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Stability of Autonomous Systems in the Plane

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Abstract

We present techniques for determining stability of autonomous systems. Lyapunov's two methods are emphasized as well as physical applications. LaSalle's theorem, an extension of Lyapunov's second method, is also featured.

Contents

1 Introduction

The study of dynamical systems has found applications in many scientific fields. Ranging from applied mathematics and physics to biology, medicine and sociology. Stability is a central concept in all of these studies. Our model (i.e. dynamical system) of a certain phenomena will always be influenced by external disturbances and uncertainties. What happens if the system is perturbed ever so slightly? Will it change behavior drastically or will it settle down again and behave 'nice' despite being meddled with? If an aircraft is suddenly hit by a gust of wind we don't want major differences from the route the plane had yesterday when it cruised the same trajectory in no wind. This would signify *stability*. A less dramatic example would be the motions of a pendulum. If we let it start near its vertical down position it will eventually reach this point and stay there. We could also release it near the vertical up position which would result in the pendulum quickly leaving that *unstable* point and instead start narrowing in on the stable point underneath. Although it could be argued that it was Newton who first touched on the subject *dynamical stability theory* in his *Principia Mathematica* when analyzing the motion of the moon; a more formal analysis was made by Lagrange in his famous treatise on analytical mechanics. Here Lagrange states that if a conservative system has an isolated minimum point of its potential energy then that point is 'stable'. A precise definition of stability is not given however. Lagrange's work was further developed by Dirichlet, Poisson and Poincaré. All these mathematicians have different stability definitions named after them in the modern literature[1]. The most complete (and most used nowadays) framework concerning stability theory is the work by the russian mathematician Aleksandr Lyapunov. We will describe his two methods for determining stability. The first, *Linearization*, is an approximation of solutions to differential equations while the second, *Lyapunov functions*, is more novel. This method lets us draw conclusions about the solutions without actually solving the differential equations. A natural continuation to the latter is *LaSalle's theorem* which extends Lyapunov's second method. The present study also highlights the subject's close ties with real physical systems.

[1]See [2] for an extensive treatment of stability analysis.

2 Linearization

Virtually all interesting phenomena that one can study using dynamical systems are nonlinear. However, with nonlinearity follows complexity and these systems can be very difficult to solve. We start this section with a brief discussion about the linear case. We then show that information can be obtained about the nonlinear system by approximating it with a linear one.

2.1 The general solution of linear autonomous plane systems

A linear autonomous system in the plane has the form

$$x'(t) = a_{11}x + a_{12}y + b_1 \tag{1}$$
$$y'(t) = a_{21}x + a_{22}y + b_2 \tag{2}$$

The main goal is to determine the *critical points* where both right-hand sides are zero. If (x_0, y_0) is a critical point, then the constant functions $x(t) \equiv x_0, y(t) \equiv y_0$ form a solution to (1) and (2).

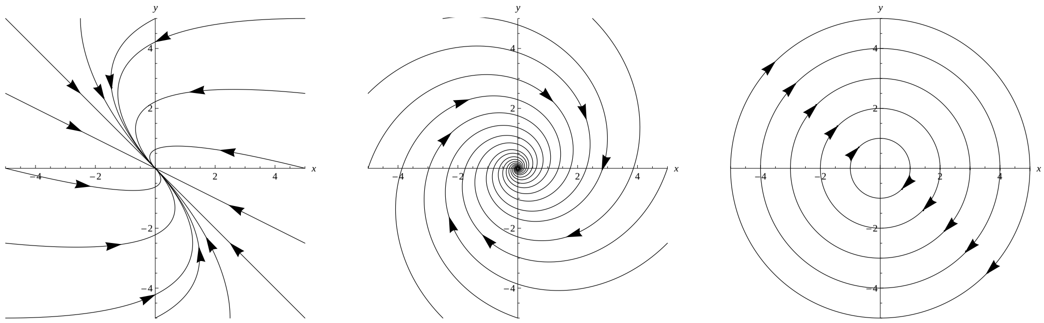
By a simple translation (where the critical point is moved to the origin) and by writing the above system in matrix form we have the following system

$$\dot{x} = Ax \text{ with } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

To find the solution to this system of differential equations we solve the *characteristic equation*

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

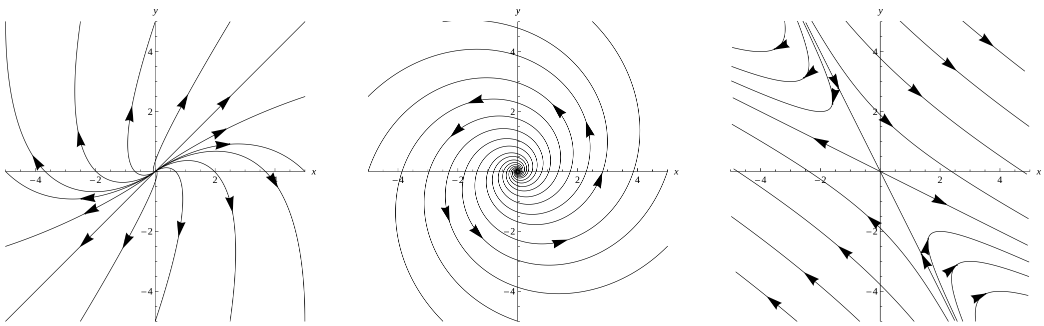
which gives us the eigenvalues λ_1 and λ_2. These will then give us the eigenvectors. If we plot the points $(x(t), y(t))$ of the solutions, for different initial values, in the $xy-$plane as t varies we get various types of trajectory configurations in the *phase plane*. The type of stability depends on the roots of the caracteristic equation. For example if λ_1 and λ_2 are real, distinct and negative we get the phase plane in figure 1 (a).



(a) Asymptotically stable node *(b) Asymptotically stable spiral* *(c) Stable center*

Figure 1: Stable equilibrium points

Figure 1 shows two types of stability for a critical point. *Asymptotically stable critical point* when the trajectories move towards the origin and *stable critical point* when the trajectories move clockwise or anti-clockwise around the origin. Loosely speaking, the critical point is stable if all trajectories that get sufficiently close to the point stay close. It is asymptotically stable if the trajectories approach the origin as $t \to \infty$. Figure 2 shows unstable behavior i.e. the trajectories move away from the critical point. The definitions of stability presented next are due to Lyapunov.



(a) Unstable node *(b) Unstable spiral* *(c) Unstable saddle*

Figure 2: Unstable critical points

Definition 1. Stability of a critical point Consider the nonlinear autonomous system

$$\frac{dx}{dt} = f(x, y) \tag{3}$$

$$\frac{dy}{dt} = g(x, y) \tag{4}$$

A critical point (x_0, y_0) is stable if, given any $\epsilon > 0$, there exists a $\delta > 0$ such that every solution $x = \phi(t)$, $y = \psi(t)$ of the system that satisfies

$$\sqrt{(\phi(0) - x_0)^2 + (\psi(0) - y_0)^2} < \delta$$

at $t = 0$ also satisfies

$$\sqrt{(\phi(t) - x_0)^2 + (\psi(t) - y_0)^2} < \epsilon$$

for all $t \geq 0$. If (x_0, y_0) is stable and there exists an $\eta > 0$ such that every solution $x = \phi(t)$, $y = \psi(t)$ that satisfies

$$\sqrt{(\phi(0) - x_0)^2 + (\psi(0) - y_0)^2} < \eta$$

at $t = 0$ also satisfies

$$\lim_{t \to \infty} \phi(t) = x_0 \text{ and } \lim_{t \to \infty} \psi(t) = y_0$$

then the critical point is asymptotically stable. A critical point that is not stable is unstable.

One might think that the property $\lim_{t\to\infty} \phi(t) = x_0$ and $\lim_{t\to\infty} \psi(t) = y_0$ is sufficient for stability of (3) and (4). There are however odd systems that show trajectories starting close to the critical point then travel away before turning back again. We see examples of these in the phase plane for the nonlinear system

$$\frac{dx}{dt} = x^3 - 2xy^2$$
$$\frac{dy}{dt} = 2x^2y - y^3$$

Such critical points are therefore unstable.

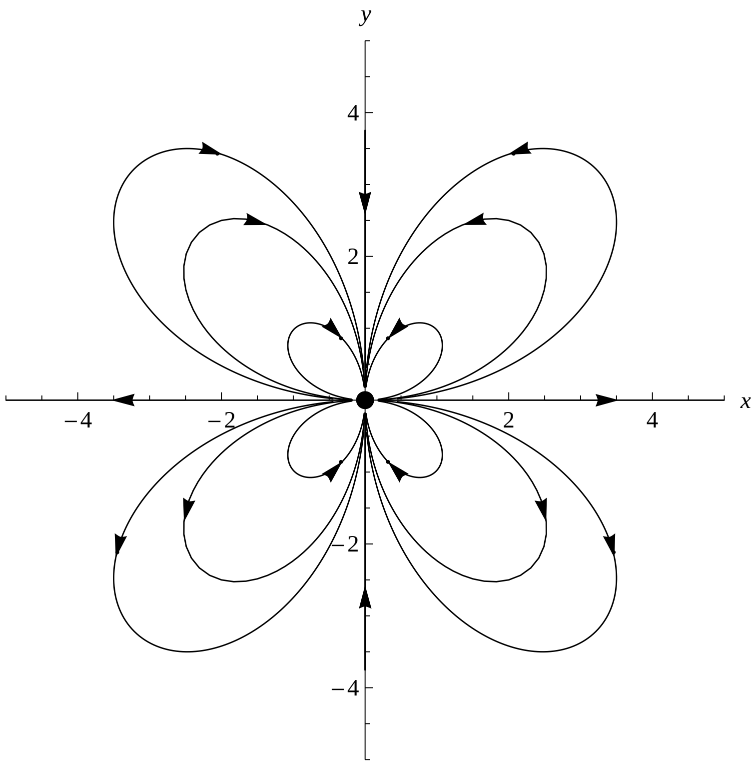


Figure 3: Unstable critical point

2.2 Linearization near critical points

We return to the nonlinear autonomous system with slightly different notation

$$\dot{x_1} = f_1(x_1, x_2) \tag{5}$$

$$\dot{x_2} = f_2(x_1, x_2) \tag{6}$$

Let $p = (p_1, p_2)$ be an equilibrium point of this system and suppose that the functions f_1 and f_2 are continuously differentiable. We can the expand these into their Taylor series about the point $p = (p_1, p_2)$. This results in

