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On Generalised Ramsey Numbers for Two Sets of Cycles

av

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Abstract

Given s non-empty sets $\mathscr{G}_1, \ldots, \mathscr{G}_s$ of graphs, the generalised Ramsey number $R(\mathscr{G}_1, \ldots, \mathscr{G}_s)$ is defined as the least positive integer n, such that whenever each edge of the complete graph K_n on n vertices is coloured with one of the colours $c_1, \ldots, c_s$, K_n contains a c_i-coloured G_i, for some $i \in \{1, \ldots, s\}$ and some $G_i \in \mathscr{G}_i$.

In this thesis, we first prove some basic, general properties of generalised Ramsey numbers, among others that they always exist. We then compute a number of (in fact, uncountably many) two colour generalised Ramsey numbers, such that $\mathscr{G}_1$ and $\mathscr{G}_2$ are sets of cycles. This generalises previous results of Erdős, Faudree, Rosta, Rousseau, and Schelp from the 1970s.

Above all, we determine all generalised Ramsey numbers $R(\mathscr{G}_1, \mathscr{G}_2)$ such that $\mathscr{G}_1 \cup \mathscr{G}_2$ contains a cycle of length 3, 4, or 5. Furthermore, we give a conjecture for the general case. We also prove some results on graphs that contain no cycle of odd length, except possibly a number of 3-cycles.

Sammanfattning

För s icke-tomma mängder $\mathscr{G}_1, \ldots, \mathscr{G}_s$ av grafer definieras det generaliserade Ramseytalet $R(\mathscr{G}_1, \ldots, \mathscr{G}_s)$ som det minsta positiva heltalet n, sådant att om varje kant i den kompletta grafen K_n på n hörn färgas med någon av färgerna $c_1, \ldots, c_s$, så innehåller K_n garanterat en c_i-färgad G_i, för något $i \in \{1, \ldots, s\}$ och något $G_i \in \mathscr{G}_i$.

I det här arbetet bevisar vi först några grundläggande, allmänna egenskaper hos generaliserade Ramseytal, bland andra att de alltid existerar. Därefter beräknar vi ett antal generaliserade Ramseytal för två färger, sådana att $\mathscr{G}_1$ och $\mathscr{G}_2$ är mängder av cykler, vilket generaliserar tidigare resultat av Erdős, Faudree, Rosta, Rousseau och Schelp från 1970-talet.

Framför allt bestämmer vi alla generaliserade Ramseytal $R(\mathscr{G}_1, \mathscr{G}_2)$ sådana att $\mathscr{G}_1 \cup \mathscr{G}_2$ innehåller en cykel av längd 3, 4 eller 5. Vidare ger vi en förmodan för det allmänna fallet. Vi bevisar också några resultat om grafer som inte innehåller någon cykel av udda längd, förutom möjligen ett antal 3-cykler.

Acknowledgements

I would like to thank my advisor Jörgen Backelin for suggesting the topic and for his dedication and support.

Contents

Preface

Much of Sections 1.1, 1.2, and 1.3 previously appeared as part of my bachelor thesis [9].

Section 1.3: The results on ordinary (that is, non-generalised) Ramsey numbers are previously known, but the proofs are my own, except the proof of Ramsey's theorem (Theorem 1.3.4). The results on generalised Ramsey numbers are almost certainly previously known, but I have not been able to find them in the literature.

Section 1.4: The alternative view of generalised Ramsey numbers stems from [1] and personal communication with its author.

Chapter 2: All results in this chapter are, to the best of my knowledge, new, except when the opposite is explicitly stated.

1 Introduction

1.1 Introductory example

The following example is often used to introduce Ramsey theory (named after the English mathematician Frank Ramsey (1903-1930)): Suppose that at a party, any two people either know each other or do not know each other. What is the least number of people that must be present at the party in order to guarantee the existence of three people who mutually know each other or three people who mutually do not know each other? This may be modeled with graphs: Let the vertices represent the people at the party and draw an edge between two vertices if and only if these two people know each other. Equivalently, one may draw a red edge between two vertices if the two people know each other and a blue edge otherwise. The above question may now be rephrased thus: What is the least number of vertices that a graph must contain in order to guarantee the existence of a 3-clique (three vertices with an edge between any two of them) or three independent vertices (three vertices with no edge between any two of them)? and What is the least number of vertices that a complete red-blue graph (a number of vertices with an edge, red or blue, between any two of them) must contain in order to guarantee the existence of a red 3-clique or a blue 3-clique? respectively. Let $R(3,3)$ denote the requested number of people/vertices. We now show that $R(3,3) = 6$.

Proposition 1.1.1. *$R(3,3) = 6$.*

Proof. $R(3,3) \geq 6$: We have to show that there is a complete red-blue graph on 5 vertices with no monochromatic 3-clique. Such a graph exists:

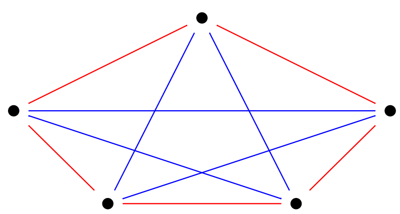


Figure 1: $R(3,3) \geq 6$.

$R(3,3) \leq 6$: We have to show that each complete red-blue graph on 6 vertices contains a monochromatic 3-clique. Let the vertices be v, a, b, c, d, and e. At least three of the edges va, vb, vc, vd, and ve are the same colour; say that (at least) va, vb, and vc are red. If ab, ac, or bc is red, then we have a red 3-clique (vab, vac, or vbc, respectively). On the other hand, if ab, ac, and bc are all blue, then we have a blue 3-clique (abc). $\square$

It is natural to proceed by trying to answer the following, more general question: What is the least number of vertices that a complete red-blue graph must contain in order to guarantee the existence of a given red subgraph or a given blue subgraph (the two subgraphs need not be the same)? In general, this is a very hard problem; for instance, even the number $R(5,5)$, where the red and the blue subgraph are both 5-cliques, is unknown (one only knows that it lies between 43 and 49). Nevertheless, many of these so called (ordinary) Ramsey numbers are known, for instance $R(C_n, C_k)$, where the red and the blue subgraph are an n- and a k-cycle, respectively. For more known values of (ordinary) Ramsey numbers, see [12].

One may generalise these Ramsey numbers by means of the following, still more general question: What is the least number of vertices that a complete red-blue graph must contain in order to guarantee the existence of a red subgraph belonging to a given set of graphs or a blue subgraph belonging to a given set of graphs (the two sets need not be the same)? This is the question to which this thesis is devoted.

In Section 1.3 we prove some basic, general properties of generalised Ramsey numbers, among others that they always exist, for any number of colours. In Section 1.4 we give an alternative view of generalised Ramsey numbers. In Chapter 2, finally, we compute a number of (in fact, uncountably many) generalised Ramsey numbers for two sets Γ_1 and Γ_2 of cycles. Above all, we determine all generalised Ramsey numbers $R(\Gamma_1, \Gamma_2)$ such that $\Gamma_1 \cup \Gamma_2$ contains a cycle of length 3, 4, or 5. Furthermore, we give a conjecture for the general case. We also prove some results on "almost bipartite graphs," by which we shall mean graphs that contain no cycle of odd length, except possibly a number of 3-cycles.

1.2 Definitions and notation

In this section we define, above all, the graph theoretical notions used in this thesis. Throughout the thesis, $G_1, \ldots, G_s$ and $\mathscr{G}_1, \ldots, \mathscr{G}_s$ denote non-empty (uncoloured) graphs and non-empty sets of non-empty (uncoloured) graphs, respectively.

Definition 1.2.1. If X is a set, let $|X|$ be the number of elements of X if X is finite, and ∞ otherwise, let $2^X = \{A \subseteq X\}$, and let $\binom{X}{k} = \{A \subseteq X \mid |A| = k\}$. If A and B are two sets, let $A - B = \{x \in A \mid x \notin B\}$ and let (as usual) $A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$ (the latter with the obvious generalisation for more than two sets). Also, if n is a positive integer, let $A^n = A \times \cdots \times A$ (n times).

Definition 1.2.2. Let $\mathbf{R}$ be the real numbers, let $\mathbf{Z}$ be the integers and let $\mathbf{N}$ be the non-negative integers. If $x \in \mathbf{R}$, let $\lfloor x \rfloor = \max\{n \in \mathbf{Z} \mid n \leq x\}$

and let $\lceil x \rceil = \min\{n \in \mathbf{Z} \mid n \geq x\}$. If $a \in \mathbf{Z}$, let $a \bmod n$ be the least non-negative integer congruent to a modulo n. If $n \in \mathbf{N}$, let $[n] = \{1, 2, \dots, n\}$ (thus $[0] = \emptyset$). If $a \leq b$ are integers, let $[a, b] = \{a, a+1, \dots, b\}$, and if $a > b$ are integers, let $[a, b] = \emptyset$. $x \equiv a$ means that $x \equiv a \pmod 2$.

Definition 1.2.3. A *graph* G is an ordered pair (V, E), where V is a finite set and $E \subseteq \binom{V}{2}$; the elements of V are called *vertices* and the elements of E are called *edges*. (Thus all graphs in this thesis are finite, simple, and undirected.) If G is a graph, let $V_G = V(G)$ and $E_G = E(G)$ denote its *vertex set* and its *edge set*, respectively. Note that we often write $v \in G$ and $|G|$ instead of $v \in V(G)$ and $|V(G)|$, respectively. Also note that we often write uv or vu for the edge $\{u, v\} = \{v, u\}$.

Let $G = (V_G, E_G)$ be a graph. Two vertices u and v of G are said to be *adjacent* to one another, or *neighbours*, if $uv \in E_G$. A vertex v and an edge xy are said to be *incident* if $v \in \{x, y\}$. Two edges are called *independent* if they have no vertex in common. A graph $H = (V_H, E_H)$ is said to be a *subgraph* of G, written $H \subseteq G$, if $V_H \subseteq V_G$ and $E_H \subseteq E_G$; H is said to be an *induced subgraph* of G if, moreover, $E_H = \binom{V_H}{2} \cap E_G$. The *complement* of G is the graph $(V_G, \binom{V_G}{2} - E_G)$.

A graph $G = (V, E)$ is said to be *bipartite* if V is the disjoint union of two subsets V_1 and V_2, such that all edges $e \in E$ are of the form $e = v_1 v_2$, where $v_1 \in V_1$ and $v_2 \in V_2$; G is said to be *complete bipartite* if, moreover, $v_1 v_2 \in E$, for all $v_1 \in V_1$ and all $v_2 \in V_2$. The graph $K_{p,q}$ is complete bipartite with $|V_1| = p$ and $|V_2| = q$. G is said to be *m-regular* if each vertex of G has precisely m neighbours. The graph K_n consists of n vertices and all $\binom{n}{2}$ possible edges; it is called the *complete* graph on n vertices, or the *n-clique* (note that K_n is $(n-1)$-regular).

Definition 1.2.4. Two graphs G_1 and G_2 are said to be *isomorphic* if there is a bijection $V(G_1) \xrightarrow{\varphi} V(G_2)$, such that $uv \in E(G_1)$ if and only if $\varphi(u)\varphi(v) \in E(G_2)$, for all $u, v \in V(G_1)$. Let the *isomorphism class* $[G]$ of a given graph G consist of all graphs isomorphic to G, and let $\mathscr{P}$ denote the set of all isomorphism classes of graphs. We usually do not distinguish between isomorphic graphs (in other words, we often identify G with $[G]$). Thus for instance, we talk about *the* complete graph on n vertices.

Definition 1.2.5. Let

$$V = \{x_1, x_2, \dots, x_n\} \quad \text{and} \quad E = \{x_1x_2, x_2x_3, \dots, x_{n-1}x_n\},$$

where n is a positive integer and $x_i \neq x_j$ for all $i \neq j$. Then (V, E) is called a *path* of length $n - 1$, denoted $P_n = x_1 x_2 \cdots x_n$, and if $n \geq 3$, then $(V, E \cup \{x_n x_1\})$ is called a *cycle* of length n, or an *n-cycle*, denoted $C_n = x_1 x_2 \cdots x_n x_1$.

If $P = x_1x_2\cdots x_n$, let $P' = x_nx_{n-1}\cdots x_1$. Formally, $P = P'$, but this is still a useful definition. If $P = x_1x_2\cdots x_n$ and $n \geq 3$, let $P^\circ = x_2x_3\cdots x_{n-1}$; $x_2, x_3, \ldots, x_{n-1}$ are the *inner vertices* of P.

A *j-chord* of a cycle $C = x_1x_2\cdots x_nx_1$ is an edge of the form x_ix_{i+j}, where $j \bmod n \notin \{1, n-1\}$. Vertex indices are always interpreted modulo the length of the cycle that we are considering at the moment. For instance, $x_{11} = x_3$ in a cycle of length 8.

Finally, let

$$V = \{x_1, x_2, y_1, y_2, \ldots, y_n\} \quad \text{and} \quad E = \{x_1x_2\} \cup \{x_iy_j \mid (i,j) \in [2] \times [n]\},$$

where n is a positive integer and $y_i \neq y_j$ for all $i \neq j$. Then (V, E) is called a *tower* of height n, denoted $T_n = x_1x_2|y_1y_2\cdots y_n$. A tower T in a graph G is called *maximal* if the height of T, denoted $\mathrm{ht}(T)$, is maximal among the heights of the towers in G.

Definition 1.2.6. A graph G is said to be *almost bipartite* if it contains no cycle of odd length, except possibly a number of 3-cycles. A graph G on $n \geq 3$ vertices is called *Hamiltonian* if it contains a cycle of length n, and *pancyclic* if it contains cycles of all lengths between 3 and n.

Definition 1.2.7. Let s be a positive integer. An *s-colouring* ρ of a set X is a function $X \xrightarrow{\rho} \{c_1, \ldots, c_s\}$; $\rho(x)$ is called the *colour* of x ($x \in X$). An *s-edge colouring* of a graph (V, E) is an s-colouring of E; the edge coloured graph is denoted (V, E, ρ) and is said to be a *colour graph.*

Let $G = (V_G, E_G, \rho_G)$ be a colour graph. Two vertices u and v of G are said to be *c_i adjacent* to one another, or *c_i neighbours*, if $uv \in E_G$ and $\rho_G(uv) = c_i$. The *c_i-coloured subgraph* G_{c_i} of G is the (uncoloured) graph $(V_G, \{e \in E_G \mid \rho_G(e) = c_i\})$. A subgraph $H = (V_H, E_H, \rho_H)$ of G is said to have the *induced colouring* if $\rho_H(e) = \rho_G(e)$ for all $e \in E_H$.

If $V \subseteq V(G)$, let $G[V]$ denote the induced subgraph on V with the induced colouring. In case G is an uncoloured graph, then we use the same notation for the induced subgraph on V. Also, if $H \subseteq G$, let $H + V = G[V(H) \cup V]$ and let $H - V = G[V(H) - V]$; moreover, if $v \in G$, let $H \pm v = H \pm \{v\}$.

A colour graph (V, E, ρ) is said to be *red-blue* if $s = 2$ and $\{c_1, c_2\} = \{\text{red}, \text{blue}\}$. Throughout the thesis, we shall assume this to be the case when $s = 2$; furthermore, red will always be the first colour and blue will always be the second. In order to simplify notation, we shall often also assume that $\{c_1, \ldots, c_s\} = [s]$ for arbitrary s.

Definition 1.2.8. Let n and s be positive integers. We write

$$n \to (G_1, \ldots, G_s)$$

if, for each s-edge colouring ρ of K_n, there is an $i \in [s]$, such that the c_i-coloured subgraph $(K_n)_{c_i}$ contains a subgraph isomorphic to G_i; we often express this as (K_n, ρ) (or just K_n) containing a c_i-coloured G_i. $n \to (t_1, \ldots, t_s)$ has the same meaning as $n \to (K_{t_1}, \ldots, K_{t_s})$.

The *ordinary Ramsey number* $R(G_1, \ldots, G_s)$ denotes the least positive integer n such that $n \to (G_1, \ldots, G_s)$; here, $R(t_1, \ldots, t_s) = R(K_{t_1}, \ldots, K_{t_s})$. Since $R(G_1, \ldots, G_s)$ only depends on the isomorphism classes of $G_1, \ldots, G_s$, one may define a function, called the *ordinary Ramsey function*, from $\mathscr{P}^s$ to the set of all positive integers, by $([G_1], \ldots, [G_s]) \mapsto R(G_1, \ldots, G_s)$.

