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The étale homotopy type

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Abstract

Artin-Mazur associated to every locally noetherian scheme X a certain invariant, the étale homotopy type. This invariant captures a lot of information, for one thing, it can be used to compute the sheaf cohomology of X for any locally constant sheaf. Recently, Harpaz-Schlank constructed a relative étale homotopy type to unify some classical obstruction theories in diophantine geometry. Later, Barnea-Schlank put this in a model categorical framework and showed that we can construct many new invariants closely related to the étale homotopy type of a scheme. In this thesis, we study the classical Étale Homotopy type of Artin-Mazur and compute it for some simple cases. This thesis should be seen as a preparation for a future master's thesis on Harpaz-Schlank's construction.

" What if the man could see Beauty Itself, pure, unalloyed, stripped of mortality, and all its pollution, stains, and vanities, unchanging, divine...the man becoming in that communion, the friend of God, himself immortal...would that be a life to disregard? "

-Plato

Acknowledgements

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Contents

1 Introduction

This thesis started in the summer of 2012 when Andreas Holmstrom told me about some very interesting new work that had been done on the étale homotopy type and suggested that it might make a good thesis subject. I became quickly intrigued and after the initial hurdles of abstraction had been overcome, I was introduced to a fantastic part of mathematics, blending abstraction with concrete arithmetical applications. This was originally intended to be a paper that covered both a relative version of the étale homotopy type and Artin-Mazur's classical construction, however, I later decided to split it in two, so that the latter becomes part of my master's thesis. This paper does not claim any new results, but simply tries to give the author's perspective on some classical constructions in Algebraic Geometry, much to help his own understanding of the concepts. This thesis can be seen as an introduction to étale homotopy, where I have included the most relevant material for understanding the relative étale homotopy type.

The first part covers some categorical constructions as well as some background on simplicial sets. All of the material here is standard, but the reader might want to spend some time on the part on simplicial sets, since these are crucial for our understanding of the étale homotopy type. The second chapter covers the constructions leading up to the étale homotopy type. The part on hypercoverings should be read carefully and the same can be said on the last chapter, concering the étale homotopy type.

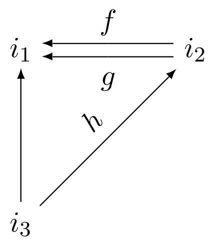
2 Background

2.1 Pro-Objects

We start out by reviewing some well-known results regarding pro-objects, which will be neccessary to understand the later parts of the thesis.

Definition 1. A category I is cofiltered when:

1. I is non-empty
2. For any two objects $i_1, i_2 \in I$,we can find an object $i_3 \in I$ such that we have morphisms $i_3 \to i_1$, $i_3 \to i_2$.
3. For any two parallell morphisms $f, g : i_1 \rightrightarrows i_2$, there exists an object $i_3 \in I$ and a morphism h:$i_3 \to i_2$ such that



commutes, i.e $f \circ h = g \circ h$.

An ordered set $(I, \leq)$ is codirected exactly when the associated category I is cofiltered. Dualizing the above definition, we get a notion of a filtered category. We will sometimes say that a category is cofiltrant / filtrant as another way of saying that it is cofiltered / filtered.

Definition 2. Let C be a category. We have an associated category, Pro(C), with objects consisting of functors $F : I \to C$, where I is small and cofiltered. The morphisms between two objects $F : I \to C$, $G : J \to C$ are

$$\mathrm{Hom}_{Pro(C)}(F, G) = \varprojlim_{j}(\varinjlim_{i} \mathrm{Hom}(F(i), G(j))).$$

We call the objects of $Pro(C)$ pro-objects. A morphism between pro-objects $F : I \to C$, $G : J \to C$ is thus given by specifying for each $j \in J$ a morphism $F(i) \to G(j)$ for some i, which are compatible with the morphisms in J. We think of pro-objects as placeholders for projective limits, and with this viewpoint, the reason why we define morphisms as we do becomes evident. To be more precise, for functors $\alpha : I \to C$, $\beta : J \to C$, with I and J cofiltrant and small, we can take the projective limit in the category $Fct(C, Set)^{opp}$ (which is guaranteed to exist) of $k_c(\alpha) = Hom_C(\alpha, -)$ which is a functor $I \to Fct(C, Set)^{opp}$, and similarily with $k_c(\beta)$.

$$\begin{aligned} Hom_{Fct(C,Set)^{opp}}(\varprojlim_i k_c(\alpha), \varprojlim_j k_c(\beta)) &= \varprojlim_j Hom_{Fct(C,Set)^{opp}}(\varinjlim_i k_c(\alpha(i)), k_c(\beta(j))) \\ &= \varprojlim_j \varinjlim_i Hom_{Fct(C,Set)^{opp}}(k_c(\alpha(i)), k_c(\beta)(j)) \\ &= \varprojlim_j \varinjlim_i Hom_C(\alpha(i), \beta(j)) \end{aligned}$$

by Yoneda and the interaction of colimits with hom-functors.

