SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Topics from discrete geometry and their continuous analogues

av

Isak Trygg Kupersmidt

2012 - No 25

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

Topics from discrete geometry and their continuous analogues

Isak Trygg Kupersmidt

Självständigt arbete i matematik 15 högskolepoäng, Grundnivå

Handledare: Paul Vaderlind

2012

Topics from discrete geometry and their continuous analogues.

Isak Trygg Kupersmidt

Abstract

This bachelor thesis investigates the interesting relation between theorems from algebraic topology and discrete geometry. The theorems that are covered are some versions of the Borsuk-Ulam theorem, Tucker's lemma, Sperner's lemma, Brouwer's fixed point theorem, as well as the discrete and continuous Ham Sandwich theorem together with some interesting extensions and the polynomial Ham Sandwich theorem.

Emphasis is put on easy and intuitive proofs using discrete geometry and algebra. The thesis also contains a brief introduction to some basic concepts from discrete geometry.

Contents

Acknowledgements

I would like to thank my adviser Dr. Paul Vaderlind for his guidance and support through the whole process and for introducing me to the subject. I would also like thank Dr. Joakim Arnlind for offering helpful comments on the thesis.

Chapter 1

Introduction

An interesting phenomena in algebraic topology is that every theorem seems to have a combinatorial interpretation. This means that there for every topological theorem exists an equivalent theorem concerning combinatorial properties of similar combinatorial structures.

Two of the most useful theorems from algebraic topology is the Borsuk-Ulam theorem and Brouwer's fixed point theorem. They both have many interesting applications and corollaries, some of which will be investigated here.

The relation between combinatorial and topological theorems is not a new discovery. In the beginning of the 20^{th} century the Albert W. Tucker was attempting to prove the equivalences between two theorems from different branches of mathematics, Sperner's lemma and the Borsuk-Ulams theorem. Sperner's lemma is a widely known result from discrete geometry about labeling of simplices while the Borsuk-Ulam theorem is a continuous theorem from algebraic topology. One reason for finding such relations is that while theorems in algebraic topology tend have advanced proofs and uses specific concepts from topology, discrete geometry geometry uses more fundamental ideas. By proving theorems from algebraic topology using discrete geometry the results become more accessible to non-topologists, something important as there are many useful results in algebraic topology.

In this thesis some topics from discrete geometry together with their continuous analogs in topology will be discussed together with some interesting corollaries, the Ham Sandwich thorems. The proofs in the thesis mainly follows others results and all sources can be found in the bibliography.

We will start with an introduction to the notation and concepts used in the thesis before moving on to the theorems.

Chapter 2

Concepts and notation

2.1 Important concepts and notation

The proofs and theorems in this thesis use concepts from mainly discrete geometry together with some algebraic topology. Concepts that are of use in specific contexts are introduced where they are being used while more fundamental ideas are introduced here together with some notation.

Notation. *A boldface character, i.e.* $\boldsymbol{x}$ *denotes a point in* $\mathbb{R}^n$*, while a normal character, i.e.* x*, denotes a real number. We usually write* $\boldsymbol{x} = (x_1, x_2, x_3, ..., x_n)$ *where* x_i *is the* i^{th} *component of* $\boldsymbol{x}$*. If* $f : \mathbb{R}^n \mapsto \mathbb{R}^m$*,* $f(\boldsymbol{x})_j$ *denotes the* j*:th component of* $f(\boldsymbol{x})$*.*

Notation. *A centered dot,* $\boldsymbol{x} \cdot \boldsymbol{a}$ *denotes the standard scalar product of* $\boldsymbol{x}$ *and* $\boldsymbol{a}$ *where* $\boldsymbol{x}$ *and* $\boldsymbol{a}$ *are vectors in* $\mathbb{R}^n$*. The scalar product is defined as follows.*

$$\boldsymbol{x} \cdot \boldsymbol{a} = (x_1, x_2, ..., x_n) \cdot (a_1, a_2, ..., a_n) = (x_1a_1, x_2a_2, ..., x_na_n)$$

The same notation will sometimes be used for normal multiplication.

Definition. *Two geometrical objects are said to be* ***homeomorphic*** *if there exists a continuous bijection between them with a continuous inverse.*

A more intuitive description of homeomorphism might be to say that two geometric objects are homeomorphic if they can be deformed into each other using continuous deformation where the objects are allowed to pass through themselves. For example, a sphere is homeomorphic to a cube, a circle is homeomorphic to a simple knot, but a plane and a sphere are not homeomorphic to each other.

The idea to regard many different shapes as the same with respect to some rule, for example homeomorphism, is used in all branches of mathematics.

For a more complete definition of homeomorphism, please see [1] or another book on algebraic topology.

Definition. *A* ***hyperplane*** H *is a (n-1)-dimensional subspace of* $\mathbb{R}^n$ *defined as*

$$H = \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{x} \cdot \boldsymbol{a} = b, \boldsymbol{a} \in \mathbb{R}^n\},$$

where we choose an $\boldsymbol{a}$ *such that* $b > 0$*. For* $n = 3$ *, a hyperplane is a normal two dimensional plane in* $\mathbb{R}^3$*.*

Definition. *A* ***half-space*** H^+ *or* H^- *is a subset of* $\mathbb{R}^n$ *defined using a hyperplane* H*. Let* H *be as above, then*

$$\begin{aligned} H^+ &= \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{x} \cdot \boldsymbol{a} > b, \boldsymbol{a} \in \mathbb{R}^n\} \\ H^- &= \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{x} \cdot \boldsymbol{a} < b, \boldsymbol{a} \in \mathbb{R}^n\} \end{aligned}$$

H^+ *is called the upper halfspace defined by* H*, and* H^- *is called the lower half space defined by* H*.*

Definition. *A function* $f : A \mapsto B$ *where* A *and* B *are metric spaces (a space with a distance function) and* d_A *and* d_B *are their distance functions respectively, is* ***uniformly continuous*** *if for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$d_A(x, y) \leq \delta, x, y \in A \quad \Rightarrow \quad d_B(f(x), f(y)) \leq \varepsilon.$$

This means that there for every ε*, exists a* δ *such that if two points in* A *are within* δ *of each other, their images under* f *are at most a distance of* ε *apart.*

Note that every continuous function defined on bounded sets are uniformly continuous.

2.1.1 Something about proofs

Note. *Many of the proofs in the thesis make use of the following fact: if* $A \Rightarrow B$ *we also have that* $\neg B \Rightarrow \neg A$*, meaning that if you prove that* A *implies* B *you have also proven that if* B *is false, so must* A *be. This is very useful in proving theorems declaring the nonexistence of different kinds of functions.*

Definition. *A* ***constructive*** *proof of a theorem is a proof that also supplies an algorithm for finding the objects in the proof.*

2.2 Introduction to discrete geometry

Discrete geometry is the mathematical study of discrete geometrical objects and their combinatorial properties. Most commonly, discrete geometry studies finite sets of points, lines and simplices. A typical question in discrete geometry could be "how many lines are required to form three bounded regions in a plane?" or "is it always possible to bisect two point sets in $\mathbb{R}^2$ using a line?".

Discrete geometry focuses on combinatorial properties such as number of objects or dimensions while ideas such as distance, volume, angle, and curvature usually are left out. It has big overlaps with fields such as graph theory, combinatorial topology, and combinatorics.

Discrete geometry is not only useful for answering typical geometrical questions, but can be used to prove theorems from many different fields of mathematics. Proofs making use of discrete geometry often have the properties that they are intuitive and easy to understand as they usually make use of only fundamental mathematical concepts.

In this section some of the basic concepts of discrete geometry as well as some of the techniques that will be used later on will be discussed. All geometric objects are assumed to be placed in $\mathbb{R}^n$ if not stated otherwise.

2.2.1 Basic concepts in discrete geometry

The fundamental building block of discrete geometry is the simplex. An n-dimensional simplex, called an n-simplex, can be said to be the simplest shape in n dimensions that still has a volume (or the dimension equivalent). They are defined as follows.

Definition. *A **n-simplex** is a set of $n+1$ affinely independent points and the smallest convex hull that contains them. The points are called the vertices of the simplex.*

For example, a 1-simplex is a line, a 2-simplex is a triangel, and a 3-simplex is a tetrahedron. They are illustrated in figure 2.1 below.

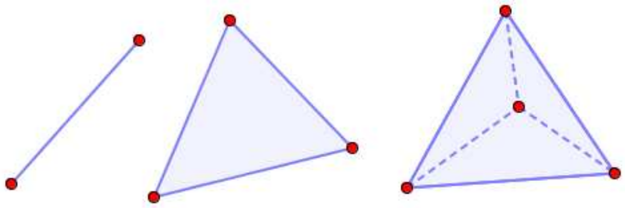

Figure 2.1: A 1-simplex, a 2-simplex, and a 3-simplex.

Every simplex is built up by a number of smaller simplices. As seen in the illustration above, a 2-simplex consists of 3 1-simplices and a 3-simplex consists of 4 2-simplices. More generally, a n-simplex consists of $n+1$ $(n-1)$-simplices. This is used to define the notion of faces.

Definition. *A **face** of a simplex is the convex hull of a subset of the vertices. Since each such subset also will be affinely independent, each face is a simplex. Every n-simplex contains exactly $n+1$ $(n-1)$-simplices.*

Simplices can be combined together to simplical complexes to form more advanced shapes. When we replace a shape with a simplical complexes we say that we have triangulated it. We make the following definitions.