It should be noted that it is not obvious that for each positive integer s and all graphs $G_1, \ldots, G_s$, there is a positive integer n such that $n \to (G_1, \ldots, G_s)$. In the next section, however, we prove this to be the case (see Theorem 1.3.4), whence the ordinary Ramsey function is well-defined.

Definition 1.2.9. Let n and s be positive integers. We write

$$n \to (\mathscr{G}_1, \ldots, \mathscr{G}_s)$$

if, for each s-edge colouring ρ of K_n, there is an $i \in [s]$, such that the c_i-coloured subgraph $(K_n)_{c_i}$ contains a subgraph isomorphic to some $G_i \in \mathscr{G}_i$; we often express this as (K_n, ρ) (or just K_n) containing a c_i-coloured G_i.

The *generalised Ramsey number* $R(\mathscr{G}_1, \ldots, \mathscr{G}_s)$ denotes the least positive integer n such that $n \to (\mathscr{G}_1, \ldots, \mathscr{G}_s)$. Since $R(\mathscr{G}_1, \ldots, \mathscr{G}_s)$ only depends on the isomorphism classes of the graphs in $\mathscr{G}_1, \ldots, \mathscr{G}_s$, one may define a function, called the *generalised Ramsey function*, from $(2^{\mathscr{P}} - \{\emptyset\})^s$ to the set of all positive integers, by $([\mathscr{G}_1], \ldots, [\mathscr{G}_s]) \mapsto R(\mathscr{G}_1, \ldots, \mathscr{G}_s)$, where by definition, $[\mathscr{G}_i] = \{[G_i] \mid G_i \in \mathscr{G}_i\}$.

It should be noted that it is not obvious that for each positive integer s and all non-empty sets of graphs $\mathscr{G}_1, \ldots, \mathscr{G}_s$, there is a positive integer n such that $n \to (\mathscr{G}_1, \ldots, \mathscr{G}_s)$. However, this is easily proved to be the case (see Corollary 1.3.7), whence the generalised Ramsey function is well-defined.

Also note that if $\mathscr{G}_i = \{G_i\}$ for some $i \in [s]$, then we often write $n \to (\mathscr{G}_1, \ldots, G_i, \ldots, \mathscr{G}_s)$ instead of $n \to (\mathscr{G}_1, \ldots, \{G_i\}, \ldots, \mathscr{G}_s)$; similarly, we let $R(\mathscr{G}_1, \ldots, G_i, \ldots, \mathscr{G}_s) = R(\mathscr{G}_1, \ldots, \{G_i\}, \ldots, \mathscr{G}_s)$.

Definition 1.2.10. Recall that $\mathscr{G}_1$ and $\mathscr{G}_2$ denote non-empty sets of non-empty (uncoloured) graphs. A complete red-blue graph G is said to be $(\mathscr{G}_1, \mathscr{G}_2)$-avoiding if G contains neither a red subgraph belonging to $\mathscr{G}_1$ nor a blue subgraph belonging to $\mathscr{G}_2$.

Let P be a property such that for each (uncoloured) graph G, G fulfils P if and only if all graphs isomorphic to G do.[1] Then a complete red-blue

[1] Formally, one may define a *graph property* as a class of (uncoloured) graphs that is closed under isomorphism (see [4]).

graph G is said to be *red P-fulfilling* (*blue P-fulfilling*) if its red subgraph G_{red} (its blue subgraph G_{blue}) fulfils P. In order to illustrate this concept, consider the complete red-blue graphs given in Figure 2. They are both blue bipartite and red (as well as blue) almost bipartite; the second one is also blue complete bipartite.

Let $\mathscr{C} = \{C_k \mid k \geq 3\}$, let $\mathscr{C}_o = \{\text{odd cycles}\} = \{C_k \mid k \equiv 1\}$, and let $\mathscr{C}_e = \{\text{even cycles}\} = \{C_k \mid k \equiv 0\}$. Also, for each integer $m \geq 3$, let $\mathscr{C}_{\leq m} = \{C_k \mid k \leq m\}$ and $\mathscr{C}_{\geq m} = \{C_k \mid k \geq m\}$. Finally, if $\Gamma \subseteq \mathscr{C}$, let

$$\min(\Gamma) = \begin{cases} \min\{k \mid C_k \in \Gamma\} & \text{if } \Gamma \text{ is non-empty} \\ \infty & \text{otherwise,} \end{cases}$$

and for each $i \in [2]$, let $\gamma^i = \min(\Gamma_i)$ and $\gamma_e^i = \min(\Gamma_i \cap \mathscr{C}_e)$.

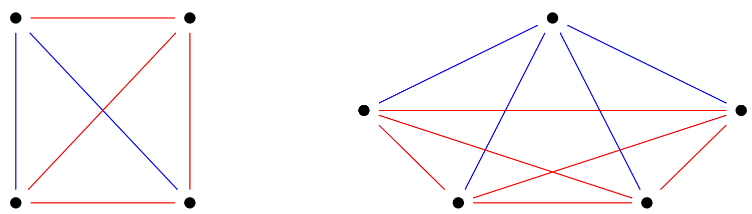


Figure 2: Two blue bipartite and red almost bipartite graphs.

Definition 1.2.11. A complete red-blue graph G is said to have a *Hamiltonian r-partition* $(P_{(1)}, \ldots, P_{(r)})$ if $V(G)$ is the disjoint union of r subsets $V_1, \ldots, V_r$, such that for each $i \in [r]$ with $|V_i| \geq 1$, $G[V_i]$ contains a red path $P_{(i)}$ of length $|V_i| - 1$.

In order to make sense of some of the proofs in this thesis, we need the following notation: Let G be a complete red-blue graph, let H be a blue K_4 or a blue tower in G, and let $v \in H$. Then a vertex $x \in G$ is said to be $\mathrm{RA}(v)$ if x is red adjacent to each vertex of H, except possibly v.

1.3 Basic properties of the generalised Ramsey function

Proposition 1.3.1. *Let s be a positive integer. Then*

$$n \to (G_1, \ldots, G_s) \qquad \textit{if and only if} \qquad n \to (G_1, \ldots, G_s, K_2).$$

Proof. ($\Rightarrow$) Fix an arbitrary $(s+1)$-edge colouring of K_n. If it contains an $(s+1)$-coloured edge, then we have an $(s+1)$-coloured K_2. If not, then we have (by assumption) an i-coloured G_i for some $i \in [s]$.

($\Leftarrow$) Fix an arbitrary s-edge colouring of K_n. Since it contains no $(s+1)$-coloured K_2, we have (by assumption) an i-coloured G_i for some $i \in [s]$. $\square$

Proposition 1.3.2. *Let s be a positive integer and let σ be a permutation of $[s]$. Then*

$$n \to (G_1, \ldots, G_s) \qquad \text{if and only if} \qquad n \to (G_{\sigma(1)}, \ldots, G_{\sigma(s)}).$$

Proof. ($\Rightarrow$) Fix an arbitrary s-edge colouring ρ of K_n. We have to show that (K_n, ρ) contains an i-coloured $G_{\sigma(i)}$ for some $i \in [s]$. Consider the s-edge colouring $\rho' = \sigma \circ \rho$ of K_n. Since $n \to (G_1, \ldots, G_s)$, (K_n, ρ') contains a j-coloured G_j for some $j \in [s]$, and since σ is surjective, (K_n, ρ') contains a $\sigma(i)$-coloured $G_{\sigma(i)}$ for some $i \in [s]$. Thus (K_n, ρ) must have contained a $G_{\sigma(i)}$ in the colours that σ maps to $\sigma(i)$, that is an i-coloured $G_{\sigma(i)}$ (since σ is injective).

($\Leftarrow$) For each $i \in [s]$, let $H_i = G_{\sigma(i)}$; note that $H_{\sigma^{-1}(i)} = G_i$. Since $n \to (H_1, \ldots, H_s)$, the $\Rightarrow$ part implies that $n \to (H_{\sigma^{-1}(1)}, \ldots, H_{\sigma^{-1}(s)})$, that is $n \to (G_1, \ldots, G_s)$. $\square$

Proposition 1.3.3. *Let s be a positive integer, and for each $i \in [s]$, let H_i be a non-empty subgraph of G_i. Then $R(H_1, \ldots, H_s) \leq R(G_1, \ldots, G_s)$ (provided the right hand side exists).*

Proof. By definition of $R(G_1, \ldots, G_s)$, $R(G_1, \ldots, G_s) \to (G_1, \ldots, G_s)$, which obviously implies that $R(G_1, \ldots, G_s) \to (H_1, \ldots, H_s)$. Thus and by definition of $R(H_1, \ldots, H_s)$, $R(H_1, \ldots, H_s) \leq R(G_1, \ldots, G_s)$. $\square$

Let s be a positive integer. Note that in case $E(G_i) = \emptyset$ for some $i \in [s]$, then

$$R(G_1, \ldots, G_s) = \min_{i \in [s]} \{ |V(G_i)| \mid E(G_i) = \emptyset \} \geq 1.$$

Thus for arbitrary graphs $G_1, \ldots, G_s$,

$$R(G_1, \ldots, G_s) \geq \min_{i \in [s]} \{ |V(G_i)| \} \geq 1$$

(provided the left hand side exists).

Theorem 1.3.4 (Ramsey's theorem)**.**

(a) $R(t) = t$, *for all* $t \geq 2$.

(b) If $s \geq 2$, $t_i \geq 2$ *for all* $i \in [s]$, *and* $t_j = 2$ *for some* $j \in [s]$, *then*

$$R(t_1, \ldots, t_s) = R(t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_s).$$

(c) If $s \geq 1$ *and* $t_i \geq 3$ *for all* $i \in [s]$, *then*

$$R(t_1, \ldots, t_s) \leq \left(\sum_{i=1}^{s} R(t_1, \ldots, t_i - 1, \ldots, t_s) \right) - s + 2.$$

(d) *For each positive integer s and all integers $t_1, \ldots, t_s \geq 2$, there is an integer $n \geq 2$ such that $n \to (t_1, \ldots, t_s)$; thus $R(t_1, \ldots, t_s)$ always exists.*

(e) *For each positive integer s and all graphs $G_1, \ldots, G_s$, there is a positive integer n such that $n \to (G_1, \ldots, G_s)$; thus $R(G_1, \ldots, G_s)$ always exists.*

Remark. In parts (b) and (c), the right hand sides are assumed to exist. In part (d), we prove this to be the case.

Proof. (a) This is an obvious result.

(b) Use Propositions 1.3.1 and 1.3.2.

(c) The result follows directly from part (a) in case $s = 1$. Thus, from now on, assume that $s \geq 2$. Let

$$n = \left(\sum_{i=1}^{s} R(t_1, \ldots, t_i - 1, \ldots, t_s) \right) - s + 2$$

and fix an arbitrary s-edge colouring ρ of K_n. Take a vertex $x \in K_n$, and for each $i \in [s]$, define $\Gamma_i^x = \{y \in K_n \mid \rho(xy) = i\}$. Then for some $j \in [s]$, $|\Gamma_j^x| \geq R(t_1, \ldots, t_j - 1, \ldots, t_s)$. (Suppose not, that is suppose that $|\Gamma_i^x| \leq R(t_1, \ldots, t_i - 1, \ldots, t_s) - 1$, for all $i \in [s]$. Then

$$\sum_{i=1}^{s} |\Gamma_i^x| \leq \left(\sum_{i=1}^{s} R(t_1, \ldots, t_i - 1, \ldots, t_s) \right) - s = n - 2,$$

which contradicts the fact that $\sum_{i=1}^{s} |\Gamma_i^x| = n - 1$, the number of neighbours of x.) By definition of $R(G_1, \ldots, G_s)$, $K_n[\Gamma_j^x]$ contains either an i-coloured K_{t_i} for some $i \in [s] - \{j\}$, or a j-coloured $K_{t_j - 1}$. In the former case, we are done, and in the latter case, $K_n[\Gamma_j^x \cup \{x\}]$ contains a j-coloured K_{t_j}.

(d) Use parts (a) through (c) and induction.

(e) Use part (d) and Proposition 1.3.3. □

Proposition 1.3.5. *Let s be a positive integer and let σ be a permutation of $[s]$. Then*

$$n \to (\mathscr{G}_1, \ldots, \mathscr{G}_s) \qquad \text{if and only if} \qquad n \to (\mathscr{G}_{\sigma(1)}, \ldots, \mathscr{G}_{\sigma(s)}).$$

Proof. This is proved in the same way as Proposition 1.3.2. □

Proposition 1.3.6. *Let s be a positive integer, and for each $i \in [s]$, let $\mathscr{H}_i$ be a non-empty subset of $\mathscr{G}_i$. Then $R(\mathscr{G}_1, \ldots, \mathscr{G}_s) \leq R(\mathscr{H}_1, \ldots, \mathscr{H}_s)$ (provided the right hand side exists).*

Proof. By definition of $R(\mathscr{H}_1, \ldots, \mathscr{H}_s)$, $R(\mathscr{H}_1, \ldots, \mathscr{H}_s) \to (\mathscr{H}_1, \ldots, \mathscr{H}_s)$, which obviously implies that $R(\mathscr{H}_1, \ldots, \mathscr{H}_s) \to (\mathscr{G}_1, \ldots, \mathscr{G}_s)$. Thus and by definition of $R(\mathscr{G}_1, \ldots, \mathscr{G}_s)$, $R(\mathscr{G}_1, \ldots, \mathscr{G}_s) \leq R(\mathscr{H}_1, \ldots, \mathscr{H}_s)$. □

Corollary 1.3.7. *Let s be a positive integer, and for each $i \in [s]$, let $G_i \in \mathscr{G}_i$. Then $R(\mathscr{G}_1, \ldots, \mathscr{G}_s) \leq R(G_1, \ldots, G_s)$; in particular, $R(\mathscr{G}_1, \ldots, \mathscr{G}_s)$ always exists.* □

Since we now know that $R(\mathscr{G}_1, \ldots, \mathscr{G}_s)$ always exists, the following result is an immediate consequence of Proposition 1.3.5.

Proposition 1.3.8. *Let s be a positive integer and let σ be a permutation of $[s]$. Then*

$$R(\mathscr{G}_1, \ldots, \mathscr{G}_s) = R(\mathscr{G}_{\sigma(1)}, \ldots, \mathscr{G}_{\sigma(s)}).$$ □

1.4 An alternative view of generalised Ramsey numbers

In this section, we give an alternative, equivalent definition of generalised Ramsey numbers, which perhaps makes this generalisation of the ordinary Ramsey numbers appear more natural. The idea stems from [1] and personal communication with its author. We begin by recalling the notion of a poset, and some related concepts.

Definition 1.4.1. If P is a set and $\leq$ is a binary relation on P, then $(P, \leq)$ is said to be a *partially ordered set*, or a *poset*, if the following properties hold, for all elements $x, y, z \in P$:

(i) $x \leq x$ (*reflexivity*);

(ii) $x = y$ if $x \leq y$ and $y \leq x$ (*antisymmetry*); and

(iii) $x \leq z$ if $x \leq y$ and $y \leq z$ (*transitivity*).

Naturally, $x \geq y$, $x < y$, and $x > y$ have the same meaning as $y \leq x$, $x \leq y$ and $x \neq y$, and $y < x$, respectively. Two elements $x, y \in P$ are *comparable* if $x \leq y$ or $y \leq x$; otherwise they are *incomparable*.

Let $(P, \leq)$ be a poset. A subset $Q \subseteq P$ is a *chain* if any two elements of Q are comparable, an *antichain* if any two distinct elements of Q are incomparable, an *order ideal* if $y \in Q$ whenever $x \in Q$ and $y \leq x$, and a *dual order ideal*, or a *filter*, if $y \in Q$ whenever $x \in Q$ and $y \geq x$. An element $x \in Q \subseteq P$ is *maximal* (*minimal*) in Q if there is no element $y \in Q$ such that $y > x$ ($y < x$). Define

$$\lfloor Q \rfloor = \{y \in P \mid y \leq x \text{ for some } x \in Q\} \tag{1}$$

and

(2) $$\lceil Q \rceil = \{y \in P \mid y \geq x \text{ for some } x \in Q\}$$

By transitivity, (1) is an order ideal and (2) is a filter. The subset Q is said to *generate* (1) and (2), respectively. In case $Q = \{x\}$, then x is said to generate the *principal order ideal* $\lfloor x \rfloor = \{y \in P \mid y \leq x\}$ and the *principal filter* $\lceil x \rceil = \{y \in P \mid y \geq x\}$, respectively. (Note that $\emptyset$ generates the empty order ideal and the empty filter, respectively.)