2.2 Categorical constructions

2.2.1 Kan Extensions

Let J, I and C be categories, and $\varphi : J \to I$ be a functor. For a functor $F : I \to C$, we can then naturally form a functor $\varphi_* F : J \to C$ by $\varphi_* F(j) = F(\varphi(j))$. It is now natural to ask the converse question - Given a functor $G : J \to C$, is there a way to extend it to a functor from I to C? The answer is, in some favourable cases yes, and it is done by Kan Extension.

Definition 3. Let J, I and C be categories and $\varphi : J \to I$, $F : J \to I$ be functors. If the functor taking $G \in Fct(I, C)$ to $Hom_{Fct(J,C)}(F, \varphi_* G)$ is representable, we call its representative the left Kan extension of F along φ, denoted $Lan_\varphi F$. We will then for every G have an adjunction

$$Hom_{Fct(J,C)}(F, \varphi_* G) \cong Hom_{Fct(I,C)}(Lan_\varphi F, G).$$

In the same way, if we demand that the functor $Hom_{Fct(J,C)}(\varphi_* G, F)$ be representable, we get the right Kan Extension of F along φ, $Ran_\varphi F$. In this case, for $G \in Fct(I, C)$ we then have an adjunction $Hom_{Fct(J,C)}(\varphi_* G, F) \cong Hom_{Fct(I,C)}(G, Ran_\varphi F)$. A natural question is to ask, how do we construct Kan Extensions? The following theorem gives some criteria, but first, we need a definition.

Definition 4. Let J,I be categories and $\varphi : J \to I$ a functor. For $i \in I$, we define $\varphi \downarrow i$, to be the category with objects $f : \varphi(j) \to i$ where $f \in Hom_I(\varphi(j), i)$ and morphisms between $f_1 : \varphi(j_1) \to i$, $f_2 : \varphi(j_2) \to i$ given by $h \in Hom_J(j_1, j_2)$ such that $f_1 = f_2\varphi(h)$. In the same way, we define the category $\varphi \uparrow i$ to have as objects $f : i \to \varphi(j)$ where $f \in Hom_I(i, \varphi(j))$ and morphisms between $f_1 : i \to \varphi(j_1)$ and $f_2 : i \to \varphi(j_2)$ given by $h \in Hom_J(j_1, j_2)$ such that $f_2 = \varphi(h)f_1$.

Theorem 5. *(i) Let $\varphi : J \to I$ be a functor and $\beta \in Fct(J, C)$. Let us assume that*

$$\varinjlim_{(\varphi(j)\to i)\in\varphi\downarrow i} \beta(j)$$

exists for any $i \in I$. Then

$$Lan_\varphi \beta$$

exists and

$$Lan_\varphi \beta \cong \varinjlim_{(\varphi(j)\to i)\in\varphi\downarrow i} \beta(j).$$

Thus, if C admits small inductive limits and J is small, the left Kan Extension of any functor $\beta : J \to C$ exists.
(ii) Let $\varphi : J \to I$ be a functor and $\beta \in Fct(J, C)$. Let us assume that

$$\varprojlim_{(i\to\varphi(j))\varphi\uparrow i} \beta(j)$$

exists for any $i \in I$. Then $Ran_\varphi \beta$ exists and

$$Ran_\varphi \beta \cong \varprojlim_{(i\to\varphi(j))\uparrow i} \beta(j).$$

Thus, if C admits small projective limits and J is small, the right Kan Extension of any functor $\beta : J \to C$ exists.

Proof. See [5] p. 52 □

Mac Lane famously proclaimed that "All concepts are Kan Extensions" so it seems fitting that we at least provide two examples. Recall that if $F : J \to C$ is a functor, for $N \in Ob(C)$, a cone from N to F is a natural transformation $const(N) \to F$, where $const(N) : J \to C$ is the constant functor. The limit of a functor F can be defined as a universal cone $\psi : const(limF) \to F$ such that any other cone factors uniquely through it.