$$\dot{x_1} = f_1(p_1, p_2) + a_{11}(x_1 - p_1) + a_{12}(x_2 - p_2) + H.O.T.$$

$$\dot{x_2} = f_2(p_1, p_2) + a_{21}(x_1 - p_1) + a_{22}(x_2 - p_2) + H.O.T.$$

where H.O.T. denotes higher order terms of the expansion. Since our concern is trajectories near the point $p = (p_1, p_2)$ we define

$$y_1 = x_1 - p_1 \text{ and } y_2 = x_2 - p_2$$

and rewrite the equations in vector form as

$$\dot{y} = Ay$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$$

is the *Jacobian matrix* of $f(x)$ evaluated at the equilibrium point. This procedure is known as the *linearization* of the system (5) - (6) and is the principle for Lyapunov's first method. It is moreover true that if the origin of the linearized system is a stable/unstable node, a stable/unstable spirale or a saddle point; then the same behavior will be found in the trajectories for the nonlinear system.

Hartman-Grobman theorem *Provided that no eigenvalues of the linearization has real part equal to zero, the conclusions drawn as to the type of critical point will be the same as for the corresponding nonlinear system.*[2]

Example 1 Find and classify the equilibrium points of the system

$$\dot{x_1} = x_1^2 - x_2^2 + 1$$

$$\dot{x_2} = x_2 - x_1^2 + 5$$

The top equation equals zero if $x_1^2 = x_2^2 - 1$. This inserted in the second makes $x_2^2 - x_2 - 6 = 0$. We have two solutions for x_2 which generates four equilibrium points: $(2\sqrt{2}, 3), (-2\sqrt{2}, 3), (\sqrt{3}, -2)$ and $(-\sqrt{3}, -2)$.

The Jacobian matrix to be evaluated at each of these points in turn is

$$J(x_1, x_2) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2x & -2y \\ -2x & 1 \end{pmatrix}$$

At $(2\sqrt{2}, 3)$ equations (5) - (6) becomes

$$\begin{pmatrix} \dot{y_1} \\ \dot{y_2} \end{pmatrix} = \begin{pmatrix} 4\sqrt{2} & -6 \\ -4\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

[2]The proof is beyond the scope of this paper but a discussion regarding this theorem can be found in [7].

with $y_1 = x_1 - 2\sqrt{2}$ and $y_2 = x_2 - 3$. The eigenvalues of the coefficient matrix (the evaluated Jacobian) are found by solving

$$\begin{vmatrix} 4\sqrt{2}-\lambda & -6 \\ -4\sqrt{2} & 1-\lambda \end{vmatrix} = 0$$

where the roots are $\lambda = 3.328 \pm 6,274$. That is, we have distinct roots of opposite sign which indicates a saddle point at $(2\sqrt{2}, 3)$. This procedure is then repeated for every equilibrium point and we find that we have another saddle at $(-\sqrt{3}, -2)$ and two spirals, one stable at $(-2\sqrt{2}, 3)$ and one unstable at $(\sqrt{3}, -2)$. The phase plane is shown in figure 4.

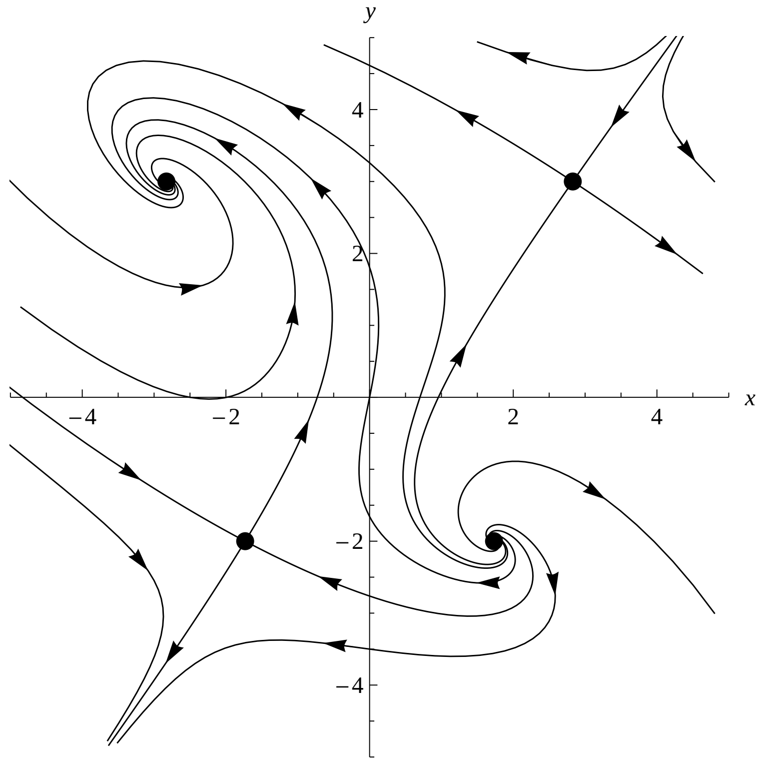


Figure 4: Phase plane of example 1

The reasoning above did not mention the case when the linearized system predicts a centre. If the Jacobian evaluated at a specific equilibrium point has pure imaginary eigenvalues we cannot be sure what kind of equilibrium point we have as the next example shows.

Example 2 Find and classify the equilibrium points of the system

$$\dot{x} = -y - x\sqrt{x^2 + y^2} \tag{7}$$

$$\dot{y} = x - y\sqrt{x^2 + y^2} \tag{8}$$

This system has only one equilibrium point, at the origin. The Jacobian we end up with looks a bit more intimidating now

$$J(x,y) = \begin{pmatrix} -\sqrt{x^2+y^2} - \frac{x^2}{\sqrt{x^2+y^2}} & -1 \\ 1 & -\sqrt{x^2+y^2} - \frac{y^2}{\sqrt{x^2+y^2}} \end{pmatrix}$$

However, there will be no singularities as we approach the origin because

$$\frac{x^2}{\sqrt{x^2+y^2}} \leq \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2} \to (0,0), \text{ as } (x,y) \to (0,0)$$

precisely because the functions (7) - (8) are continuously differentiable so the nonlinear terms can be discarded. The linearized system is

$$\begin{aligned} \dot{x} &= -y \\ \dot{y} &= x \end{aligned}$$

That makes the Jacobian evaluated at the origin

$$J(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and the eigenvalues are $\lambda = \pm i$. Hence, the linearized system has a center at the origin. This is unfortunately not true for the original nonlinear system. We can see that this is so by switching to polar coordinates (r, θ) given by $x = r\cos\theta$ and $y = r\sin\theta$. The system (7) - (8) now becomes, in terms of r and θ

$$\begin{aligned} \dot{r}\cos\theta - \dot{\theta}r\sin\theta &= -r\sin\theta - r^2\cos\theta \\ \dot{r}\sin\theta + \dot{\theta}r\cos\theta &= r\cos\theta - r^2\sin\theta \end{aligned}$$

By multiplying these lines with $\cos\theta$ and $\sin\theta$ respectively followed by the reverse multiplication we can solve for $\dot{r}$ and $\dot{\theta}$ and obtain

$$\dot{r} = -r^2, \ \dot{\theta} = 1$$

The radius vector is shrinking as time progresses. The origin is in fact a stable spiral. We can also see that the rotation is counterclockwise.

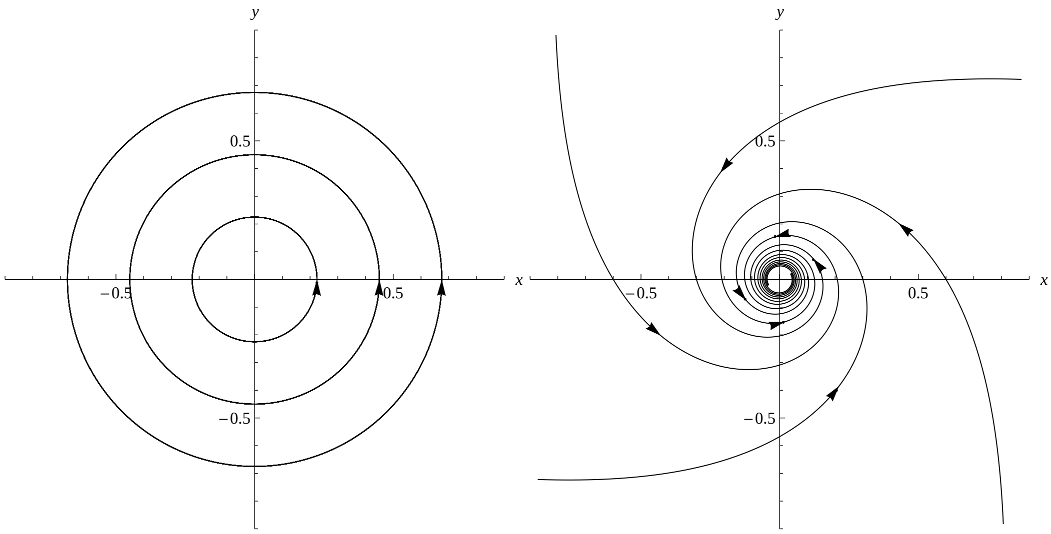


Figure 5: The linearized system and the nonlinear system from example 2

Example 2 shows us that linearization can fail. There are cases when this approach provides no information about the stability of an equilibrium point. In the preceding example we were able to solve the system by changing coordinates, but in general an explicit solution may prove impossible to find. However, using other methods will occasionally help us determine the behavior of the trajectories without actually solving the equations. This is the topic of the next section.