A poset $(P, \leq)$ is said to satisfy the *ascending chain condition* or ACC (the *descending chain condition* or DCC) if there is no infinite sequence $(x_i)_{i\geq 1}$ in P such that $x_1 < x_2 < \cdots$ $(x_1 > x_2 > \cdots)$.

Proposition 1.4.2. *Given a poset $(P, \leq)$ that satisfies ACC, there is a natural bijection φ between the set of antichains A and the set of order ideals I, given by $\varphi(A) = \lfloor A \rfloor$ and whose inverse is given by*

$$\varphi^{-1}(I) = \{x \in I \mid x \textit{ is maximal in } I\}.$$

Similarly, given a poset $(P, \leq)$ that satisfies DCC, there is a natural bijection ψ between the set of antichains A and the set of filters J, given by $\psi(A) = \lceil A \rceil$ and whose inverse is given by

$$\psi^{-1}(J) = \{x \in J \mid x \textit{ is minimal in } J\}.$$

Proof. By symmetry, it suffices to prove the second part of the proposition. We know that $\lceil A \rceil$ is a filter. We thus have to prove that (i) $\psi^{-1}(J)$ is an antichain, (ii) $\psi^{-1}(\psi(A)) = A$, and (iii) $\psi(\psi^{-1}(J)) = J$.

(i): We are done if $|\psi^{-1}(J)| \leq 1$. Thus, take $x \neq y$ in $\psi^{-1}(J) \subseteq J$. Since $x \in \psi^{-1}(J)$ and $y \in J$, $y < x$ does not hold. Similarly, $x < y$ does not hold. Thus x and y are incomparable.

(ii): $\psi^{-1}(\psi(A)) \subseteq A$: Take $z \in \psi^{-1}(\psi(A))$. By definition, z is minimal in $\lceil A \rceil$. In particular, $z \in \lceil A \rceil$, whence $z \geq x$ for some $x \in A$. Since z is minimal in $\lceil A \rceil$, $z = x$, whence $z \in A$.

$\psi^{-1}(\psi(A)) \supseteq A$: Take $z \in A$. We have to show that z is minimal in $\lceil A \rceil$. Suppose not, that is suppose that there is an element $y \in \lceil A \rceil$ such that $y < z$. Since $y \in \lceil A \rceil$, $y \geq x$ for some $x \in A$. By transitivity, $z > x$, which contradicts the fact that A is an antichain.

(iii): $\psi(\psi^{-1}(J)) \subseteq J$: Take $y \in \psi(\psi^{-1}(J))$. By definition, $y \geq x$ for some $x \in \psi^{-1}(J)$, that is $y \geq x$ for some (minimal) element x in J. Since J is a filter, $y \in J$.

$\psi(\psi^{-1}(J)) \supseteq J$: Take $y \in J$. We have to show that $y \geq x$ for some $x \in \psi^{-1}(J)$ or, equivalently, that

(3) $$y \geq x \text{ for some minimal element } x \text{ in } J.$$

If y is minimal in J, then (3) holds for $x = y$. If not, then there is an element $y_1 \in J$ such that $y_1 < y$. If y_1 is minimal in J, then (3) holds for $x = y_1$. If not, then there is an element $y_2 \in J$ such that $y_2 < y_1$, and so forth. Since $(P, \leq)$ satisfies DCC, we eventually reach a minimal element y_n in J such that $y_n < y_{n-1} < \cdots < y_1 < y$. Thus and by transitivity, (3) holds for $x = y_n$. □

Now, recall that $\mathscr{P}$ is the set of all isomorphism classes of graphs. We can make the set $\mathscr{P}$ into a poset $(\mathscr{P}, \leq)$ by defining $[G_1] \leq [G_2]$ in $\mathscr{P}$ if G_1 is isomorphic to a subgraph of G_2. Note that if $[G_1] > [G_2]$, then $|V(G_1)| + |E(G_1)| > |V(G_2)| + |E(G_2)|$. Thus $(\mathscr{P}, \leq)$ satisfies DCC.

Recall also, that the generalised Ramsey number $R(\mathscr{G}_1, \ldots, \mathscr{G}_s)$ is the least positive integer n, such that for each s-edge colouring of K_n, there is an $i \in [s]$ such that $(K_n)_{c_i}$ contains a subgraph isomorphic to some $G_i \in \mathscr{G}_i$. We are now ready to give the alternative definition:

Definition 1.4.3. The *generalised Ramsey number* $R(\mathscr{G}_1, \ldots, \mathscr{G}_s)$ is the least positive integer n, such that for each s-edge colouring of K_n, there is an $i \in [s]$ such that $(K_n)_{c_i}$ belongs to the filter $\lceil [\mathscr{G}_i] \rceil$.

In particular, the *ordinary Ramsey number* $R(G_1, \ldots, G_s)$ is the least positive integer n, such that for each s-edge colouring of K_n, there is an $i \in [s]$ such that $(K_n)_{c_i}$ belongs to the *principal* filter $\lceil [G_i] \rceil$.

Finally, for each $i \in [s]$, let $\mathscr{A}_i$ be a set of graphs such that

$$[\mathscr{A}_i] = \psi^{-1}(\lceil [\mathscr{G}_i] \rceil).$$

Since $(\mathscr{P}, \leq)$ satisfies DCC, it follows from Proposition 1.4.2 that $[\mathscr{A}_i]$ is an antichain and

$$\lceil [\mathscr{A}_i] \rceil = \psi([\mathscr{A}_i]) = \psi(\psi^{-1}(\lceil [\mathscr{G}_i] \rceil)) = \lceil [\mathscr{G}_i] \rceil.$$

Hence, $R(\mathscr{A}_1, \ldots, \mathscr{A}_s) = R(\mathscr{G}_1, \ldots, \mathscr{G}_s)$. Thus, when dealing with generalised Ramsey numbers, one may always take $[\mathscr{G}_1], \ldots, [\mathscr{G}_s]$ to be antichains in $\mathscr{P}$ (with each $\mathscr{G}_i$ containing no two isomorphic graphs). For instance, since $[K_4] \geq [C_3]$, $R(C_4, \{C_3, C_6, K_4\}) = R(C_4, \{C_3, C_6\})$. As to the two sets of cycles case (see Chapter 2), note that each subset of $[\mathscr{C}]$ is an antichain in $(\mathscr{P}, \leq)$.

2 The two sets of cycles case

In this chapter, we investigate generalised Ramsey numbers for two sets of cycles, that is generalised Ramsey numbers of the form $R(\Gamma_1, \Gamma_2)$, where (Γ_1, Γ_2) is a pair of non-empty sets of cycles.

2.1 Preliminaries and previously known results

In this section, we first define a number of sets and colourings that will be needed in the proofs to come. We then present some results which, to the best of the author's knowledge, include all previously known generalised Ramsey numbers for two sets of cycles.

2.1.1 Preliminaries

For pairs (n, k) of integers such that $n \geq k \geq 3$, define

$$\begin{aligned}
\Delta_0 &= \{(3,3),(4,4)\},\\
\Delta_1 &= \{(n,k) \mid n \equiv 0 \text{ and } k \equiv 0\} - \{(4,4)\},\\
\Delta_2 &= \{(n,k) \mid n \equiv 1,\ k \equiv 0, \text{ and } 2n \geq 3k\},\\
\Delta_3 &= \{(n,k) \mid n \equiv 1,\ k \equiv 0, \text{ and } 2n \leq 3k\}, \text{ and}\\
\Delta_4 &= \{(n,k) \mid k \equiv 1\} - \{(3,3)\}.
\end{aligned}$$

Also, for pairs (Γ_1, Γ_2) of non-empty sets of cycles, define

$$\begin{aligned}
\mathfrak{A}^0 &= \{(\Gamma_1,\Gamma_2) \mid C_3 \text{ or } C_4 \in \Gamma_1 \cap \Gamma_2 \text{ and } C_5 \notin \Gamma_1 \cup \Gamma_2\},\\
\mathfrak{A}^1_{red} &= \{(\Gamma_1,\Gamma_2) \mid 0 \equiv \gamma^2 \geq \gamma^1_e \text{ and } (\gamma^2,\gamma^1_e) \neq (4,4)\},\\
\mathfrak{A}^2_{red} &= \{(\Gamma_1,\Gamma_2) \mid 1 \equiv \gamma^2 \geq 3\gamma^1_e/2\},\\
\mathfrak{A}^3_{red} &= \{(\Gamma_1,\Gamma_2) \mid 1 \equiv \gamma^1 \leq 3\gamma^2_e/2,\ 0 \equiv \gamma^2 < \gamma^1, \text{ and } \gamma^1_e \geq 2\gamma^2\},\\
\mathfrak{A}^4_{red} &= \{(\Gamma_1,\Gamma_2) \mid \gamma^2 \geq 4 \text{ and } 1 \equiv \gamma^1 \leq \gamma^2 \leq \gamma^1_e/2\},\\
\mathfrak{A}^1_{blue} &= \{(\Gamma_1,\Gamma_2) \mid 0 \equiv \gamma^1 \geq \gamma^2_e \text{ and } (\gamma^1,\gamma^2_e) \neq (4,4)\},\\
\mathfrak{A}^2_{blue} &= \{(\Gamma_1,\Gamma_2) \mid 1 \equiv \gamma^1 \geq 3\gamma^2_e/2\},\\
\mathfrak{A}^3_{blue} &= \{(\Gamma_1,\Gamma_2) \mid 1 \equiv \gamma^2 \leq 3\gamma^1_e/2,\ 0 \equiv \gamma^1 < \gamma^2, \text{ and } \gamma^2_e \geq 2\gamma^1\}, \text{ and}\\
\mathfrak{A}^4_{blue} &= \{(\Gamma_1,\Gamma_2) \mid \gamma^1 \geq 4 \text{ and } 1 \equiv \gamma^2 \leq \gamma^1 \leq \gamma^2_e/2\}.
\end{aligned}$$

Finally, define

$$\mathfrak{B}_1 = \bigcup_{j=1}^{4} \mathfrak{A}^j_{red} \quad \text{and} \quad \mathfrak{B}_2 = \bigcup_{j=1}^{4} \mathfrak{A}^j_{blue},$$

and for each $i \in [2]$, let

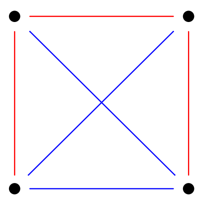
$$\mathfrak{B}'_i = \mathfrak{B}_i \cap \{(\Gamma_1,\Gamma_2) \mid \min(\Gamma_1 \cup \Gamma_2) \geq 6\}.$$

One can show that

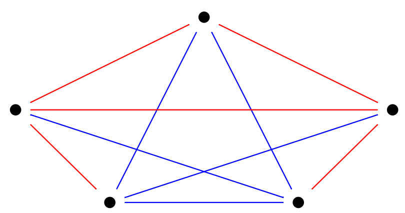
$$\begin{aligned}
\mathfrak{B}_1 = \big\{(\Gamma_1,\Gamma_2) \mid\, & 0 \equiv \gamma^2 \geq \max(6,\gamma_e^1),\ \gamma^2 \geq 3\gamma_e^1/2, \text{ or} \\
& \big(\gamma^1 \equiv 1,\ \gamma_e^1 \geq 2\gamma^2, \text{ and } (0 \equiv \gamma^2 \geq 2\gamma^1/3 \text{ or } \gamma^2 \geq \max(4,\gamma^1))\big)\big\}, \\
\mathfrak{B}_2 = \big\{(\Gamma_1,\Gamma_2) \mid\, & 0 \equiv \gamma^1 \geq \max(6,\gamma_e^2),\ \gamma^1 \geq 3\gamma_e^2/2, \text{ or} \\
& \big(\gamma^2 \equiv 1,\ \gamma_e^2 \geq 2\gamma^1, \text{ and } (0 \equiv \gamma^1 \geq 2\gamma^2/3 \text{ or } \gamma^1 \geq \max(4,\gamma^2))\big)\big\}, \\
\mathfrak{B}_1' = \Big\{(\Gamma_1,\Gamma_2) \mid\, & \gamma^1 \geq 6 \text{ and } \Big(0 \equiv \gamma^2 \geq \gamma_e^1,\ \gamma^2 \geq 3\gamma_e^1/2, \text{ or} \\
& \big(\gamma^1 \equiv 1,\ \gamma_e^1 \geq 2\gamma^2, \text{ and } (0 \equiv \gamma^2 \geq 2\gamma^1/3 \text{ or } \gamma^2 \geq \gamma^1)\big)\Big)\Big\}, \text{ and} \\
\mathfrak{B}_2' = \Big\{(\Gamma_1,\Gamma_2) \mid\, & \gamma^2 \geq 6 \text{ and } \Big(0 \equiv \gamma^1 \geq \gamma_e^2,\ \gamma^1 \geq 3\gamma_e^2/2, \text{ or} \\
& \big(\gamma^2 \equiv 1,\ \gamma_e^2 \geq 2\gamma^1, \text{ and } (0 \equiv \gamma^1 \geq 2\gamma^2/3 \text{ or } \gamma^1 \geq \gamma^2)\big)\Big)\Big\}.
\end{aligned}$$

We now turn to the colourings. Note that Colourings 3, 4, 5, and 6 were used to prove the lower bounds in Theorem 2.1.1, when (n,k) belongs to Δ_0, $\Delta_1 \cup \Delta_2$, Δ_3, and Δ_4, respectively.

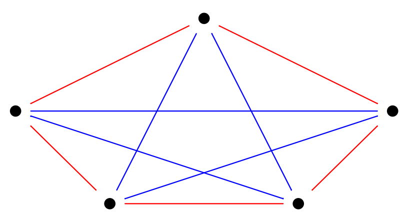
- **Colouring 1:**



- **Colouring 2:**



- **Colouring 3:**



- **Colouring 4:** The complete red-blue graph with vertex set

$$\{x_1, \dots, x_{n-1}, y_1, \dots, y_{k/2-1}\}$$

and

$$\begin{cases} \rho(x_i x_j) = \rho(y_i y_j) = \text{red} \\ \rho(x_i y_j) = \text{blue}. \end{cases}$$

- **Colouring 5:** The complete red-blue graph with vertex set

$$\{x_1, \dots, x_{k-1}, y_1, \dots, y_{k-1}\}$$

and

$$\begin{cases} \rho(x_i x_j) = \rho(y_i y_j) = \text{blue} \\ \rho(x_i y_j) = \text{red}. \end{cases}$$

- **Colouring 6:** The complete red-blue graph with vertex set

$$\{x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1}\}$$

and

$$\begin{cases} \rho(x_i x_j) = \rho(y_i y_j) = \text{red} \\ \rho(x_i y_j) = \text{blue}. \end{cases}$$

2.1.2 Previously known results

The $|\Gamma_1| = |\Gamma_2| = 1$ subcase was proved independently by Rosta [13] and by Faudree and Schelp [6]. A new, simpler proof was given by Károlyi and Rosta [11]. The second formula is due to Schwenk (see [10]).

Theorem 2.1.1. *Let $n \geq k \geq 3$ be integers. Then*

$$R(C_n, C_k) = \begin{cases} 6 & \text{if } (n,k) \in \Delta_0 \\ n + k/2 - 1 & \text{if } (n,k) \in \Delta_1 \cup \Delta_2 \\ 2k - 1 & \text{if } (n,k) \in \Delta_3 \\ 2n - 1 & \text{if } (n,k) \in \Delta_4 \end{cases}$$

or, equivalently,

$$R(C_n, C_k) = \max\big(6, n + k/2 - 1, (2k-1)(n - 2\lfloor n/2 \rfloor), (2n-1)(k - 2\lfloor k/2 \rfloor)\big).$$

Furthermore, we have the following results of Erdős, Faudree, Rousseau, and Schelp:

Theorem 2.1.2 ([5, Theorem 3]). *For all $n \geq 2$,*

$$R(\mathscr{C}_{\leq m}, K_n) = \begin{cases} 2n & \text{if } n < m < 2n-1 \\ 2n-1 & \text{if } m \geq 2n-1. \end{cases}$$

Corollary 2.1.3.

$$R(\mathscr{C}_{\leq m}, C_3) = \begin{cases} 6 & \text{if } m = 4 \\ 5 & \text{if } m \geq 5. \end{cases} \qquad \square$$

Remark. Of course, $R(\mathscr{C}_{\leq 3}, C_3) = R(C_3, C_3) = 6$.

