagrange dual can be applied directly as in Section 4.

Case 1, $h_i \equiv 0$: The primal problem now has a DC objective function and convex constraints,

$$(P_{DC'}) \qquad \inf_{\substack{g_i(x) \leq 0 \\ i=1,\ldots m, \\ x \in X}} \{g(x) - h(x)\}$$

where $g, h : \mathbb{R}^n \to \overline{\mathbb{R}}$, $g_i : \mathbb{R}^n \to \overline{\mathbb{R}}, i = 1, \ldots, m$ are proper and convex functions, and X is a nonempty convex subset of $\mathbb{R}^n$. Suppose the assumptions made above hold. Then h is proper, convex and lower semicontinuous and hence $h(x) = \sup_{x^* \in \mathrm{dom}(h^*)} \{x^{*T}x - h^*(x^*)\}$. Therefore,

$$(P_{DC'}) \qquad \inf_{x^* \in \mathrm{dom}(h^*)} \inf_{\substack{g_i(x) \leq 0 \\ i=1,\ldots m, \\ x \in X}} \{g(x) - x^{*T}x + h^*(x^*)\}$$

As before, the inner infimum is a convex optimization problem. Fixing $x^* \in \operatorname{dom}(h^*)$ and letting $f(x) = g(x) - x^{*T}x + h^*(x^*)$, we evaluate the problem,

$$(P_{x^*}) \qquad \inf_{\substack{g_i(x)\le 0\\ i=1,\dots,m,\\ x\in X}} \{f(x)\}$$

For a $p \in \mathbb{R}^n$ and a $q \geq 0$ in $\mathbb{R}^m$, the Fenchel-Lagrange dual of (P_{x^*}) is

$$(D_{x^*}) \qquad \sup_{\substack{p\in\mathbb{R}^n\\ q\ge 0}} \{-f^*(p) - \left(\sum_{i=1}^m q_i g_i\right)_X^* (-p)\}$$

Since $f^*(p) = g^*(p + x^*) - h^*(x^*)$, the dual problem becomes

$$(D_{x^*}) \qquad \sup_{\substack{p\in\mathbb{R}^n\\ q\ge 0}} \{h^*(x^*) - g^*(p + x^*) - \left(\sum_{i=1}^m q_i g_i\right)_X^* (-p)\}$$

Weak duality holds between (P_{x^*}) and (D_{x^*}). For strong duality we must rewrite the constraint qualification as

$$(CQ_0) \qquad \exists x' \in \bigcap_{i=1}^m \operatorname{ri}(\operatorname{dom}(g_i)) \cap \operatorname{ri}(\operatorname{dom}(g)) \cap \operatorname{ri}(X) : \begin{cases} g_i(x') \le 0 & i \in L \\ g_i(x') < 0 & i \in N \end{cases}$$

where L and N are defined the same as above.

The dual problem to $(P_{DC'})$ is

$$(D_{DC'}) \qquad \inf_{x^*\in\operatorname{dom}(h^*)} \sup_{\substack{p\in\mathbb{R}^n\\ q\ge 0}} \{h^*(x^*) - g^*(p + x^*) - \left(\sum_{i=1}^m q_i g_i\right)_X^* (-p)\}$$

Weak duality holds and if (CQ_0) is satisfied then strong duality holds as well.

Another way to look at the problem is to note that if $h_i \equiv 0$ for $i = 1, \dots, m$ then

$$h_i^*(y_i^*) = \begin{cases} 0 & y_i^* = 0 \\ +\infty & y_i^* \neq 0 \end{cases}$$

Thus $\operatorname{dom}(h_i^*) = \{0\}$ for $i = 1, \dots, m$. Recalling the dual problem derived for the original problem,

$$(D_{DC}) \qquad \inf_{\substack{x^*\in\operatorname{dom}(h^*)\\ y^*\in\prod_{i=1}^m \operatorname{dom}(h_i^*)}} \sup_{\substack{p\in\mathbb{R}^n\\ q\ge 0}} \left\{h^*(x^*) - g^*(p + x^*) + \sum_{i=1}^m q_i h_i^*(y_i^*) - \left(\sum_{i=1}^m q_i g_i\right)_X^* \left(\sum_{i=1}^m q_i y_i^* - p\right)\right\}$$

it is clear that since $y^* = (0, \dots, 0)$,

$$(D_{DC}) \qquad \inf_{x^*\in\operatorname{dom}(h^*)} \sup_{\substack{p\in\mathbb{R}^n\\ q\ge 0}} \left\{h^*(x^*) - g^*(p + x^*) + \sum_{i=1}^m q_i(0) - \left(\sum_{i=1}^m q_i g_i\right)_X^* \left(\sum_{i=1}^m q_i(0) - p\right)\right\}$$

which is the dual problem we derived above. Next we consider the case where h is zero.

Case 2, $h \equiv 0$: In this case the primal problem has a convex objective function and DC constraint functions,

$$(P_{DC''}) \qquad \inf_{\substack{g_i(x)-h_i(x)\leq 0\\ i=1,\ldots m,\, x\in X}} \{g(x)\}$$

where $g : \mathbb{R}^n \to \overline{\mathbb{R}}$, $g_i, h_i : \mathbb{R}^n \to \overline{\mathbb{R}}, i = 1, \ldots, m$ are proper and convex functions, and X is a nonempty convex subset of $\mathbb{R}^n$. Again, the original assumptions hold. By Lemma 6.1, the primal problem becomes

$$(P_{DC''}) \qquad \inf_{y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)} \inf_{\substack{g_i(x)-y^{*T}x+h_i^*(y^*)\leq 0\\ i=1,\ldots,m,\, x\in X}} \{g(x)\}$$

Treating the inner infimum as a separate convex optimization problem, we fix $y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)$ and evaluate the problem,

$$(P_{y^*}) \qquad \inf_{\substack{g_i(x)-y^{*T}x+h_i^*(y^*)\leq 0\\ i=1,\ldots,m,\, x\in X}} \{g(x)\}$$

or equivalently,

$$(P_{y^*}) \qquad \inf_{\substack{f_i(x)\leq 0\\ i=1,\ldots,m,\\ x\in X}} \{g(x)\}$$

where $f_i(x) = g_i(x) - y^{*T}x + h_i^*(y^*)$. For a $p \in \mathbb{R}^n$ and a $q \geq 0$ in $\mathbb{R}^m$, the Fenchel-Lagrange dual of (P_{y^*}) is

$$(D_{y^*}) \qquad \sup_{\substack{p\in\mathbb{R}^n\\ q\geq 0}} \{-g^*(p) - \left(\sum_{i=1}^{m} q_i f_i\right)_X^* (-p)\}$$

Since

$$\left(\sum_{i=1}^{m} q_i f_i\right)_X^* (-p) = \left(\sum_{i=1}^{m} q_i g_i\right)_X^* \left(\sum_{i=1}^{m} q_i y_i^* - p\right) - \sum_{i=1}^{m} q_i h_i^*(y_i^*)$$

the dual problem becomes

$$(D_{y^*}) \qquad \sup_{\substack{p\in\mathbb{R}^n\\ q\geq 0}} \{-g^*(p) - \left(\sum_{i=1}^{m} q_i g_i\right)_X^* \left(\sum_{i=1}^{m} q_i y_i^* - p\right) + \sum_{i=1}^{m} q_i h_i^*(y_i^*)\}$$

Weak duality holds between (P_{y^*}) and (D_{y^*}) and strong duality holds under the original constraint qualification (CQ_{y^*}). From (D_{y^*}) we get the dual problem to $(P_{DC''})$,

$$(D_{DC''}) \qquad \inf_{y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)} \sup_{\substack{p\in\mathbb{R}^n\\ q\geq 0}} \{-g^*(p) - \left(\sum_{i=1}^{m} q_i g_i\right)_X^* \left(\sum_{i=1}^{m} q_i y_i^* - p\right) + \sum_{i=1}^{m} q_i h_i^*(y_i^*)\}$$

We can also look at the original dual problem directly. If $h \equiv 0$ then $\mathrm{dom}(h^*) = \{0\}$. Thus the dual problem becomes,

$$(D_{DC}) \qquad \inf_{y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)} \sup_{\substack{p\in\mathbb{R}^n\\ q\geq 0}} \left\{h^*(0) - g^*(p+0) + \sum_{i=1}^{m} q_i h_i^*(y_i^*) - \left(\sum_{i=1}^{m} q_i g_i\right)_X^* \left(\sum_{i=1}^{m} q_i y_i^* - p\right)\right\}$$

which is $(D_{DC''})$ found above. Again, weak duality holds and strong duality holds under (CQ_{y^*}).

The method developed in this problem will be used to find a dual problem to a few more optimization problems. The next problem is more complex, but can be rewritten in order to apply our process of finding a dual problem.

6.2 DC fractional programming with DC constraints

Consider the following fractional programming problem with DC functions as presented in [13],

$$(P^0_{FP}) \qquad \inf_{\substack{\phi_i(x)-\psi_i(x)\leq 0\\ i=1,\ldots,m,\, x\in X}} \left\{ \frac{g(x)-h(x)}{u(x)-v(x)} \right\}$$

where $X \subseteq \mathbb{R}^n$ is convex, g and h are proper and convex, u and v are proper and concave and ϕ_i and ψ_i are proper and convex for $i=1,\ldots,m$. Further assume that $f-g$ and $-u+v$ are proper, that $u-v$ is positive in the feasible region $\mathcal{F}(P^0_{FP}) = \{x \in X \mid \phi_i(x) - \psi_i(x) \leq 0,\, i=1,\ldots,m,\}$, that ψ_i are subdifferentiable on $\mathcal{F}(P^0_{FP})$ for $i=1,\ldots,m$, and that

$$\bigcap_{i=1}^{m} (\phi_i - \psi)^{-1}(-\mathbb{R}_+) \cap X \cap \operatorname{dom}(g-h) \neq \emptyset$$

where $(\phi_i - \psi)^{-1}(-\mathbb{R}_+) = \{x \in \mathbb{R}^n \mid (\phi_i - \psi)(x) \leq 0\}$, so that $\mathcal{F}(P^0_{FP}) \neq \emptyset$.

We need to convert (P^0_{FP}) via the well-know Dinkelbach transformation, in [9], and then derive a form of the problem to which Fenchel-Lagrange duality can be applied. The Dinkelback transformation leads to the following problem:

$$(P_{FP}) \qquad \inf_{\substack{\phi_i(x)-\psi_i(x)\leq 0\\ i=1,\ldots,m,\, x\in X}} \{g(x)-h(x)-\lambda u(x)+\lambda v(x)\}$$

for some $\lambda \in \mathbb{R}$. The connection between these two optimization problems is described in the following result:

Lemma 6.2. $v(P^0_{FP}) \geq \lambda \Leftrightarrow v(P_{FP}) \geq 0$

Proof. Suppose $v(P^0_{FP}) \geq \lambda$. Then

$$\frac{g(x)-h(x)}{u(x)-v(x)} \geq \inf_{\substack{\phi_i(x)-\psi_i(x)\leq 0\\ i=1,\ldots,m, x\in X}} \left\{ \frac{g(x)-h(x)}{u(x)-v(x)} \right\} \geq \lambda$$

which implies that $g(x)-h(x) \geq \lambda u(x) - \lambda v(x)$. Subtracting the right-hand side and taking the infimum gives

$$\inf_{\substack{\phi_i(x)-\psi_i(x)\leq 0\\ i=1,\ldots,m,\, x\in X}} \{g(x)-h(x)-\lambda u(x)+\lambda v(x)\} \geq 0$$

The other direction is done similarly. Let $v(P_{FP}) \geq 0$. Then,

$$g(x)-h(x)-\lambda u(x)+\lambda v(x) \geq \inf_{\substack{\phi_i(x)-\psi_i(x)\leq 0\\ i=1,\ldots,m,\, x\in X}} \{g(x)-h(x)-\lambda u(x)+\lambda v(x)\} \geq 0$$

and hence $\frac{g(x)-h(x)}{u(x)-v(x)} \geq \lambda$. Again, taking the infimum leads to the desired result. □

Using Lemma 6.2, we can focus the primal problem (P_{FP}). We must explore two separate cases: $\lambda < 0$ and $\lambda \geq 0$. For negative λ, (P_{FP}) can be seen as a DC programming problem by noticing that $g+\lambda v$ and $h+\lambda u$ are convex. In the other case, $g-\lambda u$ and $h-\lambda v$ are convex, again making the objective function DC.