3 Energy Function Methods

Several differential equations arise from problems in mechanics. Therefore it is natural to consider the *energy* and the effect this quantity has on the solutions to the system. Indeed, the stability of a critical point can often be determined by analyzing the system as if it was a physical problem. This *energy-method* will, as we shall see, shed some light on the case when the corresponding linear equation predicts a center. These ideas can then be generalized to systems that are completely unrelated to mechanics. We start this section with Newton's second law where the force is conservative i.e. the work done on a particle is independent of the path taken. If the acceleration is written as vdv/dx, then $F = ma$ becomes

$$F(x) = mv\frac{dv}{dx}.$$

Separating variables and integrating from a given point x_0 where the velocity is v_0 to an arbitrary point x where the velocity is v gives us the following

$$\int_{x_0}^{x} F(s)ds = \int_{v_0}^{v} mudu \Rightarrow \int_{x_0}^{x} F(s)ds = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2.$$

The initial velocity v_0 corresponding to the initial position x_0 is a reference point that we are free to pick, generating a *constant.* We define $E \equiv \frac{1}{2}mv_0^2$ and get

$$E = \frac{1}{2}mv^2 - \int_{x_0}^{x} F(s)ds.$$

The first term on the right is the familiar *kinetic energy* and it is customary to refer to $V(x)$ as the *potential energy* where

$$V(x) \equiv -\int_{x_0}^{x} F(s)ds.$$

In a system with a conservative force the mechanical energy is conserved and this is just what we will find by writing the total energy as

$$E = \frac{1}{2}mv^2 + V(x).$$

Newton's second law is the condition that makes the total energy constant during the motion of the particle, because

$$\frac{dE}{dt} = mv\frac{dv}{dt} + \frac{dV}{dx}\frac{dx}{dt} = v(m\frac{dv}{dt} - F(x)) = 0.$$

The constant m is slightly annoying so we'll get rid of it by dividing throughout, setting $g(x) \equiv \frac{1}{m}\frac{dV}{dx}$ and use $a = \frac{d^2x}{dt^2}$ to obtain the *standard form of the differential equation for a conservative system*

$$\frac{d^2x}{dt^2} + g(x) = 0;$$

and the equivalent *phase plane system*

$$\begin{aligned} \dot{x} &= v, \\ \dot{v} &= -g(x); \end{aligned}$$

the *potential function*

$$G(x) \equiv \int g(x)dx + C;$$

and *energy function*

$$E(x,v) = \frac{1}{2}v^2 + G(x). \tag{9}$$

As the total energy is constant the phase plane trajectories of the system can be viewed as the *level curves*

$$E(x,v) = k, \quad k \text{ a constant.} \tag{10}$$

Inspecting the potential function $G(x)$ will yield much information since $G^{'}(a) = g(a) = 0$ actually implies that the point $(a,0)$ is a critical point. Furthermore, if we place the graph of $G(x)$ directly above the phase plane the relative extrema for the potential function will end up over the corresponding critical points. To see how these graphs relate, look at the level curves for the energy function. Solving for v gives

$$v = \pm\sqrt{2(k - G(x))}$$

The $\pm$ sign tells us that the phase plane has to be symmetric with respect to the horizontal axis and that the velocity only exists if the quantity under the radical is ≥ 0. If $G(x)$ has a strict local maximum at $x = x_0$, the trajectories in the phase plane will touch the x-axis at x_0 but for $k > G(x_0)$ they will not. If $k < G(x_0)$ there is an intervall about x_0 where there are *no* points of the curve $E(x,v) = k$ as the velocity becomes imaginary. Thus we have a saddle point at x_0. The level curves near $(x_0, 0)$ in the phase plane resembling hyperbolas.

If $G(x)$ has a strict local minimum at x_1, a level curve $k_1 > G(x_1)$ results in *two* curves that join up to produce a closed curve. This occurs for any k satisfying $G(x_1) < k \leq k_1$ so the critical point is encircled by closed trajectories. Hence, this is a center! This is a conclusion drawn without any approximations. We are therefore free to trust the linearization if it predicts a center at a certain point $(x,0)$ for a conservative dynamical system

$$\dot{x} = y, \qquad \dot{y} = -g(x)$$

provided that the same point is a local minimum of the potential function

$$G(x) \equiv \int g(x)dx + C$$

Example 3 Use the energy method to determine the critical points for the equation

$$\frac{d^2x}{dt^t} + x - x^4 = 0.$$

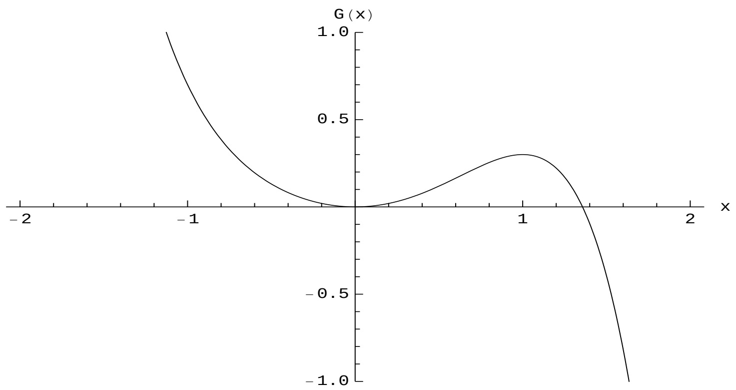
This equation can be rewritten as a conservative system (as if it described a physical problem originating from Newtons second law)

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -g(x) = -(x - x^4). \end{aligned}$$

The potential function is

$$G(x) = \frac{1}{2}x^2 - \frac{1}{5}x^5.$$

We get the extreme values for $G(x)$ by solving $G^{'}(x) = g(x) = x - x^4 = 0$. Here $G(x)$ has a minimum at $x = 0$ and a maximum at $x = 1$. The critical point $(0, 0)$ is a center and the critical point $(1, 0)$ is a saddle point. We place the graphs in the way described above and get figure 6:



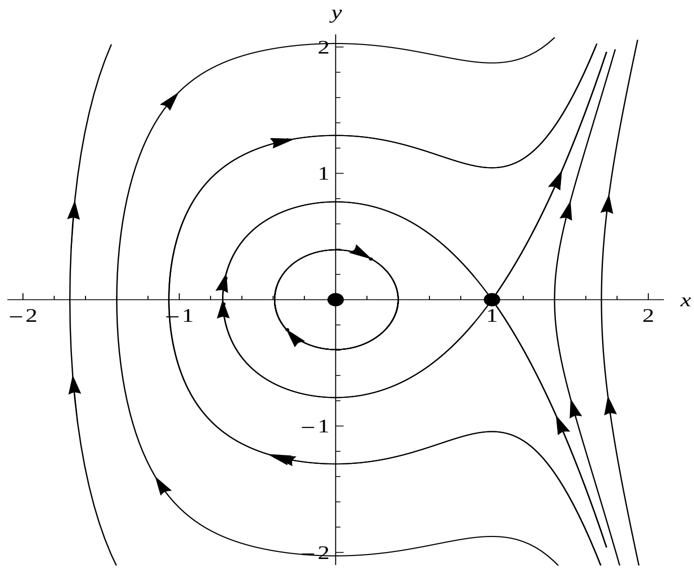


Figure 6: Phase plane for $\frac{d^2x}{dt^t} + x - x^4 = 0$

Linearization in this example gives us the Jacobian matrix

$$J(x_1, x_2) = \begin{pmatrix} 0 & 1 \\ -1 + 4x^2 & 0 \end{pmatrix}$$

which, when evaluated at $(0, 0)$ and $(1, 0)$, gives us the eigenvalues $\lambda = \pm i$ and $\lambda = \pm\sqrt{3}$ respectively. That is, a center at the origin and a saddle at $(1, 0)$.

Example 4 Use the energy method to determine the critical points for the equation

$$\frac{d^2x}{dt^2} + x - x^3 = 0.$$

This is also a conservative system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -g(x) = -(x - x^3); \end{aligned}$$

with the potential function

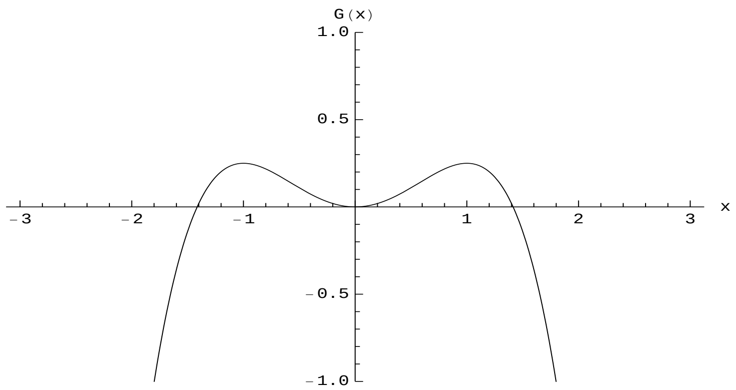
$$G(x) = \frac{1}{2}x^2 - \frac{1}{4}x^4.$$

The local maxima and minima of $G(x)$ are where $G^{'}(x) = g(x) = x - x^3 = 0$, that is at $x = 0, -1, 1$. The phase plane has critical points at $(0, 0)$, $(-1, 0)$, and $(1, 0)$. The first one is a center because the potential function has a local minimum for $x = 0$ while the last two constitutes saddle points as $G(x)$ has local maxima at $x = \pm 1$. The graphs are shown in figure 7. Once again we can show that linearization gives the same result. The Jacobian matrix

$$J(x_1, x_2) = \begin{pmatrix} 0 & 1 \\ -1 + 3x^2 & 0 \end{pmatrix}$$

evaluated at the three critical points $(0, 0)$, $(-1, 0)$, and $(1, 0)$ yields the eigenvalues

$$\begin{aligned} \lambda &= \pm i && \text{(center)} \\ \lambda &= \pm\sqrt{2} && \text{(saddle)} \\ \lambda &= \pm\sqrt{2} && \text{(saddle)} \end{aligned}$$