Theorem 2.1.4 ([7, Theorem 2]). *For all $m \geq 3$ and all $n \geq 2$,*

$$R(\mathscr{C}_{\geq m}, K_n) = (m-1)(n-1) + 1.$$

Corollary 2.1.5. *For all $m \geq 3$,*

$$R(\mathscr{C}_{\geq m}, C_3) = 2m - 1. \qquad \square$$

2.2 The red and blue Ramsey numbers

In this section, we define some numbers whose definitions are similar to that of generalised Ramsey numbers. We then determine all such numbers for two sets of cycles. They will turn out to be very closely related to generalised Ramsey numbers for two sets of cycles (see Theorem 2.3.2 and Conjecture 2.3.1).

Definition 2.2.1. Recall that $\mathscr{G}_1$ and $\mathscr{G}_2$ denote non-empty sets of non-empty (uncoloured) graphs. Let the *red Ramsey number* $R_{red}(\mathscr{G}_1, \mathscr{G}_2)$ (the *red complete Ramsey number* $R_{redcomp}(\mathscr{G}_1, \mathscr{G}_2)$) be the least positive integer n, such that each red bipartite (red complete bipartite) graph on n vertices contains a red subgraph belonging to $\mathscr{G}_1$ or a blue subgraph belonging to $\mathscr{G}_2$. The *blue Ramsey number* $R_{blue}(\mathscr{G}_1, \mathscr{G}_2)$ and the *blue complete Ramsey number* $R_{bluecomp}(\mathscr{G}_1, \mathscr{G}_2)$ are defined analogously.

Proposition 2.2.2.

$$R_{redcomp}(\mathscr{G}_1, \mathscr{G}_2) \leq R_{red}(\mathscr{G}_1, \mathscr{G}_2) = R(\mathscr{G}_1 \cup \mathscr{C}_o, \mathscr{G}_2) \leq R(\mathscr{G}_1, \mathscr{G}_2)$$

and

$$R_{bluecomp}(\mathscr{G}_1, \mathscr{G}_2) \leq R_{blue}(\mathscr{G}_1, \mathscr{G}_2) = R(\mathscr{G}_1, \mathscr{G}_2 \cup \mathscr{C}_o) \leq R(\mathscr{G}_1, \mathscr{G}_2).$$

In particular, the red (complete) and blue (complete) Ramsey numbers always exist. $\square$

Corollary 2.2.3.

$$R(\mathscr{G}_1, \mathscr{G}_2) \geq \max(R_{red}(\mathscr{G}_1, \mathscr{G}_2), R_{blue}(\mathscr{G}_1, \mathscr{G}_2)). \qquad \square$$

Lemma 2.2.4. *Let* (Γ_1, Γ_2) *be a pair of non-empty sets of cycles. Then*

$$R_{redcomp}(\Gamma_1, \Gamma_2) = \begin{cases} \gamma^2 + \gamma_e^1/2 - 1 & \text{if } 2\gamma^2 > \gamma_e^1 \\ 2\gamma^2 - 1 & \text{if } 2\gamma^2 \leq \gamma_e^1 \end{cases} \tag{4}$$

and

$$R_{bluecomp}(\Gamma_1, \Gamma_2) = \begin{cases} \gamma^1 + \gamma_e^2/2 - 1 & \text{if } 2\gamma^1 > \gamma_e^2 \\ 2\gamma^1 - 1 & \text{if } 2\gamma^1 \leq \gamma_e^2 \end{cases} \tag{5}$$

or, equivalently,

$$R_{redcomp}(\Gamma_1, \Gamma_2) = \min(\gamma^2 + \gamma_e^1/2 - 1, 2\gamma^2 - 1)$$

and

$$R_{bluecomp}(\Gamma_1, \Gamma_2) = \min(\gamma^1 + \gamma_e^2/2 - 1, 2\gamma^1 - 1).$$

Remarks. Actually, one may extend the definition of $R_{redcomp}(\Gamma_1, \Gamma_2)$ to include the case $\Gamma_1 = \emptyset$, and (4) will still hold (note that $\gamma^1 = \gamma_e^1 = \infty$). Similarly, one may extend the definition of $R_{bluecomp}(\Gamma_1, \Gamma_2)$ to include the case $\Gamma_2 = \emptyset$, and (5) will still hold (now note that $\gamma^2 = \gamma_e^2 = \infty$). The analogous remarks apply to Proposition 2.2.5. Also note that $\gamma^i + \gamma_e^j/2 - 1 = 2\gamma^i - 1$ when $2\gamma^i = \gamma_e^j$.

Proof. The two statements (4) and (5) are symmetric. Thus we only have to prove (5). In order to simplify notation, let $n = \gamma^1$ and $k = \gamma_e^2$.

Assume first that $2n > k$. $R_{bluecomp}(\Gamma_1, \Gamma_2) \geq n + k/2 - 1$: Colouring 4 is blue complete bipartite on $n + k/2 - 2$ vertices, and contains no red cycle of length at least n, no blue cycle of length at least k, and no odd blue cycle.

$R_{bluecomp}(\Gamma_1, \Gamma_2) \leq n + k/2 - 1$: Let G be an arbitrary blue complete bipartite graph on $n + k/2 - 1$ vertices; say that $G_{blue} = K_{p,q}$. Then either $\max(p, q) \geq n$ or $\min(p, q) \geq k/2$. In the former case, G contains a red C_n, and in the latter case, G contains a blue C_k.

Assume now that $2n \leq k$. $R_{bluecomp}(\Gamma_1, \Gamma_2) \geq 2n - 1$: Colouring 6 is blue complete bipartite on $2n - 2$ vertices, and contains no red cycle of length at least n, no blue cycle of length at least k, and no odd blue cycle.

$R_{bluecomp}(\Gamma_1, \Gamma_2) \leq 2n - 1$: Let G be an arbitrary blue complete bipartite graph on $2n - 1$ vertices; say that $G_{blue} = K_{p,q}$. Then $\max(p, q) \geq n$, whence G contains a red C_n. $\square$

Proposition 2.2.5. *Let (Γ_1, Γ_2) be a pair of non-empty sets of cycles. Then*

$$R_{red}(\Gamma_1,\Gamma_2) = \begin{cases} \gamma^2 + \gamma^1_e/2 - 1 & \text{if } 2\gamma^2 > \gamma^1_e \text{ and } (\gamma^2, \gamma^1_e) \neq (3,4) \\ 2\gamma^2 - 1 & \text{if } 2\gamma^2 \leq \gamma^1_e \text{ or } (\gamma^2, \gamma^1_e) = (3,4) \end{cases} \tag{6}$$

and

$$R_{blue}(\Gamma_1,\Gamma_2) = \begin{cases} \gamma^1 + \gamma^2_e/2 - 1 & \text{if } 2\gamma^1 > \gamma^2_e \text{ and } (\gamma^1, \gamma^2_e) \neq (3,4) \\ 2\gamma^1 - 1 & \text{if } 2\gamma^1 \leq \gamma^2_e \text{ or } (\gamma^1, \gamma^2_e) = (3,4) \end{cases} \tag{7}$$

or, equivalently,

$$R_{red}(\Gamma_1,\Gamma_2) = \begin{cases} 5 & \text{if } (\gamma^2, \gamma^1_e) = (3,4) \\ \min(\gamma^2 + \gamma^1_e/2 - 1, 2\gamma^2 - 1) & \text{otherwise} \end{cases}$$

and

$$R_{blue}(\Gamma_1,\Gamma_2) = \begin{cases} 5 & \text{if } (\gamma^1, \gamma^2_e) = (3,4) \\ \min(\gamma^1 + \gamma^2_e/2 - 1, 2\gamma^1 - 1) & \text{otherwise.} \end{cases}$$

Remark. Note that $R_{red}(\Gamma_1,\Gamma_2) = R_{redcomp}(\Gamma_1,\Gamma_2)$, unless $(\gamma^2, \gamma^1_e) = (3,4)$, in which case $R_{red}(\Gamma_1,\Gamma_2) = R_{redcomp}(\Gamma_1,\Gamma_2) + 1$. Of course, the analogous remark applies to $R_{blue}(\Gamma_1,\Gamma_2)$.

Proof. The two statements (6) and (7) are symmetric. Thus we only have to prove (7). In order to simplify notation, let $n = \gamma^1$ and $k = \gamma^2_e$.

The lower bounds follow from Proposition 2.2.2 and Colouring 1.

We now turn to the upper bounds. $2n > k$ and $(n,k) \neq (3,4)$: Since $2n > k$ and $(n,k) \neq (3,4)$, $n \geq 4$. Let G be an arbitrary blue bipartite graph on $n + k/2 - 1$ vertices; say that $G_{blue} \subseteq K_{p,q}$. W.l.o.g., assume that $p \geq q$. In order to obtain a contradiction, assume G is (Γ_1,Γ_2)-avoiding.

Either $p \geq n$ or $q \geq k/2$. Were $p \geq n$, G would contain a red C_n, whence $q \geq k/2$. Were G blue complete bipartite, G would contain a blue C_k, whence there is at least one red edge between the red K_p and the red K_q. Were there two independent red edges between K_p and K_q, G would contain a red C_n (since $n \geq 4$), whence all red edges between K_p and K_q have a common vertex x. Thus, were $q \geq k/2 + 1$, G would contain a blue C_k, whence $q = k/2$ and $p = n - 1$. (Regardless whether x belongs to K_p or to K_q, G would contain a blue $K_{k/2+1,k/2}$.)

Assume first that $x \in K_{k/2}$. Were there at least two red edges between K_{n-1} and x, G would contain a red C_n, whence there is only one red edge between K_{n-1} and x. Thus and since $n \geq 4$, there are at least two blue edges between K_{n-1} and x, say v_1x and v_2x, and v_1xv_2 can be extended to a blue C_k, contrary to the hypothesis.

Assume now that $x \in K_{n-1}$. Then all edges between $K_{n-1} - x$ and $K_{k/2}$ are blue, whence G contains a blue C_k, unless $n - 1 = k/2$, in which case $x \in K_{k/2}$ (which we have already treated).

$2n \leq k$ or $(n, k) = (3, 4)$: Let G be an arbitrary blue bipartite graph on $2n - 1$ vertices; say that $G_{blue} \subseteq K_{p,q}$. Then $\max(p, q) \geq n$, whence G contains a red C_n. $\square$

Given a pair (Γ_1, Γ_2) of non-empty sets of cycles, let

$$\mathfrak{m} = \mathfrak{m}(\Gamma_1, \Gamma_2) = \max(R_{red}(\Gamma_1, \Gamma_2), R_{blue}(\Gamma_1, \Gamma_2)).$$

We shall see that the generalised Ramsey number $R(\Gamma_1, \Gamma_2)$ often equals $\mathfrak{m}$ (Theorem 2.3.2), and we conjecture that there are no exceptions besides the ones enumerated in the theorem (Conjecture 2.3.1). The following result is an immediate consequence of Proposition 2.2.5.

Corollary 2.2.6. *Let (Γ_1, Γ_2) be a pair of non-empty sets of cycles. Then*

$$\mathfrak{m} = \max(5, \min(\gamma^2 + \gamma^1_e/2 - 1, 2\gamma^2 - 1), \min(\gamma^1 + \gamma^2_e/2 - 1, 2\gamma^1 - 1)). \qquad \square$$

2.3 Main theorem and a conjecture

Let us first give the conjecture. We shall then prove that the conjecture holds for many pairs of non-empty sets of cycles (see Theorem 2.3.2).

Conjecture 2.3.1. *Let (Γ_1, Γ_2) be a pair of non-empty sets of cycles. Then*

$$(8) \quad R(\Gamma_1, \Gamma_2) = \begin{cases} \mathfrak{m} + 1 & \text{if } C_3 \text{ or } C_4 \in \Gamma_1 \cap \Gamma_2 \text{ and } C_3 \text{ or } C_5 \notin \Gamma_1 \cup \Gamma_2 \\ \mathfrak{m} & \text{otherwise} \end{cases}$$
$$= \begin{cases} 6 & \text{if } C_3 \text{ or } C_4 \in \Gamma_1 \cap \Gamma_2 \text{ and } C_3 \text{ or } C_5 \notin \Gamma_1 \cup \Gamma_2 \\ \mathfrak{m} & \text{otherwise.} \end{cases}$$

Remark. Note that C_3 or $C_4 \in \Gamma_1 \cap \Gamma_2$ is equivalent to $\min(\Gamma_1 \cap \Gamma_2) \leq 4$.

The lower bounds follow from Corollary 2.2.3 and Proposition 2.3.4 (see below). For all pairs (Γ_1, Γ_2) such that (8) holds, note that since $\mathfrak{m}$ only depends on γ^1, γ^1_e, γ^2, and γ^2_e, so does $R(\Gamma_1, \Gamma_2)$, unless $(\gamma^1, \gamma^2) \in \{(3, 3), (4, 3), (3, 4)\}$. In particular, if the conjecture is true, then this applies for all pairs (Γ_1, Γ_2).

We are now ready to state the main result of this thesis:

Theorem 2.3.2. *Let* (Γ_1, Γ_2) *be a pair of non-empty sets of cycles such that either* $\min(\Gamma_1 \cup \Gamma_2) \leq 5$ *or* $(\Gamma_1, \Gamma_2) \in \mathfrak{B}_1' \cup \mathfrak{B}_2'$*. Then*

$$R(\Gamma_1, \Gamma_2) = \begin{cases} \mathfrak{m} + 1 & \textit{if } C_3 \textit{ or } C_4 \in \Gamma_1 \cap \Gamma_2 \textit{ and } C_3 \textit{ or } C_5 \notin \Gamma_1 \cup \Gamma_2 \\ \mathfrak{m} & \textit{otherwise} \end{cases}$$
$$= \begin{cases} 6 & \textit{if } C_3 \textit{ or } C_4 \in \Gamma_1 \cap \Gamma_2 \textit{ and } C_3 \textit{ or } C_5 \notin \Gamma_1 \cup \Gamma_2 \\ \mathfrak{m} & \textit{otherwise.} \end{cases}$$

Theorem 2.3.2 is an immediate consequence of Propositions 2.3.3 through 2.3.10 (see below). We shall devote the rest of this chapter to the proofs of these propositions.

Proposition 2.3.3. *Let* (Γ_1, Γ_2) *be a pair of non-empty sets of cycles such that* $(\Gamma_1, \Gamma_2) \in \mathfrak{B}_1 \cup \mathfrak{B}_2$*. Then*

$$R(\Gamma_1, \Gamma_2) = \mathfrak{m} = \begin{cases} R_{red}(\Gamma_1, \Gamma_2) & \textit{if } (\Gamma_1, \Gamma_2) \in \mathfrak{B}_1 \\ R_{blue}(\Gamma_1, \Gamma_2) & \textit{if } (\Gamma_1, \Gamma_2) \in \mathfrak{B}_2. \end{cases}$$

Proposition 2.3.4. *Let* (Γ_1, Γ_2) *be a pair of non-empty sets of cycles such that* C_3 *or* $C_4 \in \Gamma_1 \cap \Gamma_2$*. Then* $R_{red}(\Gamma_1, \Gamma_2) = R_{blue}(\Gamma_1, \Gamma_2) = 5$ *and*

$$R(\Gamma_1, \Gamma_2) = \begin{cases} 5 & \textit{if } C_3 \textit{ and } C_5 \in \Gamma_1 \cup \Gamma_2 \\ 6 & \textit{otherwise.} \end{cases}$$

Proposition 2.3.5. *Let* (Γ_1, Γ_2) *be a pair of non-empty sets of cycles such that* $C_4 \in \Gamma_i \not\ni C_3$ *and* $C_3 \in \Gamma_j \not\ni C_4$*, where* $i \in [2]$ *and* $j = 3 - i$*. Then*

$$R(\Gamma_1, \Gamma_2) = \mathfrak{m} = \begin{cases} 6 & \textit{if } C_6 \in \Gamma_j \\ 7 & \textit{otherwise.} \end{cases}$$

Proposition 2.3.6. *Let* (Γ_1, Γ_2) *be a pair of non-empty sets of cycles such that* $\gamma^i \geq 5$ *and* $\gamma^j = 3$*, where* $i \in [2]$ *and* $j = 3 - i$*. Then*

$$R(\Gamma_1, \Gamma_2) = \mathfrak{m} = \begin{cases} R_{blue}(\Gamma_1, \Gamma_2) & \textit{if } i = 1 \\ R_{red}(\Gamma_1, \Gamma_2) & \textit{if } i = 2. \end{cases}$$

Remark. Note the following special case of Proposition 2.3.6: $R(C_n, C_3) = 2n - 1$, for all $n \geq 5$.