Case 1, $\lambda \geq 0$: In this case, we also assume that

$$\bigcap_{i=1}^{m} \operatorname{ri}(\operatorname{dom}(\phi_i)) \cap \operatorname{ri}(\operatorname{dom}(g)) \cap \operatorname{ri}(\operatorname{dom}(-u)) \cap \operatorname{ri}(X) \neq \emptyset$$

and that $h-\lambda v$ is lower semicontinuous on the feasible set. Define $G(x) := g(x) - \lambda u(x)$ and $H(x) := h(x) - \lambda v(x)$. Then (P_{FP}) becomes

$$(P_{FP}) \qquad \inf_{\substack{\phi_i(x)-\psi_i(x)\leq 0\\ i=1,\ldots,m,\, x\in X}} \{G(x)-H(x)\}$$

Note that this is the exact form of the first DC problem above and because of our assumptions, we can use the results from 6.2. This leads to the dual problem of (P_{FP}),

$$(D_{FP}) \qquad \inf_{\substack{x^*\in\mathrm{dom}(H^*)\\ y^*\in\prod_{i=1}^m \mathrm{dom}(\psi_i^*)}} \sup_{\substack{p\in\mathbb{R}^n\\ q\geq 0}} \left\{ H^*(x^*) - G^*(p+x^*) + \sum_{i=1}^m q_i\psi_i^*(y_i^*) - \left(\sum_{i=1}^m q_i\phi_i\right)_X^* \left(\sum_{i=1}^m q_i y_i^* - p\right)\right\}$$

or equivalently,

$$(D_{FP}) \qquad \inf_{\substack{x^*\in\mathrm{dom}((h-\lambda v)^*)\\ y^*\in\prod_{i=1}^m \mathrm{dom}(\psi_i^*)}} \sup_{\substack{p\in\mathbb{R}^n\\ q\geq 0}} \left\{ (h-\lambda v)^*(x^*) - (g-\lambda u)^*(p+x^*) + \sum_{i=1}^m q_i\psi_i^*(y_i^*) \right.$$
$$\left. - \left(\sum_{i=1}^m q_i\phi_i\right)_X^* \left(\sum_{i=1}^m q_i y_i^* - p\right)\right\}$$

By our assumption, $\mathrm{ri}(\mathrm{dom}(g)) \cap \mathrm{ri}(\mathrm{dom}(-u)) \neq \emptyset$. Hence by Lemma 3.1 and the fact that for a function $f(x)$,

$$(\lambda f)^*(p^*) = \sup\{p^{*T}x - \lambda f(x)\} = \lambda\sup\{\frac{1}{\lambda}p^{*T}x - f(x)\} = \lambda(f^*)(\frac{1}{\lambda}p^*)$$

we can rewrite $(g-\lambda u)^*(p+x^*)$ as

$$\begin{aligned}(g-\lambda u)^*(p+x^*) &= \inf_{p_1+p_2=p+x^*}\{g^*(p_1) + (-\lambda u)^*(p_2)\}\\ &= \inf_{p_1+\lambda p_2=p+x^*}\{g^*(p_1) + \lambda(-u)^*(p_2)\}\end{aligned}$$

Putting this into the dual problem changes it to,

$$(D_{FP}) \qquad \inf_{\substack{x^*\in\mathrm{dom}((h-\lambda v)^*)\\ y^*\in\prod_{i=1}^m \mathrm{dom}(\psi_i^*)}} \sup_{\substack{p_1,p_2\in\mathbb{R}^n\\ q\geq 0}} \left\{ (h-\lambda v)^*(x^*) - g^*(p_1) - \lambda(-u)^*(p_2) + \sum_{i=1}^m q_i\psi_i^*(y_i^*) \right.$$
$$\left. - \left(\sum_{i=1}^m q_i\phi_i\right)_X^* \left(\sum_{i=1}^m q_i y_i^* - p_1 - \lambda p_2 + x^*\right)\right\}$$

As before, weak duality holds. For strong duality we have the following constraint qualification,

$$(CQ'_{y^*}) \qquad \exists x' \in \bigcap_{i=1}^m \mathrm{ri}(\mathrm{dom}(\phi_i)) \cap \mathrm{ri}(\mathrm{dom}(g)) \cap \mathrm{ri}(\mathrm{dom}(-u)) \cap \mathrm{ri}(X), \text{ such that}$$
$$\begin{cases}\phi_i(x') - x'^T y_i^* + \psi_i^*(y_i^*) \leq 0 & i \in L\\ \phi_i(x') - x'^T y_i^* + \psi_i^*(y_i^*) < 0 & i\in N\end{cases}$$

where $L = \{i \in \{1,\ldots,m\} \,|\, \phi_i \text{ is affine}\}$ and $N = \{1,\ldots,m\}\setminus L$.

Propostion 6.3. Suppose (CQ'_{y^*}) is satisfied for all $y^* \in \prod_{i=1}^m \mathrm{dom}(\psi_i^*)$. Then strong duality holds between (P_{FP}) and (D_{FP}).

We leave out the proofs for the fractional programming problems, as they are very similar to the proofs in 6.1. Interested readers may look at [5], [13], and [14] for a proofs and a more detailed analysis of problems from 6.2 and 6.3.

Next we explore the case where λ is negative.

Case 2, $\lambda < 0$**:** Assume for this case that

$$\bigcap_{i=1}^{m} \mathrm{ri}(\mathrm{dom}(\phi_i)) \cap \mathrm{ri}(\mathrm{dom}(g)) \cap \mathrm{ri}(\mathrm{dom}(-v)) \cap \mathrm{ri}(X) \neq \emptyset$$

and that $h + \lambda u$ is lower semicontinuous on the feasible set. Define $\bar{G}(x) := g(x) + \lambda v(x)$ and $\bar{H}(x) := h(x) + \lambda u(x)$. Using the results from the first case the primal problem

$$(\bar{P}_{FP}) \qquad \inf_{\substack{\phi_i(x)-\psi_i(x)\leq 0 \\ i=1,\ldots,m,\, x\in X}} \left\{ \bar{G}(x) - \bar{H}(x) \right\}$$

leads to the dual problem

$$(\bar{D}_{FP}) \qquad \inf_{\substack{x^*\in\mathrm{dom}((h+\lambda u)^*) \\ y^*\in \prod_{i=1}^{m}\mathrm{dom}(\psi_i^*)}} \sup_{\substack{p_1,p_2\in\mathbb{R}^n \\ q\geq 0}} \left\{ (h{+}\lambda u)^*(x^*) - g^*(p_1) + \lambda(-v)^*(p_2) + \sum_{i=1}^{m} q_i\psi_i^*(y_i^*) \right.$$
$$\left. - \left(\sum_{i=1}^{m} q_i\phi_i\right)_X^* \left(\sum_{i=1}^{m} q_i y_i^* - p_1 + \lambda p_2 + x^*\right) \right\}$$

Weak duality holds and for strong duality we have the constraint qualification,

$$(CQ''_{y^*}) \qquad \exists x' \in \bigcap_{i=1}^{m} \mathrm{ri}(\mathrm{dom}(\phi_i)) \cap \mathrm{ri}(\mathrm{dom}(g)) \cap \mathrm{ri}(\mathrm{dom}(-v)) \cap \mathrm{ri}(X), \text{ such that}$$
$$\begin{cases} \phi_i(x') - x'^T y_i^* + \psi_i^*(y_i^*) \leq 0 & i \in L \\ \phi_i(x') - x'^T y_i^* + \psi_i^*(y_i^*) < 0 & i \in N \end{cases}$$

where $L = \{i \in \{1,\ldots,m\} \mid \phi_i \text{ is affine}\}$ and $N = \{1,\ldots,m\} \setminus L$.

Propostion 6.4. Suppose (CQ''_{y^*}) is satisfied for all $y^* \in \prod_{i=1}^{m}\mathrm{dom}(\psi_i^*)$. Then strong duality holds between $(\bar{P}_{FP})$ and $(\bar{D}_{FP})$.

There are many problems that are special cases of (P^0_{FP}) and the following couple of problems will discuss these. One such problem is looked at in [14]. Consider a fractional programming problem with DC objective functions and convex constraints,

$$(P^0_{FP'}) \qquad \inf_{\substack{\phi_i(x)\leq 0 \\ i=1,\ldots,m, \\ x\in X}} \left\{ \frac{g(x) - h(x)}{u(x) - v(x)} \right\}$$

where $X \subseteq \mathbb{R}^n$ is convex, g, h, and ϕ_i, $i = 1,\ldots,m$, are proper and convex, and u and v are proper. Further assume that $f - g$ and $-u + v$ are proper, that $u - v$ is positive in the feasible region $\mathcal{F}(P^0_{FP'}) = \{x \in X \mid \phi_i(x) \leq 0,\, i = 1,\ldots,m,\}$, and that

$$\bigcap_{i=1}^{m} (\phi_i)^{-1}(-\mathbb{R}_+) \cap X \cap \mathrm{dom}(g - h) \neq \emptyset$$

where $\phi_i^{-1}(-\mathbb{R}_+) = \{x \in \mathbb{R}^n \mid \phi_i(x) \leq 0\}$, so that $\mathcal{F}(P^0_{FP}) \neq \emptyset$. By the Dinkelbach transformation, we get the dual problem,

$$(P_{FP'}) \qquad \inf_{\substack{\phi_i(x)\leq 0 \\ i=1,\ldots,m, \\ x\in X}} \{g(x) - h(x) - \lambda u(x) + \lambda v(x)\}$$

for some $\lambda \in \mathbb{R}$. By Lemma 6.2 it follows that,

Lemma 6.3. $v(P^0_{FP'}) \geq \lambda \Leftrightarrow v(P_{FP'}) \geq 0$

As before, there are two cases, $\lambda \geq 0$ and $\lambda < 0$.

Case 1, $\lambda \geq 0$: For this case, assume that

$$\bigcap_{i=1}^{m} \mathrm{ri}(\mathrm{dom}(\phi_i)) \cap \mathrm{ri}(\mathrm{dom}(g)) \cap \mathrm{ri}(\mathrm{dom}(-u)) \cap \mathrm{ri}(X) \neq \emptyset$$

and that $h - \lambda v$ is lower semicontinuous on the feasible set. Defining $G(x) := g(x) - \lambda u(x)$ and $H(x) := h(x) - \lambda v(x)$ changes the primal problem to

$$(P_{FP'}) \qquad \inf_{\substack{\phi_i(x) \leq 0 \\ i=1,\ldots,m, \\ x \in X}} \{G(x) - H(x)\}$$

Now the problem is in the form of one of the subcases of the first DC problem, $(P_{DC'})$. Therefore, with the results of that case, the dual problem to $(P_{FP'})$,

$$(D_{FP'}) \qquad \inf_{x^* \in \mathrm{dom}(H^*)} \sup_{\substack{p \in \mathbb{R}^n \\ q \geq 0}} \{H^*(x^*) - G^*(p + x^*) - \left(\sum_{i=1}^{m} q_i \phi_i\right)_X^* (-p)\}$$

Because $\mathrm{ri}(\mathrm{dom}(g)) \cap \mathrm{ri}(\mathrm{dom}(-u)) \neq 0$, $G^*(p + x^*) = \inf_{p_1 + \lambda p_2 = p + x^*} \{g^*(p_1) + \lambda(-u)^*(p_2)\}$. Then the dual problem becomes,

$$\begin{aligned}(D_{FP'}) \qquad \inf_{x^* \in \mathrm{dom}((h - \lambda v)^*)} \sup_{\substack{p_1, p_2 \in \mathbb{R}^n \\ q \geq 0}} \{&(h - \lambda v)^*(x^*) - g^*(p_1) - \lambda(-u)^*(p_2) \\ &- \left(\sum_{i=1}^{m} q_i \phi_i\right)_X^* (x^* - p_1 - \lambda p_2)\}\end{aligned}$$

Weak duality holds and for strong duality we have the following constraint qualification,

$$(CQ'_0) \qquad \exists x' \in \bigcap_{i=1}^{m} \mathrm{ri}(\mathrm{dom}(\phi_i)) \cap \mathrm{ri}(\mathrm{dom}(g)) \cap \mathrm{ri}(\mathrm{dom}(-u)) \cap \mathrm{ri}(X), \text{ such that}$$
$$\begin{cases} \phi_i(x') \leq 0 & i \in L \\ \phi_i(x') < 0 & i \in N \end{cases}$$

where $L = \{i \in \{1, \ldots, m\} \mid \phi_i \text{ is affine}\}$ and $N = \{1, \ldots, m\} \setminus L$.

Propostion 6.5. Suppose (CQ'_0) is satisfied. Then strong duality holds between $(P_{FP'})$ and $(D_{FP'})$. That is $v(P_{FP'}) = v(D_{FP'})$.

Case 2, $\lambda < 0$: Assume that

$$\bigcap_{i=1}^{m} \mathrm{ri}(\mathrm{dom}(\phi_i)) \cap \mathrm{ri}(\mathrm{dom}(g)) \cap \mathrm{ri}(\mathrm{dom}(-v)) \cap \mathrm{ri}(X) \neq \emptyset$$

and that $h + \lambda u$ is lower semicontinuous on the feasible set. Define $\tilde{G}(x) := g(x) + \lambda v(x)$ and $\tilde{H}(x) := h(x) + \lambda u(x)$. Like the last case, the primal problem becomes,

$$(\tilde{P}_{FP'}) \qquad \inf_{\substack{\phi_i(x) \leq 0 \\ i=1,\ldots,m, \\ x \in X}} \{\tilde{G}(x) - \tilde{H}(x)\}$$

By the results from problem $(P_{DC'})$, the dual problem to $(\tilde{P}_{FP'})$,

$$(\tilde{D}_{FP'}) \qquad \inf_{x^* \in \mathrm{dom}(\tilde{H}^*)} \sup_{\substack{p \in \mathbb{R}^n \\ q \geq 0}} \{\tilde{H}^*(x^*) - \tilde{G}^*(p + x^*) - \left(\sum_{i=1}^{m} q_i \phi_i\right)_X^* (-p)\}$$

or more precisely,

$$(\tilde{D}_{FP'}) \qquad \inf_{x^* \in \mathrm{dom}((h+\lambda u)^*)} \sup_{\substack{p_1,p_2 \in \mathbb{R}^n \\ q \geq 0}} \{(h+\lambda u)^*(x^*) - g^*(p_1) + \lambda(-v)^*(p_2) - \left(\sum_{i=1}^m q_i \phi_i\right)^*_X (x^* - p_1 + \lambda p_2)\}$$

Weak duality holds as does strong duality under the following constraint qualification,

$$(\tilde{CQ}_0') \qquad \exists x' \in \bigcap_{i=1}^m \mathrm{ri}(\mathrm{dom}(\phi_i)) \cap \mathrm{ri}(\mathrm{dom}(g)) \cap \mathrm{ri}(\mathrm{dom}(-v)) \cap \mathrm{ri}(X), \text{ such that } \begin{cases} \phi_i(x') \leq 0 & i \in L \\ \phi_i(x') < 0 & i \in N \end{cases}$$

where $L = \{i \in \{1, \dots, m\} \,|\, \phi_i \text{ is affine}\}$ and $N = \{1, \dots, m\} \setminus L$.

Propostion 6.6. Suppose $(\tilde{CQ}_0')$ is satisfied. Then strong duality holds between $(\tilde{P}_{FP'})$ and $(\tilde{D}_{FP'})$.

6.3 Fractional programming problem

In [5], Boţ looks at a fractional programming problem with convex constraints. Via the Dinkelbach transformation he turns it into a DC programming problem and then applies a similar method as what has been done so far to find its dual problem. Here, we will first address a similar problem, but with DC constraint functions and then look at the other primal problem from [5] as a special case of it. Thus, consider the fractional programming problem,

$$(P^0_{FP_0}) \qquad \inf_{\substack{g_i(x) - h_i(x) \leq 0 \\ i=1,\dots,m,\, x \in X}} \left\{ \frac{g(x)}{h(x)} \right\}$$

where $X \subseteq \mathbb{R}^n$ is nonempty and convex, $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ is proper and convex, $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ is concave, proper and lower semicontinuous over $\mathcal{F}(P^0_{FP_0})$ (the feasible set of the problem), and $g_i, h_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \dots, m$, are proper and convex functions such that

$$\bigcap_{i=1}^m (g_i - h_i)(-\mathbb{R}_+) \cap \mathrm{dom}(g) \cap X \neq \emptyset$$

where $(g_i - h_i)(-\mathbb{R}_+) = \{x \in \mathbb{R}^n \,|\, (g_i - h_i)(x) \leq 0\}$. Moreover, assume that $h(x) > 0$ for all feasible x to the primal problem, that h_i is subdifferentiable on $\mathcal{F}(P^0_{FP_0})$, and that

$$\bigcap_{i=1}^m \mathrm{ri}(\mathrm{dom}(g_i)) \cap \mathrm{ri}(\mathrm{dom}(g)) \cap \mathrm{ri}(\mathrm{dom}(-h)) \cap \mathrm{ri}(X)$$

As mentioned above, we use the Dinkelbach transformation to get the optimization problem,

$$(P_{FP_0}) \qquad \inf_{\substack{g_i(x) - h_i(x) \leq 0 \\ i=1\dots,m,\, x \in X}} \{g(x) - \lambda h(x)\}$$

where λ is an arbitrary real number. then we have the following lemma.