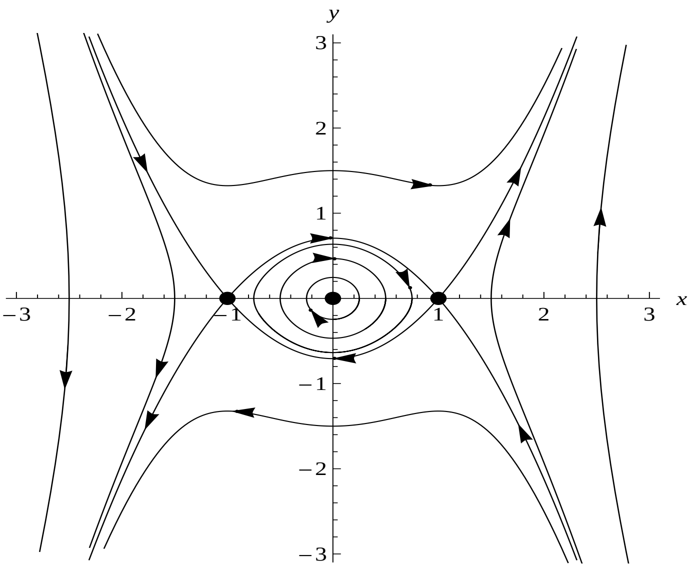


Figure 7: Phase plane for $\frac{d^2x}{dt^t} + x - x^3 = 0$

Example 5 Determine the critical points for the equation

$$\frac{d^2x}{dt^2} + x^2 - x^4 = 0.$$

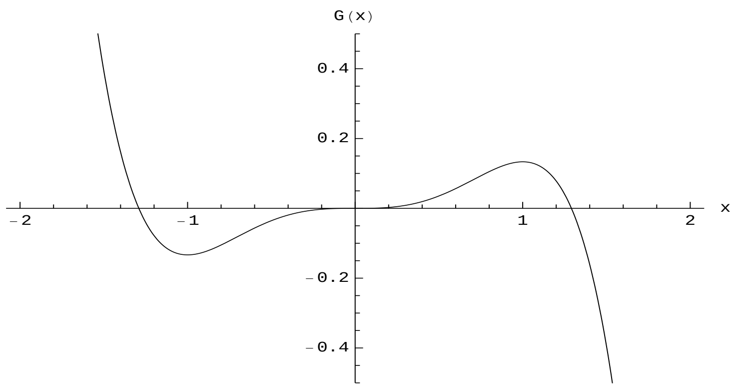
Rewritten as a conservative system this looks like

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= x^4 - x^2\end{aligned}$$

The potential function is

$$G(x) = \int (x^2 - x^4)dx = \frac{1}{3}x^3 - \frac{1}{5}x^5.$$

Now we get an *inflection point* at the origin for $G(x)$ as shown below in figure 8. Having dealt with both maxima and minima this is the third and final possibility we need to worry about.



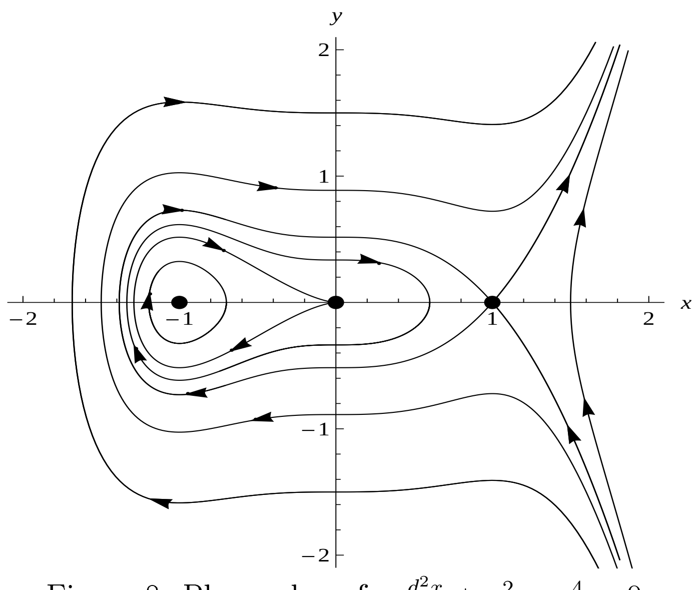


Figure 8: Phase plane for $\frac{d^2x}{dt^t} + x^2 - x^4 = 0$

If we try to determine the stability at $(0,0)$ by linearization we get the Jacobian matrix

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

which has $\lambda = \pm 0$. This would give us a *line* in the phase plane made up of critical points and that is not what we see. The phase plane at the bottom surely indicates that the origin is an unstable critical point. To convince ourselves, notice that $G^{'}(x) = g(x)$ is strictly increasing on the interval $[-1, 0]$. Now, if $x(t)$ satisfies $-1 < x(t_0) < 0$ then the acceleration $\frac{d^2x}{dt^t} = -g(x)$ must be negativ. If $\frac{dx}{dt}(t_0) = y(t_0) < 0$ then $x(t)$ will decrease to -1, *no matter how close the point* $(x(t_0, y(t_0))$ *is to* $(0,0)$. Therefore the origin must be unstable as there are trajectories leaving it. The same reasoning could be made concerning a situation where $G^{'}(x) = g(x)$ is strictly *decreasing* in a closed interval to the right of a critical point. As a simple analogy we can think of a ball rolling along the potential energy graph. Inflection points and local maximums are obviously unstable points for the ball while local minimums are stable. This is sometimes presented as *Lagrange's stability theorem*, stated in his *Méchanique Analytique* and proved later by Dirichlet.

Lagrange's stability theorem *If in a certain rest position* x_0, *where* $G^{'}(x_0) = 0$, *a conservative mechanical system has minimum potential energy, then this position corresponds to stable equilibrium; if the rest position does not correspond to minimum potential energy, then the equilibrium is unstable.*[3]

3.1 Pendulums

Ever since Galileo started using a pendulum for time keeping it has been at the heart of many areas in science. It has helped us making sense of navigation, allowing measuring the earths gravity, even exploring vibrating atoms and building robots. The pendulum is a nonlinear system that does not have an explicit analytic solution (although this is usually glossed over by the small-angle approximation $\sin\phi \approx \phi$) but the phase plane approach and energy method will prove helpful here.

Example 6 The nonlinear pendulum has the equation $\ddot{\theta} + \frac{g}{l}\sin\theta$ where g is the gravitational constant and l is the length of the pendulum. We rewrite this as

[3] A proof is given in [1].

$$\ddot{x} + \omega^2 \sin x = 0. \tag{11}$$

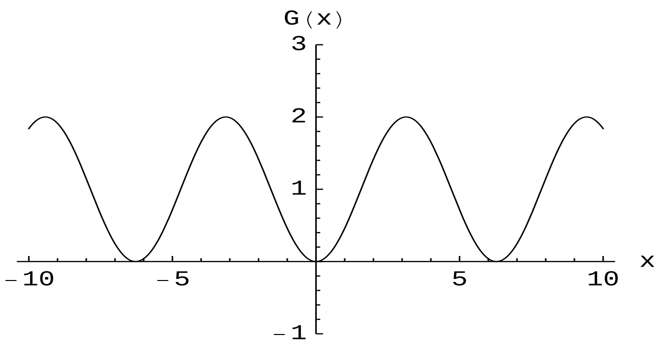
This is an undamped pendulum so there is no loss of energy in the system. The phase plane system is

$$\dot{x} = y,$$
$$\dot{y} = -g(x) = -\omega^2 \sin x.$$

The potential function is

$$G(x) = -\omega^2 \cos x + C$$

where the constant $C = \omega^2$ as the potential energy equals zero at $(0, 0)$. The phase plane is shown in figure 9, setting $\omega^2 = 1$ for simplicity.



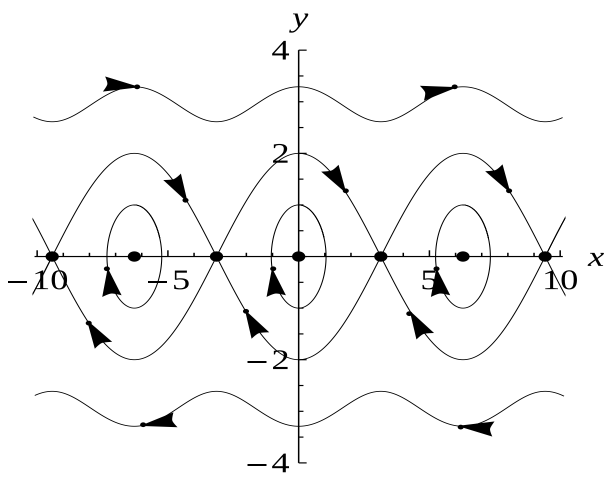


Figure 9: Phase plane for $\frac{d^2x}{dt^t} + \sin x = 0$

The wiggly trajectories at the top and bottom of the phase plane represent the pendulum swinging over the top counter clockwise and clockwise respectively. The inner circles are the usual pendulum movements, back and forth around the equilibrium, not having enough energy to swirl. Separating these two types of pendulum motions are the *separatrices*. These represent the pendulum starting in the inverted position and making a full swirl in either direction to finish in the same unstable upside down position. The time to make this revolution is actually infinite and this is the special case when the total energy E *exactly* equals the maximum of the potential function. This example shows once again that where the potential function has a local minimum or local maximum we get a center and saddle point respectively.