Proposition 2.3.7. *Let* (Γ_1, Γ_2) *be a pair of non-empty sets of cycles such that* $\gamma^i = 5$ *and* $\gamma^j = 4$*, where* $i \in [2]$ *and* $j = 3 - i$*. Then*

$$R(\Gamma_1, \Gamma_2) = \mathfrak{m} = \begin{cases} 6 & \textit{if } C_6 \in \Gamma_i \\ 7 & \textit{otherwise.} \end{cases}$$

Proposition 2.3.8. *Let* (Γ_1, Γ_2) *be a pair of non-empty sets of cycles such that* $\gamma^i \geq 6$ *and* $\gamma^j = 4$*, where* $i \in [2]$ *and* $j = 3 - i$*. Then*

$$R(\Gamma_1, \Gamma_2) = \mathfrak{m} = \gamma^i + 1.$$

Proposition 2.3.9. *Let* (Γ_1, Γ_2) *be a pair of non-empty sets of cycles such that* $\gamma^1 = \gamma^2 = 5$*. Then*

$$R_{red}(\Gamma_1, \Gamma_2) = \begin{cases} 7 & \text{if } \gamma_e^1 = 6 \\ 8 & \text{if } \gamma_e^1 = 8 \\ 9 & \text{if } \gamma_e^1 \geq 10, \end{cases} \tag{9}$$

$$R_{blue}(\Gamma_1, \Gamma_2) = \begin{cases} 7 & \text{if } \gamma_e^2 = 6 \\ 8 & \text{if } \gamma_e^2 = 8 \\ 9 & \text{if } \gamma_e^2 \geq 10, \end{cases} \tag{10}$$

and

$$R(\Gamma_1, \Gamma_2) = \mathfrak{m} = \begin{cases} 7 & \text{if } \max(\gamma_e^1, \gamma_e^2) = 6 \\ 8 & \text{if } \max(\gamma_e^1, \gamma_e^2) = 8 \\ 9 & \text{if } \max(\gamma_e^1, \gamma_e^2) \geq 10. \end{cases}$$

Proposition 2.3.10. *Let* (Γ_1, Γ_2) *be a pair of non-empty sets of cycles such that* $\gamma^i \geq 6$ *and* $\gamma^j = 5$*, where* $i \in [2]$ *and* $j = 3 - i$*. Then*

$$R(\Gamma_1, \Gamma_2) = \mathfrak{m} = \begin{cases} R_{blue}(\Gamma_1, \Gamma_2) & \text{if } i = 1 \\ R_{red}(\Gamma_1, \Gamma_2) & \text{if } i = 2. \end{cases}$$

Remark. Note the following special case of Proposition 2.3.10: $R(C_n, C_5) = 2n - 1$, for all $n \geq 6$.

We end this section by proving Proposition 2.3.3:

Proof of Proposition 2.3.3. Consider Colourings 4, 5, and 6. Each one of them is a complete red-blue graph whose red or blue subgraph equals a complete bipartite (uncoloured) graph $K_{p,q}$. Since $K_{p,q}$ and its complement contain precisely the cycles of even length at most $2\min(p, q)$ and of length at most $\max(p, q)$, respectively, Colourings 3, 4, 5, and 6 contain precisely the following red and blue cycles:

- Colouring 3: The red cycle C_5 and the blue cycle C_5.
- Colouring 4: The red cycles $C_3, C_4, \dots, C_{n-1}$ and the blue cycles $C_4, C_6, \dots, C_{k-2}$ if $k \geq 6$ (no blue cycle exists if $k = 4$).

- Colouring 5: The red cycles $C_4, C_6, \ldots, C_{2k-2}$ and the blue cycles $C_3, C_4, \ldots, C_{k-1}$.
- Colouring 6: The red cycles $C_3, C_4, \ldots, C_{n-1}$ and the blue cycles $C_4, C_6, \ldots, C_{2n-2}$.

Thus and by Corollary 1.3.7 and the upper bounds in Theorem 2.1.1, the following statements hold, for each subset $\Phi \subseteq \mathscr{C}_{\geq n}$, each subset $\Psi \subseteq \mathscr{C}_{\geq k}$, each subset $\Phi_0 \subseteq \mathscr{C}_{\geq 2k-1}$, each subset $\Psi_0 \subseteq \mathscr{C}_{\geq 2n-1}$, each subset $\Omega \subseteq \mathscr{C}_o$, and all subsets $\Omega_1, \Omega_2 \subseteq \mathscr{C} - \{C_5\}$:

$$\begin{aligned} R(\{C_n\} \cup \Omega_1, \{C_k\} \cup \Omega_2) &= 6 && \text{if } (n,k) \in \Delta_0, \\ R(\{C_n\} \cup \Phi, \{C_k\} \cup \Psi \cup \Omega) &= n + k/2 - 1 && \text{if } (n,k) \in \Delta_1 \cup \Delta_2, \\ R(\{C_n\} \cup \Phi_0 \cup \Omega, \{C_k\} \cup \Psi) &= 2k - 1 && \text{if } (n,k) \in \Delta_3, \text{ and} \\ R(\{C_n\} \cup \Phi, \{C_k\} \cup \Psi_0 \cup \Omega) &= 2n - 1 && \text{if } (n,k) \in \Delta_4. \end{aligned}$$

Thus and by symmetry,

$$(11) \qquad R(\Gamma_1, \Gamma_2) = \begin{cases} 6 & \text{if } (\Gamma_1, \Gamma_2) \in \mathfrak{A}^0 \\ \gamma^2 + \gamma_e^1/2 - 1 & \text{if } (\Gamma_1, \Gamma_2) \in \mathfrak{A}^1_{red} \cup \mathfrak{A}^2_{red} \\ 2\gamma^2 - 1 & \text{if } (\Gamma_1, \Gamma_2) \in \mathfrak{A}^3_{red} \cup \mathfrak{A}^4_{red} \\ \gamma^1 + \gamma_e^2/2 - 1 & \text{if } (\Gamma_1, \Gamma_2) \in \mathfrak{A}^1_{blue} \cup \mathfrak{A}^2_{blue} \\ 2\gamma^1 - 1 & \text{if } (\Gamma_1, \Gamma_2) \in \mathfrak{A}^3_{blue} \cup \mathfrak{A}^4_{blue}. \end{cases}$$

Thus, if $(\Gamma_1, \Gamma_2) \in \mathfrak{B}_1$, then $R(\Gamma_1, \Gamma_2) = R_{red}(\Gamma_1, \Gamma_2)$, and if $(\Gamma_1, \Gamma_2) \in \mathfrak{B}_2$, then $R(\Gamma_1, \Gamma_2) = R_{blue}(\Gamma_1, \Gamma_2)$. By Corollary 2.2.3, this completes the proof. □

2.4 Preparatory results

The following two lemmas, due to Károlyi and Rosta, will be used in the proof of Proposition 2.3.7 (see Section 2.7).

Lemma 2.4.1 ([11, Lemma 3.1])**.** *Let $n \geq k \geq 4$, let $n \geq 5$, let $k \equiv 0$, and let G be a complete red-blue graph on $n + k/2 - 1$ vertices. Then G contains either a monochromatic cycle of length at least n or a blue C_k.*

Lemma 2.4.2 ([11, Lemma 3.3])**.** *Let $n \geq k \geq 3$, let $k \geq 4$ if $0 \equiv n \geq 6$, and let G be a complete red-blue graph such that G contains a blue C_n; moreover, assume that $|G| \geq 2n - 1$ if $n \equiv 0$ and $k \equiv 1$. Then G contains either a red C_n or a blue C_k.*

We shall now prove four lemmas, the first of which will be used in the proof of Proposition 2.3.6 (see Section 2.6); the other three will be used in the proof of Proposition 2.3.10 (see Section 2.8). Note that in Lemmas 2.4.3 and 2.4.5, the conditions $n \geq 6$ and $n \geq 7$, respectively, are necessary.

Lemma 2.4.3. *Let G be a complete red-blue graph and let $n \geq 6$. If G contains a C_n all of whose chords are red, then the red subgraph G_{red} is pancyclic.*

Proof. Let $C = x_1x_2 \cdots x_nx_1$ be a C_n all of whose chords are red. If $n \equiv 1$, then $x_1x_3 \cdots x_nx_2x_4 \cdots x_{n-1}x_1$ is a red C_n and $x_1x_3 \cdots x_nx_2x_4 \cdots x_{n-3}x_1$ is a red C_{n-1}, and if $n \equiv 0$, then $x_1x_3 \cdots x_{n-1}x_2x_nx_{n-2} \cdots x_4x_1$ is a red C_n and $x_1x_3 \cdots x_{n-1}x_2x_{n-2}x_{n-4} \cdots x_4x_1$ is a red C_{n-1}. Now, define a sequence $(a_i)_{i\geq 1}$ such that for all $j \geq 0$, $a_{5j+1} = x_{5j+1}$, $a_{5j+2} = x_{5j+3}$, $a_{5j+3} = x_{5j+5}$, $a_{5j+4} = x_{5j+2}$, and $a_{5j+5} = x_{5j+4}$. For each $k \in [3, n-2]$, let $P_{(k)}$ be the path whose vertices are the first k elements of the sequence (a_i). Then $P_{(k)}x_1$ is a red C_k, unless $k = n-2$ and $n \equiv 0 \pmod 5$, in which case $P_{(k-2)}x_{n-1}x_{n-3}x_1$ is a red C_k. $\square$

Definition 2.4.4. Two 2-chords e_1 and e_2 of a cycle $C = x_1x_2 \cdots x_nx_1$ are called *crossing* if there exists an $i \in [n]$ such that $e_1 = x_ix_{i+2}$ and $e_2 = x_{i+1}x_{i+3}$. If such an i does not exist, they are called *non-crossing.*

Lemma 2.4.5. *Let G be a complete red-blue graph and let $n \geq 7$. If G contains a C_n all of whose chords are red, except possibly one 2-chord or two crossing 2-chords, then the red subgraph G_{red} is pancyclic.*

Proof. Let $C = x_1x_2 \cdots x_nx_1$ be a C_n all of whose chords are red, except possibly x_ix_{i+2} and $x_{i+1}x_{i+3}$ for some $i \in [n]$. W.l.o.g., assume that x_1x_3 and x_2x_4 are either red or blue, and that the remaining chords are red. If $n \equiv 1$, then $x_1x_5x_3x_7x_9 \cdots x_nx_2x_6x_4x_8x_{10} \cdots x_{n-1}x_1$ is a red C_n and $x_1x_5x_3x_7x_9 \cdots x_nx_2x_6x_8 \cdots x_{n-1}x_1$ is a red C_{n-1} (note that $x_8 = x_1$ in case $n = 7$), and if $n \equiv 0$, then $x_1x_5x_3x_7x_9 \cdots x_{n-1}x_2x_nx_{n-2} \cdots x_4x_1$ is a red C_n and $x_1x_5x_3x_7x_9 \cdots x_{n-1}x_2x_{n-2}x_{n-4} \cdots x_4x_1$ is a red C_{n-1}. Now, define a sequence $(a_i)_{i\geq 1}$ such that $a_1 = x_1$, $a_2 = x_4$, $a_3 = x_6$, $x_4 = x_2$, $a_5 = x_5$, $a_6 = x_3$, and for all $j \geq 1$, $a_{5j+2} = x_{5j+2}$, $a_{5j+3} = x_{5j+4}$, $a_{5j+4} = x_{5j+6}$, $a_{5j+5} = x_{5j+3}$, and $a_{5j+6} = x_{5j+5}$. For each $k \in [3, n-2]$, let $P_{(k)}$ be the path whose vertices are the first k elements of the sequence (a_i). Then $P_{(k)}x_1$ is a red C_k, unless $k = n-2$ and $n \equiv 1 \pmod 5$, in which case $P_{(k-2)}x_{n-1}x_{n-3}x_1$ is a red C_k. $\square$

Remark. We have recently found the following result by Bondy [2], from which Lemma 2.4.3, but not Lemma 2.4.5, follows as a (non-immediate) corollary: *Let G be Hamiltonian with n vertices and at least $n^2/4$ edges. Then G either is pancyclic or equals $K_{n/2,n/2}$.*

Lemma 2.4.6. *Let G be a blue almost bipartite graph, let $n \geq 7$, and assume that there is a vertex $x \in G$ such that $G - x$ contains a red C_{n-1}. Furthermore, assume that*

(1) $|G| = n + 1$*,* $n \geq 8$*, and* $G - x$ *contains a blue* K_4*;*

(2) $|G| = n + 1$*,* G *contains no blue* K_4*, and* $G - x$ *contains a blue tower* T *which is maximal in* G*; or*

(3) $|G| = n + 2$*,* G *contains no blue* K_4*, and* $G - x$ *contains a blue tower* T *which is maximal in* G *and of height at least 3.*

Then G *contains a red* C_n*.*

Remark. The complete red-blue graph on 8 vertices such that its red subgraph equals $K_{4,4}$ shows that in Case 1, the condition $n \geq 8$ is necessary. This has made the proof of Proposition 2.3.10, Subcase 1a, somewhat more complicated than that of Subcase 1b.

Proof. Let $C = x_1 x_2 \cdots x_{n-1}$ be a red C_{n-1} in $G - x$, and let y be the only vertex or one of the two vertices of $G - x - V(C)$.

Claim 1. *If there is a* $j \in [3, n-4]$*, such that for some* $i \in [n-1]$*,* xx_i *and* xx_{i+j} *are red, then* G *contain a red* C_n*.*

Proof. In order to obtain a contradiction, suppose that G does not contain a red C_n. Since $j \in [3, n-4]$, x_{i-1}, x_i, x_{i+1}, x_{i+j-1}, x_{i+j}, and x_{i+j+1} are all different, and since G contains no red C_n, x is blue adjacent to x_{i-1}, x_{i+1}, x_{i+j-1}, and x_{i+j+1}. Were $x_{i-1}x_{i+j-1}$ or $x_{i+1}x_{i+j+1}$ red,

$$x_{i-1}x_{i+j-1}x_{i+j-2} \cdots x_i x x_{i+j} x_{i+j+1} \cdots x_{i-1}$$

or

$$x_{i+1}x_{i+j+1}x_{i+j+2} \cdots x_i x x_{i+j} x_{i+j-1} \cdots x_{i+1},$$

respectively, would be a red C_n, whence $x_{i-1}x_{i+j-1}$ and $x_{i+1}x_{i+j+1}$ are blue. Were $x_{i-1}x_{i+1}$ blue, $x_{i+1}x_{i+j+1}xx_{i+j-1}x_{i-1}$ would be a blue C_5, whence $x_{i-1}x_{i+1}$ is red. Were $x_i x_{i+j-1}$ and $x_i x_{i+j+1}$ blue,

$$x_{i+j-1}x_i x_{i+j+1} x_{i+1} x x_{i+j-1}$$

would be a blue C_5, whence $x_i x_{i+j-1}$ or $x_i x_{i+j+1}$ is red. Thus

$$x_i x_{i+j-1} x_{i+j-2} \cdots x_{i+1} x_{i-1} x_{i-2} \cdots x_{i+j} x x_i$$

or

$$x_i x_{i+j+1} x_{i+j+2} \cdots x_{i-1} x_{i+1} x_{i+2} \cdots x_{i+j} x x_i,$$

respectively, is a red C_n, contrary to the hypothesis. □

Thus, from now on, assume that no such j exists.