Example 6. *(All limits are Kan Extensions) Let C be a categoy $F : I \to C$ some functor, $T : I \to 1$ the unique functor from I to the terminal category 1. Then, suppose that the right Kan Extension of F along T exists. A functor $X : 1 \to C$ is easily identified with an object in C. We have that since the functor $Hom_{FctI,C}(T_*, F)$ is representable, for $X : 1 \to C$, we have $Hom_{Fct(I,C)}(T_*X, F) \cong Hom_{Fct1,C}(X, Ran_T F)$. This translates to that $Ran_T F$ is the limit of F, since we can identify the left side of the adjunction as a cone to F. The right hand side then simply says that for each such cone there is an unique morphism to the cone $Ran_T F$, that is, $Ran_T F$ is the universal cone and as such, the limit of F. A similar argument works for colimits, assuming that the left Kan extension of F along T exists.*

Example 7. *(Induction is a Kan Extension) Let us try out the above formula in a simple and manageable case. Let G be a finite group considered as a category with one object, and H a subgroup of G. Let $G - Vec_k$ be the category of vector spaces over the field k with a G-representation, and $H - Vec_k$ the analogous definition for H.We can identify a representation of G with a functor $F : G \to Vec_k$, into the category of vector spaces. We have a natural inclusion functor $i : H \to G$ and the restriction functor $Res : G - Vec_k \to H - Vec_k$ is given by, for a functor $F : G \to Vec_k$ by i_*F, precomposition. Now, take the left Kan Extension of the representation $F : H \to Vec_k$ along $i : H \to G$.*

I claim that this is the induced representation of F, $Ind_H^G F$. Applying the above formula, we have that $Lan_I F(G) \cong \varinjlim_{i(H)\to G} F(H)$. We see that the category $i_*H \to G$ can be seen as a category consisting of separated components, where two elements $g_1 : G \to G$, $g_2 : G \to G$ lie in the same component iff they lie in the same coset in G/H. Now, this is easily shown to be equivalent to the discrete category with $[G : H]$ objects. We then verify that, taking the colimit over this category, $Lan_I F(G) \cong \oplus_{g\in G/H} F(G)_g$, one for each coset .For each $g \in G$ and g_j a representative of a coset of G/H, there is a $h \in H$ and a coset representative g_i such that $gg_j = g_i h$. Then, for $x_{g_j} \in F(G)_{g_j}$ the action of g on x_j is given by $gx_j = hx_i$, were $x_i \in F(G)_{g_i}$. It is an easy verification to show that $\oplus_{g\in G/H} F(g)_g$ with this action is the correct colimit and thus, the left Kan Extension of F along Res, which we have shown to be isomorphic to the induced representation.

2.2.2 Localization

We briefly introduce the concept of localization. The reader further interested in the subject should consult Kashiwara-Schapira chap 7. Let us say that we have a category C and some certain class of morphisms $\mathscr{M}$ of C. A localization of C by $\mathscr{M}$ should be seen as an universal way of turning all the morphisms in $\mathscr{M}$ to isomorphisms. It is very useful, for example, to construct an associated homotopy category out of a category of weak equivalences. If we formulate this with universal properties, we get the following definition.

Definition 8. A localization of the category C by $\mathscr{M}$ is a category $C_{\mathscr{M}}$ and a functor $Q : C \to C_{\mathscr{M}}$ such that :
(i) For all $m \in \mathscr{M}$, $Q(m)$ is an isomorphism.
(ii) For any other category A and a functor $F : C \to A$ such that for all $m \in \mathscr{M}$, $F(m)$ is an isomorphism, there exists a functor $F_{\mathscr{M}} : C_{\mathscr{M}} \to A$ and a natural isomorphism $F_{\mathscr{M}} \circ Q \cong F$.
(iii) The natural map $(-) \circ Q : Fct(C_{\mathscr{M}}, A) \to Fct(C, A)$ is fully faithful.

Example 9. *(Localization of a ring) Let R be a ring. R can be seen as a category with one object (just as groups) and such that the homset $Hom_R(R, R)$ is enriched over the category of abelian groups, that is, the hom-set is an abelian group and the composition is bilinear. Let $S \subset R$ be a multiplicative set and consider the $S^{-1}R$ as a category, with the natural functor $Q : R \to S^{-1}R$. This is not the localization of R with (as a category) by S. It does however satisfy some properties, which we shall investigate further. For all $s \in S$. $Q(s) = s/1 \in S^{-1}R$ has an inverse, and as such is an isomorphism. (ii) of the above just refers to the universal property of localization of rings. (iii) is not however always true. Indeed, for rings, consider any multiplicative subset S containing 0. Then localizing in S, $S^{-1}R = 0$, the trivial ring. Then $Hom(0, T)$ is empty, but there is no reason for $Hom(R, T)$ to be, and we can have for two $f, g \in Hom(R, T)$ a $h : R \to R$ such that $f = gh$, and clearly this h can not in this case come from a morphism $Hom(0, T) = \emptyset$.*