Example 7 A more realistic pendulum is the nonlinear one, given by the equation

$$\ddot{x} + b\dot{x} + \omega^2 \sin x = 0 \tag{12}$$

and the phase plane equations

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\omega^2 \sin x - by. \end{aligned}$$

This is a *non-conservative* system so we can no longer use the fact that the energy remains constant along the trajectories in the phase plane. However, if we use the energy function $E(x, y) = \frac{1}{2}y^2 + G(x)$ from the preceding example and write

$$\frac{dE}{dt} = y\dot{y} + g(x)\dot{x} = \frac{dx}{dt}\left(\frac{d^2x}{dt^2} + g(x)\right) = -b\dot{x}.$$

The energy is obviously *decreasing* if $b > 0$ and *increasing* if $b < 0$. The energy-method is not applicable here but linearization will work as we don't expect any centers where the motion is perpetual. The critical points are $y = 0$ and $x = \pm n\pi$ where $n \in Z$. The Jacobian matrix is

$$J(x, y) = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos x & -b \end{pmatrix}$$

If n is an even number we get

$$J(x, y) = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -b \end{pmatrix}$$

which generates the eigenvalues

$$\lambda = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - \omega^2}$$

This is two complex eigenvalues with negative real part which tells us that we have an asymptotically stable spiral at these points. The corresponding Jacobian if n is an odd number is

$$J(x, y) = \begin{pmatrix} 0 & 1 \\ \omega^2 & -b \end{pmatrix}$$

with eigenvalues

$$\lambda = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} + \omega^2}.$$

This consequently represents unstable saddle points. This is confirmed when we look at the phase plane which looks a little different from the nonlinear pendulum without a *friction constant* b. If we take $b = 0.1$ the phase plane from the last example starts tilting as the trajectories now zoom in on the equilibrium points. There are no endless motion anymore (no separatrices) as energy is constantly being lost.

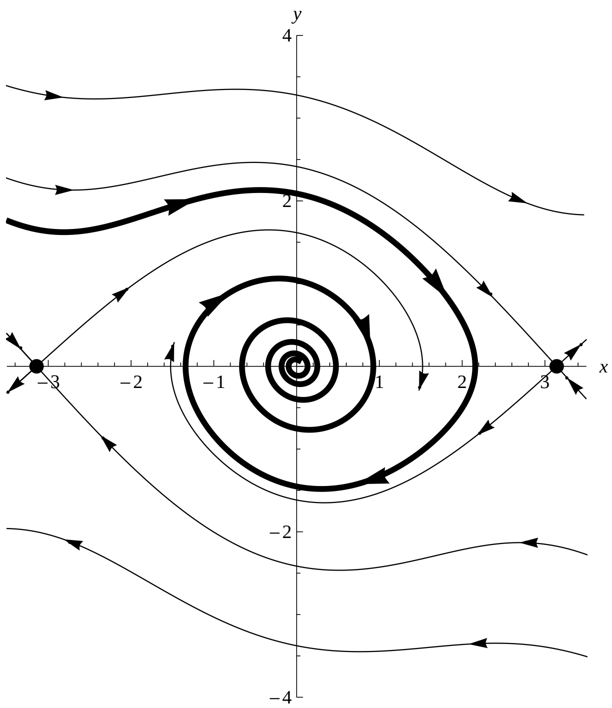


Figure 10: Phase plane for $\ddot{x} + 0.1\dot{x} + \omega^2 \sin x = 0$

As b increases so does the tilting. Figure 11 shows the system with $b = 0.6$.

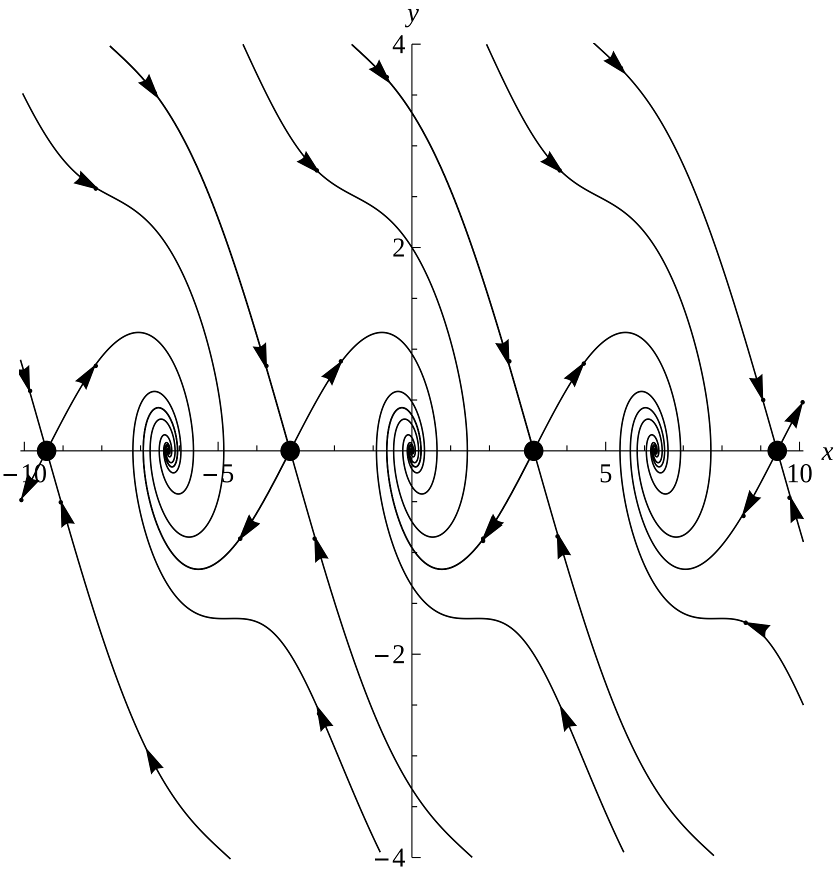


Figure 11: Phase plane for $\ddot{x} + 0.6\dot{x} + \omega^2 \sin x = 0$

4 The Lipschitz condition

In order for the differential equation $\dot{x} = f(t, x)$ to be a tool for describing a physical system it has to fulfill some fundamental properties. These include existence and uniqueness of solutions for the initial-value problem

$$\dot{x} = f(t, x) \qquad x(t_0) = x_0 \tag{13}$$

This can be accomplished by imposing the so-called *Lipschitz condition*, where the function satisfies the inequality

$$|f(t, x) - f(t, y)| \leq L\,|x - y| \tag{14}$$

These functions are said to be *Lipschitz in x* (or just *Lipschitz*) and the constant L is called a *Lipschitz constant.*

Example 8 The function $f(x) = \sqrt{x}$ is Lipschitz on $[1, \infty)$ but not on the intervall $[0, 1]$

$$\begin{aligned}|f(x_1) - f(x_2)| = |\sqrt{x_1} - \sqrt{x_2}| &= \left| \frac{(\sqrt{x_1} - \sqrt{x_2})(\sqrt{x_1} + \sqrt{x_2})}{\sqrt{x_1} + \sqrt{x_2}} \right| \\ &= \frac{1}{\sqrt{x_1} + \sqrt{x_2}} |x_1 - x_2| \leq \frac{1}{2} |x_1 - x_2|\end{aligned}$$

for all $x_1, x_2 \in [1, \infty)$. We say that $f(x) = \sqrt{x}$ is *locally Lipschitz* on the domain $D = (1, \infty)$ (open and connected set) because each point in D has a neighborhood where f satisfies (14). On the second interval, $[0, 1]$, the Lipschitz condition cannot be satisfied however as the expression $\frac{1}{\sqrt{x_1}+\sqrt{x_2}}$ is unbounded. Hence there does not exist a Lipschitz constant in this case. We say that f is not *globally Lipschitz* on R.

The Lipschitz condition is sometimes introduced via Picard's existence theorem. This theorem shows existence and uniqueness by constructing a sequence of functions that converges to a solution. Some versions of the proof[4] uses the fact that $f(t, x)$ and $\frac{\partial f}{\partial x}(t, x)$ are continuous functions on a compact domain. The compactness results in the latter being bounded i.e. there exists a constant K such that $\left|\frac{\partial f}{\partial x}(t, x)\right| \leq K$. This fact can be used to strengthen Picard's theorem by assuming that $\frac{\partial f}{\partial x}(t, x)$ satisfies the Lipschitz condition while relaxing the continuity condition. The main point here being that *a continuously differentiable function is Lipschitz.*

Proof [5]

By the fundamental theorem of calculus we have for all $(t, x), (t, y) \in R$, where R is a compact domain

$$f(t, x) - f(t, y) = \int_y^x \frac{\partial f}{\partial s}(t, s) ds$$

hence we get

[4] See [8]

[5] This proof actually deals with the more simple rectangle-domain for the variable x rather than the general *convex* domain although the result is still the same. A general proof is given in [4].

$$
\begin{aligned}
|f(t,x)-f(t,y)| &= \left|\int_y^x \frac{\partial f}{\partial s}(t,s)ds\right| \\
&\le \left|\int_y^x \left|\frac{\partial f}{\partial s}(t,s)\right| ds\right| \\
&\le \left|\int_y^x K ds\right| \\
&= K\,|x-y|
\end{aligned}
$$

The premise that $\frac{\partial f}{\partial x}(t,x)$ is only assumed to be bounded makes existence proofs using Lipschitz conditions stronger than Picard's theorem because the partial derivative doesn't have to be continuous anymore. In fact, a Lipschitz function doesn't have to be (continuously) differentiable.

Example 9 The function $f(x) = |x|$ is Lipschitz as

$$|f(x) - f(y)| = ||x| - |y|| \le |x - y|$$

so the condition is met for any (x, y) with Lipschitz constant $L = 1$. The function is not differentiable at $x = 0$.

If one should drop the Lipschitz condition and just stress that the function $f(t, x)$ be continuous then the uniqueness part of (13) suffers.