Lemma 6.4.

$$v(P^0_{FP_0}) \geq \lambda \Leftrightarrow v(P_{FP_0}) \geq 0$$

There are two cases to consider. If $\lambda \geq 0$, then (P_{FP_0}) has a convex objective function with DC constraint functions. On the other hand, if λ is negative, then the objective function is also DC. We start with $\lambda \geq 0$.

Case 1, $\lambda \geq 0$: As mentioned, the constraints of the primal problem are DC functions. Thus by Lemma 6.1, the primal problem can be written as,

$$(P_{FP_0}) \qquad \inf_{y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)} \inf_{\substack{g_i(x) - {y^*}^T x + h_i^*(y^*) \leq 0 \\ i=1,\ldots,m,\, x \in X}} \{g(x) - \lambda h(x)\}$$

Looking at the inner infimum separately, we have the convex optimization problem,

$$(P_{y^*}) \qquad \inf_{\substack{g_i(x) - {y^*}^T x + h_i^*(y^*) \leq 0 \\ i=1,\ldots,m, x \in X}} \{g(x) - \lambda h(x)\}$$

for a fixed $y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)$. Let $f(x) = g(x) - \lambda h(x)$ and $f_i(x) = g_i(x) - {y^*}^T x + h_i^*(y^*)$. Then the convex optimization problem becomes

$$(P_{y^*}) \qquad \inf_{\substack{f_i(x) \leq 0 \\ i=1,\ldots,m, \\ x \in X}} \{f(x)\}$$

The Fenchel-Lagrange dual of this is,

$$(D_{y^*}) \qquad \sup_{\substack{p \in \mathbb{R}^n \\ q \geq 0}} \{-f^*(p) - \left(\sum_{i=1}^{m} q_i f_i\right)_X^* (-p)\}$$

where $q \in \mathbb{R}^n$. Since

$$-f^*(p) = -\sup\{p^T - (g - \lambda h)(x)\} = \sup\{-p^T x - (\lambda h - g)(x)\} = (\lambda h - g)^*(-p)$$

and

$$\left(\sum_{i=1}^{m} q_i f_i\right)_X^* (-p) = \left(\sum_{i=1}^{m} q_i g_i\right)_X^* \left(\sum_{i=1}^{m} q_i y_i^* - p\right) - \sum_{i=1}^{m} q_i h_i^*(y_i^*)$$

which makes the dual problem,

$$(D_{y^*}) \qquad \sup_{\substack{p \in \mathbb{R}^n \\ q \geq 0}} \{(\lambda h - g)^*(-p) - \left(\sum_{i=1}^{m} q_i g_i\right)_X^* \left(\sum_{i=1}^{m} q_i y_i^* - p\right) + \sum_{i=1}^{m} q_i h_i^*(y_i^*)\}$$

Notice that (D_{y^*}) can be slightly altered if we recall that $\mathrm{ri}(\mathrm{dom}(g)) \cap \mathrm{ri}(\mathrm{dom}(-h)) \cap \mathrm{ri}(X) \neq \emptyset$. Then by Lemma 3.1,

$$\begin{aligned} f^*(p) &= \inf_{p_1 + p_2} \{g^*(p_1) + (-\lambda h)^*(p_2)\} \\ &= \inf_{p_1 + \lambda p_2} \{g^*(p_1) + \lambda(-h)^*(p_2)\} \end{aligned}$$

The dual subproblem now becomes,

$$(D_{y^*}) \qquad \sup_{\substack{p_1, p_2 \in \mathbb{R}^n \\ q \geq 0}} \{-g^*(p_1) - \lambda(-h)^*(p_2) - \left(\sum_{i=1}^{m} q_i g_i\right)_X^* \left(\sum_{i=1}^{m} q_i y_i^* - p_1 - \lambda p_2\right) + \sum_{i=1}^{m} q_i h_i^*(y_i^*)\}$$

Reconsidering the fractional programming problem, (P_{FP_0}). The outer infimum can be attached to the dual problem (D_{y^*}) to give,

$$\begin{aligned} (D_{FP_0}) \qquad \inf_{y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)} \sup_{\substack{p_1, p_2 \in \mathbb{R}^n \\ q \geq 0}} \{ &- g^*(p_1) - \lambda(-h)^*(p_2) \\ &- \left(\sum_{i=1}^{m} q_i g_i\right)_X^* \left(\sum_{i=1}^{m} q_i y_i^* - p_1 - \lambda p_2\right) + \sum_{i=1}^{m} q_i h_i^*(y_i^*)\} \end{aligned}$$

The constraint qualification needed for strong duality is,

$$(\hat{CQ}_{y^*}) \qquad \exists x' \in \bigcap_{i=1}^{m} \mathrm{ri}(\mathrm{dom}(g_i)) \cap \mathrm{ri}(\mathrm{dom}(g)) \cap \mathrm{ri}(\mathrm{dom}(-h)) \cap \mathrm{ri}(X) : \\ \begin{cases} g_i(x') - x'^T y_i^* + h_i^*(y_i^*) \le 0 & i \in L \\ g_i(x') - x'^T y_i^* + h_i^*(y_i^*) < 0 & i \in N \end{cases}$$

where $L = \{i \in \{1, \dots, m\} \,|\, g_i \text{ is affine}\}$ and $N = \{1, \dots, m\} \setminus L$.

Propostion 6.7. Let $\lambda \ge 0$ and suppose $(\hat{CQ}_{y^*})$ is satisfied. Then strong duality holds between (P_{FP_0}) and (D_{FP_0}), i.e. $v(P_{FP_0}) = v(D_{FP_0})$.

Case 2, $\lambda < 0$: In this case, the problem has DC functions for constraints and for the objective function. Using the results from problem (P_{DC}), we get that the dual problem to (P_{FP_0}) when $\lambda < 0$ is,

$$(\bar{D}_{FP_0}) \qquad \inf_{\substack{z^* \in \mathrm{dom}((\lambda h)^*) \\ y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)}} \sup_{\substack{p \in \mathbb{R}^n \\ q \ge 0}} \{(\lambda h)^*(z^*) - g^*(p + z^*) - \left(\sum_{i=1}^{m} q_i g_i\right)_X^* \left(\sum_{i=1}^{m} q_i y_i^* - p\right) + \sum_{i=1}^{m} q_i h_i^*(y_i^*)\}$$

or if we set $x^* = \frac{1}{\lambda} z^*$,

$$(\bar{D}_{FP_0}) \qquad \inf_{\substack{x^* \in \mathrm{dom}(h^*) \\ y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)}} \sup_{\substack{p \in \mathbb{R}^n \\ q \ge 0}} \{\lambda h^*(x^*) - g^*(p + \lambda x^*) - \left(\sum_{i=1}^{m} q_i g_i\right)_X^* \left(\sum_{i=1}^{m} q_i y_i^* - p\right) + \sum_{i=1}^{m} q_i h_i^*(y_i^*)\}$$

As for all the problems in this paper, weak duality holds, i.e, $v(P_{FP_0}) \ge v(\bar{D}_{FP_0})$. For strong duality, we adjust our constraint qualification,

$$(CQ_{y^*}) \qquad \exists x' \in \bigcap_{i=1}^{m} \mathrm{ri}(\mathrm{dom}(g_i)) \cap \mathrm{ri}(\mathrm{dom}(g)) \cap \mathrm{ri}(X) : \begin{cases} g_i(x') - x'^T y_i^* + h_i^*(y_i^*) \le 0 & i \in L \\ g_i(x') - x'^T y_i^* + h_i^*(y_i^*) < 0 & i \in N \end{cases}$$

where $L = \{i \in \{1, \dots, m\} \,|\, g_i \text{ is affine}\}$ and $N = \{1, \dots, m\} \setminus L$.

Propostion 6.8. Suppose $\lambda < 0$ and (CQ_{y^*}) is satisfied. Then strong duality holds between (P_{FP_0}) and $(\bar{D}_{FP_0})$, i.e. $v(P_{FP_0}) = v(\bar{D}_{FP_0})$.

In [5], Boţ looks at the following fraction programming problem,

$$(P^0_{FP'_0}) \qquad \inf_{\substack{g_i(x) \le 0 \\ i = 1, \dots, m \\ x \in X}} \left\{ \frac{g(x)}{h(x)} \right\}$$

where $X \subseteq \mathbb{R}^n$ is nonempty and convex, $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ is proper and convex, $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ is concave such that $-h$ is proper and lower semicontinuous over $\mathcal{F}(P_{FP'_0})$ (the feasible set of the problem), and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, m$, are convex functions such that

$$\bigcap_{i=1}^{m} (g_i)^{-1}(-\mathbb{R}_+) \cap X \cap \mathrm{dom}(g) \neq \emptyset$$

where $(g_i)^{-1}(-\mathbb{R}_+) = \{x \in \mathbb{R}^n \,|\, g_i(x) \le 0\}$. He further assumes that $h(x) > 0$ for all feasible points of $(P^0_{FP'_0})$.

This is clearly just the problem $(P^0_{FP_0})$ where $h_i \equiv 0$ for $i = 1, \dots, m$. Thus, following the same method as above, the given problem can be associated with another primal problem using the Dinkelbach transformation and is justified by Lemma 6.2. Hence define,

$$(P_{FP'_0}) \qquad \inf_{\substack{g_i(x)\leq 0\\ i=1,\dots,m,\\ x\in X}} \{g(x) - \lambda h(x)\}$$

where λ is an arbitrary real number.

Unlike before, we will only look at one case, since for $\lambda \geq 0$, $(P_{FP'_0})$ is simply a convex optimization problem. In [5], Boţ deals with this case and also what will be presented here, the case were $\lambda < 0$ and $(P_{FP'_0})$ is a DC programming problem.

By Lemma 3.2, the problem can be rewritten into

$$(P_{FP'_0}) \qquad \inf_{x^*\in\mathrm{dom}((-h)^*)} \inf_{\substack{g_i(x)\leq 0\\ i=1,\dots,m,\\ x\in X}} \{g(x) - \lambda x^{*T}x - \lambda(-h)^*(x^*)\}$$

As in previous problems, notice that the inner infimum is a convex optimization problem. Hence, for a fixed $x^* \in \mathrm{dom}((-h)^*)$, consider

$$(P_{x^*}) \qquad \inf_{\substack{g(x)\leq 0\\ x\in X}} \{g(x) - \lambda(-\hat{h})(x)\}$$

where $\hat{h}(x) = x^{*T}x - (-h)^*(x^*)$. If we further assume that $\mathrm{ri}(\mathrm{dom}(g)) \cap \mathrm{ri}(\mathrm{dom}(-h) \cap \mathrm{ri}(X)$ then the Fenchel-Lagrange dual problem of (P_{x^*}) is

$$(D_{x^*}) \qquad \sup_{\substack{p_1,p_2\in\mathbb{R}^n\\ q\geq 0}} \{-g^*(p_1) + \lambda(-\hat{h})^*(p_2) - \left(\sum_{i=1}^m q_i g_i\right)^*_X (-p_1 + \lambda p_2)\}$$

for $q \in \mathbb{R}^n$. Looking at $\hat{h}$ we get that

$$\begin{aligned}
(-\hat{h})^*(p_2) &= \sup\{p_2^T x - (-x^{*T}x + (-h)^*(x^*))\}\\
&= \sup\{(p_2 + x^{*T})^T x\} - (-h)^*(x^*))\\
&= \begin{cases} -(-h)^*(x^*)) & p_2 = -x^* \\ +\infty & \text{otherwise}\end{cases}
\end{aligned}$$

Hence, letting $p = p_1$ the dual problem becomes,

$$(D_{x^*}) \qquad \sup_{\substack{p\in\mathbb{R}^n\\ q\geq 0}} \{-g^*(p) - \lambda(-h)^*(x^*) - \left(\sum_{i=1}^m q_i g_i\right)^*_X (-p - \lambda x^*)\}$$

Now the dual problem to the original primal problem can be defined as

$$(D_{FP'_0}) \qquad \inf_{x^*\in\mathrm{dom}((-h)^*)} \sup_{\substack{p\in\mathbb{R}^n\\ q\geq 0}} \{-g^*(p) - \lambda(-h)^*(x^*) - \left(\sum_{i=1}^m q_i g_i\right)^*_X (-p - \lambda x^*)\}$$

As before, weak duality holds. For strong duality we present,

$$(CQ'_0) \qquad \exists x' \in \cap\,\mathrm{ri}(\mathrm{dom}(g)) \cap \mathrm{ri}(X) : \begin{cases} g_i(x') \leq 0 & i \in L \\ g_i(x') < 0 & i \in N\end{cases}$$

where $L = \{i \in \{1, \dots, m\} \,|\, g_i \text{ is affine}\}$ and $N = \{1, \dots, m\} \setminus L$.

Propostion 6.9. Suppose (CQ'_0) is satisfied for all $x^* \in \mathrm{dom}((-h)^*)$. Then strong duality holds between (P_{FP}) and (D_{FP}).

6.4 DC programming problem containing a composition with a linear continuous operator

In [10], Fang, Li, and Yang look at a DC programming problem that has two DC functions in the objective function, one of which is composed with a linear continuous operator.

$$(P_A^0) \qquad \inf_{x \in X} \{g_1(x) - g_2(x) + h_1(Ax) - h_2(Ax)\}$$

where $g_1, g_2, h_1, h_2 : \mathbb{R}^n \to \overline{\mathbb{R}}$ are proper convex functions and $A \in \mathbb{R}^{n \times n}$ is linear continuous operator such that $A(\mathrm{dom}(g_1 - g_2)) \cap \mathrm{dom}(h_1 - h_2) \neq \emptyset$. They evaluate it using Fenchel duality (in fact, they use two types of Fenchel dual problems).