Definition. *A **simplicial complex** is a union of simplices such that every two simplices meet in a simplex.*

Definition. *A **triangulation** of a geometric object $A \subset \mathbb{R}^n$ is a set of simplices $T = \{\sigma_i\}$ such that:*

- $\cup_i \sigma_i = A$
- *For any $\sigma_j, \sigma_k \in A$, $\sigma_j \cap \sigma_k$ is either empty or a face of both σ_j och σ_k.*

The notation $f : T \mapsto A$ where T is a triangulation and A is any set, will mean that f is a function from the vertices of the simplices in T, not from the simplices themselves.

In two dimensions the triangulation of an area is the division of the area in to a number of triangles. It has been shown that all shapes with a continuous boundary can be triangulated, but some requires an infinite amount of triangles. Triangulation is generalized to higher dimensions by replacing triangles with the corresponding simplex. A triangulation of a set is thus a simplical complex.

For a more complete discussion and definition of triangulation, please see [1] or another book on discrete geometry or topology.

We also make the following definition.

Definition. *A triangulation T is called **symmetric** if for every vertex $a \in T$, $-a$ is also a vertex in T.*

A common method in discrete geometry is to use labelings of simplices or simplical complexes. This means that a value is assigned to every vertex and will be described as a function from the vertices to some set of values. As simplical complexes can be interpreted as discrete versions of continuous objects, a labeling of the complex can thus be seen as a discrete counterpart to a function from points on the continuous object to some set.

Definition. *A simplex is said to have a **complementary edge** under a labeling function if it contains two adjacent vertices whose labels sum to zero, meaning that they have the same value but opposite sign. A triangulation has a complementary edge if it contains a simplex with at least one.*

Many of the theorems that will be discussed later concerns the n-sphere, usually denoted by S^n, and the triangulation of it. It is though not possible triangulate an ordinary sphere using a finite number of simplices which makes many of the later proofs much more complicated. Due to that, only the combinatorial n-sphere, Σ^n, will be discussed. We make the following definition.

Definition. *Σ^n denotes the **combinatorial sphere** which is defined as*

$$\Sigma^n = \{\boldsymbol{x} \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} |x_i| = 1\}$$

*We also define the **combinatorial ball** Ω^n as*

$$\Omega^n = \{\boldsymbol{x} \in \mathbb{R}^n : \sum_{i=0}^{n} |x_i| \leq 1\}.$$

Σ^n can be viewed as a n-dimentional octahedron and is illustrated for $n = 1$ and $n = 2$ below.

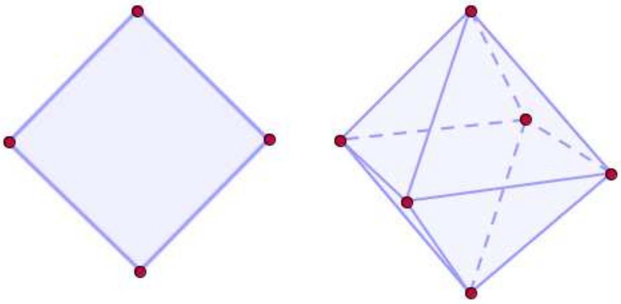

Figure 2.2: Illustration of Σ^1 and Σ^2.

As Σ^n is homeomorphic S^n, and Ω^n is homeomorphic to B^n, every result we will find in this thesis regarding Σ^n and Ω^n also applies for S^n and B^n respectively. For a more complete discussion of this please see [1].

Chapter 3

Tucker's lemma, the Borsuk-Ulam theorem and Fan's N+1 theorem

The first theorem that will be introduced is the Borsuk-Ulam theorem, a famous result from algebraic topology. A short sketch of the proof of the theorem will also be given. We will then look at Tucker's lemma and Fan's N+1 theorem which are two combinatrial theorems equivalent to the Borsuk-Ulam theorem. After the theorems are introduced the equivalence of the theorems will be proved.

3.1 The Borsuk-Ulam theorem

The Borsuk-Ulam theorem is one of the most useful theorems from algebraic topology with interesting applications in many different fields of mathematics. The theorem is named after Stanislaw Ulam, who conjectured it, and Karol Borsuk, who proved it in 1933. The Borsuk-Ulam theorem is interesting for many reasons. It can, first of all, be stated in many different but equivalent ways, and it has many interesting corollaries. For example the Ham Sandwich theorems which concerns measure theory, discrete geometry, and theory of polynomial. The Borsuk-Ulam theorem will later be used to prove those theorems, but first we will look at four of its many versions. These versions are chosen since they are easy to understand, even for an high school student, and have direct consequences for the later theorems that will be investigated.

Theorem 1. ***The Borsuk-Ulam Theorem (BU1).***
For every continuous mapping $f : \Sigma^n \mapsto \mathbb{R}^n$ *there exists a point* $\boldsymbol{a} \in \Sigma^n$ *such*

that $f(\boldsymbol{a}) = f(-\boldsymbol{a})$.

This is usually popularized by saying that there always are two antipodal places on earth with the same temperature and air pressure. In this case $n = 2$, and we have assumed that the function from every place on earth to the temperature and air pressure is continuous. Another way of exemplifying it for 2-spheres is to say that if you take a ball and deflate it, and lay it flat on the floor, there will always be two points lying on top of each other that were antipodal before the deflation.

Many of the theorems and the coming proofs uses what is called antipodal functions, which are continuous functions that maps antopodal points to antipodal points. Formally they are defined as follows.

Definition. *A function $f : A \mapsto B$ is called an **antipodal function** if it is continuous and maps antipodal points in A to antipodal points in B, meaning that $f(\boldsymbol{x}) = -f(-\boldsymbol{x})$ for all $\boldsymbol{x} \in A$.*

Antipodal functions are sometimes called odd functions when defined from $\mathbb{R}^n$, but looking at spheres the notion of antipodality makes more sense.

Using antipodal functions, some of the equivalent versions to $(BU1)$ can now be given.

Theorem 2. ***The Borsuk-Ulam Theorem (BU2).***
For every antipodal mapping $f : \Sigma^n \mapsto \mathbb{R}^n$ there exists a point $\boldsymbol{a} \in \Sigma^n$ such that $f(\boldsymbol{a}) = 0$.

Theorem 3. ***The Borsuk-Ulam Theorem (BU3).***
There exists no antipodal mapping $f : \Sigma^n \mapsto \Sigma^{n-1}$.

Theorem 4. ***The Borsuk-Ulam Theorem (BU4).***
There exists no continuous mapping $f : \Omega^n \mapsto \Sigma^{n-1}$ satisfying $f(\boldsymbol{x}) = -f(-\boldsymbol{x})$ for all $\boldsymbol{x} \in \Sigma^{n-1}$. Note that $\Sigma^{n-1} = \partial\Omega^n \subset \mathbb{R}^n$.

These different theorems may appear quite dissimilar from each other, but as we soon will see the proofs for their equivalences are very short, and sometimes close to trivial. Most continuous functions from $\mathbb{R}^n$ to itself may not have the required properties for the theorems to apply, but it is often possible to construct new functions with these properties using other functions. I.e. let $f : \Sigma^n \mapsto \mathbb{R}^n$ be a continuous function and consider the functions $h(x) = f(x) - f(-x)$ and $g(x) = \frac{f(x)}{|f(x)|}$ with $(g(x) \neq 0)$. It is easy to verify that h is an antipodal mapping and must thus have a zero, which gives us some insight about f, and that $g : \Sigma^n \mapsto \Sigma^{n-1}$. We will now use such constructions to see why all these statements are equivalent.

3.1.1 Proof of the equivalence of the different versions

The equivalence of the different statements will be shown in the following order by showing implications in both directions.

$$(BUA1) \Leftrightarrow (BUA2) \Leftrightarrow (BUA3) \Leftrightarrow (BUA4)$$

Proof. $(BU1) \Rightarrow (BU2)$.
(BU1) gives that every continuous mapping $f : \Sigma^n \to \mathbb{R}^n$ has a point $\boldsymbol{a} \in \Sigma^n$ such that $f(\boldsymbol{a}) = f(-\boldsymbol{a})$. But if f also is antipodal, then $f(\boldsymbol{x}) = -f(-\boldsymbol{x})$ for all $\boldsymbol{x}$. This can only be true if $f(\boldsymbol{a}) = f(-\boldsymbol{a}) = -f(-\boldsymbol{a})$, and $f(-\boldsymbol{a}) = -f(-\boldsymbol{a})$ only has the solution $f(\boldsymbol{a}) = 0$ and we get that $(BU1) \Rightarrow (BU2)$ □

Proof. $(BU2) \Rightarrow (BU1)$.
Let $g : \Sigma^n \mapsto \mathbb{R}^n$ be a continuous function and let $f(\boldsymbol{x}) = g(\boldsymbol{x}) - g(-\boldsymbol{x})$. It is clear that $f(\boldsymbol{x})$ is an antipodal mapping. (BU2) states that there is a point $\boldsymbol{a} \in \Sigma^n$ such that $f(\boldsymbol{a}) = g(\boldsymbol{a}) - g(-\boldsymbol{a}) = 0$. But this is only true if $g(\mathbf{a}) = g(-\mathbf{a})$, and thus $(BU2) \Rightarrow (BU1)$. □

Proof. $(BU2) \Rightarrow (BU3)$.
We start by noting that an antipodal mapping $h : \Sigma^n \mapsto \Sigma^{n-1}$ is also a map $h : \Sigma^n \mapsto R^n$ as $\Sigma^{n-1} \subset R^n$. As Σ^{n-1} does not contain the origin, h can not have a zero which contradict (BU2) and thus h can not exist, giving that $(BU2) \Rightarrow (BU3)$. □

Proof. $(BU3) \Rightarrow (BU2)$.
Assume to show contradiction that there is an antipodal map $f : \Sigma^n \mapsto \mathbb{R}^n$ without a zero. As $f(x)$ has no zeroes we can define an antipodal function $g(\boldsymbol{x})$ like this:

$$g(\boldsymbol{x}) = \frac{f(\boldsymbol{x})}{|f(\boldsymbol{x})|}.$$

It is clear that $g : \Sigma^n \mapsto \Sigma^{n-1}$, composing it with the obvious homeomorphism (which may be choosen to be antipodal) it is clear that we get a map $\Sigma^n \mapsto \Sigma^{n-1}$. But such a map can not exist according to $(BU3)$ and thus can f not exist either, which gives $(BU3) \Rightarrow (BU2)$. □

For the last two proofs a projection function from Σ^n to Ω^n will be used. Will call it θ and it is defined as

$$\theta(\boldsymbol{x}) = \theta(x_1, x_2, x_3, ..., x_{n-1}, x_n) = (x_1, x_2, x_3, ..., x_{n-1})$$

Further more, let θ_N denote the same map, but defined only from the upper hemisphere of Σ^n. The upper hemisphere of Σ^n is defined as

$$\{\boldsymbol{x} \in \mathbb{R}^{n+1} | x_1 + x_2 + x_3 + ... + x_n \leq 1 \, and \, x_i \geq 0\}.$$

This makes θ_N is bijective and thus it has an inverse, θ_N^{-1}.