Example 10 The initial-value problem

$$\dot{x} = 3x^{2/3}, \quad x(0) = 0$$

has both $x(t) = t^3$ and $x(t) \equiv 0$ as solutions. As the function $3x^{2/3}$ is continuous in x continuity is not sufficient to guarantee uniqueness of the solution.

Example 11 Every function that is Lipschitz is uniformly continuous. Since $|f(x) - f(y)| \le L\,|x - y|$, we can choose for any ϵ a corresponding $\delta(\epsilon) = \frac{\epsilon}{2L}$. This makes

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| \le \frac{\epsilon}{2} < \epsilon$$

If we write the Lipschitz condition

$$\frac{|f(x) - f(y)|}{|x - y|} \le L$$

and interpret this geometrically it says that any line segment that joins two arbitrary points on the graph of f cannot have a slope greater than $\pm L$. So any function that displays an infinite slope at any point is not Lipschitz. This of course excludes all discontinuous functions from being Lipschitz.

What this has been leading up to is the following important theorem:[6]

Theorem 1 Let $f(t,x)$ be piecewise continuous in t and[7] locally Lipschitz in x for all $t \geq t_0$ and all x in a domain $D \subset R^n$. Let W be a compact subset of D, $x_0 \in W$, and suppose it is known that every solution of

$$\dot{x} = f(t,x), \quad x(t_0) = x_0$$

lies entirely in W. Then there is a unique solution that is defined for all $t \geq t_0$.

5 Lyapunov stability

Aleksandr Lyapunov's doctoral thesis *The General Problem of the Stability of Motion* marks the beginning of modern stability theory. His methods provide a powerful framework for analyzing nonlinear dynamical systems. *Lyapunov functions* lets us draw conclusions about the behavior of the dynamical system without actually solving it explicitly. The method presented here is called *Lyapunov's second method* or *Lyapunov's direct method* because it allows us to *directly* apply it to the differential equation without any knowledge about the solutions. In the following theorem we state without loss of generality that the equilibrium point is at the origin. As we have seen; any equilibrium can always be shifted to $(0,0)$ via a change of variables.

To establish stability the main idea is to find a new function $V(x,y)$ whose level curves encircle the equilibrium point and whose values decrease along the trajectories of the system. The bowls in Figure 12 is the function V and in the two examples we plot $V(x(t),y(t))$ versus the solution trajectory $(x(t),y(t))$ which lie in the horizontal plane.

[6] For a proof see [4].

[7] The function is only piecewise continuous in t to allow for abrupt step changes with time e.g. electrical signals.

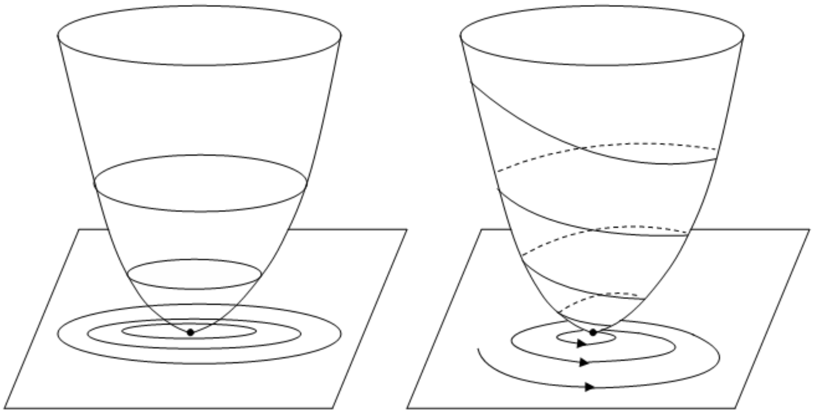


Figure 12: Stable and asymptotically stable equilibrium point

We start with the autonomous system

$$\dot{x} = f(x) \tag{15}$$

where $f : D \to R^n$ is a locally Lipschitz map from a domain $D \subset R^n$ into R^n.

Lyapunov's stability theorem Let $x = 0$ be an equilibrium point for (15) and $D \subset R^n$ be a domain containing $x = 0$. Let $V : D \to R$ be a continuously differentiable function such that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\} \tag{16}$$

$$\dot{V}(x) \leq 0 \text{ in } D \tag{17}$$

Then $x = 0$ is stable. Furthermore, if

$$\dot{V}(x) < 0 \text{ in } D - \{0\} \tag{18}$$

then $x = 0$ is asymptotically stable.

Proof Given $\epsilon > 0$, choose $r \in (0, \epsilon]$ such that

$$B_r = \{x \in R^n : |x| \leq r\} \subset D$$

Let $\alpha = \min_{|x|=r} V(x)$. Then, $\alpha > 0$ by (16). Take $\beta \in (0, \alpha)$ and let

$$\Omega_\beta = \{x \in B_r \,|V(x) \leq \beta\}$$

Then Ω_β is the interior of B_r. Any trajectory starting at $t = 0$ inside Ω_β stays in Ω_β as the derivative $\dot{V}(x(t))$ is negative semidefinite, so

$$\dot{V}(x(t)) \leq 0 \Rightarrow V(x(t)) \leq V(x(0)) \leq \beta, \ \ \forall t \geq 0$$

Because Ω_β is a compact set, we conclude from Theorem 1 that (15) has a unique solution defined for all $t \geq 0$ whenever $x(0) \in \Omega_\beta$. As $V(x)$ is continuous and $V(0) = 0$, there is $\delta > 0$ such that

$$|x| \leq \delta \Rightarrow V(x) < \beta$$

Then,

$$B_\delta \subset \Omega_\beta \subset B_r$$

and

$$x(0) \in B_\delta \Rightarrow x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \Rightarrow x(t) \in B_r$$

Therefore,

$$|x(0)| < \delta \Rightarrow |x(t)| < r \leq \epsilon, \ \ \forall t \geq 0.$$

This shows that the equilibrium point $x = 0$ is stable.

If $\dot{V} < 0$, i.e. *negative definite*, we know that the origin is stable from the previous arguments. If it can be proven that $V(x(t)) \to 0$ (implying that $x(t) \to 0$ because V is positive definite everywhere, except at the origin where $V(0) = 0$) as $t \to \infty$, this will be sufficient to prove asymptotic stability. Suppose to the contrary that there exists a solution $x(t)$ to (15) that does *not* approach the origin as $t \to \infty$. Since $V(x(t))$ is monotonically decreasing and apparently bounded from below by some constant L and as $V : D \to R$ is continuous there exists an $h > 0$ such that $V(x) < L$ for all $x \in B_h$. Hence the trajectory $x(t)$ is trapped inside the annulus $\bar{A} \equiv \{x : h \leq |x| \leq r\}$. Since this is a compact set the continuous function $\dot{V}(x)$ has a maximum value $\gamma(< 0)$ over this set. For this trajectory, forever moving inside $\bar{A}$, we have

$$\dot{V}(x(t)) \leq \gamma.$$

Integrating

$$\int_0^t \dot{V}(x(s))ds \leq \int_0^t \gamma ds$$

and it follows that

$$V(x(t)) \leq V(x(0)) + \gamma(t-0).$$

Since $\gamma < 0$, the right-hand side must eventually become negative for t sufficiently large. This contradicts the fact that V is positive definite. The assumption that there exists a solution that doesn't approach the origin is false. Hence, we have $V(x(t)) \to 0$ as $t \to \infty$.

The principal limitation with this method is that there are no general procedures for constructing a Lyapunov function. There are however several good approaches for finding a candidate.

Example 12 For the system

$$\begin{aligned} \dot{x} &= -x + xy^3, \\ \dot{y} &= -3x^2y^2 - y^3 \end{aligned}$$

we want to prove that its only equilibrium point $(0,0)$ is an asymptotically stable critical point. This can be achieved by constructing a suitable Lyapunov function of the form $V(x,y) = ax^{2m} + by^{2n}$, which will guarantee positive definiteness. This gives us

$$\dot{V}(x,y) = -2max^{2m} + 2max^{2m}y^3 - 6nbx^2y^{2n+1} - 2nby^{2n+2}.$$

By choosing $m = n = 1$ and $a = 3$ and $b = 1$ the cross product terms, which are sign indefinite, cancels and we get a Lyapunov function

$$V(x,y) = 3x^2 + y^2 \text{ with } \dot{V}(x,y) = -6x^2 - 2y^4.$$

This is a positive definite function with a negative definite time derivative, hence $(0,0)$ is a (globally)[8] asymptotically stable critical point for the system.

Example 13 In the section dealing with linearization we had the following system

$$\dot{x} = -y - x\sqrt{x^2 + y^2}, \tag{19}$$

$$\dot{y} = x - y\sqrt{x^2 + y^2} \tag{20}$$

where we had to revert to using polar coordinates to prove asymptotic stability. If we rewrite (19) - (20) as

[8] For a full description of *the region of attraction*, see [2], [4] or [7].

$$\dot{x} = -y - xf(x,y),$$
$$\dot{y} = x - yf(x,y)$$

and use the Lyapunov function candidate $V(x,y) = x^2 + y^2$ we get

$$\dot{V} = 2x(-y - xf(x,y)) + 2y(x - yf(x,y)) = -2(x^2+y^2)f(x,y).$$

Because $f(x,y) = \sqrt{x^2+y^2} > 0$ for all (x,y) except the origin (the point we are interested in) it is clear that $\dot{V}$ is negative definite. The conclusion is that the origin is (globally) asymptotically stable.