Inspired by their work, this section looks at the Fenchel-Lagrange duals of two similar problems. These two problems will be slightly more complex versions of the above primal problem due to the addition of constraints. The first of these simply includes convex constraints,

$$(P_A') \qquad \inf_{\substack{\phi_i(x) \leq 0 \\ i=1,\ldots,m, \\ x \in X}} \{g_1(x) - g_2(x) + h_1(Ax) - h_2(Ax)\}$$

where $g_1, g_2, h_1, h_2, \phi_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, for $i = 1, \ldots, m$, are proper convex functions and $A \in \mathbb{R}^{n \times n}$ is linear continuous operator such that $A(\mathrm{dom}(g_1 - g_2)) \cap \mathrm{dom}(h_1 - h_2) \neq \emptyset$. To reformulate problem (P_A'), define $G(x) = g_1(x) + h_1(Ax)$ and $H(x) = g_2(x) + h_2(Ax)$. Then we get,

$$(P_A') \qquad \inf_{\substack{\phi_i(x) \leq 0 \\ i=1,\ldots,m, \\ x \in X}} \{G(x) - H(x)\}$$

where $G, H : \mathbb{R}^n \to \overline{\mathbb{R}}$ are convex and proper functions. At this point, we assume further that $H(x) = g_2(x) + h_2(Ax)$ is lower semicontinuous and that

$$\bigcap_{i=1}^{m} \mathrm{ri}(\mathrm{dom}(\phi_i)) \cap \mathrm{ri}(\mathrm{dom}(g_1)) \cap \mathrm{ri}(A^{-1}(\mathrm{dom}(h_1))) \cap \mathrm{ri}(X) \neq \emptyset$$

Then we can follow the same method as in the problems from 6.1, i.e., the primal problem can be written as

$$(P_A') \qquad \inf_{x^* \in \mathrm{dom}(H^*)} \inf_{\substack{\phi_i(x) \leq 0 \\ i=1,\ldots,m, \\ x \in X}} \{G(x) - x^{*T}x + H^*(x^*)\}$$

For a fixed $x^* \in \mathrm{dom}(H^*)$, the inner infimum is a convex optimization problem,

$$(P_{x^*}) \qquad \inf_{\substack{\phi_i(x) \leq 0 \\ i=1,\ldots,m, \\ x \in X}} \{G(x) - x^{*T}x + H^*(x^*)\}$$

with Fenchel-Lagrange dual,

$$(D_{x^*}) \qquad \sup_{\substack{p \in \mathbb{R}^n \\ q \geq 0}} \{H^*(x^*) - G(p + x^*) - \left(\sum_{i=1}^{m} q_i \phi_i\right)_X^* (-p)\}$$

Now, since we assumed that $\mathrm{ri}(\mathrm{dom}(g_1)) \cap \mathrm{ri}(A^{-1}(\mathrm{dom}(h_1))) \neq \emptyset$, then by Lemma 3.1,

$$\begin{aligned} G^*(p + x^*) &= \inf_{p_1 + p_2 = p + x^*} \{g_1^*(p_1) + h_1^*(A^{*-1}p_2)\} \\ &= -\sup_{p_1 + p_2 = p + x^*} \{-g_1^*(p_1) - h_1^*(A^{*-1}p_2)\} \end{aligned}$$

Then (D_{x^*}) changes to,

$$(D_{x^*}) \qquad \sup_{\substack{p_1,p_2\in\mathbb{R}^n \\ q\geq 0}} \{(g_2+h_2\circ A)^*(x^*) - g_1^*(p_1) - h_1^*(A^{*-1}p_2) - \left(\sum_{i=1}^m q_i\phi_i\right)^*_X (x^* - p_1 - p_2)\}$$

The dual of the primal problem is,

$$(D'_{A_1}) \qquad \inf_{x^*\in\mathrm{dom}(g_2+h_2\circ A)} \sup_{\substack{p_1,p_2\in\mathbb{R}^n \\ q\geq 0}} \{(g_2+h_2\circ A)^*(x^*) - g_1^*(p_1) - h_1^*(A^{*-1}p_2) - \left(\sum_{i=1}^m q_i\phi_i\right)^*_X (x^* - p_1 - p_2)\}$$

Since weak duality holds between (P_{x^*}) and (D_{x^*}), it also holds for (P'_A) and (D'_{A_1}). To attain strong duality, we need the following constraint qualification,

$$(CQ_0) \qquad \exists x' \in \bigcap_{i=1}^m \mathrm{ri}(\mathrm{dom}(\phi_i)) \cap \mathrm{ri}(\mathrm{dom}(g_1)) \cap \mathrm{ri}(A^{-1}(\mathrm{dom}(h_1))) \cap \mathrm{ri}(X) : \begin{cases} \phi_i(x') \leq 0 & i \in L \\ \phi_i(x') < 0 & i \in N \end{cases}$$

where as before, $L = \{i \in \{1,\ldots,m\} \mid \phi_i \text{ is affine}\}$ and $N = \{1,\ldots,m\} \setminus L$.

Propostion 6.10. Suppose (CQ_0) is satisfied. Then strong duality holds between (P'_A) and (D'_{A_1}).

Proof. To begin, for a fixed $x^* \in \mathrm{dom}(H^*)$, the problem

$$(P_{x^*}) \qquad \inf_{\substack{\phi_i(x)\leq 0 \\ i=1,\ldots,m, \\ x\in X}} \{G(x) - x^{*T}x + H^*(x^*)\}$$

has the Fenchel-Lagrange dual,

$$(D_{x^*}) \qquad \sup_{\substack{p\in\mathbb{R}^n \\ q\geq 0}} \{H^*(x^*) - G(p+x^*) - \left(\sum_{i=1}^m q_i\phi_i\right)^*_X (-p)\}$$

The effective domain of the objective function in the primal problem is $\mathrm{dom}(G) = \mathrm{dom}(g_1) \cap (A^{-1}(\mathrm{dom}(h_1))$. If we let the objective function of (P_{x^*}) be F then the constraint qualification implies that

$$\exists x' \in \bigcap_{i=1}^m \mathrm{ri}(\mathrm{dom}(\phi_i)) \cap \mathrm{ri}(\mathrm{dom}(F)) \cap \mathrm{ri}(X) : \begin{cases} \phi_i(x') \leq 0 & i \in L \\ \phi_i(x') < 0 & i \in N \end{cases}$$

By Theorem 4.3, strong duality holds between (P_{x^*}) and (D_{x^*}). Furthermore, taking the infimum of both problems over x^* proves that strong duality holds between (P'_A) and (D'_{A_1}). □

The problem can be approached in another way. Given the problem

$$(P'_A) \qquad \inf_{\substack{\phi_i(x)\leq 0 \\ i=1,\ldots,m, \\ x\in X}} \{g_1(x) - g_2(x) + h_1(Ax) - h_2(Ax)\}$$

with the above assumptions, further suppose that g_2 and h_2 are lower semicontinuous. Then

$$g_2(x) = x^{*T}x - g_2^*(x^*) \qquad \text{and} \qquad h_2(Ax) = (Ax)^T y^* - h_2^*(y^*)$$

for $x^* \in \mathrm{dom}(g_2^*)$ and $y^* \in \mathrm{dom}(h_2^*)$. Using this the problem becomes,

$$(P'_A) \qquad \inf_{\substack{x^*\in\mathrm{dom}(g_2^*) \\ y^*\in\mathrm{dom}(h_2^*)}} \inf_{\substack{\phi_i(x)\leq 0 \\ i=1,\ldots,m, \\ x\in X}} \{g_1(x) - x^{*T}x + g_2^*(x^*) + h_1(Ax) - (Ax)^T y^* + h_2^*(y^*)\}$$

Fix $x^* \in \mathrm{dom}(g_2^*)$ and $y^* \in \mathrm{dom}(h_2^*)$. The inner infimum is now convex,

$$(P_{x^*,y^*}) \qquad \inf_{\substack{\phi_i(x)\leq 0\\ i=1,\ldots,m,\\ x\in X}} \{g_1(x) - x^{*T}x + g_2^*(x^*) + h_1(Ax) - (Ax)^T y^* + h_2^*(y^*)\}$$

To determine the Fenchel-Lagrange dual of (P_{x^*,y^*}), we must find the (negative) conjugate of its objective function. Thus we want to look at this conjugate,

$$\begin{aligned}
&\sup\{p^T x - (g_1(x) - x^{*T}x + g_2^*(x^*) + h_1(Ax) - (Ax)^T y^* + h_2^*(y^*))\}\\
=&\sup\{p^T x - g_1(x) + x^{*T}x - h_1(Ax) + x^T(A^*y^*)\} - g_2^*(x^*) - h_2^*(y^*)\\
=&\sup\{(p + x^* + A^*y^*)^T x - g_1(x) - h_1(Ax)\} - g_2^*(x^*) - h_2^*(y^*)\\
=&(g_1 + h_1 \circ A)^*(p + x^* + A^*y^*) - g_2^*(x^*) - h_2^*(y^*)
\end{aligned}$$

where A^* is the adjoint of A. From this if follows that the Fenchel-Lagrange dual of (P_{x^*,y^*}) is

$$(D_{x^*,y^*}) \qquad \sup_{\substack{p\in\mathbb{R}^n\\ q\geq 0}} \{-(g_1 + h_1 \circ A)^*(p + x^* + A^*y^*) + g_2^*(x^*) + h_2^*(y^*) - \left(\sum_{i=1}^m q_i\phi_i\right)_X^*(-p)\}$$

Therefore, the dual problem to (P'_A) is,

$$(D'_A) \qquad \inf_{\substack{x^*\in\mathrm{dom}(g_2^*)\\ y^*\in\mathrm{dom}(h_2^*)}} \sup_{\substack{p\in\mathbb{R}^n\\ q\geq 0}} \{-(g_1 + h_1 \circ A)^*(p + x^* + A^*y^*) + g_2^*(x^*) + h_2^*(y^*) - \left(\sum_{i=1}^m q_i\phi_i\right)_X^*(-p)\}$$

Weak duality holds between (P'_A) and (D'_A) and for strong duality, we use the following constraint qualification,

$$(CQ_0) \qquad \exists x' \in \bigcap_{i=1}^m \mathrm{ri}(\mathrm{dom}(\phi_i)) \cap \mathrm{ri}(\mathrm{dom}(g_1)) \cap \mathrm{ri}(A^{-1}(\mathrm{dom}(h_1))) \cap \mathrm{ri}(X) : \begin{cases}\phi_i(x') \leq 0 & i \in L\\ \phi_i(x') < 0 & i \in N\end{cases}$$

where as before, $L = \{i \in \{1,\ldots,m\} \mid \phi_i \text{ is affine}\}$ and $N = \{1,\ldots,m\} \setminus L$.

Propostion 6.11. Suppose (CQ_0) is satisfied. Then strong duality holds between (P'_A) and (D'_A).

Can the problem be further complicated and still yield results? As before, the next problem adds constraints to the unconstrained problem (P_A^0). This time, however, consider the problem with DC constraint functions,

$$(P_A) \qquad \inf_{\substack{\phi_i(x)-\psi_i\leq 0\\ i=1,\ldots,m,\, x\in X}} \{g_1(x) - g_2(x) + h_1(Ax) - h_2(Ax)\}$$

where $g_1, g_2, h_1, h_2, \phi_i, \psi_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, for $i = 1,\ldots,m$, are proper convex functions and $A \in \mathbb{R}^{n\times n}$ is linear continuous operator such that $A(\mathrm{dom}(g_1 - g_2)) \cap \mathrm{dom}(h_1 - h_2) \neq \emptyset$. Again, suppose that g_2 and h_2 are lower semicontinuous.

As done above we can reformulate the problem by using biconjugates. Moreover, we can use the results of Lemma 6.1 and further change the problem to

$$(P_A) \qquad \inf_{\substack{x^*\in\mathrm{dom}(g_2^*)\\ y^*\in\mathrm{dom}(h_2^*)\\ z^*\in\prod_{i=1}^m \mathrm{dom}(\psi_i^*)}} \inf_{\substack{\phi_i(x)-z_i^*x+\psi_i^*(z_i^*)\leq 0\\ i=1,\ldots,m,\, x\in X}} \{g_1(x) - x^{*T}x + g_2^*(x^*) + h_1(Ax) - (Ax)^T y^* + h_2^*(y^*)\}$$

If we fix $x^* \in \mathrm{dom}(g_2^*)$, $y^* \in \mathrm{dom}(h_2^*)$ and $z^* \in \prod_{i=1}^m \mathrm{dom}(\psi_i^*)$, the inner infimum can be evaluated separately as a convex optimization problem,

$$(P_{x^*,y^*,z^*}) \qquad \inf_{\substack{\phi_i(x)-z_i^*x+\psi_i^*(z_i^*)\leq 0\\ i=1,\ldots,m,\, x\in X}} \{g_1(x) - x^{*T}x + g_2^*(x^*) + h_1(Ax) - (Ax)^T y^* + h_2^*(y^*)\}$$

The Fenchel-Lagrange dual to (P_{x^*,y^*,z^*}) is,

$$(D_{x^*,y^*,z^*}) \qquad \sup_{\substack{p\in\mathbb{R}^n\\ q\geq 0}} \{-(g_1 + h_1 \circ A)^*(p + x^* + A^*y^*) + g_2^*(x^*) + h_2^*(y^*) - \left(\sum_{i=1}^m q_i\phi_i\right)_X^* \left(\sum_{i=1}^m q_i z_i^* - p\right) + \sum_{i=1}^m q_i\psi_i^*(z_i^*)\}$$

So to (P_A), we associate the dual problem,

$$(D_A) \qquad \inf_{\substack{x^*\in\mathrm{dom}(g_2^*)\\ y^*\in\mathrm{dom}(h_2^*)\\ z^*\in\prod_{i=1}^m \mathrm{dom}(\psi_i^*)}} \sup_{\substack{p\in\mathbb{R}^n\\ q\geq 0}} \{-(g_1+h_1 \circ A)^*(p + x^* + A^*y^*) + g_2^*(x^*) + h_2^*(y^*) - \left(\sum_{i=1}^m q_i\phi_i\right)_X^* \left(\sum_{i=1}^m q_i z_i^* - p\right) + \sum_{i=1}^m q_i\psi_i^*(z_i^*)\}$$

$$(CQ_{z^*}) \qquad \exists x' \in \bigcap_{i=1}^m \mathrm{ri}(\mathrm{dom}(\phi_i)) \cap \mathrm{ri}(\mathrm{dom}(g_1)) \cap \mathrm{ri}(A^{-1}(\mathrm{dom}(h_1))) \cap \mathrm{ri}(X) : \begin{cases} \phi_i(x') - z_i^*x + \psi_i^*(z_i^*) \leq 0 & i \in L \\ \phi_i(x') - z_i^*x + \psi_i^*(z_i^*) < 0 & i \in N \end{cases}$$

where as before, $L = \{i \in \{1, \dots, m\} \mid \phi_i \text{ is affine}\}$ and $N = \{1, \dots, m\} \setminus L$.