Case 1. Let K be a blue K_4 in $G-x$. Since $x \notin K$, x is blue adjacent to at most one vertex of K. Thus and since $n \geq 8$, x is red adjacent to exactly two vertices of C. Thus xy is red, $y \in K$, and x is blue adjacent to some vertex $v \in K$. Of course, y is blue adjacent to v, but not to some other blue neighbour z of x, since then $xzyuvx$ would be a blue C_5 for some $u \in K$. In particular, y is red adjacent to two consecutive vertices of C, whence G contains a red C_n.

Case 2. We consider two similar subcases, depending on $\mathrm{ht}(T)$.

$\mathrm{ht}(T) = 1$: Since $x \notin T$, x is blue adjacent to at most one vertex of T. Thus x is red adjacent to at least one vertex of C. Thus, w.l.o.g., among the vertices of C, only x_2 is red adjacent to x, only x_2 and x_4 are red adjacent to x, or $n = 7$ and only x_2, x_4, and x_6 are red adjacent to x. In the former case, we obtain a red C_n in the same way as in Case 1 (replacing K with T). In the latter cases, x_1x_3 or x_3x_5 is red (otherwise $G[\{x, x_1, x_3, x_5\}]$ would be a blue K_4 or $xx_3|x_1x_5$ would be a blue tower of height 2), whence G contains a red C_n.

$\mathrm{ht}(T) \geq 2$: Since $x \notin T$, x is blue adjacent to at most one vertex of T. Thus x is red adjacent to at least two vertices of C. Thus, w.l.o.g., among the vertices of C, only x_2 and x_4 are red adjacent to x, or $n = 7$ and only x_2, x_4, and x_6 are red adjacent to x. In the former case, we obtain a red C_n in the same way as in Case 1 (replacing K with T). In the latter case, x_1x_3, x_3x_5, or x_5x_1 is red (otherwise $G[\{x, x_1, x_3, x_5\}]$ would be a blue K_4), whence G contains a red C_n.

Case 3. The proof of Case 2, $\mathrm{ht}(T) \geq 2$, applies to this case as well. □

Lemma 2.4.7. *Let G be a blue almost bipartite graph on n vertices. Then precisely one of the following statements holds:*

(i) G has a Hamiltonian 2-partition;

(ii) G_{blue} is a tower; and

(iii) $n \leq 5$ and G contains a blue K_4.

Proof. It is easy to see that at most one of the three statements holds. Thus we have to prove that at least one of them holds.

If G is blue bipartite with parts V_1 and V_2, let $P_{(1)}$ and $P_{(2)}$ be red paths on V_1 and on V_2, respectively. Then $(P_{(1)}, P_{(2)})$ is a Hamiltonian 2-partition of G. Thus, from now on, assume that G is not blue bipartite. We use induction on n.

Base cases. $n \leq 4$. If $n \leq 2$, then (i) holds, and if $n = 3$, then either (i) or (ii) holds. $n = 4$: If G contains at least two red edges, then (i) holds, and if G contains exactly one red edge, then (ii) holds. Otherwise (iii) holds.

Induction step. Assume that the statement holds for $n = p$, for some $p \geq 4$. We have to show that the statement holds for $n = p + 1$. In order to obtain a contradiction, suppose that this is not the case.

Take $x \in G$. By the induction hypothesis, (1) $G - x$ has a Hamiltonian 2-partition, (2) $(G - x)_{blue}$ is a tower, or (3) $p \leq 5$ and $G - x$ contains a blue K_4.

Case 1. Let $(P_{(1)}, P_{(2)}) = (x_1 x_2 \cdots x_{n_1}, y_1 y_2 \cdots y_{n_2})$ be a Hamiltonian 2-partition of $G - x$. If possible, choose $P_{(1)}$ and $P_{(2)}$ so that $n_1, n_2 \geq 2$. Were xx_1, xx_{n_1}, xy_1, xy_{n_2}, x_1y_1, $x_1y_{n_2}$, $x_{n_1}y_1$, or $x_{n_1}y_{n_2}$ red, G would have a Hamiltonian 2-partition, whence they are all blue. Since G contains no blue C_5, $x_1 = x_{n_1}$ or $y_1 = y_{n_2}$; say that $y_1 = y_{n_2} = y$.

$p = 4$: Were x_1x_3 blue, $G[\{x, y, x_1, x_3\}]$ would be a blue K_4, whence x_1x_3 is red. If xx_2 or yx_2 is red, then $(x_1x_3x_2x, y)$ or $(x_1x_3x_2y, x)$, respectively, is a Hamiltonian 2-partition of G, and if they are both blue, then $G_{blue} = xy|x_1x_2x_3$ is a blue tower, contrary to the hypothesis.

$p \geq 5$: Were y red adjacent to some x_i, $(x_1x_2 \cdots x_{i-1}, yx_ix_{i+1} \cdots x_{n_1})$ (if $i \neq 2$) or $(x_1x_2y, x_3x_4 \cdots x_{n_1})$ (if $i = 2$) would be a Hamiltonian 2-partition of $G - x$, whence all yx_i are blue. Were x red adjacent to all inner vertices of $P_{(1)}$, $(x_1x_2xx_3x_4 \cdots x_{n_1}, y)$ would be a Hamiltonian 2-partition of G, whence xx_ℓ is blue for some $x_\ell \in P^\circ_{(1)}$. Were $x_1x_{n_1}$ blue, $xx_\ell yx_1x_{n_1}x$ would be a blue C_5, whence $x_1x_{n_1}$ is red. Were xx_i red for some $x_i \in P^\circ_{(1)}$, $(xx_ix_{i+1} \cdots x_{n_1}x_1x_2 \cdots x_{i-1}, y)$ would be a Hamiltonian 2-partition of G, whence all xx_i, $i \in [n_1]$, are blue. Were x_ix_j blue for some i and j, then for some k, $xx_ix_jyx_kx$ would be a blue C_5, whence all x_ix_j are red. Thus $G_{blue} = xy|x_1x_2 \cdots x_{n_1}$ is a tower, contrary to the hypothesis.

Case 2. Let $(G - x)_{blue} = u_1u_2|v_1v_2 \cdots v_{p-2}$. If both xu_1 and xu_2 are blue, then $G_{blue} = u_1u_2|xv_1v_2 \cdots v_{p-2}$ is a tower, and if xu_i is red for some $i \in [2]$, then for some j, $(u_ixv_jv_{j+1} \cdots v_{p-2}v_1v_2 \cdots v_{j-1}, u_{3-i})$ is a Hamiltonian 2-partition of G, again contrary to the hypothesis.

Case 3. Note that $p = 5$. Let $K = G[\{u_1, u_2, u_3, u_4\}]$ be a blue K_4 in $G - x$, and let y be the vertex of $G - x - V(K)$. Then, w.l.o.g., either both x and y are RA(u_1) or x is RA(u_1) while y is RA(u_2). In either case, $(u_1, u_2xu_3yu_4)$ is a Hamiltonian 2-partition of G, once again contrary to the hypothesis. $\square$

2.5 The $\max(\gamma^1, \gamma^2) \leq 4$ subcase

Proof of Proposition 2.3.4. It is easy to see that

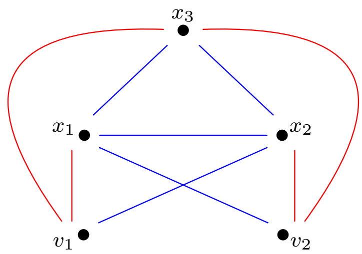
$$R_{red}(\Gamma_1, \Gamma_2) = R_{blue}(\Gamma_1, \Gamma_2) = 5.$$

Thus and by Corollary 2.2.3, $R(\Gamma_1, \Gamma_2) \geq 5$. By Corollary 1.3.7 and since $R(C_3, C_3) = R(C_4, C_4) = 6$, $R(\Gamma_1, \Gamma_2) \leq 6$.

If $C_3 \notin \Gamma_1 \cup \Gamma_2$, then Colouring 2 shows that $R(\Gamma_1, \Gamma_2) \geq 6$, and if $C_5 \notin \Gamma_1 \cup \Gamma_2$, then Colouring 3 shows that $R(\Gamma_1, \Gamma_2) \geq 6$. Thus, from now on, assume that both C_3 and $C_5 \in \Gamma_1 \cup \Gamma_2$.

$C_3 \in \Gamma_1 \cap \Gamma_2$: We have to show that $R(\Gamma_1, \Gamma_2) \leq 5$. Thus, let G be an arbitrary complete red-blue graph on 5 vertices. In order to obtain a contradiction, assume G is (Γ_1, Γ_2)-avoiding. Were some vertex of G incident with at least three edges of the same colour, G would contain a red C_3 or a blue C_3 (see the proof of Proposition 1.1.1), whence each vertex of G is incident with at most two edges of same colour. Since each vertex is incident with a total of four edges, each vertex has two red and two blue neighbours. Thus G_{red} and G_{blue} are 2-regular, whence G contains a red C_5 and a blue C_5, contrary to the hypothesis.[2]

$C_3 \notin \Gamma_1 \cap \Gamma_2$: Then $C_4 \in \Gamma_1 \cap \Gamma_2$ and, w.l.o.g., either both C_3 and $C_5 \in \Gamma_1$ or $C_3 \in \Gamma_1$ while $C_5 \in \Gamma_2$. In either case, we have to show that $R(\Gamma_1, \Gamma_2) \leq 5$. Thus, let G be an arbitrary complete red-blue graph on 5 vertices. In order to obtain a contradiction, assume G is (Γ_1, Γ_2)-avoiding. Were each vertex of G incident with at most two edges of the same colour, we would obtain, in the same way as above, a red C_5 and a blue C_5, whence some vertex of G is incident with at least three edges of the same colour. Thus and since G contains no red C_3, G contains a blue C_3, say $C = x_1x_2x_3x_1$; let v_1 and v_2 be the vertices of $G - V(C)$. Were v_1 or v_2 blue adjacent to at least two vertices of C, G would contain a blue C_4, whence both v_1 and v_2 are red adjacent to at least two vertices of C. Were v_1 and v_2 red adjacent to the same two vertices of C, G would contain a red C_4 whence, w.l.o.g., we have the following figure:



Now, if v_1v_2 is red, then $v_1v_2x_3v_1$ is a red C_3, and if v_1v_2 is blue, then $v_1v_2x_1x_2v_1$ is a blue C_4, contrary to the hypothesis. □

[2]We have to prove that a 2-regular graph on 5 vertices is a C_5: The vertex v has two neighbours, say v_1 and v_2. v_2 has one neighbour other than v, say v_3. Were $v_3 = v_1$, either the remaining two vertices would have at most one neighbour each or v, v_1, or v_2 would have more than two neighbours, whence $v_3 \neq v_1$. v_3 has one neighbour other than v_2, say v_4. Were $v_4 = v$, v would have more than two neighbours, and were $v_4 = v_1$, either the remaining vertex would have no neighbours or v, v_1, v_2, or v_3 would have more than two neighbours, whence $v_4 \notin \{v, v_1\}$. v_4, finally, has one neighbour other than v_3. This neighbour must be v_1, since otherwise v or v_2 would have more than two neighbours.

Proof of Proposition 2.3.5. W.l.o.g., assume that $i = 1$.

It is easy to see that $R_{red}(\Gamma_1, \Gamma_2) = 5$ and

$$R_{blue}(\Gamma_1, \Gamma_2) = \begin{cases} 6 & \text{if } C_6 \in \Gamma_2 \\ 7 & \text{otherwise.} \end{cases}$$

Thus and by Corollary 2.2.3,

$$R(\Gamma_1, \Gamma_2) \geq \begin{cases} 6 & \text{if } C_6 \in \Gamma_2 \\ 7 & \text{otherwise.} \end{cases}$$

We now turn to the upper bounds. By Corollary 1.3.7, $R(\Gamma_1, \Gamma_2) \leq R(C_4, C_3) = 7$. Thus, from now on, assume that $C_6 \in \Gamma_2$. We have to show that $R(\Gamma_1, \Gamma_2) \leq 6$. Thus, let G be an arbitrary complete red-blue graph on 6 vertices. In order to obtain a contradiction, assume G is (Γ_1, Γ_2)-avoiding. Since $R(C_3, C_3) = R(C_4, C_4) = 6$, G contains a red C_3 and a blue C_4, say $C = x_1x_2x_3x_4x_1$; let v_1 and v_2 be the vertices of $G - V(C)$. Then x_1x_3 and x_2x_4 are red and, w.l.o.g., either (1) $v_1v_2x_4v_1$ or (2) $v_2x_2x_4v_2$ is a red C_3.

Case 1. Since G contains no red C_4, x_2v_1 is blue, since G contains no blue C_3, x_3v_1 is red, and since G contains no red C_4, x_3v_2 is blue. Now, if x_2v_2 is red, then $x_2v_2v_1x_4x_2$ is a red C_4, and if x_2v_2 is blue, then $x_2v_2x_3x_2$ is a blue C_3, contrary to the hypothesis.

Case 2. Since G contains no red C_4, either v_1x_2 or v_1x_4 is blue. In either case, v_1x_1 and v_1x_3 are red (since G contains no blue C_3). Thus $C_{(1)} = x_1v_1x_3x_1$ and $C_{(2)} = x_2v_2x_4x_2$ are red 3-cycles. Since G contains no red C_4, at most one of the edges between $C_{(1)}$ and $C_{(2)}$ is red, whence G contains a blue C_6, again contrary to the hypothesis. □

2.6 The $C_3 \in \Gamma_1 \cup \Gamma_2$ subcase

Proof of Proposition 2.3.6. W.l.o.g., assume that $i = 1$ and $\gamma^1 = n$.

For the lower bound, see Proposition 2.2.2.

$R(\Gamma_1, \Gamma_2) \leq R_{blue}(\Gamma_1, \Gamma_2)$: Let G be an arbitrary complete red-blue graph on $R_{blue}(\Gamma_1, \Gamma_2)$ vertices, and assume G is (Γ_1, Γ_2)-avoiding. Were G blue bipartite, G would contain a red subgraph belonging to Γ_1 or a blue subgraph belonging to Γ_2, whence G contains an odd blue cycle. Let $C_0 = u_1u_2 \cdots u_{2k+1}u_1$ be a shortest odd blue cycle in G; note that $k \geq 2$. Were some chord of C_0 blue, G would contain an odd blue cycle shorter than C_0, whence all chords of C_0 are red.

We shall show that G contains a red C_n, contradicting the (Γ_1, Γ_2)-avoidance of G. We do so by proving the following, stronger result:

Statement 1. *For each subset $V \subseteq V(G)$, $G[V]$ either contains a red $C_{|V|}$ or is blue bipartite.*

How does Statement 1 imply that G contains a red C_n? Consider the case $|V| = n$; note that $n \leq |V(G)|$. If $2k+1 \leq n$, choose V so that $G[V]$ contains C_0. Then $G[V]$ is not blue bipartite, whence (by Statement 1) $G[V]$ contains a red $C_{|V|} = C_n$. On the other hand, if $2k+1 > n$, then by Lemma 2.4.3, $G[V(C_0)]$, and thus G, contains a red C_n. We now turn to the proof of Statement 1, which is by induction on $|V|$:

Base cases. $|V| \leq 2k$. Since $G[V] \subseteq G$ and $2k+1$ is the length of a shortest odd blue cycle in G, $G[V]$ contains no odd blue cycle, whence $G[V]$ is blue bipartite.

Induction step. Assume that the statement holds for each subset $V \subseteq V(G)$ with $|V| \leq p$, for some $p \in [2k, |V(G)|-1]$. We have to show that for each subset $V \subseteq V(G)$ with $|V| = p+1$, $G[V]$ either contains a red $C_{|V|}$ or is blue bipartite. Thus, let $V \subseteq V(G)$ with $|V| = p+1$. If $G[V]$ is not blue bipartite, then $G[V]$ contains an odd blue cycle. Let $C = x_1x_2\cdots x_{2m+1}x_1$ be a shortest odd blue cycle in $G[V]$; note that $m \geq k$. As for C_0, all chords of C are red. In particular, the m-chords of C form a red C_{2m+1}, say C'. If $V(C) = V$, then C' is a red C_{p+1}. If not, then we may construct a red C_{p+1} by replacing one or two edges of C' (that is, one or two m-chords of C) with one or two red paths, respectively:

Consider the non-empty subgraph $H = G[V] - V(C)$ of $G[V]$. By the induction hypothesis, H either contains a red C_{p-2m} or is blue bipartite. In the former case, let $P = v_1v_2\cdots v_{p-2m}$ be a red P_{p-2m} in H. In the latter case, $V - V(C)$ is the disjoint union of two subsets V_1 and V_2, which we may assume to be non-empty, such that $G[V_1]$ and $G[V_2]$ are red cliques. Then, let $r = |V_1|$ and let $P_{(1)} = v_1v_2\cdots v_r$ and $P_{(2)} = v_{r+1}v_{r+2}\cdots v_{p-2m}$ be a red $P_{|V_1|}$ and a red $P_{|V_2|}$, respectively, in H.