Proof. $(BU3) \Rightarrow (BU4)$.
Assume to show contradiction that there is an antipodal mapping $h : \Omega^n \mapsto \Sigma^{n-1}$. Define $q(\boldsymbol{x}) = h(\theta(\boldsymbol{x}))$. Thus $q : \Sigma^n \mapsto \Sigma^{n-1}$, and as h is antipodal, so must q be. But (BU3) states that there is no such mapping, giving that the starting assumption is false and we have that $(BU3) \Rightarrow (BU4)$. □

Proof. $(BU4) \Rightarrow (BU3)$.
If there is an antipodal continuous map $f : \Sigma^n \mapsto \Sigma^{n-1}$ then the map $g : \Omega^n \mapsto \Sigma^{n-1}$ defined by $g(\boldsymbol{x}) = f(\theta_N^{-1}(\boldsymbol{x}))$ would be antipodal on Ω^n which contradict (BU4) and we get that $(BU4) \Rightarrow (BU3)$. □

We have now seen that the different statements are equivalent. There exists more versions of the theorem, mainly concerning coverings of spheres.

3.1.2 Sketch of the proof of the Borsuk-Ulam theorem

There exists many proofs of the Borsuk-Ulam theorem. In a newer one found in [10], it is first shown that if there exists an antipodal map $f : \Sigma^n \mapsto \Sigma^{n-1}$, then there must exist an antipodal map from Σ^{n-1} to Σ^{n-2}. It is then shown that there exists no antipodal map from Σ^2 to Σ^1, which by induction gives (BU3).

A more well-known proof is by using homotopy to show that any antipodal function $f : \Sigma^n \mapsto \mathbb{R}^n$ must have a zero. A complete proof together with a more complete discussion can be found in [1], but we give a short overview of it here.

We want to show that the antipodal map $f : \Sigma^n \mapsto \mathbb{R}^n$ has a zero. Let θ be the same projection map as above and define

$$F(\boldsymbol{x}, t) = t \cdot \theta(\boldsymbol{x}) = (1 - t) f(\boldsymbol{x})$$

with $0 \leq t \leq 1$.

For fixed t, $F(\boldsymbol{x}, t)$ thus "interpolates" between $\theta(\boldsymbol{x}) = F(x, 0)$ and $f(\boldsymbol{x}) = F(\boldsymbol{x}, 1)$. The domain of F is $\Sigma^n \times [0, 1]$, which can be visualized as a $n + 2$ dimensional cylinder. As both f and θ are antipodal, we see that $F(\boldsymbol{x})$ must also be antipodal with respect to x, meaning that $F(\boldsymbol{x}, t) = -F(-\boldsymbol{x}, t)$. This can be interpreted as opposite points on the cylinder is mapped to opposite points. It is also clear that F is continuous.

That two functions can be continuously deformed into each other like this is called that they are homotopic. Using some homotopy gives us that $F^{-1}(0)$ must consist of closed paths, or paths starting and ending at the top or the bottom of the cylinder. θ has two zeroes, and this means that $F(\boldsymbol{x}, t)$ have two zeroes when $t = 0$. If we follow the path of the zero set from one of those zeroes, it must either meet the path of the other zero, or end at the top of the cylinder. As F is antipodal the two path cannot meet, giving that both must end at the top, i.e. when $t = 1$. This gives that $F(x, 1) = f(x)$ must have a zero. We illustrate this in figure 3.1 below.

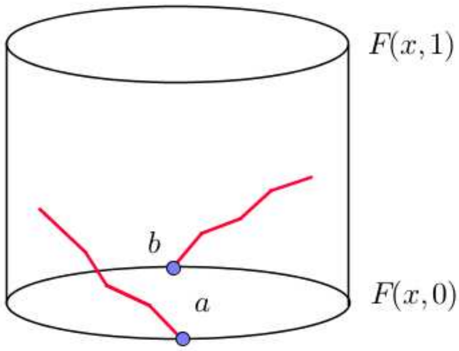

Figure 3.1: Illustration of $F^{-1}(0)$ (red) of $F(x, t)$ starting at the two points a and b.

As $F(x, t)$ is antipodal the two paths cannot meet, since if $F(\boldsymbol{a}, t) = 0$ we also have that $F(-\boldsymbol{a}, t) = -F(\boldsymbol{a}, t) = 0$, but (a, t) and $-(a, t)$ are on different paths.

3.2 Tucker's lemma

Tucker's lemma is a combinatorial analog to the Borsuk-Ulam theorem. Instead of functions from Σ^n to $\mathbb{R}^n$, it considers functions from an triangulation to a discrete set of real numbers.

The theorem was introduced by Alfred Tucker who tried to prove the equivalence between Sperner's lemma and Brouwer's fixed point theorem, a theorem that will be covered later. After showing the equivalence of the Borsuk-Ulam theorem and Tucker's lemma we will look in to Brouwer's fixed point theorem and show that Tucker's lemma in fact does imply Brouwer's theorem indirectly.

Theorem 5. ***Tucker's lemma***.
Let T be a symmetric triangulation of Σ^n. Then any antipodal labeling

$$\lambda : T \mapsto \{\pm 1, \pm 2, \pm 3, ..., \pm n\}$$

must assign opposite values to at least one pair of adjacent vertices. We call such a labeling a ***Tucker labeling*** *and say that T has a complementary edge under λ.*

Remember that a function from a triangulation by convention is a function from the vertices of the simplexes in the triangulation and that a symmetric triangulation is a triangulation where if a is a vertex of a simplex in the triangulation, so is $-a$.

The easiest way to understand this theorem might be to compare it to the Borsuk-Ulam theorem. Like (BU2), Tucker's lemma concerns an antipodal mapping that easily can be extended continuously to the whole sphere.

Tucker's lemma states the existence of two adjacent vertices in the triangulation with opposite labels under λ. This means that the two vertices are mapped to opposite sides of zero, and if we make the triangulation finer and finer the vertices will come closer and closer to each other. This means that for every ε we can find two vertices that are withing ε distance from each other and that are mapped to different sides of zero. If we then extend λ to the whole sphere we know that λ must be zero somewhere on the edge connecting those two points.

Later, a similar argument will be used to prove the equivalence of Tucker's lemma and the Borsuk-Ulam theorem by constructing Tucker labelings using arbitrary functions from Σ^n to $\mathbb{R}^n$, and the other way around.

3.3 Fan's N+1 theorem

Fan's N+1 theorem is another combinatorial analog to the Borsuk-Ulam theorem. As can be expected, the theorem is also very similar to Tucker's lemma. Fan introduced it in 1958 and it is in many ways a better analogue to the Borsuk-Ulam theorem than Tucker's lemma. While Tucker's lemma has a long history and is widely known, Fan's N+1 theorem is more useful for proving some of the later theorems, and this motivates its introduction.

We start by making the following definition.

Definition. *A simplex $\sigma = \{s_1, s_2, s_3, ..., s_m\}$ is said to be **alternating** under a labeling λ if the following conditions holds.*

- *If the vertices are indexed such that $|\lambda(s_i)| \leq |\lambda(s_{i+1})|$, then $|\lambda(s_1)| < |\lambda(s_2)| < |\lambda(s_3)| < ... < |\lambda(s_m)|$.*
- *If $\lambda(s_i)$ is positive then $\lambda(s_{i+1})$ is negative, and the other way around.*

Fan's N+1 theorem states the existence of at least one such simplex in every symmetric triangulation of S^n with a Tucker-like labeling. Formally it is stated as follows.

Theorem 6. ***Fan's N+1 theorem*** *Let T be a symmetric triangulation of Σ^n and let λ be any antipodal labeling*

$$\lambda : T \mapsto \{\pm 1, \pm 2, \pm 3, ..., \pm n, \pm(n+1)\}$$

*such that T has no complementary edges under λ. Then there exists at least one alternating simplex in T. We call such a labeling a **Fan labeling**.*

Comparing Tucker's lemma to Fan's, some interesting differences reveal them self. If λ is a antipodal labeling function from a symmetric triangulation T of Σ^n such that

$$\lambda : T \mapsto \{\pm 1, \pm 2, \pm 3, ..., \pm n\}$$

we get that there must be an complementary edge, but if λ maps to

$$\{\pm 1, \pm 2, \pm 3, ..., \pm n, \pm(n+1)\}$$

we either get an complementary edge and/or an alternating simplex.