Propostion 6.12. Suppose (CQ_{z^*}) is satisfied for all $z^* \in \prod_{i=1}^m \mathrm{dom}(\psi_i^*)$. Then strong duality holds between (P_A) and (D_A).

Proof. For fixed $x^* \in \mathrm{dom}(g_2^*)$, $y^* \in \mathrm{dom}(h_2^*)$ and $z^* \in \prod_{i=1}^m \mathrm{dom}(\psi_i^*)$, consider the problem,

$$(P_{x^*,y^*,z^*}) \qquad \inf_{\substack{\phi_i(x)-z_i^*x+\psi_i^*(z_i^*)\leq 0\\ i=1,\dots,m,\, x\in X}} \{g_1(x) - x^{*T}x + g_2^*(x^*) + h_1(Ax) - (Ax)^Ty^* + h_2^*(y^*)\}$$

Its Fenchel-Lagrange dual is,

$$(D_{x^*,y^*,z^*}) \qquad \sup_{\substack{p\in\mathbb{R}^n\\ q\geq 0}} \{-(g_1 + h_1 \circ A)^*(p + x^* + A^*y^*) + g_2^*(x^*) + h_2^*(y^*) - \left(\sum_{i=1}^m q_i\phi_i\right)_X^* \left(\sum_{i=1}^m q_i z_i^* - p\right) + \sum_{i=1}^m q_i\psi_i^*(z_i^*)\}$$

Define $f(x) = g_1(x) - x^{*T}x + g_2^*(x^*) + h_1(Ax) - (Ax)^Ty^* + h_2^*(y^*)$ and $f_i(x) = \phi_i(x) - z_i^*x + \psi_i^*(z_i^*)$. Notice that $\mathrm{dom}(f) = \mathrm{dom}(g_1) \cap (A^{-1}\mathrm{dom}(h_1))$ and $\mathrm{dom}(f_i) = \mathrm{dom}(\phi_i)$. Then by the constraint qualification,

$$\exists x' \in \bigcap_{i=1}^m \mathrm{ri}(\mathrm{dom}(f_i)) \cap \mathrm{ri}(\mathrm{dom}(f)) \cap \mathrm{ri}(X) : \begin{cases} f_i(x') \leq 0 & i \in L \\ f_i(x') < 0 & i \in N \end{cases}$$

This further implies that strong duality hold between (P_{x^*,y^*,z^*}) and (D_{x^*,y^*,z^*}). That is, there exist $\bar{p}$ and $\bar{q}$ such that

$$\begin{aligned}
v(P_{x^*,y^*,z^*}) &= \inf_{\substack{\phi_i(x)-z_i^* x+\psi_i^*(z_i^*)\le 0\\ i=1,\dots,m,\, x\in X}} \{g_1(x) - x^{*T}x + g_2^*(x^*) + h_1(Ax) - (Ax)^T y^* + h_2^*(y^*)\} \\
&= \inf_{\substack{f_i(x)\le 0\\ i=1,\dots,m,\\ x\in X}} \{f(x)\} = -(f)^*(\bar{p}) - \left(\sum_{i=1}^m \bar{q}_i f_i\right)_X^* (-\bar{p}) \\
&= -(g_1 + h_1 \circ A)^*(\bar{p} + x^* + A^* y^*) + g_2^*(x^*) + h_2^*(y^*) \\
&\quad - \left(\sum_{i=1}^m \bar{q}_i \phi_i\right)_X^* \left(\sum_{i=1}^m \bar{q}_i z_i^* - \bar{p}\right) + \sum_{i=1}^m \bar{q}_i \psi_i^*(z_i^*) = v(D_{x^*,y^*,z^*})
\end{aligned}$$

Taking the infimum over x^*, y^*, and z^* of (D_{x^*,y^*,z^*}) gives (D_A). Likewise taking the same infimum of (P_{x^*,y^*,z^*}) and recalling that this is equivalent to

$$(P_A) \qquad \inf_{\substack{\phi_i(x)-\psi_i\le 0\\ i=1,\dots,m,\, x\in X}} \{g_1(x) - g_2(x) + h_1(Ax) - h_2(Ax)\}$$

shows that strong duality holds between (P_A) and (D_A). □

Given the work done in this section, we want to sum up the main results in a formal way. In the spirit of the previous literature published on this topic thus far, we give such a summary by presenting some Farkas-type results that are based on each pair of primal and dual problems above. The next section outlines the approach by looking at the results of a convex problem and its Fenchel-Lagrange dual. The section after will then address the work done here, in Section 6.

7 Farkas-Type Results

The following two sections will be dedicated to discussing the Farkas-type results for the convex optimization problem of Section 4 and how this may be used in the problems presented in Section 6 above. Recall the convex optimization problem with convex inequality constraints,

$$(P) \qquad \inf_{\substack{g_i(x)\leq 0\\ i=1,\dots,m,\\ x\in X}} \{f(x)\}$$

and the Fenchel-Lagrange dual to (P),

$$(D_{FL}) \qquad \sup_{\substack{p\in\mathbb{R}^n\\ q\geq 0}} \{-f^*(p) - (q^T g)^*_X(-p)\}$$

Furthermore, recall the constraint qualification needed for strong duality,

$$(CQ) \qquad \exists x' \in \bigcap_{i=1}^{m} \operatorname{ri}(\operatorname{dom}(g_i)) \cap \operatorname{ri}(\operatorname{dom}(f)) \cap \operatorname{ri}(X) : \begin{cases} g_i(x') \leq 0 & i \in L \\ g_i(x') < 0 & i \in N \end{cases}$$

where $L = \{i \in \{1,\dots,m\} \,|\, g_i \text{ is affine}\}$ and $N = \{1,\dots,m\} \setminus L$. Using these problems and the (CQ), we can formulate the following theorem:

Theorem 7.1. Suppose the constraint qualification (CQ) is satisfied. Then the following are equivalent:
 (i) $x \in X, g_i(x) \leq 0, i = 1,\dots,m, \Rightarrow\ f(x) \geq 0$
 (ii) There exist $p \in \mathbb{R}^n$ and $q \in \mathbb{R}^m$, $q \geq 0$ such that $f^*(p) + (q^T g)^*_X(-p) \leq 0$

Proof. ((i) $\Rightarrow$ (ii)): Let (i) be true, so that the objective function for (P) is greater than or equal to 0 for all the feasible points of the problem. Then $v(P) \geq 0$. Furthermore, since the constraint qualification (CQ) is satisfied, by Theorem 4.3, the dual problem has a solution and $v(P) = v(D_{FL})$. That is, $\exists p \in \mathbb{R}^n$, $q \in \mathbb{R}^m$, $q \geq 0$ such that

$$v(P) = v(D_{FL}) = -f^*(p) - (q^T g)^*_X(-p) \geq 0$$

implying that

$$f^*(p) + (q^T g)^*_X(-p) \leq 0$$

Thus (ii) holds.

((ii) $\Rightarrow$ (i)): Next suppose (ii) holds. Then there exist $p \in \mathbb{R}^n$ and $q \in \mathbb{R}^m$, $q \geq 0$ such that $-f^*(p) - (q^T g)^*_X(-p) \geq 0$. Now, by weak duality and the fact that $v(D_{FL})$ is greater than or equal to $-f^*(p) - (q^T g)^*_X(-p)$, we get

$$v(P) \geq v(D_{FL}) \geq -f^*(p) - (q^T g)^*_X(-p) \geq 0$$

$v(P) \geq 0$ implies that for all $x \in X$ such that $g_i(x) \leq 0$, for $i = 1,\dots,m$, we have $f(x) \geq 0$. That is, (i) is true. □

It should be noted that (CQ) was not used in proving (ii) $\Rightarrow$ (i). The next theorem of alternatives follows directly from Theorem 7.1.

Theorem 7.2. Suppose the constraint qualification (CQ) is satisfied. Then either the inequality system
 (i) $x \in X,\ g_i(x) \leq 0, i = 1,\dots,m, f(x) < 0$
has a solution or the system
 (ii) $f^*(p) + (q^T g)^*_X(-p) \leq 0, p \in \mathbb{R}^n, q \geq 0$
has a solution, but never both.

In the next section, the problems from Section 6.1 will be considered in a similar format as above.

8 Results for DC and Fractional Programming problems

This portion of the text illustrates the results from the fractional and DC programming problems in the form of Farkas-type theorems. We start with the first problem in Section 6.

8.1 DC objective function and inequality constraints

Recall the DC programming problem of 6.1,

$$(P_{DC}) \qquad \inf_{\substack{g_i(x)-h_i(x)\leq 0 \\ i=1,\dots,m,\, x\in X}} \{g(x)-h(x)\}$$

where $g, h : \mathbb{R}^n \to \overline{\mathbb{R}}, g_i, h_i : \mathbb{R}^n \to \overline{\mathbb{R}}, i = 1, \dots, m$ are proper and convex functions, and X is a nonempty convex subset of $\mathbb{R}^n$ and its dual problem,

$$(D_{DC}) \qquad \inf_{\substack{x^*\in\mathrm{dom}(h^*) \\ y^*\in\prod_{i=1}^m \mathrm{dom}(h_i^*)}} \sup_{\substack{p\in\mathbb{R}^n \\ q\geq 0}} \left\{ h^*(x^*) - g^*(p+x^*) + \sum_{i=1}^m q_i h_i^*(y_i^*) - \left(\sum_{i=1}^m q_i g_i\right)_X^* \left(\sum_{i=1}^m q_i y_i^* - p\right) \right\}$$

Also, recall the constraint qualification required for strong duality between these two problems,

$$(CQ_{y^*}) \qquad \exists x' \in \bigcap_{i=1}^m \mathrm{ri}(\mathrm{dom}(g_i)) \cap \mathrm{ri}(\mathrm{dom}(g)) \cap \mathrm{ri}(X) : \begin{cases} g_i(x') - x'^T y_i^* + h_i^*(y_i^*) \leq 0 & i \in L \\ g_i(x') - x'^T y_i^* + h_i^*(y_i^*) < 0 & i \in N \end{cases}$$

where $L = \{i \in \{1, \dots, m\} \,|\, g_i \text{ is affine}\}$ and $N = \{1, \dots, m\} \setminus L$. This information can be used to develop the following theorem,

Theorem 8.1. Suppose the constraint qualification (CQ_{y^*}) is satisfied for all $y^* \in \prod_{i=1}^m \mathrm{dom}(h_i^*)$. Then the following are equivalent:

(i) $x \in X, g_i(x) - h_i(x) \leq 0, i = 1, \dots, m, \Rightarrow\ g(x) - h(x) \geq 0$

(ii) for all $x^* \in \mathrm{dom}(h^*)$ and for all $y^* \in \prod_{i=1}^m \mathrm{dom}(h_i^*)$, there exist $p \in \mathbb{R}^n$ and $q \in \mathbb{R}^m$, $q \geq 0$ such that

$$-h^*(x^*) + g^*(p+x^*) - \sum_{i=1}^m q_i h_i^*(y_i^*) + \left(\sum_{i=1}^m q_i g_i\right)_X^* \left(\sum_{i=1}^m q_i y_i^* - p\right) \leq 0$$

Proof. ((i) $\Rightarrow$ (ii)): Let (i) be true and let $x^* \in \mathrm{dom}(h^*)$ and $y^* \in \prod_{i=1}^m \mathrm{dom}(h_i^*)$. From (i) we get that $v(P_{DC}) \geq 0$. Now recall the alternative form of (P_{DC}),

$$(P_{DC}) \qquad \inf_{\substack{x^*\in\mathrm{dom}(h^*) \\ y^*\in\prod_{i=1}^m \mathrm{dom}(h_i^*)}} \inf_{\substack{g_i(x)-y^{*T}x+h_i^*(y^*)\leq 0 \\ i=1,\dots,m,\, x\in X}} \{g(x) - x^{*T}x + h^*(x^*)\}$$

Having fixed x^* and y^*, we also defined the inner infimum as the subproblem (P_{x^*,y^*}). Then $v(P_{DC}) \geq 0$ implies that $v(P_{x^*,y^*}) \geq 0$. Now, since the constraint qualification is satisfied, by Proposition 6.1, the dual problem has a solution and $v(P_{x^*,y^*}) = v(D_{x^*,y^*})$. Hence there exist $p \in \mathbb{R}^n$ and $q \in \mathbb{R}^m$, $q \geq 0$ such that

$$h^*(x^*) - g^*(p+x^*) + \sum_{i=1}^m q_i h_i^*(y_i^*) - \left(\sum_{i=1}^m q_i g_i\right)_X^* \left(\sum_{i=1}^m q_i y_i^* - p\right) \geq 0$$

Thus since (CQ_{y^*}) holds for all $y^* \in \prod_{i=1}^m \mathrm{dom}(h_i^*)$, (ii) is proven to be true.

((ii) $\Rightarrow$ (i)): Next suppose (ii) is true and fix $x^* \in \mathrm{dom}(h^*)$ and $y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)$. Then there exist $p \in \mathbb{R}^n$, $q \in \mathbb{R}^m$, $q \geq 0$ such that the equation in (ii) is true and thus that

$$\sup_{\substack{p\in\mathbb{R}^n \\ q\in\mathbb{R}^m}} \left\{ h^*(x^*) - g^*(p+x^*) + \sum_{i=1}^{m} q_i h_i^*(y_i^*) - \left(\sum_{i=1}^{m} q_i g_i\right)_X^* \left(\sum_{i=1}^{m} q_i y_i^* - p\right) \right\} \geq 0$$

Now, since this argument holds for all x^* and y^*, we know that $v(D_{DC})$ is greater than or equal to 0. By weak duality, $v(P) \geq v(D_{FL}) \geq 0$ and (i) must be true. $\square$

An immediate consequence of Theorem 8.1 is the following theorem of alternatives.

Theorem 8.2. Suppose the constraint qualification (CQ_{y^*}) holds true for all $y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)$. Then either the inequality system

(i) $x \in X$, $g_i(x) - h_i(x) \leq 0, i = 1, \ldots, m, g(x) - h(x) < 0$

has a solution or the following systems

$$(\mathrm{ii}_{x^*,y^*})\; h^*(x^*) - g^*(p+x^*) + \sum_{i=1}^{m} q_i h_i^*(y_i^*) - \left(\sum_{i=1}^{m} q_i g_i\right)_X^* \left(\sum_{i=1}^{m} q_i y_i^* - p\right) \geq 0, p \in \mathbb{R}^n, q \geq 0$$

where $x^* \in \mathrm{dom}(h^*)$ and $y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)$, has a solution, but never both.