Example 10. *(Derived category) Let R be any commutative ring and $Ch(R)$ the category of chain complexes in R. Let $HoCh(R)$ be the category with the same objects, but* *where $Hom_{HoCh(R)}(C_\bullet, D_\bullet) = Hom_{Ch(R)}(C_\bullet, D_\bullet)/\sim$, where $\sim$ means that we identify maps of chain complexes that are chain homotopic.Now, it can be shown that the quasi-isomorphisms in $Hom_{HoCh(R)}$, i,.e the equivalence classes of maps inducing isomorphisms on homology groups forms a multiplicative system (see definition below) and we can thus localize to form the derived category. We only mention this, but won't delve deeper into this highly technical subject.*

By usual abstract nonsense, if it exists it is unique up to equivalence of categories . A localization with respect to some class of morphisms is not guaranteed to exist. For rings we have that localization is only defined for a multiplicative set. We have a similar construction for categories where the localization exists.

Definition 11. A family $\mathscr{M}$ of morphisms is a right multiplicative system if:
(i) All isomorphisms are in $\mathscr{M}$.
(ii) $\mathscr{M}$ is closed under composition.
(iii) Given morphism $f : X \to Y$ and $g : X \to Z$ with $g \in \mathscr{M}$ we can find t and s with $t \in \mathscr{M}$ such that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle t} \\ Z & \xrightarrow{s} & W \end{array}$$

commutes.
(iv) Given a morphism $m \in \mathscr{M}$ and parallel morphisms $f, g : X \rightrightarrows Y$ such that $f \circ m = g \circ m$, we can find a $t : Y \to Z$ in $\mathscr{M}$ such that $t \circ f = t \circ g$.

We get a similar notion of a left multiplicative system by reversing the arrows. Now, let $\mathscr{M}$ be a right-multiplicative system. We define $\mathscr{M}^Y$, for $Y \in C$ as the category which has objects morphisms $s : Y \to Y'$ where $s \in \mathscr{M}$ and morphisms are the obvious ones - namely for two objects $s : Y \to Y'$, $s' : Y \to Y''$, a morphism is a map $g : Y' \to Y''$ such that $g \circ s = s'$.. Let us form a new category $C_{\mathscr{M}}$ as follows. The objects are the same as in C, and

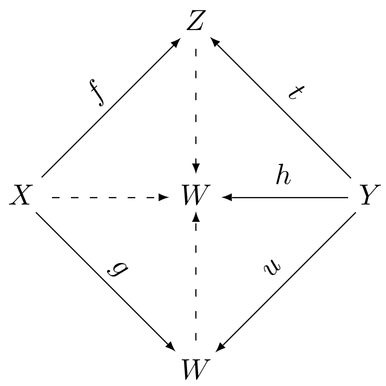
$$Hom_{C_{\mathscr{M}}}(X, Y) = \varinjlim_{Y \to Y' \in \mathscr{M}^Y} Hom_C(X, Y').$$

It can easily be shown that the category $\mathscr{M}^Y$ is filtrant for any $Y \in C$. To help the reader get a feeling for multiplicative systems, we will give some details on how to prove the following lemma:

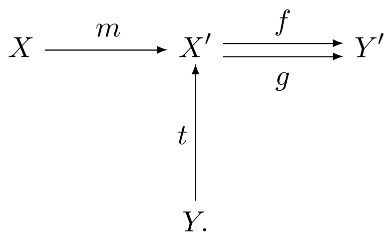
Lemma 12. *Assume that $\mathscr{M}$ is a right multiplicative system. Then if $m : X \to X'$, $m \in \mathscr{M}$, we have that composition with m gives us an isomorphism*

$$\varinjlim_{(Y \to Y') \in \mathscr{M}^Y} Hom_C(X', Y') \cong \varinjlim_{(Y \to Y') \in \mathscr{M}^Y} Hom_C(X, Y').$$