3.4 Equivalence of Tucker's lemma, the Borsuk-Ulam theorems and Fan's N+1 theorem

One of the main points in this thesis is the relationship between Tucker's lemma, the Borsuk-Ulam theorems and Fan's N+1 theorem. Before investigating some of their curious corollaries we will show that they implies each other.

3.4.1 Tucker's lemma and the Borsuk-Ulam theorem

It is now time to establish the first relationship between a discrete and a continuous theorem. The method for doing so might appear strange, but is very similar to those we will use later. The idea is to use a Tucker labeling, as defined using the function f from (BU1), to create a sequence $\{\boldsymbol{x}_i\}_{i=0}^{\infty}$ such that $f(\boldsymbol{x}_i) \to 0$ when $i \to \infty$. When the triangulation with the Tucker labeling is made finer and finer we will see that the two vertices defining the complementary edge of the triangulation will define the required sequence.

Proof. Tucker's lemma ⇒ Borsuk-Ulam theorem (BU2).
We want to show that every continuous function $f : \Sigma^n \mapsto \mathbb{R}^n$ has a zero. If there exists a sequence of points $\{\boldsymbol{x}\}_{i=0}^{\infty}$ in Σ^n such that $|f(\boldsymbol{x}_i)| \leq \frac{1}{i}$ for all i then $f(x)$ must have a zero as Σ^n is compact. Here the supremum norm will be used instead of the quadratic norm to define length of a vector.

For every i let T_i be a new symmetric triangulation of the sphere such that the image of every vertex is within the distance $\frac{1}{i}$ from the image of the vertices adjacent to it. As f is a continuous function from a compact space it is uniformly continuous, so this is always possible as is easy verified by the definition of uniform continuity.

Let $g : \mathbb{R}^n \mapsto \mathbb{R}$ be such that $g(\boldsymbol{x})$ is the index of the the component of $\boldsymbol{x}$ with the largest absolute value. For example, $g(7, 8, -9) = 3$.

Let $\Phi(\boldsymbol{x}) = g(\boldsymbol{x}) \cdot sgn_{g(\boldsymbol{x})}(\boldsymbol{x})$ where $sgn_{g(\boldsymbol{x})}(\boldsymbol{x})$ denotes the sign of the $g(\boldsymbol{x})$:th component of $\boldsymbol{x}$. This means that Φ maps a vector to the index of it's largest component times the sign of the component. For example, $\Phi(7, 8, -9) = g(7, 8, -9) \cdot sgn(-9) = 3 \cdot (-1) = -3$.

For all i we label every simplex σ in T_i with $\Phi(\sigma)$. This is a Tucker labeling as antipodal points will have opposite signs and $\Phi : \Sigma^n \mapsto \{\pm 1, \pm 2, \pm 3..., \pm n\}$.

Tucker's lemma gives that there must be two adjacent vertices with opposite label in T. Let $\mathbf{a}_i$ denote the vertex with a positive label. As the image of the two points under f must be within the distance $\frac{1}{i}$ from each other and have opposite sign of their largest component, their image must also be within the distance $\frac{1}{i}$ from the origin. Thus $\{\mathbf{a}_i\}_{i=0}^{\infty}$ is the required sequence and we get that f must have a zero. $\square$

The next proof goes the other way around. Using a Tucker labeling we create a continuous function $f : \Sigma^n \mapsto \Omega^n$. We then show that when $f(\boldsymbol{x}) = 0$, two adjacent vertices of the triangulation must have opposite labels.

Proof. Borsuk-Ulam theorem (BU2) $\Rightarrow$ Tucker's lemma.
Let T be a triangulation of Σ^n and let λ be a Tucker labeling of T. Define $f : T \mapsto \mathbb{R}^n$ as follows.

$$f(\boldsymbol{x}) = \begin{cases} \text{the } \lambda(\boldsymbol{x})^{th} \text{ unit vector} & \text{if } \lambda(\boldsymbol{x}) > 0 \\ -f(-\boldsymbol{x}) & \text{if } \lambda(\boldsymbol{x}) < 0 \end{cases} \tag{3.1}$$

We now extend f linearly to the rest of the sphere in such a way that any simplex is mapped to the simplex spanned by the image of it's vertices giving $f : \Sigma^n \mapsto \mathbb{R}^n$. As vertices with opposite labels are mapped to opposite points in $\mathbb{R}^n$ the same thing is true for any points in Σ^n. The function f is thus an antipodal map. As f is antipodal $(BU2)$ states that it must have a zero, giving that some of the simplices in $\mathbb{R}^n$ must contain the origin. But a simplex with its vertices at plus/minus unit vectors, meaning that the vertices are of the form $(0, 0, ..., 1, ..., 0, 0)$ or $(0, 0, ..., -1, ..., 0, 0)$, can only contain the origin if it has at least two vertices with opposite label. And if two vertices have opposite labels under f they must also have it under λ, meaning that there must exist a pair of adjacent vertices with opposite labels in T. Note that every vertex of a simplex has a common edge with every other vertex of that simplex. $\square$

The idea that a simplex with its vertices at unit vectors must have opposite labels to contain the origin is very intuitive if given a little thought. One way to think about it is that if the simplex does not have any vertices with opposite labels, then the simplex is contained at one side of a hyperplane through the origin and does not contain it.

With the relation between the Borsuk-Ulam theorems and Tucker's lemma established it is now time to look at their relations to Fan's N+1 theorem.

3.4.2 Fan's N+1 theorem and the Borsuk-Ulam theorem

This proof is somewhat similar to the previous: to prove that Fan's N+1 theorem implies the Borsuk-Ulams theorem we create a Fan labeling using any continuous function $f : \Sigma^n \mapsto \mathbb{R}^n$ and show that for finer and finer triangulations of Σ^n we get a sequence of alternating simplices that converge towards a single point. We then show that it implies that f has a zero. To show the other direction we will use an argument involving the dot product to show that when a function created from an arbitrary labeling of Σ^n has a zero, there must be an alternating simplex.

Proof. The Borsuk-Ulam theorem (BU2) $\Rightarrow$ Fan's N+1 theorem.
Let T be a symmetric triangulation of Σ^n with an arbitrary Fan labeling λ such that there are no complementary edges (in particular it is symmetric). Let $w_i \in \mathbb{R}^{n+1}$ be the point in $\mathbb{R}^{n+1}$ with n as its i:th coordinate and -1 everywhere else and define w_{-i} as $-w_i$. Further more, define

$$\begin{aligned} W_+ &= \{w_1, w_2, w_3, ..., w_n, w_{n+1}\} \\ W_- &= \{-w_1, -w_2, -w_3, ..., -w_n, -w_{n+1}\} \\ W &= W_+ \cup W_-. \end{aligned}$$

From the definition of w_i it follows directly that

$$w_i \cdot (1, 1, ..., 1) = 0.$$

This implies that all w_i lies in a hyperplane H in $\mathbb{R}^{n+1}$, defined as follows.

$$H = \{\boldsymbol{x} \in \mathbb{R}^{n+1} : \boldsymbol{x} \cdot (1, 1, ..., 1) = 0\}$$

Now define a function $h : T \mapsto H$ in the following way.

$$h(\boldsymbol{x}) = \begin{cases} -w_{\lambda(\boldsymbol{x})} & \textit{if } \lambda(\boldsymbol{x}) \textit{ is even} \\ w_{\lambda(\boldsymbol{x})} & \textit{if } \lambda(\boldsymbol{x}) \textit{ is odd} \end{cases} \tag{3.2}$$

Where $w_{-i} = -w_i$. We extend h linearly to the rest of the sphere by mapping any simplex to the simplex spanned by the image of its vertices under h in a continuous way. It follows directly from (3.2) if $\lambda(\boldsymbol{x})$ is even then so is $\lambda(-\boldsymbol{x})$ and the converse. Thus is $h(\boldsymbol{x}) = w_i$, then $h(-\boldsymbol{x}) = w_{-i} = -w_i$, and if $h(\boldsymbol{x}) = -w_i$ then $h(-\boldsymbol{x}) = -w_{-i} = w_i$ and it is clear that $h(\boldsymbol{x}) = -h(-\boldsymbol{x})$ meaning that h is antipodal.

The Borsuk-Ulam theorem (BU2) thus gives that there must be a point $\mathbf{a} \in H \subset \mathbb{R}^{n+1}$ such that $h(\boldsymbol{a}) = 0$. As T does not have any complementary edges $\boldsymbol{a}$ must be in some n-simplex, σ, and not on an edge. Note that n is the number of dimensions of the sphere.

We now prove that σ must be an alternating n-simplex. If we can show that $h(\sigma) = W_-$ or $h(\sigma) = W_+$ we are done as definition (3.2) implies that σ must then be alternating under λ. To see this consider an alternating simplex Δ where the even vertices have negative sign. The even vertices will thus be mapped the set $\{w_1, w_3, w_5, ...\}$ and the odd vertices are mapped to the set $\{w_2, w_4, w_6, ...\}$. As h is a bijection and Δ is a n-simplex it is clear that $h(\Delta) = W_+$. If the odd vertices of Δ would have have negatives sign it follows analogously $h(\Delta) = W_-$. As a h is a bijection this argument works the other way around and it is follows that Δ is alternating if and only if $h(\Delta) = W_+$ or W_-.

Let K denote the set of the labels of σ under h. We can thus write $K = \{w_i\}_{i\in A} \cup \{-w_i\}_{i\in B}$ where A and B are disjoint subsets of $\{1, 2, 3, ..., n, n+1\}$ as and that K contains $n+1$ points or less.