Next we briefly present the cases of (P_{DC}), $h \equiv 0$ and $h_i \equiv 0$ for $i = 1, \ldots, m$ and give theorems for each case.

Case 1, $h_i \equiv 0$**:** In this case the primal problem was

$$(P_{DC'}) \qquad \inf_{\substack{g_i(x)\leq 0 \\ i=1,\ldots,m, \\ x\in X}} \{g(x) - h(x)\}$$

where $g, h : \mathbb{R}^n \to \overline{\mathbb{R}}, g_i : \mathbb{R}^n \to \overline{\mathbb{R}}, i = 1, \ldots, m$ are proper and convex functions and X is a nonempty convex subset of $\mathbb{R}^n$, with dual problem,

$$(D_{DC'}) \qquad \inf_{x^*\in\mathrm{dom}(h^*)} \sup_{\substack{p\in\mathbb{R}^n \\ q\geq 0}} \{h^*(x^*) - g^*(p+x^*) - \left(\sum_{i=1}^{m} q_i g_i\right)_X^* (-p)\}$$

For strong duality, we used the constraint qualification,

$$(CQ_0) \qquad \exists x' \in \bigcap_{i=1}^{m} \mathrm{ri}(\mathrm{dom}(g_i)) \cap \mathrm{ri}(\mathrm{dom}(g)) \cap \mathrm{ri}(X) : \begin{cases} g_i(x') \leq 0 & i \in L \\ g_i(x') < 0 & i \in N \end{cases}$$

where as before, $L = \{i \in \{1, \ldots, m\} \,|\, g_i \text{ is affine}\}$ and $N = \{1, \ldots, m\} \setminus L$. These results lead to our next theorem,

Theorem 8.3. Suppose the constraint qualification (CQ_0) is satisfied. Then the following are equivalent:

(i) $x \in X, g_i(x) \leq 0, i = 1, \ldots, m, \Rightarrow\ g(x) - h(x) \geq 0$

(ii) for all $x^* \in \mathrm{dom}(h^*)$, there exist $p \in \mathbb{R}^n$ and $q \in \mathbb{R}^m$, $q \geq 0$ such that

$$h^*(x^*) - g^*(p+x^*) - \left(\sum_{i=1}^{m} q_i g_i\right)_X^* (-p) \geq 0$$

The proof for Theorem 8.3 is omitted as it follows directly from Theorem 8.1. As before, Theorem 8.3 can be expressed as a theorem of alternatives.

Theorem 8.4. Suppose the constraint qualification (CQ_0) holds true. Then either the inequality system
(i) $x \in X$, $g_i(x) \leq 0, i = 1, \ldots, m, g(x) - h(x) < 0$
has a solution or the following systems

$$(\mathrm{ii}_{\,x^*})\; h^*(x^*) - g^*(p + x^*) - \left(\sum_{i=1}^{m} q_i g_i\right)^*_X (-p) \geq 0,\; p \in \mathbb{R}^n, q \geq 0$$

for $x^* \in \mathrm{dom}(h^*)$, has a solution, but never both.

Case 2, $h \equiv 0$: Consider the case where the primal problem is

$$(P_{DC''}) \qquad \inf_{\substack{g_i(x) - h_i(x) \leq 0 \\ i=1,\ldots,m,\, x \in X}} \{g(x)\}$$

where $g : \mathbb{R}^n \to \overline{\mathbb{R}}, g_i, h_i : \mathbb{R}^n \to \overline{\mathbb{R}}, i = 1, \ldots, m$ are proper and convex functions, and X is a nonempty convex subset of $\mathbb{R}^n$ and its dual problem,

$$(D_{DC''}) \qquad \inf_{y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)} \sup_{\substack{p \in \mathbb{R}^n \\ q \geq 0}} \left\{ -g^*(p) + \sum_{i=1}^{m} q_i h_i^*(y_i^*) - \left(\sum_{i=1}^{m} q_i g_i\right)^*_X \left(\sum_{i=1}^{m} q_i y_i^* - p\right) \right\}$$

In this case, the constraint qualification (CQ_{y^*}) used for the original primal and dual problems, (P_{DC}) and (D_{DC}) gives strong duality. With this, we present the following theorem,

Theorem 8.5. Suppose the constraint qualification (CQ_{y^*}) is satisfied for all $y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)$. Then the following are equivalent:
(i) $x \in X, g_i(x) - h_i(x) \leq 0, i = 1, \ldots, m, \Rightarrow\; g(x) \geq 0$
(ii) $\forall y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)$, there exist $p \in \mathbb{R}^n$ and $q \in \mathbb{R}^m$, $q \geq 0$ such that

$$g^*(p) - \sum_{i=1}^{m} q_i h_i^*(y_i^*) + \left(\sum_{i=1}^{m} q_i g_i\right)^*_X \left(\sum_{i=1}^{m} q_i y_i^* - p\right) \leq 0$$

Once again, the proof is omitted as it follows immediately from Theorem 8.1. As a consequence of Theorem 8.5, we present the following.

Theorem 8.6. Suppose the constraint qualification (CQ_{y^*}) holds true for all $y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)$. Then either the inequality system
(i) $x \in X$, $g_i(x) - h_i(x) \leq 0, i = 1, \ldots, m, g(x) < 0$
has a solution or the follwing systems

$$(\mathrm{ii}_{\,y^*})\; g^*(p) - \sum_{i=1}^{m} q_i h_i^*(y_i^*) + \left(\sum_{i=1}^{m} q_i g_i\right)^*_X \left(\sum_{i=1}^{m} q_i y_i^* - p\right) \leq 0,\; p \in \mathbb{R}^n, q \geq 0$$

where $y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)$, has a solution, but never both.

8.2 DC fractional programming with DC constraints

In this part of the text we look at the second problem from Section 6. Recall that the main primal problem was,

$$(P^0_{FP}) \qquad \inf_{\substack{\phi_i(x)-\psi_i(x)\leq 0\\ i=1,\ldots,m,\, x\in X}} \left\{ \frac{g(x)-h(x)}{u(x)-v(x)} \right\}$$

where $X \subseteq \mathbb{R}^n$ is convex, g and h are proper and convex, u and v are proper and concave, ϕ_i and ψ_i are proper and convex for $i = 1, \ldots, m$. There were two cases to consider, $\lambda < 0$ and $\lambda \geq 0$, each with its own set of assumptions. For each case, we will present two theorems and suppose that the assumptions from earlier still hold. Proofs in the section will be left out as they are similar to the proofs of the above subsection. Interested readers can refer to [13] and [14] for further information.

Case 1, $\lambda \geq 0$: In this case the dual problem to (P^0_{FP}) was

$$(D_{FP}) \qquad \inf_{\substack{x^*\in\operatorname{dom}((h-\lambda v)^*)\\ y^*\in\prod_{i=1}^m \operatorname{dom}(\psi_i^*)}} \sup_{\substack{p_1,p_2\in\mathbb{R}^n\\ q\geq 0}} \Bigg\{ (h-\lambda v)^*(x^*) - g^*(p_1) - \lambda(-u)^*(p_2) + \sum_{i=1}^m q_i\psi_i^*(y_i^*) - \left(\sum_{i=1}^m q_i\phi_i\right)^*_X \left(\sum_{i=1}^m q_i y_i^* - p_1 - \lambda p_2 + x^*\right) \Bigg\}$$

For strong duality we used the following constraint qualification,

$$(CQ'_{y^*}) \qquad \exists x' \in \bigcap_{i=1}^m \operatorname{ri}(\operatorname{dom}(\phi_i)) \cap \operatorname{ri}(\operatorname{dom}(g)) \cap \operatorname{ri}(\operatorname{dom}(-u)) \cap \operatorname{ri}(X), \text{ such that}$$
$$\begin{cases} \phi_i(x') - x'^T y_i^* + \psi_i^*(y_i^*) \leq 0 & i \in L \\ \phi_i(x') - x'^T y_i^* + \psi_i^*(y_i^*) < 0 & i \in N \end{cases}$$

where $L = \{i \in \{1, \ldots, m\} \,|\, \phi_i \text{ is affine}\}$ and $N = \{1, \ldots, m\} \setminus L$. Now we have everything needed for the Farkas-type theorems:

Theorem 8.7. Suppose the constraint qualification (CQ'_{y^*}) is satisfied for all $y^* \in \prod_{i=1}^m \operatorname{dom}(\psi_i^*)$ and that $\lambda \geq 0$. Then the following are equivalent:

(i) $x \in X, \phi_i(x) - \psi_i(x) \leq 0, i = 1, \ldots, m, \Rightarrow \frac{g(x)-h(x)}{u(x)-v(x)} \geq \lambda$

(ii) $\forall x^* \in \operatorname{dom}((h - \lambda v)^*)$ and $\forall y^* \in \prod_{i=1}^m \operatorname{dom}(\psi_i^*)$, $\exists p_1, p_2 \in \mathbb{R}^n$, $q \in \mathbb{R}^m$, $q \geq 0$ such that

$$(h-\lambda v)^*(x^*) - g^*(p_1) - \lambda(-u)^*(p_2) + \sum_{i=1}^m q_i\psi_i^*(y_i^*) - \left(\sum_{i=1}^m q_i\phi_i\right)^*_X \left(\sum_{i=1}^m q_i y_i^* - p_1 - \lambda p_2 + x^*\right) \geq 0$$

Theorem 8.8. Suppose the constraint qualification (CQ'_{y^*}) holds true for all $y^* \in \prod_{i=1}^m \operatorname{dom}(\psi_i^*)$ and that $\lambda \geq 0$. Then either the inequality system

(i) $x \in X$, such that $\phi_i(x) - \psi_i(x) \leq 0, i = 1, \ldots, m,$ $\frac{g(x)-h(x)}{u(x)-v(x)} < \lambda$

has a solution or the following systems

$$(\text{ii}_{x^*,y^*}) \quad (h - \lambda v)^*(x^*) - g^*(p_1) - \lambda(-u)^*(p_2) + \sum_{i=1}^m q_i\psi_i^*(y_i^*) - \left(\sum_{i=1}^m q_i\phi_i\right)^*_X \left(\sum_{i=1}^m q_i y_i^* - p_1 - \lambda p_2 + x^*\right) \geq 0,\ p_1, p_2 \in \mathbb{R}^n,\ q \in \mathbb{R}^m,\ q \geq 0$$

where $x^* \in \operatorname{dom}((h - \lambda v)^*)$ and $y^* \in \prod_{i=1}^m \operatorname{dom}(\psi_i^*)$, has a solution, but never both.

Case 2, $\lambda < 0$: For the second case, the dual problem to (P^0_{FP}) was

$$(\bar{D}_{FP}) \qquad \inf_{\substack{x^*\in \mathrm{dom}((h+\lambda u)^*)\\ y^* \in \prod_{i=1}^m \mathrm{dom}(\psi_i^*)}} \sup_{\substack{p_1,p_2\in\mathbb{R}^n\\ q\geq 0}} \left\{ (h+\lambda u)^*(x^*) - g^*(p_1) + \lambda(-v)^*(p_2) + \sum_{i=1}^m q_i \psi_i^*(y_i^*) \right.$$
$$\left. - \left(\sum_{i=1}^m q_i\phi_i\right)^*_X \left(\sum_{i=1}^m q_i y_i^* - p_1 + \lambda p_2 + x^*\right)\right\}$$

and the constraint qualification for strong duality was,

$$(CQ''_{y^*}) \qquad \exists x' \in \bigcap_{i=1}^m \mathrm{ri}(\mathrm{dom}(\phi_i)) \cap \mathrm{ri}(\mathrm{dom}(g)) \cap \mathrm{ri}(\mathrm{dom}(-v)) \cap \mathrm{ri}(X), \text{ such that}$$
$$\begin{cases} \phi_i(x') - x'^T y_i^* + \psi_i^*(y_i^*) \leq 0 & i \in L \\ \phi_i(x') - x'^T y_i^* + \psi_i^*(y_i^*) < 0 & i \in N \end{cases}$$

where $L = \{i \in \{1,\dots,m\} \mid \phi_i \text{ is affine}\}$ and $N = \{1,\dots,m\} \setminus L$.

Next we present two Farkas-type theorems based on this pair of problems, (P^0_{FP}) and $(\bar{D}_{FP})$, and the constraint qualification.

Theorem 8.9. Suppose the constraint qualification (CQ''_{y^*}) is satisfied for all $y^* \in \prod_{i=1}^m \mathrm{dom}(\psi_i^*)$ and that $\lambda < 0$. Then the following are equivalent:

(i) $x \in X, \phi_i(x) - \psi_i(x) \leq 0, i = 1,\dots,m, \Rightarrow \frac{g(x)-h(x)}{u(x)-v(x)} \geq \lambda$

(ii) $\forall x^* \in \mathrm{dom}((h+\lambda u)^*)$ and $\forall y^* \in \prod_{i=1}^m \mathrm{dom}(\psi_i^*)$, there exist $p_1, p_2 \in \mathbb{R}^n$ and $q \in \mathbb{R}^m$, $q \geq 0$ such that

$$(h+\lambda u)^*(x^*) - g^*(p_1) + \lambda(-v)^*(p_2) + \sum_{i=1}^m q_i\psi_i^*(y_i^*) - \left(\sum_{i=1}^m q_i\phi_i\right)^*_X \left(\sum_{i=1}^m q_i y_i^* - p_1 + \lambda p_2 + x^*\right) \geq 0$$

As a consequence of Theorem 8.9, we have the following theorem of alternatives.

Theorem 8.10. Suppose the constraint qualification (CQ''_{y^*}) holds true for all $y^* \in \prod_{i=1}^m \mathrm{dom}(\psi_i^*)$ and that $\lambda < 0$. Then either the inequality system

(i) $x \in X, \phi_i(x) - \psi_i(x) \leq 0, i = 1,\dots,m, \frac{g(x)-h(x)}{u(x)-v(x)} < \lambda$

has a solution or the following systems

$$(\mathrm{ii}_{x^*,y^*}) \quad (h+\lambda u)^*(x^*) - g^*(p_1) + \lambda(-v)^*(p_2) + \sum_{i=1}^m q_i\psi_i^*(y_i^*)$$
$$- \left(\sum_{i=1}^m q_i\phi_i\right)^*_X \left(\sum_{i=1}^m q_i y_i^* - p_1 + \lambda p_2 + x^*\right) \geq 0,\ p_1,p_2 \in \mathbb{R}^n,\ q \in \mathbb{R}^m,\ q \geq 0$$

where $x^* \in \mathrm{dom}((h-\lambda v)^*)$ and $y^* \in \prod_{i=1}^m \mathrm{dom}(\psi_i^*)$, has a solution, but never both.