Proof. Since this is an isomorphism in Set, it suffices to show that the map $\circ m$ is bijective. We start by showing injectivity. Note that a morphism is given by an equivalence class (f, t, Y') with $t \in \mathscr{M}$, $f : X \to Y'$ and $t : Y \to Y'$. The equivalance relation is given as follows: Since the category $\mathscr{M}^Y$ is filtrant, (f, t, Z) $f \in Hom_C(X', Z)$, $t : Y \to Z$ and (g, u, W) $g \in Hom_C(X', W)$, $u : Y \to W$, $t, u \in \mathscr{M}$ are equivalent in the limit if there is some $h : Y \to U$, $h \in \mathscr{M}$ and maps filling in the dots in



and making the whole diagram commutative. With this, notion, injectivity follows from (iv). Indeed, let $f : X' \to Y'$ and $g : X' \to Y''$, with $s : Y \to Y'$, $t : Y \to Y''$, $s, t \in \mathscr{M}$, and suppose that composition with m maps them to the same equivalence class. We can, since the category is filtrant, assume that $Y' = Y''$. Then we have a commutative diagram



We can now by (iv) of the axioms find a morphism $t' : Y' \to W$, $t \in \mathscr{M}$ such that $t' \circ f = t' \circ g$. So they're equal in the limit, that is, $\circ m$ is injective. The reader should have no problem proving surjectivity using the third axiom of a right multiplicative system. □

It can be shown that we can find a composition that is both well-defined and associative and we get a resulting category, $C_{\mathscr{M}}$. This category is the localization of C with respect to $\mathscr{M}$. The objects are the same, but Hom sets are given as previously defined. We have a natural functor $Q : C \to C_{\mathscr{M}}$. Now, for each $m \in \mathscr{M}$, $m : X \to X'$ $Q(m)$ is an isomorphism. This follows from that by our previous lemma,

$$Hom_{C_{\mathscr{M}}}(X', Y) \cong Hom_{C_{\mathscr{M}}}(X, Y),$$

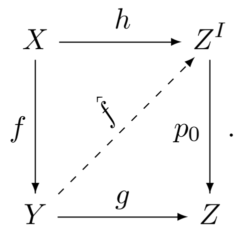
for any Y, the isomorphism given by composition with m. So, m is an isomorphism under the Yoneda embedding and since this is fully faithful, m must be an isomorphism.

Theorem 13. *$C_{\mathscr{M}}$ and Q is the localization of C with respect to $\mathscr{M}$*

Proof. [5] 7.1.16. p. 155 □

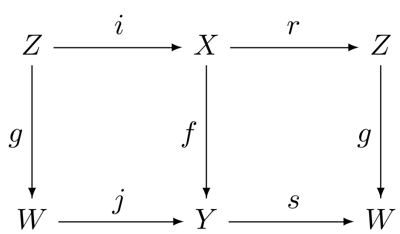
2.3 Model Categories

A model category is a certain kind of category where one can perform homotopy theory. If we try to get to grip with what is really going on when we work with homotopy in a category of convenient topological spaces (for example, compactly generated weakly Hausdorff spaces), we see that we have attached to the objects three classes of morphisms.Firstly, we have the fibrations, which as we recall, are simply maps satisfying the homotopy lifting property for **all** spaces. A cofibration $f : X \to Y$ is simply a map satsifying the homotopy extension property for all spaces, which we visualise by the diagram



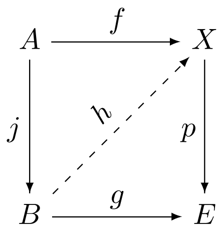
Finally, a weak equivalence $f : X \to Y$ is a map inducing isomorphisms, $f_* : \pi_n(X) \to \pi_n(Y)$ for all n and choice of basepoints. We know from basic homotopy theory that we can factor each continous map $f : X \to Y$ ([14] p. 113) as $f = p \circ i$ where p is a fibration and i is an acyclic cofibration (i.e a cofibration that is a weak equivalence) and also as $f = q \circ j$ for q an acyclic fibration and j a cofibration.

Definition 14. A model structure on a category C is a collection of three types of special morphisms, $(\mathscr{W}, \mathscr{F}, \mathscr{C})$, weak equivalences, fibrations and cofibrations respectively, that satisfy the following axioms:
(i) Each class is closed under composition and contains all identity maps.
(ii) The classes of morphisms are closed under retractions. More explicitly, if $f : X \to Y$ is a map in C, and $g : Z \to W$ is a map belonging to some class of $(\mathscr{W}, \mathscr{F}, \mathscr{C})$ such that we have i, j, r and s such that



commutes and $ri = id_Z$, $sj = id_W$, then g belongs to the same class of morphism as f.
(iii) (2 of 3-property) If f and g are morphisms such that $g \circ f$ is defined, then, if two of f,g, $g \circ f$ are weak equivalences, so is the third.