Consider first the case in which H contains a red P_{p-2m}. Since G, and thus $G[V]$, contains no blue C_3, v_1 is red adjacent to some x_i, say x_1, and v_{p-2m} is red adjacent either to x_{m+1} or to x_{m+2}, say x_{m+2}. We thus obtain a red C_{p+1} by replacing the edge x_1x_{m+2} with the red path P.

Consider now the case in which H contains a red $P_{|V_1|}$ and a red $P_{|V_2|}$. Since $G[V]$ contains no blue C_3, v_1 is red adjacent to some x_i, say x_1, v_r is red adjacent either to x_{m+1} or to x_{m+2}, say x_{m+2}, and v_{r+1} is red adjacent either to x_2 or to x_3.

$v_{r+1}x_2$ red: $v_{p-2m}x_{m+2}$ or $v_{p-2m}x_{m+3}$ is red, whence we obtain a red C_{p+1} by replacing the edges x_1x_{m+2} and $x_{m+2}x_2$ or the edges x_1x_{m+2} and x_2x_{m+3}, respectively, with the red paths $P_{(1)}$ and $P'_{(2)}$ or with the red paths $P_{(1)}$ and $P_{(2)}$, respectively.

$v_{r+1}x_3$ red: $v_{p-2m}x_{m+3}$ or $v_{p-2m}x_{m+4}$ is red, whence we obtain a red C_{p+1} by replacing the edges x_1x_{m+2} and $x_{m+3}x_3$ or the edges x_1x_{m+2} and x_3x_{m+4}, respectively, with the red paths $P_{(1)}$ and $P'_{(2)}$ or with the red paths

$P_{(1)}$ and $P_{(2)}$, respectively. □

2.7 The $C_4 \in \Gamma_1 \cup \Gamma_2$ subcase

Proof of Proposition 2.3.7. W.l.o.g., assume that $i = 1$.

It is easy to see that $R_{blue}(\Gamma_1, \Gamma_2) = 6$ and

$$R_{red}(\Gamma_1, \Gamma_2) = \begin{cases} 6 & \text{if } C_6 \in \Gamma_1 \\ 7 & \text{otherwise.} \end{cases}$$

Thus and by Corollary 2.2.3,

$$R(\Gamma_1, \Gamma_2) \geq \begin{cases} 6 & \text{if } C_6 \in \Gamma_1 \\ 7 & \text{otherwise.} \end{cases}$$

We now turn to the upper bounds. By Corollary 1.3.7, $R(\Gamma_1, \Gamma_2) \leq R(C_5, C_4) = 7$. Thus, from now on, assume that $C_6 \in \Gamma_1$. We have to show that $R(\Gamma_1, \Gamma_2) \leq 6$. Thus, let G be an arbitrary complete red-blue graph on 6 vertices. In order to obtain a contradiction, assume G is (Γ_1, Γ_2)-avoiding. By Lemma 2.4.1 (with $(n, k) = (5, 4)$), G contains a monochromatic C_5, a monochromatic C_6, or a blue C_4. Since G is (Γ_1, Γ_2)-avoiding, we have a blue C_5 or a blue C_6. In the former case, Lemma 2.4.2, with $(n, k) = (5, 4)$, yields a red C_5 or a blue C_4, and in the latter case, Lemma 2.4.2, now with $(n, k) = (6, 4)$, yields a red C_6 or a blue C_4, contrary to the hypothesis. □

Proof of Proposition 2.3.8. W.l.o.g., assume that $i = 1$.

Note that $(\Gamma_1, \Gamma_2) \in \mathfrak{A}^1_{blue} \cup \mathfrak{A}^2_{blue} \subseteq \mathfrak{B}_2$. Thus the result follows from Proposition 2.3.3 and (11). □

2.8 The $C_5 \in \Gamma_1 \cup \Gamma_2$ subcase

Proof of Proposition 2.3.9. The two statements (9) and (10) follow directly from Proposition 2.2.5. Thus and by Corollary 2.2.3,

$$R(\Gamma_1, \Gamma_2) \geq \begin{cases} 7 & \text{if } \max(\gamma_e^1, \gamma_e^2) = 6 \\ 8 & \text{if } \max(\gamma_e^1, \gamma_e^2) = 8 \\ 9 & \text{if } \max(\gamma_e^1, \gamma_e^2) \geq 10. \end{cases}$$

We now turn to the upper bounds. By Corollary 1.3.7, $R(\Gamma_1, \Gamma_2) \leq R(C_5, C_5) = 9$.

$R(\Gamma_1, \Gamma_2) \leq 7$ if $\max(\gamma_e^1, \gamma_e^2) = 6$: Let G be an arbitrary complete red-blue graph on 7 vertices, and assume G is (Γ_1, Γ_2)-avoiding. Since $R(C_4, C_4) = 6$, G contains a monochromatic C_4; say that $C = x_1x_2x_3x_4x_1$

is a blue C_4 in G, and let v_1, v_2, and v_3 be the vertices of $G - V(C)$. Since G contains no blue C_5, each v_i is red adjacent to two opposite vertices of C. W.l.o.g., assume that v_1 and v_2 are red adjacent to x_1 and x_3.

Since G contains no red C_5, v_3 is blue adjacent to two opposite vertices of the red C_4 $v_1x_1v_2x_3v_1$. Thus, (1) v_3x_1 and v_3x_3 are red, in which case v_3v_1 and v_3v_2 are blue, (2) v_3x_2 and v_3x_4 are red, and v_3x_1 and v_3x_3 are blue, or (3) v_3x_2 and v_3x_4 are red, and v_3v_1 and v_3v_2 are blue. Note that Cases 1 and 2 are symmetric. Thus we only have to consider Cases 1 and 3. In either case, note that if v_1x_2 and v_2x_4 were blue, then $v_1x_2x_1x_4v_2v_3v_1$ would be a blue C_6, whence at least one of them is red, say v_1x_2.

Case 1. Were x_2v_2 red, $x_2v_2x_1v_3x_3v_1x_2$ would be a red C_6, whence x_2v_2 is blue. Were x_4v_3 blue, $x_4v_3v_2x_2x_1x_4$ would be a blue C_5, whence x_4v_3 is red. Now, if x_4v_1 is red, then $x_4v_1x_1v_2x_3v_3x_4$ is a red C_6, and if x_4v_1 is blue, then $x_4v_1v_3v_2x_2x_1x_4$ is a blue C_6, contrary to the hypothesis.

Case 3. Were x_4v_2 red, $x_4v_2x_1v_1x_2v_3x_4$ would be a red C_6, whence x_4v_2 is blue. Now, if x_1v_3 is red, then $x_1v_3x_2v_1x_3v_2x_1$ is a red C_6, and if x_1v_3 is blue, then $x_1v_3v_2x_4x_3x_2x_1$ is a blue C_6, again contrary to the hypothesis.

$R(\Gamma_1, \Gamma_2) \leq 8$ if $\max(\gamma_e^1, \gamma_e^2) \leq 8$: Let G be an arbitrary complete red-blue graph on 8 vertices, and assume G is (Γ_1, Γ_2)-avoiding. Since $R(C_4, C_4) = 6$, G contains a monochromatic C_4; say that $C = x_1x_2x_3x_4x_1$ is a blue C_4 in G, and let v_1, v_2, v_3, and v_4 be the vertices of $G - V(C)$. Also, let $V = \{v_1, v_2, v_3, v_4\}$. As before, each v_i is red adjacent to two opposite vertices of C. Thus, w.l.o.g., (1) v_1, v_2, v_3, and v_4 are red adjacent to x_1 and x_3, (2) only v_1, v_2, and v_3 are red adjacent to x_1 and x_3, while v_4 is red adjacent to x_2 and x_4, or (3) only v_1 and v_2 are red adjacent to x_1 and x_3, while only v_3 and v_4 are red adjacent to x_2 and x_4.

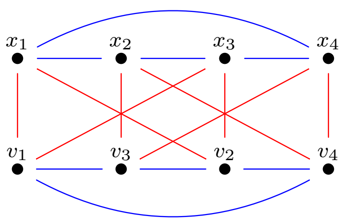
Case 1. Consider the red C_4 $v_ix_1v_jx_3v_i$. Since G contains no red C_5, v_kv_i and v_kv_j are blue. This is true for all i, j, and k, whence each v_iv_j is blue. Since G contains no blue C_5, x_2, as well as x_4, has at most one blue edge to V. Also, if both x_2 and x_4 have a blue edge to V, then it must be to the same vertex v_i; w.l.o.g., assume that $i = 1$. Then $x_1v_1x_3v_2x_2v_3x_1$ and $x_1v_1x_3v_2x_2v_3x_4v_4x_1$ are a red C_6 and a red C_8, respectively, contrary to the hypothesis.

Case 2. Note that v_4x_1 or v_4x_3 is blue, say v_4x_3. Consider the red C_4 $v_ix_1v_jx_3v_i$. Since G contains no red C_5, v_kv_i and v_kv_j are blue. This is true for all i, j, and k in [3], whence v_1v_2, v_1v_3, and v_2v_3 are blue. We now consider two subcases: (a) v_4x_1 red and (b) v_4x_1 blue.

Subcase 2a. Since G contains no red C_5, v_1v_4, v_2v_4, and v_3v_4 are blue. Then, since G contains no blue C_5, neither x_2 nor x_4 has a blue edge to V. Thus $x_1v_4x_2v_1x_3v_2x_1$ and $x_1v_4x_2v_1x_3v_2x_4v_3x_1$ are a red C_6 and a red C_8, respectively, contrary to the hypothesis.

Subcase 2b. Were x_2x_4 blue, $x_2x_4x_3v_4x_1x_2$ would be a blue C_5, whence x_2x_4 is red. Were v_1x_2 and v_2x_4 red, $v_1x_2x_4v_2x_1v_1$ would be a red C_5, whence at least one of them is blue, say v_1x_2. Were v_2v_4 blue, $v_2v_4x_1x_2v_1v_2$ would be a blue C_5, whence v_2v_4 is red. Now, note that if x_4v_3 is red, then $x_4v_3x_1v_2v_4x_4$ is a red C_5, and if x_4v_3 is blue, then $x_4v_3v_1x_2x_1x_4$ is a blue C_5, again contrary to the hypothesis.

Case 3. Consider the red C_4 $v_1x_1v_2x_3v_1 = C'$. Since G contains no red C_5, v_3 and v_4 are blue adjacent to two opposite vertices of C'. Were v_3 or v_4 blue adjacent to x_1 and x_3, we would be in a case symmetric to Case 1 or to Case 2 (with C' instead of C), whence both v_3 and v_4 are blue adjacent to v_1 and v_2:



Now, v_1x_2 or v_1x_4 is blue, say v_1x_2, and v_3x_1 or v_3x_3 is blue, say v_3x_1. Thus $v_1x_2x_3x_4x_1v_3v_1$ and $v_1x_2x_3x_4x_1v_3v_2v_4v_1$ are a blue C_6 and a blue C_8, respectively, once again contrary to the hypothesis. □

Proof of Proposition 2.3.10. W.l.o.g., assume that $i = 1$ and $\gamma^1 = n$.

For the lower bound, see Proposition 2.2.2.

$R(\Gamma_1, \Gamma_2) \leq R_{blue}(\Gamma_1, \Gamma_2)$: Let G be an arbitrary complete red-blue graph on $R_{blue}(\Gamma_1, \Gamma_2)$ vertices, and assume G is (Γ_1, Γ_2)-avoiding. Were G blue bipartite, G would contain a red subgraph belonging to Γ_1 or a blue subgraph belonging to Γ_2, whence G contains an odd blue cycle. We shall consider two cases: (1) G is blue almost bipartite and (2) G is not blue almost bipartite.

Case 1. We shall consider two subcases: (a) G contains a blue K_4 and (b) G does not contain a blue K_4. Note that $|G| \geq n + 2$.

Subcase 1a. Let $H' \subseteq G$ with $|H'| = n + 1$, such that H' contains a blue K_4, say $K = G[\{u_1, u_2, u_3, u_4\}]$. If $n = 7$, let $H = H' + v$ for some $v \in G - V(H')$, and if $n \neq 7$, let $H = H'$. We shall show that H, and thus G, contains a red C_n.

$n = 6$: Let v_1, v_2, and v_3 be the vertices of $H - V(K)$. Since G, and thus H, contains no blue C_5, each v_j is blue adjacent to at most one u_i. Thus, w.l.o.g., v_1, v_2, and v_3 are $\mathrm{RA}(u_1)$, v_1 and v_2 are $\mathrm{RA}(u_1)$ while v_3 is $\mathrm{RA}(u_3)$, or each v_j is $\mathrm{RA}(u_j)$. In each case, $v_1u_2v_3u_4v_2u_3v_1$ is a red C_6.

$n = 7$: In order to obtain a contradiction, suppose that H does not contain a red C_7. Take $x, y \in H - V(K)$. By the $n = 6$ case, $H - \{x, y\}$

contains a red C_6, say $C = x_1x_2\cdots x_6x_1$; let z be the vertex of $H - \{x, y\} - V(C)$. Then, w.l.o.g., $K = H[\{z, x_2, x_4, x_6\}]$, and xx_2 and xx_4 are red. Thus xx_1, xx_3, xx_5, x_1x_3, and x_3x_5 are blue. Since H contains no blue C_5, y is blue adjacent to at most one x_i with $i \equiv 1$ and at most one x_i with $i \equiv 0$. Thus y is red adjacent to two consecutive vertices of C, which yields a red C_7, contrary to the hypothesis.

$n \geq 8$: We use induction on n.

Base case. $n = 8$. Since $R(C_3, C_5) = 9$ and H contains no blue C_5, H contains a red C_3, say C. Take $x \in H - V(K) - V(C)$, and let v_1, v_2, v_3, and v_4 be the vertices of $H - x - V(K)$. Regardless whether K and C have one vertex or zero vertices in common, $H - x - V(K)$ contains a red edge, say v_2v_3. Consider K and v_1, v_2, and v_3. By the $n = 6$ case and w.l.o.g., we may assume that $v_1u_2v_3u_4v_2u_3v_1$ is a red C_6 in $H - x$. Now, consider the four cases v_4u_3 and v_4u_4 red, v_4u_3 blue, v_4u_4 blue and v_4v_2 red, and v_4u_4 and v_4v_2 blue. In the last case, v_2u_1 is red (otherwise $v_2v_4u_4u_2u_1v_2$ would be a blue C_5), whence $v_1u_2v_3v_2u_1v_4u_3v_1$ is a red C_7. In the other cases, it is even easier to find a red C_7 in $H - x$. Thus and by Case 1 of Lemma 2.4.6, H contains a red C_8.

Induction step. Assume that the statement holds for $n = p$, for some $p \geq 8$. We have to show that if $n = p + 1$, then H contains a red C_{p+1}.

Take $x \in H - V(K)$. By the induction hypothesis, $H - x$ contains a red C_p. Thus and by Case 1 of Lemma 2.4.6, H contains a red C_{p+1}.

Subcase 1b. Let $H' \subseteq G$ with $|H'| = n + 1$, such that H' contains a blue C_3, and thus a maximal blue tower T'. If $\mathrm{ht}(T') \geq 3$, let $H = H' + v$ for some $v \in G - V(H')$, and if $\mathrm{ht}(T') \leq 2$, let $H = H'$. In either case, let $T = a_1a_2|b_1b_2\cdots b_{\mathrm{ht}(T)}$ be a maximal blue tower in H. Using induction on n, we shall show that H, and thus G, contains a red C_n.

Base case. $n = 6$.