Now define v as the the sum of vectors in K like this:

$$v = \sum_{j\in A} w_j - \sum_{i\in B} w_i$$

Now consider the standard scalar multiplication of two vectors $w_i, w_j \in W$. It gives the following.

$$w_i \cdot w_j = \begin{cases} n(n+1) & if\ i = j \\ -n(n+1) & if\ i \neq j \end{cases} \tag{3.3}$$

Using this we calculate the product of w_k and v when $k \in A$ and when $k \in B$. For $k \in A$ we get the following.

$$\begin{aligned} w_k \cdot v &= w_k \cdot \sum_{j\in A} w_j - w_k \cdot \sum_{i\in B} w_i \\ &= \sum_{j\in A}(w_k \cdot w_j) - \sum_{i\in B}(w_k \cdot w_i) \\ &= n(n+1) - (|A|-1)(n+1) + (|B|+1)(n+1) \\ &= (n+1)(n+1-|A|+|B|) \end{aligned}$$

For $k \in B$ we get a similar result.

$$\begin{aligned} -w_k \cdot v &= \sum_{j\in B}(w_k \cdot w_j) - \sum_{i\in A}(w_k \cdot w_i) \\ &= n(n+1) - (|A|-1)(n+1) + (|B|+1)(n+1) \\ &= (n+1)(n+1-|B|+|A|) \end{aligned}$$

As K contains the origin there must be a k such that $w_k \cdot v = 0$. This means that for some k, one of the two expressions above must be equal to zero. As $|A|, |B| \leq n+1$, the only way $w_k \cdot v$ can be equal to or less than zero is if $|A| = n + 1$ and $|B| = 0$ or the converse. This gives that $K = W_+$ or $K = W_-$, and as stated above, this means that σ must be alternating under λ. □

Proof. Fan's N+1 theorem $\Rightarrow$ the Borsuk-Ulam theorem (BU1)
Let $f : \Sigma^n \mapsto \mathbb{R}^n$ be a antipodal function and assume, to show contradiction, that there exists no $\mathbf{a} \in S^n$ such that $f(\mathbf{a}) = 0$. Using f we create a new function $g : \mathbb{R}^n \mapsto \mathbb{R}^{n+1}$ defined as

$$g(\boldsymbol{x}) = (f(\boldsymbol{x}), -\sum_{i=0}^{n} f(\boldsymbol{x})_i)$$

Let H be the same hyperplane as earlier.

$$H = \{\boldsymbol{x} \in \mathbb{R}^{n+1} : \boldsymbol{x} \cdot (1, 1, ..., 1) = 0\}$$

This gives that $g : \mathbb{R}^n \mapsto H \subset \mathbb{R}^{n+1}$. We also note that $f(\mathbf{a}) = 0 \Leftrightarrow g(\mathbf{a}) = 0$, so by assumption g does not have a zero either.

Define W as in the preceding proof and let $\lambda(\boldsymbol{x})$ be a j such that w_j is as close to $g(\boldsymbol{x})$ as possible. If there exists several w_i with the same distance to $g(\boldsymbol{x})$ let $\lambda(\boldsymbol{x})$ be the index of the w_i with the smallest absolute value. We can also see that $\lambda(\boldsymbol{x}) = -\lambda(-\boldsymbol{x})$ as $g(\boldsymbol{x}) = -g(-\boldsymbol{x})$, and thus λ induces a Fan labeling of a symmetric triangulation T of Σ^n.

By Fan's N+1 theorem T must have an alternating simplex. We now make a finer and finer triangulation of Σ^n so that the alternating simplex converge to a single point, $\boldsymbol{z}$. As the vertices of the simplex converges to $\boldsymbol{z}$ it follows that there for every ε exists a ball of radius ε around $\boldsymbol{z}$ such that there for any k exists a point in the ball that are of closer to w_k than any w_j for all $j \neq k$. This means that $\boldsymbol{z}$ are of equal distance of all w_i. But the only point with that property is 0, so

$$g(\boldsymbol{z}) = 0 \quad \Rightarrow \quad h(\boldsymbol{z}) = 0.$$

□

We sum up our findings in the following theorem.

Theorem 7. *Fan's N+1 theorem, Tucker's lemma and the Borsuk-Ulam theorems are equivalent.*

Chapter 4

Corollaries

In this chapter two interesting corollaries to the theorems in the earlier chapter are introduced. They are called Sperner's lemma and Brouwer's fixed point theorem. Sperner's lemma is directly implied by the Fan's N+1 theorem while Brouwer's fixed point theorem follows from the Borsuk-Ulam theorem. Both of the theorems have many interesting applications and they are also equivalent to each other.

4.1 Sperner's Lemma

Sperner's lemma is a discrete analogue of Brouwer's fixed point theorem. Just as Tucker's lemma it considers labelings of vertices, but not of triangulations of Σ^n, but of $n - simplices$. We start by making the following definitions.

Definition. *Let $S = \{s_1, s_2, s_3, ..., s_n, s_{n+1}\}$ be n-simplex and T a triangulation of it. A labeling function λ is called* ***Sperner labeling*** *if:*

- $\lambda : T \mapsto \{1, 2, 3, ..., n, n+1\}$.
- $\lambda(s_i) \neq \lambda(s_j)$ *for all* $i \neq j$, $s_i, s_j \in S$.
- *Every $t \in T$ located on a face of S has the same label as at least one of the vertices defining the face.*

An n-simplex is said to be ***completely colored*** *or* ***completely labeled*** *if every vertex has different labels.*

With these definitions in place it is now possible to give the theorem.

Theorem 8. ***Sperner's Lemma*** *Every Sperner labeled triangulation of a n-simplex contains at least one completely colored n-simplex.*

This is easily exemplified in two dimensions. Mark each corner of a triangle with blue, red and yellow respectively. Triangulate the triangle and give all the vertices on the edges the same color as one of the two vertices defining the edge, and give every other vertex any of the three colors. The triangulation will then contain at least one completely colored triangle. The result could look like figure 4.1 below.

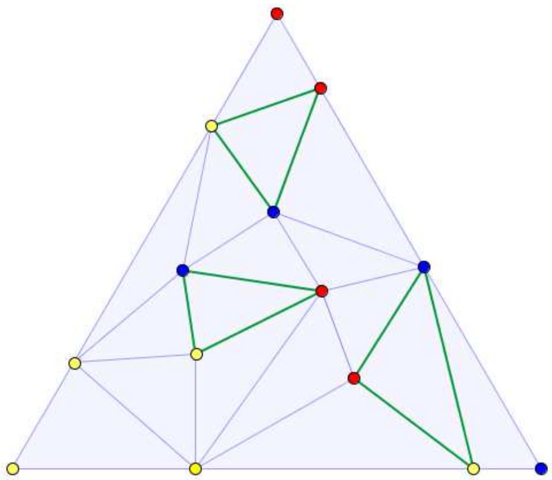

Figure 4.1: Illustration of Sperner's lemma in 2 dimensions with the completely labeled triangles marked with green.

Sperner's lemma can be extended to the statement that every Sperner labeled triangulation of Σ^n contains an odd number of completely colored simplices. The proof can be found in [9].

4.1.1 Implication from Fan's N+1 theorem

There exists direct proofs of Sperner's lemma using both Fan's N+1 theorem and Tucker's lemma. Fan's N+1 theorem will be used here as the proof is both shorter and uses more fundamental techniques. Some knowledge of group theory is required though.

Proof. Fan's N+1 theorem $\Rightarrow$ *Sperner's lemma.*
Let Δ^n be a triangulated n-simplex with a Sperner labelling λ. We extend the triangulation to the whole Σ^n by reflecting copies of the triangulation

the rest of the Σ^n. We call this triangulation T. Let G denote the group of symmetries that arise through the reflections of points to different sides of Σ^n. What is happening when a point is reflected to any of the other side of Σ^n is that some of it's components shifts signs. Every g can thus be represented as a vector with ± 1 as its entries, where g:s action on a point v is defined as $gv = (g_1 v_1, g_2 v_2, ..., g_{n+1} v_{n+1})$. For example, $g' = (-1, -1, ..., -1)$ reflects point to its antipodal point. The reflection of the vertices of a triangle in Σ^2 is illustrated in figure 4.2.

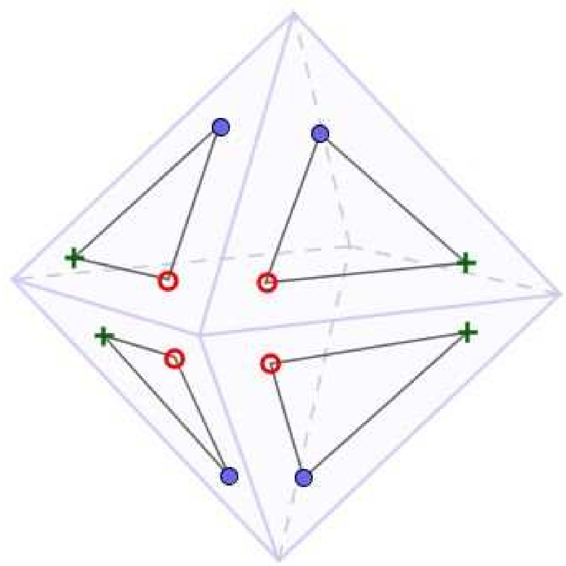

Figure 4.2: Illustration of the reflection of a simplex under G for $n = 2$.