Earlier, we also looked at the fractional programming problem where ψ_i was zero,

$$(P^0_{FP'}) \qquad \inf_{\substack{\phi_i(x)\leq 0\\ i=1,\ldots,m\\ x\in X}} \left\{\frac{g(x)-h(x)}{u(x)-v(x)}\right\}$$

which also had the two cases.

Case 1, $\lambda \geq 0$: In this case the dual problem to $(P^0_{FP'})$ was

$$(D_{FP}) \qquad \inf_{x^*\in\mathrm{dom}((h-\lambda v)^*)} \sup_{\substack{p_1,p_2\in\mathbb{R}^n\\ q\geq 0}} \Bigg\{(h-\lambda v)^*(x^*) - g^*(p_1) - \lambda(-u)^*(p_2) - \left(\sum_{i=1}^m q_i\phi_i\right)^*_X (x^*-p_1-\lambda p_2)\Bigg\}$$

For strong duality we used the following constraint qualification,

$$(CQ'_0) \qquad \exists x' \in \bigcap_{i=1}^m \mathrm{ri}(\mathrm{dom}(\phi_i)) \cap \mathrm{ri}(\mathrm{dom}(g)) \cap \mathrm{ri}(\mathrm{dom}(-u)) \cap \mathrm{ri}(X), \text{ such that } \begin{cases} \phi_i(x') \leq 0 & i\in L\\ \phi_i(x') < 0 & i \in N\end{cases}$$

where $L = \{i \in \{1,\ldots,m\} \mid \phi_i \text{ is affine}\}$ and $N = \{1,\ldots,m\}\setminus L$. Next we present the Farkas-type theorems:

Theorem 8.11. Suppose the constraint qualification (CQ'_0) is satisfied and that $\lambda \geq 0$. Then the following are equivalent:

(i) $x \in X, \phi_i(x) \leq 0, i = 1,\ldots,m, \Rightarrow \frac{g(x)-h(x)}{u(x)-v(x)} \geq \lambda$

(ii) $\forall x^* \in \mathrm{dom}((h-\lambda v)^*)$, there exist $p_1, p_2 \in \mathbb{R}^n$, $q \in \mathbb{R}^m$, $q \geq 0$ such that

$$(h-\lambda v)^*(x^*) - g^*(p_1) - \lambda(-u)^*(p_2) - \left(\sum_{i=1}^m q_i\phi_i\right)^*_X (x^*-p_1-\lambda p_2) \geq 0$$

Theorem 8.12. Suppose the constraint qualification (CQ'_0) holds true and that $\lambda \geq 0$. Then either the inequality system

(i) $x \in X$, such that $\phi_i(x) \leq 0, i = 1,\ldots,m, \frac{g(x)-h(x)}{u(x)-v(x)} < \lambda$

has a solution or the following systems

$$(\mathrm{ii}_{x^*}) \quad (h-\lambda v)^*(x^*) - g^*(p_1) - \lambda(-u)^*(p_2) - \left(\sum_{i=1}^m q_i\phi_i\right)^*_X (x^*-p_1-\lambda p_2) \geq 0,$$

$$p_1, p_2 \in \mathbb{R}^n,\ q \in \mathbb{R}^m,\ q \geq 0$$

where $x^* \in \mathrm{dom}((h-\lambda v)^*)$, has a solution, but never both.

Case 2, $\lambda < 0$: For this case, the dual problem to $(P^0_{FP'})$ was

$$(\tilde{D}_{FP'}) \qquad \inf_{x^*\in\mathrm{dom}((h+\lambda u)^*)} \sup_{\substack{p_1,p_2\in\mathbb{R}^n\\ q\geq 0}} \{(h+\lambda u)^*(x^*) - g^*(p_1) + \lambda(-v)^*(p_2) - \left(\sum_{i=1}^m q_i\phi_i\right)^*_X (x^*-p_1+\lambda p_2)\}$$

and the constraint qualification for strong duality was,

$$(\tilde{CQ}_0') \qquad \exists x' \in \bigcap_{i=1}^{m} \mathrm{ri}(\mathrm{dom}(\phi_i)) \cap \mathrm{ri}(\mathrm{dom}(g)) \cap \mathrm{ri}(\mathrm{dom}(-v)) \cap \mathrm{ri}(X), \text{ such that}$$
$$\begin{cases} \phi_i(x') \leq 0 & i \in L \\ \phi_i(x') < 0 & i \in N \end{cases}$$

where $L = \{i \in \{1, \dots, m\} \,|\, \phi_i \text{ is affine}\}$ and $N = \{1, \dots, m\} \setminus L$. Using this gives the following two Farkas-type theorems:

Theorem 8.13. Suppose the constraint qualification $(\tilde{CQ}_0')$ is satisfied and that $\lambda < 0$. Then the following are equivalent:
(i) $x \in X, \phi_i(x) \leq 0, i = 1, \dots, m, \Rightarrow \frac{g(x)-h(x)}{u(x)-v(x)} \geq \lambda$
(ii) $\forall x^* \in \mathrm{dom}((h + \lambda u)^*)$, there exist $p_1, p_2 \in \mathbb{R}^n$ and $q \in \mathbb{R}^m$, $q \geq 0$ such that

$$(h + \lambda u)^*(x^*) - g^*(p_1) + \lambda(-v)^*(p_2) - \left(\sum_{i=1}^{m} q_i \phi_i\right)_X^* (x^* - p_1 + \lambda p_2) \geq 0$$

Theorem 8.14. Suppose the constraint qualification $(\tilde{CQ}_0')$ holds true and that $\lambda < 0$. Then either the inequality system
(i) $x \in X, \phi_i(x) \leq 0, i = 1, \dots, m, \frac{g(x)-h(x)}{u(x)-v(x)} < \lambda$
has a solution or the following systems

$$(\mathrm{ii}_{x^*}) \;\; (h + \lambda u)^*(x^*) - g^*(p_1) + \lambda(-v)^*(p_2) - \left(\sum_{i=1}^{m} q_i \phi_i\right)_X^* (x^* - p_1 + \lambda p_2) \geq 0,$$

$$p_1, p_2 \in \mathbb{R}^n,\ q \in \mathbb{R}^m,\ q \geq 0$$

where $x^* \in \mathrm{dom}((h + \lambda u)^*)$, has a solution, but never both.

8.3 Fractional programming problem

Continuing the work with fractional programming, we now look at the fractional programming problem from the third part of Section 6. Recall the fractional programming problem with DC constraints,

$$(P_{FP_0}^0) \qquad \inf_{\substack{g_i(x)-h_i(x)\leq 0 \\ i=1,\dots,m,\, x\in X}} \left\{ \frac{g(x)}{h(x)} \right\}$$

where $X \subseteq \mathbb{R}^n$ is nonempty and convex, $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ is proper and convex, $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ is concave such that $-h$ is proper and lower semicontinuous over $\mathcal{F}(P_{FP_0}^0)$ (the feasible set of the problem), and $g_i, h_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \dots, m$, are proper and convex functions. As done above, two cases must be considered, $\lambda < 0$ and $\lambda \geq 0$. For each case, the same assumptions hold about the problem as in Section 6. Two theorems will be presented for each case but proofs will be omitted.

Case 1, $\lambda \geq 0$: When λ was nonnegative, the dual problem was

$$(D_{FP_0}) \qquad \inf_{y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)} \sup_{\substack{p_1,p_2 \in \mathbb{R}^n \\ q \geq 0}} \Big\{ - g^*(p_1) - \lambda(-h)^*(p_2)$$
$$- \left(\sum_{i=1}^{m} q_i g_i\right)_X^* \left(\sum_{i=1}^{m} q_i y_i^* - p_1 - \lambda p_2\right) + \sum_{i=1}^{m} q_i h_i^*(y_i^*) \Big\}$$

For strong duality we used the constraint qualification,

$$(\hat{CQ}_{y^*}) \qquad \exists x' \in \bigcap_{i=1}^{m} \operatorname{ri}(\operatorname{dom}(g_i)) \cap \operatorname{ri}(\operatorname{dom}(g)) \cap \operatorname{ri}(\operatorname{dom}(-h)) \cap \operatorname{ri}(X):$$
$$\begin{cases} g_i(x') - x'^T y_i^* + h_i^*(y_i^*) \leq 0 & i \in L \\ g_i(x') - x'^T y_i^* + h_i^*(y_i^*) < 0 & i \in N \end{cases}$$

where $L = \{i \in \{1, \ldots, m\} \,|\, g_i \text{ is affine}\}$ and $N = \{1, \ldots, m\} \setminus L$. Using this we come up with the following Farkas-type theorems:

Theorem 8.15. Suppose the constraint qualification $(\hat{CQ}_{y^*})$ is satisfied for all $y^* \in \prod_{i=1}^{m} \operatorname{dom}(h_i^*)$ and that $\lambda \geq 0$. Then the following are equivalent:

(i) $x \in X, g_i(x) - h_i(x) \leq 0, i = 1, \ldots, m, \Rightarrow \frac{g(x)}{h(x)} \geq \lambda$

(ii) $\forall y^* \in \prod_{i=1}^{m} \operatorname{dom}(h_i^*)$, there exist $p \in \mathbb{R}^n$ and $q \in \mathbb{R}^m$, $q \geq 0$ such that

$$g^*(p_1) + \lambda(-h)^*(p_2) + \left(\sum_{i=1}^{m} q_i g_i\right)_X^* \left(\sum_{i=1}^{m} q_i y_i^* - p_1 - \lambda p_2\right) - \sum_{i=1}^{m} q_i h_i^*(y_i^*) \leq 0$$

The next theorem follows immediately from Theorem 8.15.

Theorem 8.16. Suppose the constraint qualification $(\hat{CQ}_{y^*})$ holds true for all $y^* \in \prod_{i=1}^{m} \operatorname{dom}(\psi_i^*)$ and that $\lambda \geq 0$. Then either the inequality system

(i) $x \in X, g_i(x) - h_i(x) \leq 0, i = 1, \ldots, m, \frac{g(x)}{h(x)} < \lambda$

has a solution or the following systems

$$(\text{ii}_{y^*}) \quad g^*(p_1) + \lambda(-h)^*(p_2) + \left(\sum_{i=1}^{m} q_i g_i\right)_X^* \left(\sum_{i=1}^{m} q_i y_i^* - p_1 - \lambda p_2\right) - \sum_{i=1}^{m} q_i h_i^*(y_i^*) \leq 0,$$
$$p \in \mathbb{R}^n,\ q \in \mathbb{R}^m,\ q \geq 0$$

where $y^* \in \prod_{i=1}^{m} \operatorname{dom}(h_i^*)$, has a solution, but never both.

Case 2, $\lambda < 0$**:** For this case, the dual problem to $(P^0_{FP_0})$ was

$$(\bar{D}_{FP_0}) \qquad \inf_{\substack{x^* \in \operatorname{dom}(h^*) \\ y^* \in \prod_{i=1}^{m} \operatorname{dom}(h_i^*)}} \sup_{\substack{p \in \mathbb{R}^n \\ q \geq 0}} \{\lambda h^*(x^*) - g^*(p + \lambda x^*) - \left(\sum_{i=1}^{m} q_i g_i\right)_X^* \left(\sum_{i=1}^{m} q_i y_i^* - p\right) + \sum_{i=1}^{m} q_i h_i^*(y_i^*)\}$$

and the constraint qualification for strong duality was (CQ_{y^*}), the same as the constraint qualification for the first DC programming problem of 8.1. From this we get the next two theorems.

Theorem 8.17. Suppose the constraint qualification (CQ_{y^*}) is satisfied for all $y^* \in \prod_{i=1}^{m} \operatorname{dom}(h_i^*)$ and that $\lambda < 0$. Then the following are equivalent:

(i) $x \in X, g_i(x) - h_i(x) \leq 0, i = 1, \ldots, m, \Rightarrow \frac{g(x)}{h(x)} \geq \lambda$

(ii) $\forall x^* \in \operatorname{dom}(h^*)$ and $\forall y^* \in \prod_{i=1}^{m} \operatorname{dom}(h_i^*)$, there exist $p \in \mathbb{R}^n$ and $q \in \mathbb{R}^m$, $q \geq 0$ such that

$$\lambda h^*(x^*) - g^*(p + \lambda x^*) - \left(\sum_{i=1}^{m} q_i g_i\right)_X^* \left(\sum_{i=1}^{m} q_i y_i^* - p\right) + \sum_{i=1}^{m} q_i h_i^*(y_i^*) \geq 0$$

Theorem 8.18. Suppose the constraint qualification (CQ_{y^*}) holds true for all $y^* \in \prod_{i=1}^{m} \text{dom}(h_i^*)$ and that $\lambda < 0$. Then either the inequality system

(i) $x \in X, g_i(x) - h_i(x) \leq 0, i = 1, \ldots, m, \frac{g(x)}{h(x)} < \lambda$

has a solution or the following systems

$$(\text{ii}_{x^*,y^*}) \quad \lambda h^*(x^*) - g^*(p + \lambda x^*) - \left(\sum_{i=1}^{m} q_i g_i\right)_X^* \left(\sum_{i=1}^{m} q_i y_i^* - p\right) + \sum_{i=1}^{m} q_i h_i^*(y_i^*) \geq 0,$$

$$p \in \mathbb{R}^n,\ q \in \mathbb{R}^m,\ q \geq 0$$

where $x^* \in \text{dom}(h^*)$ and $y^* \in \prod_{i=1}^{m} \text{dom}(h_i^*)$, has a solution, but never both.