(iv) In a square



where the outer square commutes and j is a cofibration and p a fibration, we can find a h making all triangles commutative if either j or p is a weak equivalence.
(v) Every morphism f in C can be factored as $f = j \circ q$ where j is a cofibration and q is an acyclic fibration and $f = i \circ p$ where i is an acyclic cofibration and p a fibration.

We call an object X cofibrant if the unique map from the initial object $1 \to X$ is a cofibration, and dually if the unique map from X to the terminal object is a fibration, X is fibrant. Note that these notions only make sense when the categories have initial and terminal objects, and in some modern definitions of a model category, one requires all finite limits and colimits to exist.

Example 15. *The category of chain complexes $Ch(R)$ with increasing differential (graded by $\mathbb{N}$) of left R-modules, for R a ring is a model category if we give it the following structure: A map $f : C_\bullet \to D_\bullet$ of chain complexes is a weak equivalence if it induces isomorphisms in homology (of the complexes). A map $f : C_\bullet \to D_\bullet$ of chain complexes is a fibration if for all n, $f_n : C_n \to D_n$ is an epimorphism where the kernel is an injective module. A map $f : C_\bullet \to D_\bullet$ is a cofibration if for each $n > 0$, $f_n : C_n \to D_n$ is a monomorphism. Let us sketch how to prove that this in fact forms a model category. We start with noting that (i) is clear, we'll now start with showing (iii) and then return to (ii) later.*

(iii) We will consider this case by case. First, say that $C_\bullet$, $D_\bullet$, $E_\bullet$ are chain complexes and that $f : C_\bullet \to D_\bullet$ and $g : D_\bullet \to E_\bullet$ are weak equivalences and that additionally, f and g are either fibrations or cofibrations. Then it is clear that the induced map $gf : C_\bullet \to E_\bullet$ is a weak equivalence, since it induces isomorphism on the homology groups (follows by transitivity of the isomorphism relation or that composition of isomorphisms are isomorphisms). If now we assume that f and gf are in the same class of morphisms and additionally that $f : C_\bullet \to D_\bullet$ and $gf : C_\bullet \to E_\bullet$ are weak equivalences, so is g. Indeed, writing f_* for the induced map on homology, we have that f_* being an isomorphism amounts to it having an inverse, so we have that $(gf)_* \circ f_*^{-1} = g_* : D_\bullet \to E_\bullet$ is an isomorphism, since the composition of isomorphisms are isomorphisms. A similar case holds when g and gf lie in the same class. So we have shown (iii).

(ii) Suppose that $g : E_\bullet \to F_\bullet$ is a weak equivalence, and that $f : C_\bullet \to D_\bullet$ is a rectraction

of g, so that we have a commutative diagram :

$$\begin{array}{ccccc} C_\bullet & \xrightarrow{i} & E_\bullet & \xrightarrow{r} & C_\bullet \\ f\downarrow & & g\downarrow & & f\downarrow \\ D_\bullet & \xrightarrow{j} & F_\bullet & \xrightarrow{s} & D_\bullet \end{array}$$

such that $ri = 1$ and $sj = 1$. g is supposed to be a weak equivalence, that is, g_* is an isomorphism, so that we have an inverse g_*^{-1} and it is then routine to check that $r_* \circ g_*^{-1} \circ j_*$ is an inverse to f_* and thus, f_* is an isomorphism and f is a weak equivalence. Now, let us simply note that for R-modules, a retract of a monomorphism (or an epimorphism) is a monomorphism (resp. an epimorphism). So it is easy to verify that cofibrations are closed under retracts, we prove that fibrations also are closed under retracts. We clearly have a commutative diagram

$$\begin{array}{ccccc} ker f & \xrightarrow{k} & ker g & \xrightarrow{w} & ker f \\ \downarrow & & \downarrow & & \downarrow \\ C_\bullet & \xrightarrow{i} & E_\bullet & \xrightarrow{r} & C_\bullet \\ f\downarrow & & g\downarrow & & f\downarrow \\ D_\bullet & \xrightarrow{j} & F_\bullet & \xrightarrow{s} & D_\bullet \end{array}$$

with $ker g$ injective.Suppose now that we have a morphism $t : A_\bullet \to ker f$ and a monomorphism $u : A_\bullet \to B_\bullet$, we want to show the existence of a $v : B_\bullet \to ker f$ such that $t = vu$. We have maps $kt : A_\bullet \to ker g$ and since ker g is injective, there is a map $l : B_\bullet \to ker g$ such that $kt = lu$.Now, we have that $w \circ k = id_{ker f}$ so, $w \circ kt = t = w \circ lu$ so that $l \circ u$ is the desired map to $ker f$, thus showing that ker f is an injective in each degree.