For every simplex $\sigma \in \Sigma^n$ we define $g\sigma$ as the simplex spanned by the vertices that arise when g acts on the vertices of σ. We can now observe that $T = \{g\sigma : \sigma \in \Sigma^n, g \in G\}$ and thus extend λ to Σ^n calling it L. It is defined as follows.

$$L(gv) = g_{\lambda(v)} \cdot (-1)^{\lambda(v)+1} \cdot \lambda(v) \tag{4.1}$$

Where $g_{\lambda(v)}$ denotes the $\lambda(v)^{th}$ component of the vector representing the reflection g. Note that $|g_{\lambda(v)}| = 1$. This means that $\lambda(v)$ and $L(gv)$ have the same absolute value, but possibly different signs. For $g_e = (1, 1, 1, ..., 1)$ and a completely colored simplex σ_c we get that $L(g_e \sigma_c)$ gets an alternating labeling under L, as can verified by equation (4.1). Fan's N+1 theorem states that there always exists such a simplex, and as λ only orders positive number to vertices the only way to get a alternating simplex under L is for $g = g_e$. But g_e is the unit element of G, so the alternating simplex must be in Δ^n.

For this proof to hold we must also show that L induces a Fan labeling of Σ^n. As $\lambda : T \mapsto \{1, 2, 3, ..., n, n+1\}$ and L assigns the same labels as λ, but with possibly different signs we get that $L : \Sigma^n \mapsto \{\pm 1, \pm 2, \pm 3, .., \pm n, \pm(n+1)\}$. By construction we also get that $L(v) = L(-v)$ which is easily verified via equation (4.1) by acknowledging that $-v = (-1, -1, -1, ..., -1)v$. Thus L induce a Fan labeling of Σ^n and the proof is complete. $\square$

4.2 Brouwer's fixed point theorem

Brouwer's fixed point theorem is somewhat of a mathematical superstar with many interesting and useful applications. Due to this, much effort has been put into proving and investigating it. Many of the theorems in this thesis was found and investigated in the search for a intuitive and constructive proof of the theorem. Brouwer's fixed point theorem is also equivalent to Sperner's lemma.

Theorem 9. ***Brouwer's fixed point theorem (BFP).***
Every continuous function $f : \Omega^n \mapsto \Omega^n$ has a fixed point, meaning that there exists an $\boldsymbol{a} \in \Omega^n$ such that $f(\boldsymbol{a}) = \boldsymbol{a}$.

This is also true for all shapes in $\mathbb{R}^n$ homeomorphic to Ω^n, for example B^n.

This means that if you shake a bottle of water one point in the liquid will always return to it's original position or that if you take a paper laying on a table, crumple it, and put it back on the same spot, at least one point in the paper will be right above its original position.

4.2.1 Implication from the Borsuk-Ulam theorem

Due to the earlier shown relations between the different theorems, and the statement that Brouwer's fixed point theorem is equivalent to Sperner's lemma, we can expect that Brouwer's fixed point theorem should be implied by the Borsuk-Ulam theorem. The proof is given below.

Proof. Borsuk-Ulam theorem (BU4) $\Rightarrow$ Brouwer's fixed point theorem.
Assume, to show contradiction, that there is a function $f : \Omega^n \mapsto \Omega^n$ with no $\boldsymbol{a}$ such that $f(\boldsymbol{a}) = \boldsymbol{a}$. Define $g : \Omega^n \mapsto \Sigma^{n-1}$ like this: starting from $f(\boldsymbol{x})$ draw a line thorough $\boldsymbol{x}$ and let $g(\boldsymbol{x})$ be the first point on Σ^{n-1} that is also on the line. If there is an $\boldsymbol{a}$ such that $f(\boldsymbol{a}) = \boldsymbol{a}$, then $g(\boldsymbol{a})$ would not be well-defined. But by assumption, there is no such $\boldsymbol{a}$ so g is well-defined and continuous. For points on Σ^{n-1} it is clear that $g(\boldsymbol{x}) = \boldsymbol{x}$, g is thus antipodal on Σ^{n-1}. But according to $(BU4)$ there exists no such function, so the assumption must be false and we can conclude that $(BU4) \Rightarrow (BFP)$. □

4.3 Equivalence of Sperner's Lemma and Brouwer's Fixed Point theorem

Sperner's lemma and Brouwer's fixed point theorem is, as mentioned earlier, equivalent. The proof of that Sperner's lemma implies Brouwer's fixed point theorem is given below, while the implication in the other direction is not shown as the proofs requires some understanding of algebraic topology. One proof of this implication can be found in [3].

The proof that Sperner's lemma $\Rightarrow$ Brouwer's fixed point theorem uses a concept called barycentric coordiantes. We give an short introduction to it.

Barycentric coordinates is a coordinate system in which the location of points is described as the center of mass in a given simplex when weights are placed at the vertices of the simplex. The weights are denoted η_i and we write $\boldsymbol{x}^\eta = (\eta_1, \eta_2, ..., \eta_n, \eta_{n+1})$. The weights are then normalized so that $\eta_1 + \eta_2 + ... + \eta_n + \eta_{n+1} = 1$. A more complete definition can be found in [4]. The coordinates of some points written in barycentric coordinates is illustrated below for two dimensions.

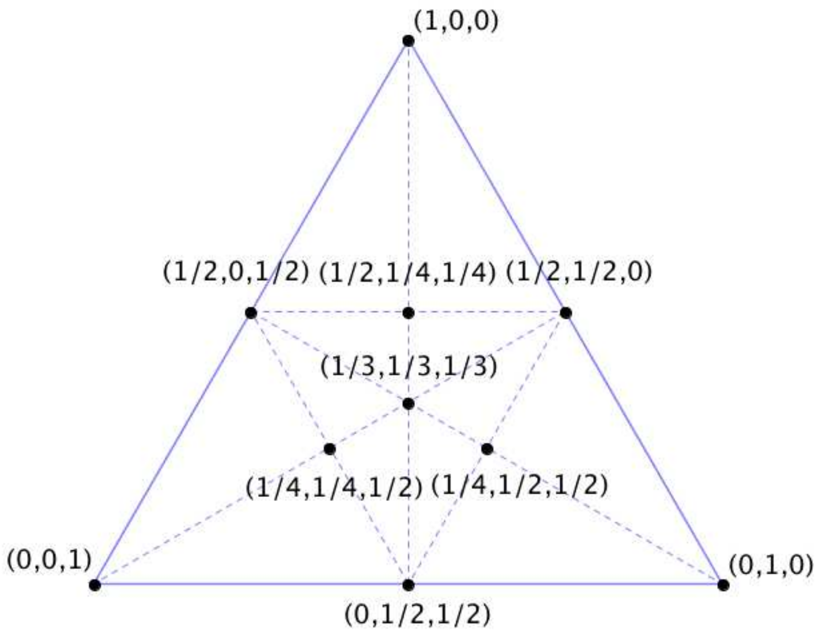

Figure 4.3: Some important points in barycentric coordinates.

Proof. Sperner's lemma $\Rightarrow$ Brouwer's fixed point theorem.
We want to show that every $f : \Omega^n \mapsto \Omega^n$ has a fixed point. For every $\varepsilon > 0$

we triangulate Ω^n such that every vertex is within distance ε from every vertex adjacent to it. We start by noting that

$$\eta_1 + \eta_2 + \eta_3 + ... + \eta_n + \eta_{n+1} =$$
$$f(\boldsymbol{x}^\eta)_1 + f(\boldsymbol{x}^\eta)_2 + f(\boldsymbol{x}^\eta)_3 + ... + f(\boldsymbol{x}^\eta)_n + f(\boldsymbol{x}^\eta)_{n+1} = 1$$

This means that there exists at least one $j \in \{1, 2, 3, ..., n, n+1\}$ such that $\eta_j \leq f(\boldsymbol{x}^\eta)_j$. We now induce a Sperner labeling of the vertices by labeling every vertex of the triangulation with one of the j:s that fulfill this in the given vertex.

Sperner's lemma now gives us that there always exists a completely colored triangle. In this case it means that there exists a simplex where the different vertices are larger than different components of their image under f. As $\varepsilon \to 0$ the complete simplex will shrink towards a point, $\boldsymbol{z}^\eta$. For each such a simplex we get that

$$\eta_1 \leq f(\boldsymbol{z}^\eta)_1, \eta_2 \leq f(\boldsymbol{z}^\eta)_2, \eta_3 \leq f(\boldsymbol{z}^\eta)_3, ..., \eta_n \leq f(\boldsymbol{z}^\eta)_n, \eta_{n+1} \leq f(\boldsymbol{z}^\eta)_{n+1}$$

This, combined with

$$f(\boldsymbol{x}^\eta)_1 + f(\boldsymbol{x}^\eta)_2 + f(\boldsymbol{x}^\eta)_3 + ... + f(\boldsymbol{x}^\eta)_n + f(\boldsymbol{x}^\eta)_{n+1} = 1,$$

gives that $f(\boldsymbol{z}^\eta) = \boldsymbol{z}^\eta$. $\boldsymbol{z}^\eta$ is thus the fixed point. □

4.4 Conclusion

We have now discussed some theorems and their relation. We sum up our findings in Figure 4.4.

$$
\begin{tikzcd}
 & \text{Tucker's lemma} \arrow[dl, Leftrightarrow] \arrow[dr, Leftrightarrow] & \\
\text{Fan's N+1 theorem} \arrow[rr, Leftrightarrow] \arrow[d, Rightarrow] & & \text{Borsuk-Ulam theorem} \arrow[d, Rightarrow] \\
\text{Sperner's lemma} \arrow[rr, Leftrightarrow] & & \text{Brouwer's fixed point theorem}
\end{tikzcd}
$$

Figure 4.4: The relation between the theorems.