5.1 Gradient Systems

This is a particular type of dynamical system where the direct method of Lyapunov is well suited. Gradient systems all have the form

$$\dot{x} = -\nabla h(x)$$

for some function $h(x)$. Suppose we are given a system $\dot{x} = f(x)$ as in (15) where, without loss of generality, the origin is a critical point. The task is now to find a function $h(x)$ such that

$$f(x) = -\nabla h(x), \;\; h(0) = 0 \;\text{ and }\; h(x) > 0 \;\; \forall x \neq 0.$$

If that succeeds we choose this function h as a Lyapunov function for the system $\dot{x} = f(x)$, i.e., $V(x) = h(x)$ which gives us a negative definite derivative because

$$\dot{V}(x) = \nabla V(x) \cdot f(x) = \nabla h(x) \cdot (-\nabla h(x)) = -\left|\nabla h(x)\right|^2 < 0, \;\; \forall x \neq 0.$$

The components of $h(x)$ are assumed to be continuously differentiable functions. If $f(x) = -\nabla h(x)$ then

$$\frac{\partial f_i}{\partial x_j} = -\frac{\partial^2 h}{\partial x_j \partial x_i} = -\frac{\partial^2 h}{\partial x_i \partial x_j} = \frac{\partial f_j}{\partial x_i}.$$

This is a necessary condition that has to be met for the system to be a gradient system. We then have to check that the other conditions hold.

Example 14 The following system is gradient

$$\dot{x}_1 = -x_1 + x_2 - x_1 x_2^2 e^{x_1^2},$$
$$\dot{x}_2 = x_1 - x_2 - x_2 e^{x_1^2}$$

because

$$\frac{\partial f_1}{\partial x_2} = 1 - 2x_1 x_2 e^{x_1^2} = \frac{\partial f_2}{\partial x_1}.$$

The critical points occur when $x_2 = \frac{x_1}{1+e^{x_1^2}}$ and when this is inserted in f_1 we get $0 = -x_1 e^{x_1^2}(1+e^{x_1^2}+x_1^2)$ which makes $(0,0)$ the only critical point for the system. The next step is to recover the function $h(x_1, x_2)$ so that $f = -\nabla h$.

$$f_2 = -\frac{\partial h}{\partial x_2} \Rightarrow \frac{\partial h}{\partial x_2} = -x_1 + x_2 + x_2 e^{x_1^2}$$

$$h(x_1, x_2) = -x_1 x_2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_2^2 e^{x_1^2} + \alpha(x_1).$$

Equating the two expressions for $\frac{\partial h}{\partial x_1}$ we get

$$-x_2 + x_1 x_2^2 e^{x_1^2} + \alpha'(x_1) = -(-x_1 + x_2 - x_1 x_2^2 e^{x_1^2})$$

from which we learn that

$$\alpha'(x_1) = x_1 \Rightarrow \alpha(x_1) = \frac{1}{2}x_1^2 + C, \text{ where } C = 0 \text{ as } h(0) = 0$$

We conclude that the function

$$h(x_1, x_2) = \frac{1}{2}(x_1 - x_2)^2 + \frac{1}{2}x_2^2 e^{x_1^2}$$

is a Lyapunov function for the system and that $(0,0)$ is (globally) asymptotically stable.

5.2 The Variable Gradient Method

If the system in $\dot{x} = f(x)$ is not gradient (which it usually is not) we can still work backwards and find a Lyapunov function through a procedure known as the *variable gradient method*. Since we are searching for a positive definite function $V(x)$ the first step is to find a function $g(x)$ that is the gradient of that function i.e.

$$g(x) = \begin{pmatrix} \frac{\partial V}{\partial x_1} \\ \frac{\partial V}{\partial x_2} \end{pmatrix} = \nabla V.$$

The derivative $\dot{V}(x)$ is given by

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = g^T(x) f(x)$$

and that has to be negative definite. Since $g(x) = \nabla V$ we obtain $V(x)$ through integration taken along the axes;

$$V(x) = \int_0^{x_1} g_1(y_1, 0) dy_1 + \int_0^{x_2} g_2(x_1, y_2) dy_2$$

as the integral here is independent of the path joining the origin and x. This last fact is also used in the proof[9] of the important fact that *a function* $g(x)$ *is a gradient of a scalar function* $V(x)$ *if and only if the Jacobian matrix* $\left[\frac{\partial g}{\partial x}\right]$ *is symmetric.*

Example 15 Consider the stability of the system

$$\begin{aligned} \dot{x}_1 &= -x_1, \\ \dot{x}_2 &= -2x_2 + x_1 x_2^2 \end{aligned}$$

the origin being an isolated equilibrium point. Assume that $g(x)$ has the form

$$g(x) = \begin{pmatrix} a x_1 + b x_2 \\ b x_1 + c x_2 \end{pmatrix}, \text{ for some constants } a, b, c.$$

This is where the trial and error begins and we choose $b = 0$, keeping the Jacobi matrix symmetric. The derivative $\dot{V}(x)$ is

$$\dot{V}(x) = \begin{pmatrix} a x_1 & c x_2 \end{pmatrix} \begin{pmatrix} -x_1 \\ -2x_2 + x_1 x_2^2 \end{pmatrix} = -a x_1^2 - 2c x_2^2 + c x_1 x_2^2$$

Take $a = c = 1$ to get

$$\dot{V}(x) = -x_1^2 - (2 - x_1 x_2) x_2^2 < 0, \text{ if } x_1 x_2 < 2.$$

The Lyapunov function is

$$V(x) = \int_0^{x_1} y_1 dy_1 + \int_0^{x_2} y_2 dy_2 = \frac{1}{2}(x_1^2 + x_2^2) > 0 \text{ for } x \neq 0.$$

This is an example of a function that is *locally* asymptotically stable in some domain $D \subset R^2$ containing the equilibrium point $x = 0$.

[9] See [2].

5.3 Mechanical systems

Finding a Lyapunov function is a matter of trial and error. However, there are cases when natural candidates exists. If the system in question is mechanical (or can be interpreted as one) a possible Lyapunov function is the total energy of the system. It is obvious that Lyapunov's second method is an extension of the energy function method but in these cases the two approaches merge.

Example 16 The system for the undamped pendulum is

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\omega^2 \sin x_1 \end{aligned}$$

and we wish to prove that the origin is stable via Lyapunov's method. The potential energy for the undamped pendulum is

$$G(x_1) = \int \omega^2 \sin x_1 dx_1 + C = -\omega^2 \cos x_1 + \omega^2$$

because the potential energy at $x_1 = 0$ is zero. The total energy (kinetic plus potential) for the undamped pendulum is

$$E(x_1, x_2) = \frac{x_2^2}{2} + \omega^2(1 - \cos x_1)$$

Here $E(0,0) = 0$ and the function is positive definite over the domain $-2\pi < x_1 < 2\pi$ and

$$\frac{dE}{dt}(x_1, x_2) = x_2 \frac{dx_2}{dt} + \omega^2 \sin x_1 \frac{dx_1}{dt} = x_2(-\omega^2 \sin x_1) + \omega^2 \sin x_1(x_2) = 0.$$

The trajectories of our Lyapunov function circle the origin and we conclude that $(0,0)$ is a stable equilibrium point.

Example 17 If we try the same Lyapunov (energy) function applied to the *damped* pendulum

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\omega^2 \sin x_1 - bx_2 \end{aligned}$$

we get

$$\frac{dE}{dt}(x_1,x_2) = x_2\frac{dx_2}{dt}+\omega^2\sin x_1\frac{dx_1}{dt} = x_2(-\omega^2\sin x_1-bx_2)+\omega^2\sin x_1(x_2) = -bx_2^2.$$

This is negative semidefinite because at any position along the x_1-axis, where $x_2 = 0$, we will get $\dot{E} = 0$. From linearization as well as from constructing the phase portrait we know that the origin is *asymptotically* stable. The energy function from the undamped case fails to show this fact. We can once again use the variable gradient method to find a better Lyapunov function. Try

$$g(x) = \begin{pmatrix} \alpha(x)x_1 + \beta(x)x_2 \\ \gamma(x)x_1 + \delta(x)x_2 \end{pmatrix}$$

where the scalar functions have to be determined. The symmetry condition of the Jacobian matrix is

$$\frac{\partial\alpha}{\partial x_2}x_1 + \beta(x) + \frac{\partial\beta}{\partial x_2}x_2 = \gamma(x) + \frac{\partial\gamma}{\partial x_1}x_1 + \frac{\partial\delta}{\partial x_1}x_2$$

and the derivative $\dot{V}(x) = g^T(x)f(x)$ becomes

$$\alpha(x)x_1x_2 + \beta(x)x_2^2 - \gamma(x)x_1\omega^2\sin x_1 - \gamma(x)x_1bx_2 - \delta(x)x_2\omega^2\sin x_1 - \delta(x)bx_2^2.$$

As $\dot{V}(x)$ has to be < 0 the sign indefinite terms has to cancel, i.e.

$$x_2\left[\alpha(x)x_1 - \gamma(x)bx_1 - \delta(x)\omega^2\sin x_1\right] = 0$$

and $x_2 \equiv 0$ is not possible because that would violate Lyapunov's definition so the bracketed terms are zero. We get

$$\dot{V}(x) = -\left[b\delta(x) - \beta(x)\right]x_2^2 - \gamma(x)x_1\omega^2\sin x_1.$$

We now simplify and take every scalar function in $g(x)$ to be a constant, everyone that is, except $\alpha(x)$. Rearranging our bracketed terms above to

$$\alpha(x)x_1 = b\gamma x_1 + \delta\omega^2\sin x_1$$

it is clear, since the right hand side only depends on x_1 that $\alpha(x) = \alpha(x_1)$. The function $g(x)$ now looks like this:

$$g(x) = \begin{pmatrix} \alpha(x)x_1 + \beta x_2 \\ \gamma x_1 + \delta x_2 \end{pmatrix} = \begin{pmatrix} b\gamma x_1 + \delta\omega^2\sin x_1 + \gamma x_2 \\ \gamma x_1 + \delta x_2 \end{pmatrix}.$$

The Lyapunov function we get is

$$\begin{aligned} V(x) = \int_0^x g^T(y)dy &= \int_0^{x_1} (b\gamma y_1 + \delta\omega^2 \sin y_1 + 0)dy_1 + \int_0^{x_2} (\gamma x_1 + \delta y_2)dy_2 \\ &= \frac{1}{2}\left[b\gamma x_1^2 + 2\gamma x_1 x_2 + \delta x_2^2\right] + \delta\omega^2(1 - \cos x_1). \end{aligned}$$

If we choose $\delta > 0$ the last term will always be positive for $-\pi < x_1 < \pi$. The first bracket can be written as

$$\frac{1}{2}\left[b\gamma\left(x_1 + \frac{1}{b}x_2\right)^2 + \left(\delta - \frac{\gamma}{b}\right)x_2^2\right]$$

showing us that for $V > 0$ and $\dot{V} < 0$ we should pick $0 < \gamma < b\delta$. Take $\gamma = \frac{b}{2}\delta$ to arrive at the Lyapunov function

$$V(x) = \frac{\delta b^2}{4}x_1^2 + \frac{\delta b}{2}x_1 x_2 + \frac{\delta}{2}x_2^2 + \delta\omega^2(1 - \cos x_1).$$

This will be a suitable Lyapunov function for the nonlinear damped pendulum using any positive constant δ.