(iv) Say that we have a commutative diagram

$$\begin{array}{ccc} C_\bullet & \xrightarrow{f} & E_\bullet \\ j\downarrow & & \downarrow p \\ D_\bullet & \xrightarrow{g} & F_\bullet \end{array}$$

where j is a cofibration that is also a weak equivalence and p a fibration. I claim that we can then find a lift h as axiom (iv) requires us to.Note that p is an injective map in all degrees, the case for $n = 0$ follows from looking at the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & kerd_C^0 & \longrightarrow & C_0 & \longrightarrow & C_1 \\ \downarrow & & \cong\downarrow & & \downarrow p_0 & & \downarrow p_1 \\ 0 & \longrightarrow & kerd_D^0 & \longrightarrow & D_0 & \longrightarrow & D_1 \end{array}$$

and applying the five lemma. Now, since ker p is an injective module we have that $E_\bullet \cong kerp \oplus F_\bullet$ and as such, the differential $\delta^n : E_n \to E_{n+1}$ can via this isomorphism be taken to the form $\delta^n(a, b) = (da + \tau c, dc)$ where d is the differential of X and τ is a map such that $d\tau + \tau d = 0$. We also have a map $q : F \to X$ which splits p. Then, $p(qgi - f) = 0$ so by the fact that kerp is injective we have an extension $h : D_\bullet \to kerq$ such that $hi = qgi - f$. Then $qg - h$ is our desired lift which shows (iv) in the case that j is a weak equivalence and a cofibration.

We will introduce two objects in our category which will play an analogous role to that of the disk and the sphere in the category of topological spaces. Let the n-disk chain complex of a R-module M be defined by $D^n(M)_k = M$ for $n \geq 0$ if $k = n$ or $n+1$, and 0 otherwise and the boundary map is the identity between two non-zero copies of M and the zero map in all other cases. The n-sphere chain complex $S^n(M)$ is 0 except when $k = n$, where it is M. Now, it is obvious that $Hom_{Ch(R)}(D^n M, C_\bullet) \cong Hom_{R-mod}(M, C_n)$ where we take $f \in Hom_{Ch(R)}(D^n M, C_\bullet)$ to f_n. If Q is an injective R-module, the I claim that $D^n(Q)$ is an injective chain complex. Indeed, this follows from the isomorphism $Hom_{Ch(R)}(C_\bullet, D^n(M)) \cong Hom_{R-mod}(C_{n+1}, M)$. It will be shown that if $Q_\bullet$ is an chain complex consisting of injectives with no homology (i.e acyclic) then we can build up $Q_\bullet$ from certain n-disk chain complexes. In fact:

Lemma 16. *Let $Q_\bullet$ be an acyclic chain complex such that each Q_n is injective. Then each module of boundaries, Imd_n $n \geq 0$ is injective and $Q_\bullet \cong \oplus_{k\geq 0} D_k(imd_k)$.*

Proof. For $k \geq 1$ let Q^k be the chain complex agreeing with Q above level k-1 and $Q^k_{k-1} = imd_{k-1}$ and and that is zero in all degrees less than k-1. Then we have that $Q^k/Q^{k+1} \cong D_k(Imd_{k-1}Q)$. Now, $Q_\bullet$ is acyclic, so that $Q_0 = Imd_0$ and we have a short exact sequence

$$0 \longrightarrow Q_0 \longrightarrow Q_1 \longrightarrow imd_1 \longrightarrow 0$$

and since Q_0 is injective, $Q_1 = Q_0 \oplus imd_1$ and as such, $Q_\bullet = Q^2 \oplus D^0(Imd_0)$ and $D^0(Imd_0)$ is injectivein each degree. A direct product is an injective R-module iff a each direct factor is injective, so that Q^2 is an injective module too. We can also check that it is acyclic as a complex and 0 in degree zero, and as such, we can repeat the argument but starting in degree one. Continuing in this way gives us $Q_\bullet \cong \oplus_{k\geq 0} D_k(imd_k)$. □

Now we are finally ready to prove the last part of (iv). Suppose we have a commutative diagram

$$\begin{array}{ccc} C_\bullet & \xrightarrow{f} & E_\bullet \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle p} \\ D_\bullet & \xrightarrow{g} & F_\bullet \end{array}$$

where j is a cofibration and p is a fibration that is also a weak equivalence. Let $Q_\bullet = kerp$, tje cpåöex that in each degree i is $kerp_i$. Then we have a short exact sequence of complexes

$$0 \longrightarrow Q_\bullet \longrightarrow E_\bullet \longrightarrow F_\bullet \longrightarrow 0$$