The proofs of the theorems equivalence have shown a clear connection between continuous and discrete theorems by extending labelings to continuous functions and the reverse. This is a very neat result that strengthen out initial claim of the existence of combinatorial interpretations of topological theorems.

Looking at this figure one is urged to ask whether there exists direct proofs in other directions as well. Does it for example exist a direct proof of Tucker's lemma from Sperner's lemma?

It is now time to leave these theorems to look at some interesting corollaries.

Chapter 5

The Ham Sandwich theorems

We now leave the subject of relations between combinatorial and continuous theorems to look at some of the interesting corollaries of the earlier theorems. We will start with the Ham Sandwich theorem for measures which is implied by the Borsuk-Ulam theorem and then show that the Ham Sandwich theorems for measures implies a number of interesting results concerning the bisection of different kinds of measures and discrete sets in $\mathbb{R}^n$.

5.1 The Ham Sandwich theorem for measures

The Ham Sandwich theorem for measures, some times called just the Ham Sandwich theorem, was introduced for three dimensions by Hugo Steinhaus in a paper from 1938 [5] and later generalized to all dimensions by Stefan Banach. The theorem states that given n not necessarily disjoint subsets of $\mathbb{R}^n$ with finite measure, there always exists a hyperplane that bisects them simultaneously with regard to the given measure. A measure is a function that assigns a value to a set that fulfill our intuitive perception of size. For example, for a measure μ of a set A, $\mu(A) \geq 0$ with equality if $A = \emptyset$, and the measure of a union of disjoint sets is equal to the sum their individual measures. That a set A has a finite measure means that $\mu(A) < \infty$. A complete introduction to measure theory can be found in [6].

We make the following formal definition of bisection.

Definition. *Let $A^+ = A \cap H^+$, where H^+ is a halfspace defined by the hyperplane H. Then H is said to* ***bisect*** *A with regard to the measure μ if $\mu(A^+) = \frac{1}{2}\mu(A)$.*

The Ham Sandwich theorem for measures is stated as follows.

Theorem 10. ***The Ham Sandwich Theorem for measures (HSTM).*** *Let $A_1, A_2, A_3, ..., A_m$ be sets in $\mathbb{R}^n$ that have finite measures with $m \leq n$. Then there exists a hyperplane that bisects all of the sets with regard to the given measure.*

The theorem got its name from a common popularization of the theorem stating that a sandwich with bread, ham and cheese can be bisected with a strait cut such that both halves contains equal amounts of bread, cheese and ham.

The proof of the theorem follows form a clever use of the Borsuk-Ulam theorem ($BUT1$) by introducing a function from Σ^{n-1} to $\mathbb{R}^{n-1}$. The function relates every point on the sphere to the measure of each set on a given side of a hyperplane uniquely defined for every set of antipodal points in Σ^n.

Proof. Let $A_1, A_2, ..., A_m$ be sets in $\mathbb{R}^n$ with $m \leq n$ such that $\mu(A_p) \leq \infty$, where $\mu(A_p)$ denotes the measure of A_p. Now place a $(n-1)$-dimensional sphere in $\mathbb{R}^n$. For every point $\boldsymbol{s}$ in Σ^{n-1} let $\vec{\boldsymbol{s}}$ denote the vector from the center of the sphere to $\mathbf{s}$.

Now define $h : \Sigma^{n-1} \to \mathbb{R}^{n-1}$ in the following way. For every $\boldsymbol{s}$ there exists exactly one hyperplane H_s that bisects A_1 and is orthogonal to $\vec{\boldsymbol{s}}$. Let $h(\boldsymbol{s}) = (\mu(A_2^s), \mu(A_3^s), ..., \mu(A_n^s))$ where $\mu(A_p^s)$ is the measure of the part of A_p on the side of the H_s from which $\vec{\boldsymbol{s}}$ points out.

As the same hyperplanes are orthogonal to $\vec{\boldsymbol{s}}$ and $-\vec{\boldsymbol{s}}$, but they point in opposite directions, we get that $h(\boldsymbol{s}) + h(-\boldsymbol{s})$ is the same thing as adding together the measure of each set from both side of the hyperplane giving that

$$h(\mathbf{s}) + h(-\mathbf{s}) = (\mu(A_2), \mu(A_3), \mu(A_4), ..., \mu(A_m))$$

We also note that h is continuous and can thus apply the Borsuk-Ulam theorem stating that there exists a point $\boldsymbol{a}$ such that $h(\boldsymbol{a}) = h(-\boldsymbol{a})$ giving

$$2h(\boldsymbol{a}) = (\mu(A_2), \mu(A_3), \mu(A_4), ..., \mu(A_m)).$$

This implies that

$$h(\boldsymbol{a}) = \frac{(\mu(A_2), \mu(A_3), \mu(A_4), ..., \mu(A_m))}{2} = (\frac{\mu(A_2)}{2}, \frac{\mu(A_3)}{2}, \frac{\mu(A_4)}{2}, ..., \frac{\mu(A_m)}{2}).$$

This shows that the hyperplane paired with $\boldsymbol{a}$ is bisecting all the sets with regard to the given measure. □

It is important to note that neither the theorem nor the proof gives us any idea of how to find such a hyperplane.

5.2 The Discrete Ham Sandwich theorem

The discrete Ham Sandwich theorem covers finite point sets $\mathbb{R}^n$. For point sets we will use a somewhat strange definition of bisection where we say that a hyperplane bisects a point set A_i when each of the half spaces defined by the hyperplane contains at most $\lfloor \frac{1}{2}|A_i| \rfloor$ of the points from A_i. $\lfloor x \rfloor$ denotes the integer part of x, i.e. $\lfloor 3.5 \rfloor = 3$ or $\lfloor -7.35 \rfloor = -7$, and $|A_i|$ denotes the number of elements of A_i.

This definition has many useful and interesting aspects. For example, we will only need to prove our theorem for sets of odd cardinality as adding a point to any of the sets will not affect whether a hyperplane bisects the set or not. It also means that if a set has odd cardinality, a hyperplane must contain at least one point from it in order to bisect it.

Theorem 11. ***The Discrete Ham Sandwich Theorem.***
Let $A_1, A_2, A_3, ..., A_m$ be finite point sets in $\mathbb{R}^n$ with $m \leq n$. Then there exists a hyperplane that bisects all of the sets simultaneously.

We make the following definition.

Definition. *A point set in $\mathbb{R}^n$ is said to be in* ***general position*** *if every hyperplane in $\mathbb{R}^n$ contains at most n points from it.*

We introduce a lemma.

Lemma.
Let $A_1, A_2, A_3, ..., A_m$ be finite point sets in $\mathbb{R}^n$ with $m \leq n$ and $A_1 \cup A_2 \cup A_3 \cup ... \cup A_m$ in general position. Then there exists a hyperplane that bisects all of the sets simultaneously.

Proof. Let $A_1, A_2, A_3, ..., A_m$ be the finite point sets in $\mathbb{R}^n$ given in the lemma above, but assume that $m = n$. Start by removing an arbitrary point from every set of even cardinality so that all sets contains an odd number of points. For every $\varepsilon > 0$ let A_i^ε arise from every point set A_i by placing a ball of n dimensions with radius ε around every point in A_i. We can then apply the Ham Sandwich theorem for measures giving that there must be a hyperplane that bisects all of the A_i'. We call that hyperplane H. As all of the sets have odd cardinality H must intersect at least one point from each of the sets. As the sets are in general position, for sufficient small ε every hyperplane intersects at most n balls, meaning that H intersects exactly one point from each set. In order of H to bisect each A_i^ε, H must pass through the center of the balls. Thus H bisects every A_i.

As this is true for $m = n$, it is also true for $m \leq n$ sets as removing some sets will not affect whether the hyperplane bisect the other sets or not. $\square$

The lemma can now be used to prove the Discrete Ham Sandwich theorem.

Proof. Now let $A_1, A_2, A_3, ..., A_m$ be the finite point sets in $\mathbb{R}^n$ from the theorem above. For every $\gamma > 0$ let $A_{i,\gamma}$ arise from A_i by moving every point by a distance of at most γ such that the union of the sets is in general position. For every γ the lemma above gives that there exists a hyperplane that bisect each A_i. Let denote that hyperplane by H_γ.

We can write $H_\gamma = \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{x} \cdot \boldsymbol{a}_\gamma = b_\gamma\}$. Due to compactness there must exist a point $(\boldsymbol{a}, b)$ such that when $\gamma \to 0$, $(\boldsymbol{a}_\gamma, b_\gamma) \to (\boldsymbol{a}, b)$. Let H be the hyperplane such that $H_\gamma \to H$ when $(\boldsymbol{a}_\gamma, b_\gamma) \to (\boldsymbol{a}, b)$.

For every γ small enough, if a point is at distance $\delta > 0$ from H, it must also be at least $\frac{1}{2}\delta$ from H_γ. Thus if the half-space determined by H contains t points from A_i, the corresponding half-space from H_γ must contains at least t points from $A_{i,\gamma}$. It follows that H bisects each A_i. $\square$

5.3 The general Position Ham Sandwich Theorem

For sufficiently separated point sets it is possible make a stronger statement than the discrete Ham Sandwich theorem. The theorem is stated as follows.

Theorem 12. ***The General position Ham Sandwich theorem.***
Let $A_1, A_2, A_3, ..., A_m$ be finite point sets in $\mathbb{R}^n$ with $m \leq n$ and $A_1 \cup A_2 \cup A_3 \cup ... \cup A_m$ in general position. Then there exists a hyperplane that bisects all of the sets and contains at most one point from each set.