$\mathrm{ht}(T) = 1$: Let v_1, v_2, v_3, and v_4 be the vertices of $H - V(T)$. Then, w.l.o.g., (α) v_1, v_2, v_3, and v_4 are RA(b_1), (β) v_1, v_2, and v_3 are RA(b_1) while v_4 is RA(a_1), (γ) v_1 and v_2 are RA(b_1) while v_3 and v_4 are RA(a_1), or (δ) v_1 and v_2 are RA(b_1), v_3 is RA(a_1), and v_4 is RA(a_2).

(α): Since H contains neither a blue K_4 nor a blue tower of height 2, $H - V(T)$ contains at least two red edges. Thus, w.l.o.g., either v_1v_2 and v_2v_3 or v_1v_2 and v_3v_4 are red. In the former case, $v_1v_2v_3a_1v_4a_2v_1$ is a red C_6, and in the latter case, $v_1v_2a_1v_3v_4a_2v_1$ is a red C_6.

(β): If v_4a_1 is red, then H may be treated in (α). Thus, suppose that v_4a_1 is blue. Similarly, if v_1b_1 or v_2b_1 is red, then H may be treated in (δ). Thus, suppose that they are both blue. Then, since H contains no blue C_5, v_1v_4 and v_2v_4 are red. Thus $v_1v_4v_2a_1v_3a_2v_1$ is a red C_6.

(γ): If v_1b_1, v_2b_1, v_3a_1, or v_4a_1 is red, then H may be treated in (δ).

Thus, suppose that they are all blue. Then, since H contains no blue C_5, v_1v_4 and v_2v_3 are red. Thus $v_1v_4a_2v_3v_2a_1v_1$ is a red C_6.

(δ): Then $v_1a_1v_4b_1v_3a_2v_1$ is a red C_6.

$\mathrm{ht}(T) = 2$: Let v_1, v_2, and v_3 be the vertices of $H - V(T)$. Then, since H contains no blue C_5, no blue K_4, and no blue tower of height 3, each v_j is blue adjacent to at most one vertex of T. Thus we obtain a red C_6 in the same way as in Subcase 1a (think of the vertices of T as vertices of a blue K_4).

$\mathrm{ht}(T) = 3$: The proof of the $\mathrm{ht}(T) = 2$ case applies to this case as well, mutatis mutandis.

$\mathrm{ht}(T) = 4$: Let v_1 and v_2 be the vertices of $H - V(T)$. Then, w.l.o.g., v_1 and v_2 are $\mathrm{RA}(a_1)$, v_1 and v_2 are $\mathrm{RA}(b_1)$, v_1 is $\mathrm{RA}(a_1)$ while v_2 is $\mathrm{RA}(a_2)$, v_1 is $\mathrm{RA}(b_1)$ while v_2 is $\mathrm{RA}(b_2)$, or v_1 is $\mathrm{RA}(a_1)$ while v_2 is $\mathrm{RA}(b_1)$. In each case, $v_1b_2b_1b_3v_2b_4v_1$ is a red C_6.

$\mathrm{ht}(T) = 5$: Let v be the vertex of $H - V(T)$. Then, w.l.o.g., v is either $\mathrm{RA}(a_1)$ or $\mathrm{RA}(b_1)$. In either case, $vb_2b_1b_3b_4b_5v$ is a red C_6.

$\mathrm{ht}(T) = 6$: Then $b_1b_2 \cdots b_6b_1$ is a red C_6.

Induction step. Assume that the statement holds for $n = p$, for some $p \geq 6$. We have to show that if $n = p + 1$, then H contains a red C_{p+1}.

If $H_{blue} = T$, then $b_1b_2 \cdots b_{p+1}b_1$ is a red C_{p+1}. Thus, from now on, assume that $H - V(T)$ is non-empty. Take $x \in H - V(T)$. By the induction hypothesis, $H - x$ contains a red C_p. Thus, if $\mathrm{ht}(T) \leq 2$, then by Case 2 of Lemma 2.4.6, H contains a red C_{p+1}, and if $\mathrm{ht}(T) \geq 3$, then by Case 3 of Lemma 2.4.6, H contains a red C_{p+1}.

Case 2. G is not blue almost bipartite. Let $C_0 = u_1u_2 \cdots u_{2k+1}u_1$ be a shortest odd blue cycle in G longer than C_3; note that $k \geq 3$. Were some j-chord of C_0 blue, with $j \notin \{2, 2k-1\}$, G would contain an odd blue cycle shorter than C_0 but longer than C_3, whence all such j-chords of C_0 are red. Furthermore, were two non-crossing 2-chords (or, equivalently, $(2k-1)$-chords) of C_0 blue, G again would contain an odd blue cycle shorter than C_0 but longer than C_3. Thus all chords of C_0 are red, except possibly one 2-chord or two crossing 2-chords.

We shall show that G contains a red C_n, contradicting the (Γ_1, Γ_2)-avoidance of G. We do so by proving the following, stronger result:

Statement 1. *For each subset $V \subseteq V(G)$, $G[V]$ either contains a red $C_{|V|}$ or is blue almost bipartite.*

How does Statement 1 imply that G contains a red C_n? Consider the case $|V| = n$; note that $n \leq |V(G)|$. If $2k+1 \leq n$, choose V so that $G[V]$ contains C_0. Then $G[V]$ is not blue almost bipartite, whence (by Statement 1) $G[V]$ contains a red $C_{|V|} = C_n$. On the other hand, if $2k + 1 > n$, then by

Lemma 2.4.5, $G[V(C_0)]$, and thus G, contains a red C_n. We now turn to the proof of Statement 1, which is by induction on $|V|$:

Base cases. $|V| \leq 2k$. Since $G[V] \subseteq G$ and $2k+1$ is the length of a shortest odd blue cycle in G longer than C_3, $G[V]$ is blue almost bipartite.

Induction step. Assume that the statement holds for each subset $V \subseteq V(G)$ with $|V| \leq p$, for some $p \in [2k, |V(G)|-1]$. We have to show that for each subset $V \subseteq V(G)$ with $|V| = p+1$, $G[V]$ either contains a red $C_{|V|}$ or is blue almost bipartite. Thus, let $V \subseteq V(G)$ with $|V| = p+1$. If $G[V]$ is not blue almost bipartite, then $G[V]$ contains an odd blue cycle longer than C_3. Let $C = x_1x_2 \cdots x_{2m+1}x_1$ be a shortest odd blue cycle in $G[V]$ longer than C_3; note that $m \geq k$. As for C_0, all chords of C are red, except possibly one 2-chord or two crossing 2-chords. In particular, the m-chords of C form a red C_{2m+1}, say C'. If $V(C) = V$, then C' is a red C_{p+1}. If not, then we may construct a red C_{p+1} by replacing one, two, three, or four edges of C' (that is, one, two, three, or four m-chords of C) with one, two, three, or four red paths, respectively:

Consider the non-empty subgraph $H = G[V] - V(C)$ of $G[V]$. By the induction hypothesis, H either contains a red C_{p-2m} or is blue almost bipartite. In the former case, H has a Hamiltonian 1-partition, and in the latter case, it follows from Lemma 2.4.7 that H either has a Hamiltonian 3-partition or is a blue K_4.

Claim 1. *If some vertex $v \in H$ is blue adjacent to x_i and to x_{i+j}, then $j \bmod (2m+1) \in \{1, 2, 2m-1, 2m\}$. In particular, x_ix_{i+j} cannot be an edge of C'.*

Proof. Were $3 \leq j \bmod (2m+1) \leq 2m-2$, $vx_ix_{i+1} \cdots x_{i+j}v$ (in case $j \bmod (2m+1) \equiv 1$) or $vx_ix_{i-1} \cdots x_{i+j}v$ (in case $j \bmod (2m+1) \equiv 0$) would be an odd blue cycle of length between 5 and $2m-1$. □

Thus, if some vertex $v \in H$ has more than two blue neighbours in C, then they have to be three and of the form x_{k-1}, x_k, and x_{k+1} for some $k \in [2m+1]$. Hence, if $H = \{v\}$, then there is an edge $x_\ell x_{\ell+m}$ of C' such that vx_ℓ and $vx_{\ell+m}$ are both red. We thus obtain a red C_{p+1} by replacing $x_\ell x_{\ell+m}$ with $x_\ell v x_{\ell+m}$. Thus, from now on, assume that $|H| \geq 2$.

Claim 2. *Let $v_1 \neq v_2$ belong to H. Then they have at most two common blue neighbours in C.*

Proof. As we just saw, if v_1 has more than two blue neighbours in C, then they have to be three and of the form x_{k-1}, x_k, and x_{k+1} for some $k \in [2m+1]$. Thus, v_1 and v_2 cannot have more than two common blue neighbours in C, since then $v_1x_{k-1}v_2x_kx_{k+1}v_1$ would be a blue C_5. □

Let $v_1 \neq v_2$ belong to H. If they have two common blue neighbours in C, then at most four of the $2m+1$ edges of C' have a vertex which is blue adjacent to both v_1 and v_2. Thus and by Claim 1, at least $2m-3 \geq 3$ of the $2m+1$ edges of C' are *good*, that is one of the edge's vertices is red adjacent to v_1, and the other one is red adjacent to v_2. Similarly, if v_1 and v_2 have at most one common blue neighbour in C, then at least $2m-1 \geq 5$ of the $2m+1$ edges of C' are good.

Claim 3.

(a) Let $K \subseteq H$ be a blue K_3 and let its vertices be v_1, v_2, and v_3. If v_1x_i is blue, then all edges between $K - v_1$ and $C - \{x_i, x_{i+3}, x_{i-3}\}$ are red.

(b) Let $K \subseteq H$ be a blue K_4 and let its vertices be v_1, v_2, v_3, and v_4. If v_1x_i is blue, then all edges between $K - v_1$ and C are red, except possibly $v_\ell x_{i+3}$ or $v_\ell x_{i-3}$ for at most one $v_\ell \in K - v_1$; if, moreover, $m \geq 4$, then all edges between $K - v_1$ and C are red.

Proof. W.l.o.g., assume that v_2x_{i+j} is blue. Then $v_1x_ix_{i+j}v_2v_3v_1$ is a blue C_5 if $j \bmod (2m+1) \in \{1, 2m\}$, $v_1x_ix_{i+1}\cdots x_{i+j}v_2v_1$ is an odd blue cycle of length between 5 and $2m-1$ if $0 \equiv j \bmod (2m+1) \in [2, 2m-4]$, and $v_1x_ix_{i-1}\cdots x_{i+j}v_2v_1$ is an odd blue cycle of length between 5 and $2m-1$ if $1 \equiv j \bmod (2m+1) \in [5, 2m-1]$. This proves part (a). In part (b), also note that $v_1x_iv_2v_3v_4v_1$ is a blue C_5 if $j \bmod (2m+1) = 0$. Thus, if $v_\ell x_{i+3}$ or $v_\ell x_{i-3}$ is blue for some $v_\ell \in K - v_1$, then all edges between $K - \{v_1, v_\ell\}$ and C are red. Finally, if $j \bmod (2m+1) \in \{3, 2m-2\}$, then $v_1x_ix_{i\pm1}x_{i\pm2}x_{i\pm3}v_2v_3v_1$ is a blue C_7, which contradicts the hypothesis if $m \geq 4$. $\square$

Now, we know that there is an $r \in [4]$ such that H has a Hamiltonian r-partition but not a Hamiltonian $(r-1)$-partition, and if $r = 4$, then H is a blue K_4. We consider two subcases: (a) $r \leq 3$ and (b) $r = 4$.

Subcase 2a. Let $(P_{(1)}, \ldots, P_{(r)}) = (v_1^1 \cdots v_{n_1}^1, \ldots, v_1^r \cdots v_{n_r}^r)$ be a Hamiltonian r-partition of H. For each $i \in [r]$, choose $k_i \in [2m+1]$ with $k_{i_1} \neq k_{i_2}$ for all $i_1 \neq i_2$, such that all $v_1^i x_j$ are red, except possibly $v_1^i x_{k_i-1}$, $v_1^i x_{k_i}$, and $v_1^i x_{k_i+1}$. By part (a) of Claim 3 and since each $v_1^{i_1} v_1^{i_2}$ is blue (otherwise H would have a Hamiltonian $(r-1)$-partition), this is always possible if $r = 3$. Since H contains no blue C_5 and $v_1^1 v_1^2$ is blue, this is also always possible if $r = 2$.

Now, for each $i \in [r]$ such that $v_1^i = v_{n_i}^i$, replace $x_{k_i+2}x_{k_i+2+m}$ with $x_{k_i+2}v_1^i x_{k_i+2+m}$, and for each $i \in [r]$ such that $v_1^i \neq v_{n_i}^i$, note that at least one of the at least three good edges of C', say $x_{\ell_i}x_{\ell_i+m}$, has not (yet) been replaced, and replace it with $x_{\ell_i}v_1^i \cdots v_{n_i}^i x_{\ell_i+m}$.

Subcase 2b. Let the vertices of H be v_1, v_2, v_3, and v_4. If all edges between H and C are red, then we obtain a red C_{p+1} by replacing x_ix_{i+m}

with the red path $x_i v_i x_{i+m}$, for each $i \in [4]$. Thus, from now on, assume that some vertex of H, say v_1, is blue adjacent to some vertex of C. As we have seen, there is a $k \in [2m+1]$ such that all $v_1 x_i$ are red, except possibly $v_1 x_{k-1}$, $v_1 x_k$, and $v_1 x_{k+1}$. In case there is only one x_i such that $v_1 x_i$ is blue, let $k = i$. Then by part (b) of Claim 3, there is a vertex $v_\ell \in H - v_1$, say v_2, such that all edges between $H - v_1$ and C are red, except possibly $v_2 x_{k+3}$ or $v_2 x_{k-3}$ if $m = 3$.

Now, replace $x_{k+2} x_{k+2+m}$, $x_{k+2+m} x_{k+1}$, $x_{k+3+m} x_{k+2}$, and $x_{k+4+m} x_{k+3}$ with $x_{k+2} v_1 x_{k+2+m}$, $x_{k+2+m} v_2 x_{k+1}$, $x_{k+3+m} v_3 x_{k+2}$, and $x_{k+4+m} v_4 x_{k+3}$, respectively. This concludes the proof of Proposition 2.3.10. $\square$

3 References

[1] Backelin, J.; *Contributions to a Ramsey Calculus*; unpublished.

[2] Bondy, J.; *Pancyclic Graphs I*; Journal of Combinatorial Theory (Series B); **11** (1971), 80-84.

[3] Chartrand, G.; Schuster, S.; *On a Variation of the Ramsey Number*; Transactions of the American Mathematical Society; **173** (1972), 353-362.

[4] Diestel, R.; *Graph Theory*; Springer; 3rd edition (2006).

[5] Erdős, P.; Faudree, R.; Rousseau, C.; Schelp, R.; *On Cycle-Complete Graph Ramsey Numbers*; Journal of Graph Theory; **2** (1978), 53-64.

[6] Faudree, R.; Schelp, R.; *All Ramsey Numbers for Cycles in Graphs*; Discrete Mathematics; **8** (1974), 313-329.

[7] Faudree, R.; Schelp, R.; *Some Problems in Ramsey Theory*; Lecture Notes in Mathematics; **642** (1978), 500-515.

[8] Graham, R.; Rothschild, B.; Spencer, J.; *Ramsey Theory*; Wiley-Interscience; 2nd edition (1990).

[9] Hansson, M.; *Ramseytalen för cykel mot cykel*; Självständiga arbeten i matematik; **2** (2012); Matematiska institutionen; Stockholms universitet.

[10] Harary, F.; *A Survey of Generalized Ramsey Theory*; Lecture Notes in Mathematics; **406** (1974), 10-17.

[11] Károlyi, G.; Rosta, V.; *Generalized and Geometric Ramsey Numbers for Cycles*; Theoretical Computer Science; **263** (2001), 87-98.

[12] Radziszowski, S.; *Small Ramsey Numbers*; The Electronic Journal of Combinatorics; Dynamic Survey 1; Revision **13** (2011).
http://www.combinatorics.org/issue/view/Surveys

[13] Rosta, V.; *On a Ramsey-Type Problem of J. A. Bondy and P. Erdős I & II*; Journal of Combinatorial Theory (Series B); **15** (1973), 94-120.

[14] Stanley, R.; *Enumerative Combinatorics, Volume 1*; Cambridge University Press; 8th printing (2007).