The last fractional programming problem we looked at was one with convex constraints,

$$(P^0_{FP_0'}) \qquad \inf_{\substack{g_i(x) \leq 0 \\ i=1,\ldots,m, \\ x \in X}} \left\{ \frac{g(x)}{h(x)} \right\}$$

We only considered the case where $\lambda < 0$, since the problem became a convex optimization problem when $\lambda \geq 0$. The dual problem to $(P^0_{FP_0'})$ when $\lambda < 0$ was

$$(D_{FP_0'}) \qquad \inf_{x^* \in \text{dom}((-h)^*)} \sup_{\substack{p \in \mathbb{R}^n \\ q \geq 0}} \left\{ -g^*(p) - \lambda(-h)^*(-x^*) - \left(\sum_{i=1}^{m} q_i g_i\right)_X^* (-p - \lambda x^*) \right\}$$

with constraint qualification,

$$(CQ_0') \qquad \exists x' \in \cap \, \text{ri}(\text{dom}(g)) \cap \text{ri}(X) : \begin{cases} g_i(x') \leq 0 & i \in L \\ g_i(x') < 0 & i \in N \end{cases}$$

where $L = \{i \in \{1, \ldots, m\} \,|\, g_i \text{ is affine}\}$ and $N = \{1, \ldots, m\} \setminus L$, for strong duality. The Farkas-type theorems associated with this set of problems are,

Theorem 8.19. Suppose the constraint qualification (CQ_0') holds and that $\lambda < 0$. Then the following are equivalent:

(i) $x \in X, g_i(x) \leq 0, \Rightarrow \frac{g(x)}{h(x)} \geq \lambda$

(ii) $\forall x^* \in \text{dom}((-h)^*)$, there exist $p \in \mathbb{R}^n$ and $q \in \mathbb{R}^m$, $q \geq 0$ such that

$$g^*(p) + \lambda(-h)^*(-x^*) + \left(\sum_{i=1}^{m} q_i g_i\right)_X^* (-p - \lambda x^*) \leq 0$$

Theorem 8.20. Suppose the constraint qualification (CQ_0') holds and that $\lambda < 0$. Then either the inequality system

(i) $x \in X, g_i(x) \leq 0, \frac{g(x)}{h(x)} < \lambda$

has a solution or the following systems

$$(\text{ii}_{x^*}) \quad g^*(p) + \lambda(-h)^*(-x^*) + \left(\sum_{i=1}^{m} q_i g_i\right)_X^* (-p - \lambda x^*) \leq 0,\ p \in \mathbb{R}^n,\ q \in \mathbb{R}^m,\ q \geq 0$$

where $x^* \in \text{dom}((-h)^*)$, has a solution, but never both.

8.4 DC programming problem containing a composition with a linear continuous operator

The fourth and last problem type discussed in Section 6 was a DC programming problem with a linear operator. We looked at two similar problems, one with convex constraints and the other with DC constraint functions. Here we will first consider the more complex of the two and then present the theorems for the other based on the results of the more complicated case. Recall that the primal problem was,

$$(P_A) \qquad \inf_{\substack{\phi_i(x)-\psi_i\leq 0\\ i=1,\dots,m,\, x\in X}} \{g_1(x) - g_2(x) + h_1(Ax) - h_2(Ax)\}$$

where $g_1, g_2, h_1, h_2, \phi_i, \psi_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, for $i = 1, \dots, m$, are proper convex functions and $A \in \mathbb{R}^{n\times n}$ is linear continuous operator. Its dual problem was

$$(D_A) \qquad \inf_{\substack{x^*\in\operatorname{dom}(g_2^*)\\ y^*\in\operatorname{dom}(h_2^*)\\ z^*\in\prod_{i=1}^m \operatorname{dom}(\psi_i^*)}} \sup_{\substack{p\in\mathbb{R}^n\\ q\geq 0}} \{-(g_1+h_1\circ A)^*(p + x^* + A^*y^*) + g_2^*(x^*) + h_2^*(y^*)$$

$$- \left(\sum_{i=1}^m q_i\phi_i\right)^*_X \left(\sum_{i=1}^m q_i z_i^* - p\right) + \sum_{i=1}^m q_i\psi_i^*(z_i^*)\}$$

For strong duality, we had

$$(CQ_{z^*}) \qquad \exists x' \in \bigcap_{i=1}^m \operatorname{ri}(\operatorname{dom}(\phi_i)) \cap \operatorname{ri}(\operatorname{dom}(g_1)) \cap \operatorname{ri}(A^{-1}(\operatorname{dom}(h_1))) \cap \operatorname{ri}(X):$$

$$\begin{cases} \phi_i(x') - z_i^*x + \psi_i^*(z_i^*) \leq 0 & i \in L \\ \phi_i(x') - z_i^*x + \psi_i^*(z_i^*) < 0 & i \in N \end{cases}$$

where as before, $L = \{i \in \{1, \dots, m\} \mid \phi_i \text{ is affine}\}$ and $N = \{1, \dots, m\} \setminus L$. Then we have the following theorem:

Theorem 8.21. Suppose the constraint qualification (CQ_{z^*}) holds for all $z^* \in \prod_{i=1}^m \operatorname{dom}(\psi_i^*)$. Then the following are equivalent:

(i) $x \in X, \phi_i(x) - \psi_i \leq 0, i = 1, \dots, m, \Rightarrow\ g_1(x) - g_2(x) + h_1(Ax) - h_2(Ax) \geq 0$

(ii) $\forall x^* \in \operatorname{dom}(g_2^*)$, $\forall y^* \in \operatorname{dom}(h_2^*)$, and $\forall z^* \in \prod_{i=1}^m \operatorname{dom}(\psi_i^*)$, $\exists p \in \mathbb{R}^n$, $q \in \mathbb{R}^m$, $q \geq 0$ such that

$$(g_1+h_1\circ A)^*(p+x^*+A^*y^*) - g_2^*(x^*) - h_2^*(y^*) + \left(\sum_{i=1}^m q_i\phi_i\right)^*_X \left(\sum_{i=1}^m q_i z_i^* - p\right) - \sum_{i=1}^m q_i\psi_i^*(z_i^*) \leq 0$$

Proof. ((i) $\Rightarrow$ (ii)): Let (i) be true and let $x^* \in \operatorname{dom}(g_2^*)$, $y^* \in \operatorname{dom}(h_2^*)$, and $z^* \in \prod_{i=1}^m \operatorname{dom}(\psi_i^*)$. From (i) we get that $v(P_A) \geq 0$. Now in Section 6 we derived an equivalent form of (P_A),

$$(P_A) \qquad \inf_{\substack{x^*\in\operatorname{dom}(g_2^*)\\ y^*\in\operatorname{dom}(h_2^*)\\ z^*\in\prod_{i=1}^m \operatorname{dom}(\psi_i^*)}} \inf_{\substack{\phi_i(x)-z_i^*x+\psi_i^*(z_i^*)\leq 0\\ i=1,\dots,m,\, x\in X}} \{g_1(x) - x^{*T}x + g_2^*(x^*) + h_1(Ax) - (Ax)^T y^* + h_2^*(y^*)\}$$

where the inner infimum was a separate convex optimization problem, (P_{x^*,y^*,z^*}). Since $v(P_A) \geq 0$, we know that $v(P_{x^*,y^*,z^*}) \geq 0$. The dual problem of (P_{x^*,y^*,z^*}) was

$$(D_{x^*,y^*,z^*}) \qquad \sup_{\substack{p\in\mathbb{R}^n\\ q\geq 0}} \{-(g_1 + h_1 \circ A)^*(p + x^* + A^*y^*) + g_2^*(x^*) + h_2^*(y^*)$$

$$- \left(\sum_{i=1}^m q_i\phi_i\right)^*_X \left(\sum_{i=1}^m q_i z_i^* - p\right) + \sum_{i=1}^m q_i\psi_i^*(z_i^*)\}$$

In the proof of Proposition 6.12, we showed that strong duality held under (CQ_{z^*}) between (P_{x^*,y^*,z^*}) and (D_{x^*,y^*,z^*}). Thus by Proposition 6.12, the dual problem, (D_{x^*,y^*,z^*}), has a solution and $v(P_{x^*,y^*,z^*}) = v(D_{x^*,y^*,z^*})$. So there exist $p \in \mathbb{R}^n$ and $q \in \mathbb{R}^m$, $q \geq 0$ such that

$$\begin{aligned}&-(g_1 + h_1 \circ A)^*(p + x^* + A^*y^*) + g_2^*(x^*) + h_2^*(y^*) - \left(\sum_{i=1}^m q_i\phi_i\right)_X^* \left(\sum_{i=1}^m q_i z_i^* - p\right) + \sum_{i=1}^m q_i\psi_i^*(z_i^*)\\ &= v(D_{x^*,y^*,z^*}) = v(P_{x^*,y^*,z^*}) \geq 0\end{aligned}$$

This, of course, implies that (ii) is true.

((ii) $\Rightarrow$ (i)): Next suppose (ii) holds. For arbitrary $x^* \in \mathrm{dom}(g_2^*)$, $y^* \in \mathrm{dom}(h_2^*)$, and $z^* \in \prod_{i=1}^m \mathrm{dom}(\psi_i^*)$, there exist $p \in \mathbb{R}^n$ and $q \in \mathbb{R}^m$, $q \geq 0$ such that the relation in (ii) is true. This implies that

$$-(g_1 + h_1 \circ A)^*(p + x^* + A^*y^*) + g_2^*(x^*) + h_2^*(y^*) - \left(\sum_{i=1}^m q_i\phi_i\right)_X^* \left(\sum_{i=1}^m q_i z_i^* - p\right) + \sum_{i=1}^m q_i\psi_i^*(z_i^*) \geq 0$$

This and weak duality imply that $v(P_A) \geq v(D_A) \geq 0$. Now, since x^*, y^*, and z^* are arbitrary, (i) must be true. $\square$

An immediate consequence of Theorem 8.23 is the next theorem of alternatives.

Theorem 8.22. Suppose the constraint qualification (CQ_{z^*}) holds for all $z^* \in \prod_{i=1}^m \mathrm{dom}(\psi_i^*)$. Then either the inequality system

(i) $x \in X$, $\phi_i(x) - \psi_i \leq 0, i = 1, \ldots, m$, $g_1(x) - g_2(x) + h_1(Ax) - h_2(Ax) < 0$

has a solution or the following systems

$$\begin{aligned}(\mathrm{ii}_{x^*,y^*,z^*})\quad &(g_1 + h_1 \circ A)^*(p + x^* + A^*y^*) - g_2^*(x^*) - h_2^*(y^*)\\ &+ \left(\sum_{i=1}^m q_i\phi_i\right)_X^* \left(\sum_{i=1}^m q_i z_i^* - p\right) - \sum_{i=1}^m q_i\psi_i^*(z_i^*) \leq 0,\ p \in \mathbb{R}^n,\ q \in \mathbb{R}^m,\ q \geq 0\end{aligned}$$

where $x^* \in \mathrm{dom}(g_2^*)$, $y^* \in \mathrm{dom}(h_2^*)$, and $z^* \in \prod_{i=1}^m \mathrm{dom}(\psi_i^*)$, has a solution, but never both.

The final problem we consider is a special case of (P_A) where ψ_i is zero. This primal problem,

$$(P'_A) \qquad \inf_{\substack{\phi_i(x) \leq 0\\ i=1,\ldots,m,\\ x \in X}} \{g_1(x) - g_2(x) + h_1(Ax) - h_2(Ax)\}$$

where $g_1, g_2, h_1, h_2, \phi_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, for $i = 1, \ldots, m$, are proper convex functions and $A \in \mathbb{R}^{n\times n}$ is linear continuous operator, has two dual problems but here we only consider one of them.

$$(D'_A) \qquad \inf_{\substack{x^* \in \mathrm{dom}(g_2^*)\\ y^* \in \mathrm{dom}(h_2^*)}} \sup_{\substack{p \in \mathbb{R}^n\\ q \geq 0}} \left\{-(g_1 + h_1 \circ A)^*(p + x^* + A^*y^*) + g_2^*(x^*) + h_2^*(y^*) - \left(\sum_{i=1}^m q_i\phi_i\right)_X^*(-p)\right\}$$

The constraint qualification used for strong duality was

$$(CQ_0) \qquad \exists x' \in \bigcap_{i=1}^m \mathrm{ri}(\mathrm{dom}(\phi_i)) \cap \mathrm{ri}(\mathrm{dom}(g_1)) \cap \mathrm{ri}(A^{-1}(\mathrm{dom}(h_1))) \cap \mathrm{ri}(X) : \begin{cases}\phi_i(x') \leq 0 & i \in L\\ \phi_i(x') < 0 & i \in N\end{cases}$$

where as before, $L = \{i \in \{1, \ldots, m\} \,|\, \phi_i \text{ is affine}\}$ and $N = \{1, \ldots, m\} \setminus L$. The next two Farkas-type theorems are a result of these problems and (CQ_0).

Theorem 8.23. Suppose the constraint qualification (CQ_0) is satisfied. Then the following are equivalent:

(i) $x \in X, \phi_i(x) \leq 0, i = 1, \ldots, m, \Rightarrow\ g_1(x) - g_2(x) + h_1(Ax) - h_2(Ax) \geq 0$

(ii) $\forall x^* \in \mathrm{dom}(g_2^*)$ and $\forall y^* \in \mathrm{dom}(h_2^*)$, there exist $p \in \mathbb{R}^n$ and $q \in \mathbb{R}^m$, $q \geq 0$ such that

$$(g_1 + h_1 \circ A)^*(p + x^* + A^*y^*) - g_2^*(x^*) - h_2^*(y^*) + \left(\sum_{i=1}^{m} q_i\phi_i\right)_X^* (-p) \leq 0$$

Theorem 8.24. Suppose the constraint qualification (CQ_0) is satisfied. Then either the inequality system

(i) $x \in X,\ \phi_i(x) \leq 0, i = 1, \ldots, m,\ g_1(x) - g_2(x) + h_1(Ax) - h_2(Ax) < 0$

has a solution or the following systems

$$(\mathrm{ii}_{x^*,y^*})\quad (g_1 + h_1 \circ A)^*(p + x^* + A^*y^*) - g_2^*(x^*) - h_2^*(y^*) + \left(\sum_{i=1}^{m} q_i\phi_i\right)_X^* (-p) \leq 0,$$

$$p \in \mathbb{R}^n,\ q \in \mathbb{R}^m,\ q \geq 0$$

where $x^* \in \mathrm{dom}(g_2^*)$ and $y^* \in \mathrm{dom}(h_2^*)$, has a solution, but never both.

9 Conclusion

Using Fenchel-Lagrange duality, we presented dual problems to some DC programming problems and fractional programming problems. The method used to find the dual problem (D_{DC}) to (P_{DC}) was applied to these various problems, including

$$(P_A) \qquad \inf_{\substack{\phi_i(x)-\psi_i\leq 0 \\ i=1,\dots,m,\, x\in X}} \{g_1(x) - g_2(x) + h_1(Ax) - h_2(Ax)\}$$

Then we presented constraint qualifications to each pair of problems which guaranteed strong duality.

The results of Section 6 led to some Farkas-type theorems for each set of problems, presented in Section 8. Thus a summary of the main results from this paper are given in the form of Theorem 8.23 and Theorem 8.24. Here we can see the relationship of the nonconvex optimizations problems to their associated Fenchel-Lagrange dual problems.

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