The last example shows how Lyapunov's direct method occasionally fails to show asymptotic stability. This is actually the case for every conservative system of the type

$$\ddot{x} + f(x) = 0$$

with the equivalent system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -f(x_1) \end{aligned}$$

where f (which we think of as a restoring force on a spring or a pendulum) is locally Lipschitz on $(-a, a)$ and satisfies

$$f(0) = 0; \;\; xf(x) > 0, \;\; \forall x \neq 0, \;\; x \in (-a, a).$$

A Lyapunov candidate

$$V(x_1, x_2) = \frac{x_2^2}{2} + \int_0^{x_1} f(s)ds$$

in the form of the total energy satisfies $V(0,0) = 0$ and $V > 0$ in the region $\{(x_1, x_2) : |x_1| < a, |x_2| < \infty\}$. The derivative however is

$$\dot{V} = x_2(-f(x_1)) + f(x_1)x_2 = 0$$

as the example for the undamped pendulum shows. This *might* be acceptable as we used this equation as an *approximation* for a pendulum, completely leaving out external forces in our model. However, any system originating from the differential equation

$$\ddot{x} + \dot{x} + f(x) = 0$$

that is, the system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -f(x_1) - x_2 \end{aligned}$$

(e.g. the damped pendulum, the motion of a mass-spring system with air resistance etc.) where f satisfies the same criteria as above, will *also* always give us a non-satisfactory answer; a negative *semidefinite* time derivative of the Lyapunov (energy) function candidate above as

$$\dot{V} = x_2(-f(x_1) - x_2) + f(x_1)x_2 = -x_2^2.$$

Lyapunov's direct method does not deliver the result we know to be true. Namely, that the origin is *asymptotically* stable for the damped pendulum as energy is continually lost. However, if we look again at the damped pendulum where $\dot{V} = -bx_2^2 \leq 0$ it is clear that

$$\dot{V} = 0 \Rightarrow \dot{x}_1 = 0 \text{ and } \ddot{x}_1 = \dot{x}_2 = -\omega^2 \sin x_1.$$

Since $\ddot{x}_1 \neq 0$ if $x_1 \neq k\pi$, for any integer k, that is, if the pendulum is not hanging straight down with zero velocity or standing straight up with zero velocity, we will have a nonzero acceleration. This will cause the velocity, $\dot{x}_1 = x_2$, to not remain at zero and $\dot{V}$ will again be strictly negative, making the Lyapunov function V to start decreasing once more. The system can only end up in one unique position if we choose $-\pi < x_1 < \pi$; hanging straight down with $\dot{x}_1 = 0$. This is intuitively correct for a pendulum loosing energy.

6 LaSalle's Theorem

The last section shows that although the variable gradient method can help us to prove asymptotic stability in difficult cases, we can adapt other arguments and still prove this fact and rid ourselves of much algebraic manipulation in the process. The strategy is to find a Lyapunov function with $\dot{V} \leq 0$ and then establish that no system trajectories can stay indefinitely at points where the functions derivative vanishes. Then the origin (the equilibrium point) is asymptotically stable. This is *Lasalle's invariance principle* for nonlinear dynamical systems. A few definitions are necessary. We start with the same prerequisites as in section 5. Let $x(t)$ be a solution to

$$\dot{x} = f(x) \tag{21}$$

where $f : D \to R^n$ is a locally Lipschitz map from a domain $D \subset R^n$ into R^n. A point p is said to be a *positive limit point* of $x(t)$ if there is a sequence $\{t_n\}$, with $t_n \to \infty$ as $n \to \infty$, such that $x(t_n) \to p$ as $n \to \infty$. The set of all positive limit points of $x(t)$ is called the *positive limit set* of $x(t)$. A set M is said to be an *invariant set* with respect to (21) if

$$x(0) \in M \Rightarrow x(t) \in M, \ \ \forall t \in R$$

That is, if a solution, at some time instant, belongs to M, then it belongs to M for all time, past or future. A set M is said to be a *positively invariant set* if

$$x(0) \in M \Rightarrow x(t) \in M, \ \ \forall t \geq 0$$

We say that $x(t)$ approaches a set M as $t \to \infty$, if for each $\epsilon > 0$ there is a $T > 0$ such that

$$dist(x(t), M) < \epsilon, \ \ \forall t > T$$

where $dist(x(t), M)$ denotes the smallest distance from a point p to any point in M. The following lemma[10] is a fundamental property of limit sets.

Lemma If a solution $x(t)$ of (21) is bounded and belongs to D for $t \geq 0$, then its positive limit set L^+ is a nonempty, compact, invariant set and $x(t)$ approaches L^+ as $t \to \infty$.

Lasalle's theorem Let $\Omega \subset D$ be a compact set that is positively invariant with respect to (21). Let $V : D \to R$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$. Let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.

Proof Let $x(t)$ be a solution of (21) starting in Ω. Since $\dot{V}(x) \leq 0$ in Ω

$$V(x(t)) - V(x(0)) = \int_0^t \dot{V}(x(s))ds \leq 0, \ \ t \geq 0$$

[10]Proven in [4].

so $V(x(t)) \leq V(x(0))$ for $t \geq 0$ implying that $V(x(t))$ is a nonincreasing function of t. Since $V(x)$ is continuous on the compact set Ω, it is bounded from below on Ω. Therefore, $V(x(t))$ has a limit a as $t \to \infty$. The positive limit set L^+ has to be in Ω because Ω is a closed set. For any $p \in L^+$, there is by definition a sequence t_n with $t_n \to \infty$ and $x(t_n) \to p$ as $n \to \infty$. By continuity of $V(x)$,

$$V(p) = V(\lim_{n\to\infty} x(t_n)) = \lim_{n\to\infty} V(x(t_n)) = \lim_{t\to\infty} V(x(t)) = a$$

on L^+, i.e. a constant function. By the lemma L^+ is an invariant set, $\dot{V}(x) = 0$ on L^+. Therefore $L^+ \subset E$. Because L^+ is invariant and M is the largest invariant set in E the conclusion is that M must contain L^+. Thus

$$L^+ \subset M \subset E \subset \Omega.$$

Since $x(t)$ is bounded, the lemma guarantees that $x(t)$ approaches L^+ as $t \to \infty$. Hence, $x(t)$ approaches M as $t \to \infty$.

The important fact to appreciate here is that $\dot{V}(x)$ doesn't have to be negative definite as in Lyapunov's theorem. We also note that this theorem can cope with systems having an *equilibrium set* (as opposed to an isolated equilibrium point). To show that $x(t) \to 0$ as $t \to \infty$ we have to ascertain that the largest invariant set in E is the origin.

Barbashin's theorem Let $x = 0$ be an equilibrium point for (21). Let $V : D \to R$ be a continuously differentiable positive definite function on a domain D containing the origin $x = 0$, such that $\dot{V}(x) \leq 0$ in D. Let $S = \left\{ x \in D \,\middle|\, \dot{V}(x) = 0 \right\}$ and suppose that no solution can stay identically in S, other than the trivial solution $x(t) \equiv 0$. Then, the origin is asymptotically stable.

Proof Lyapunov stability follows from the positive definiteness of V and $\dot{V}(x) \leq 0$. To prove asymptotic stability consider a compact and invariant set

$$\Omega_c = \{x \in D \,|\, V(x) \leq c\}$$

for some positive constant c. Let $x(t)$ be a solution starting in this set, $x(0) \in \Omega_c$. Lasalle's theorem implies that $L^+ \subseteq M$ but M is the *largest* invariant set contained in S so $M = \{0\}$. Therefore, $x(t) \to M = \{0\}$ as $t \to \infty$ establishing asymptotic stability of the zero solution $x(t) \equiv 0$ to (21).

Example 18 Consider again the nonlinear system for the damped pendulum.

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -\omega^2 \sin x_1 - b x_2.\end{aligned}$$

The total energy $V(x_1, x_2) = \frac{x_2^2}{2} + \omega^2(1 - \cos x_1)$ as a Lyapunov function candidate gave us the derivative

$$\frac{dV}{dt}(x_1, x_2) = -b x_2^2 \leq 0.$$

Choose $c > 0$ such that $D = \{(x_1, x_2) \,|V(x_1, x_2) \leq c\}$ is compact. Therefore D is a positively invariant set. Let R be the set where the derivative vanishes. That is, $R = \{(x_1, x_2) \,|x_2 = 0\}$. It is clear that $\dot{V}(x_1, x_2) < 0$ everywhere except on the line $x_2 = 0$ where it is zero. Let M be the largest invariant set contained in R. Because the origin is an equilibrium point we have $(0, 0) \in M \subset R$. If the system is to guarantee $\dot{V}(x_1, x_2) = 0$, the trajectory of the system must lie on the line $x_2 = 0$. We have the implication

$$x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow \sin x_1(t) \equiv 0.$$

In the interval $x_1 \in (-\pi, \pi)$, the largest invariant set contained in R is $M = \{(0, 0)\}$. This proves that the origin is asymptotically stable.

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