Proof. Let H_0 be the hyperplane that bisect all the sets according to $(HSTD)$. Either this hyperplane fulfills the requirements from the theorem and we are done, or it contains too many points from some of the sets. We also note that the hyperplane can contain at most n points from any set, as the sets are still in general position. As the points in H_0 are affinely independent we can adjust H_0 so that any point in it is no longer in it, but in any of the two halfspaces defined by H_0. We give a short motivation for this.

Add a number of points to the hyperplane H_0 such that it contains exactly n points. In $\mathbb{R}^n$, n points defines a hyperplane. For any point $\boldsymbol{a}_i \in H_0$ that we do not want to have in the hyperplane, let $\boldsymbol{a}_i^\varepsilon$ arise from $\boldsymbol{a}_i$ by moving it at most a distance of $\varepsilon > 0$ away from $\boldsymbol{a}_i$ into any of the halfspaces. For every other point let $\boldsymbol{a}_i = \boldsymbol{a}_i^\varepsilon$. Let H_ε be the hyperplane that is defined from all of the $\boldsymbol{a}_i^\varepsilon$. For a small enough ε the points will still be affinely independent, and H_ε will not have moved to contain any new point.

This shows that it is always possible to move H_0 so that it does not contain more than one point from any A_i, and decide into which halfspace every point should be added, without adding any new points to the hyperplane. □

This reasoning can be extended further to make a number of different statement about different cuts of point sets in general position. For example it is possible to find a cut with a hyperplane that does not contain any points if we allow sets with odd cardinality to have up to one more point in one of the half-spaces.

5.4 The Polynomial Ham Sandwich Theorem

We will conclude the thesis with the polynomial Ham Sandwich theorem. The polynomial Ham Sandwich theorem is in many ways more advanced than the earlier theorems and is mainly used in algebraic geometry. There are mainly two reasons for presenting it in this context. First of all it can be formulated in a way similar to the other Ham Sandwich theorems and, even more important, we will use the discrete Ham Sandwich theorem to prove it. One big difference from the earlier theorems is that the Polynomial Ham Sandwich theorem can be applied for any number of sets in any dimension. The theorem is stated as follows.

Theorem 13. ***The Polynomial Ham Sandwich Theorem.***
Let $A_1, A_2, A_3, ..., A_m$ be finite point sets in $\mathbb{R}^n$, and let d be the smallest integer such that $m \leq \binom{n+d-1}{d}$ and let $k = \binom{n+d-1}{d}$. Then there exists a non-zero polynomial $p(\boldsymbol{x}) = \mathbb{R}[x_1, x_2, x_3, ..., x_n]$ of at most degree k such that $p(\boldsymbol{x})$ bisects all of the sets.

That a polynomial bisects a point set means that $p(\boldsymbol{x}) > 0$ in at most $\lfloor \frac{1}{2}|A_i| \rfloor$ points in each A_i and $p(\boldsymbol{x}) < 0$ in at most $\lfloor \frac{1}{2}|A_i| \rfloor$ points in each A_i.

The proof of the theorem uses a function called the Veronese map. Using it, all sets will be mapped to a space of appropriate dimension where the discrete Ham Sandwich theorem states that there must be a hyperplane that bisects all of them. The hyperplane will then be used to construct the desired polynomial. The Veronese map is defined as follows.

Definition. *The **Veronese map** Φ_d of degree d is the map that send $\boldsymbol{x} = (x_0, x_1, x_2, ..., x_n)$ to the vector whoose coordinates are the values of all distinct monomials of degree d that can be formed with the varibles $x_0, x_1, x_2, ..., x_n$. Formally,*

$$\Phi_d : \mathbb{R}^n \mapsto \mathbb{R}^k \quad \textit{with} \quad k = \binom{n+d-1}{d}.$$

We give some examples.

- For $d = 2$ and $n = 2$ we get $k = 3$ and

$$\Phi_2 : (x_0, x_1) \mapsto (x_0^2, x_0 x_1, x_1^2)$$

- For $d = 3$ and $n = 2$ we get $k = 4$ and

$$\Phi_3 : (x_0, x_1) \mapsto (x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3)$$

- For $d = 3$ and $n = 3$ we get $k = 10$ and

$$\Phi_2 : (x_0, x_1, x_2) \mapsto (x_0^3, x_1^3, x_2^3, x_0^2x_1, x_0^2x_2, x_0x_1^2, x_1^2x_2, x_0x_2^2, x_1x_2^2, x_0x_1x_2)$$

And now the proof.

Proof. Let $A_1, A_2, A_3, ..., A_m$ be point sets in $\mathbb{R}^n$ and let $\Phi_d : \mathbb{R}^n \mapsto \mathbb{R}^k$ be the Veronese map of degree d, with d chosen such that $m \leq \binom{n+d-1}{d} = k$. Define $A_i' = \Phi_d(A_i)$ for all i. Since each A_i' is contained in $\mathbb{R}^k$ and as $m \leq k$ the discrete Ham Sandwich theorem gives that there must be a hyperplane $H \subset \mathbb{R}^k$ such that H bisects each every A_i'. We write H on the form:

$$H = \{\boldsymbol{x} \in \mathbb{R}^k | \boldsymbol{x} \cdot v = b, v \in \mathbb{R}^k\}$$

Let H^+ and H^- be the half-space defined by H. Now consider the polynomial $p(\boldsymbol{x}) = \Phi_d(\boldsymbol{x}) \cdot v - b$ for a given point $\boldsymbol{x} = \boldsymbol{a}$ in any A_i. We observe the following:

$$\begin{aligned}
\Phi_d(\boldsymbol{a}) \in H \quad &\Leftrightarrow \quad \Phi_d(\boldsymbol{a}) \cdot v = b \quad \Leftrightarrow \quad p(\boldsymbol{a}) = 0 \\
\Phi_d(\boldsymbol{a}) \in H^+ \quad &\Leftrightarrow \quad \Phi_d(\boldsymbol{a}) \cdot v > b \quad \Leftrightarrow \quad p(\boldsymbol{a}) > 0 \\
\Phi_d(\boldsymbol{a}) \in H^- \quad &\Leftrightarrow \quad \Phi_d(\boldsymbol{a}) \cdot v < b \quad \Leftrightarrow \quad p(\boldsymbol{a}) < 0
\end{aligned}$$

This means that $p(\boldsymbol{x}) > 0$ exactly when $\Phi_d(\boldsymbol{x}) > 0$ and that $p(\boldsymbol{x}) < 0$ exactly when $\Phi_d(\boldsymbol{x}) < 0$. As at most $\lfloor \frac{1}{2}|A_i| \rfloor$ points are mapped to H^+ and H^- respectively we see that $p(\boldsymbol{x}) < 0$ in at most $\lfloor \frac{1}{2}|A_i| \rfloor$ points from each A_i and that $p(\boldsymbol{x}) > 0$ in at most $\lfloor \frac{1}{2}|A_i| \rfloor$ points from each A_i. By the definition of bisection it is clear that $p(\boldsymbol{x})$ bisects every A_i. $\square$

Chapter 6

Conclusions

In the first chapters some basic discrete geometry was introduced together with some concepts and notation. We then investigated the relation between some continuous and discrete theorems, most notably the Borsuk-Ulam theorem and its discrete analogues.

One of the main questions concerning the discrete theorems is how to construct efficient algorithms for finding the points sought in the theorem, and how the algorithms then can be used to find stationary point for continuous functions. Looking at the proofs in the earlier chapters it is clear that many of them is actually creating algorithms for finding specific points.

In the later chapters the Ham Sandwich theorems was introduced and we saw that the Borsuk-Ulam theorem implied all of them in a very neat chain of implications.

The polynomial Ham Sandwich theorem showed that the earlier results do not only have a wide range of applications in discrete geometry and in algebraic topology, but also in the theory of polynomials. This raises the question whether there exists bisection-theorems in other branches of mathematics that can be proved using the Ham Sandwich theorem.

Some other open questions are if there exists equivalences where we have proven implications, and if there exists a strictly combinatorial proof that Brouwer's fixed point theorem implies Sperner's lemma.

Appendix A

Bibliography

[1] Jiří Matoušek, *Using the Borsuk-Ulam Theorem.* Springer, corrected 2nd printing, 2008.

[2] Kathryn L. Nyman and Francis Edward Su, *A Borsuk-Ulam Equivalent that Directly Imples Sperner's Lemma.* 2012.

[3] M. Yoseloff, *Topological Proofs of some Combinatorial Theorems.* J. Combinatorial Theory Ser. A 17, 1974.

[4] H. S. M. Coxeter, *Introduction to Geometry.* Wiley Classical Library, 2nd edition, 1969.

[5] Hugo Steinhaus, *A note on the ham sandwich theorem.* Mathesis Polska 9: 26-28, 1938.

[6] Walter Rudin, *Priciples of Mathematical Analysis.* McGraw-Hill, 3rd edition, 1976.

[7] Timothy Prescott and Francis Su, Advisor, *Extensions of the Borsuk-Ulam Theorem.* Harvey Mudd College, 2002.

[8] Mutiara Sondjaja, *Sperner's Lemma Implies Kakutani's Fixed Point Theorem.* Harvey Mudd College, 2008. http://www.math.hmc.edu/seniorthesis/archives/2008/, 11/4 2012.

[9] Jonathan Huang, *On Sperner's lemma and its applications.* Stanford University. http://www.stanford.edu/ jhuang11/research/old/, 11/04 2012.

[10] Shchepin, Evgeny Vital'evich, *Elementary Proof of the Borsuk-Ulam Theorem.* Steklov Mathematical Institute, Russian Academy of Science. http://www.mi.ras.ru/ scepin/, 11/04 2012.