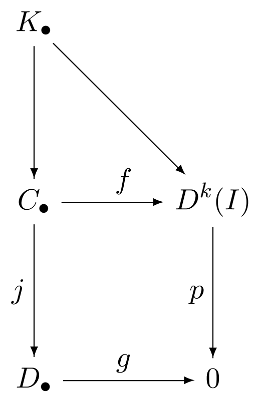
which gives rise to a long exact sequence of homology and this shows that since $E_\bullet$ and $F_\bullet$ have isomorphic homology groups, $Q_\bullet$ is an acyclic complex of injectives and we can write $Q_\bullet = \oplus_{k\geq 0} D^n(imd_k)$ and $im_d k$ is injective. We then have that $E_\bullet \cong Q_\bullet \oplus F_\bullet$.Now, drawing the diagram

$$\begin{array}{ccc} C_\bullet & \xrightarrow{f} & Q_\bullet \oplus F_\bullet \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle p} \\ D_\bullet & \xrightarrow{g} & F_\bullet \end{array}$$

we see that we can find a lift by the property of $\oplus$ being a coproduct in the category of chain complexes and $Q_\bullet$ being injective (i.e we choose g for mapping to the factor F, and to Q any lift of f and use the universal property of the coproduct to get a map to the direct sum). This completes (iv).

(v) We will first show that a map is a cofibration iff it has the left lifting property with respect to maps $D^n(I) \to 0$ with I injective. Let $K_\bullet$ be the kernel of $j : C_\bullet \to D_\bullet$ and let $k \geq 1$ be given and embed K_k in an injective module I. We know that $Hom_{Ch(R)}(K_\bullet, D_k(I)) \cong Hom_{R-mod}(K_{k+1}, I)$. Now, since I is injective, we can find a

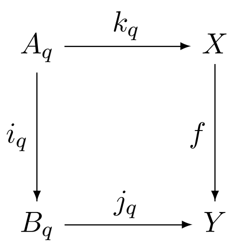
map $C_{k+1} \to I$. So we have a commutative diagram



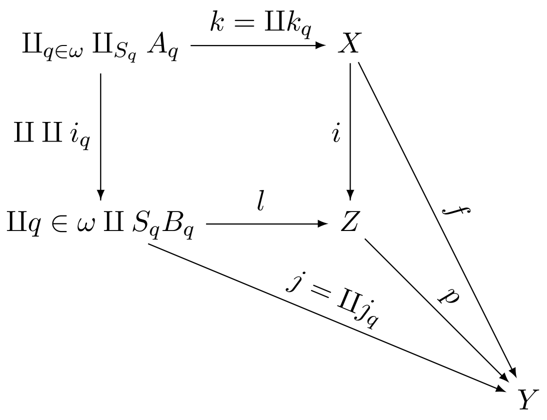
*and we can obiously not find an extension $D_\bullet \to D^k(I)$ if $C_{k+1} \neq 0$. So, $C_k = 0$ for $k > 0$, implying that i is a monomorphism in all non-zero degrees. We will now briefly introduce Quillen's small object argument since it is so immensly useful for proving the existence of factorizations. We will **not** however supply a proof, since this example is long already as it is.*

Definition 17. A weak factorization system in a category C is an ordered pair $(\mathcal{L}, \mathcal{R})$ of morphisms of C such that every morphism $f : X \to Y$ in C can be factored as $X \xrightarrow{g} U \xrightarrow{h} Z$ where $f \in \mathcal{L}$ and $g \in \mathcal{R}$ and such that $\mathcal{L}$ consists of precisely those morphisms which have the left lifting property with respect to maps in $\mathcal{R}$ and $\mathcal{R}$ consists of those maps that precisely have the right lifting property with respect to morphisms in $\mathcal{L}$.

Suppose now that we have a set $\mathcal{L}$ of morphisms in C and $f : X \to Y$ and we want to factor f as a composite map where the first is in $\mathcal{L}$ and the second is a morphism with the right lifting property with respect to maps in $\mathcal{L}$. Choose a well-ordering of $\mathcal{L}$ and an order isomorphism with some ordinal ω. For f a morphism in C and $q \in \omega$ let S_q the set of commutative squares



where $i_q \in \mathcal{L}$ correspondes to $q \in \omega$ by our well-ordering. We will now construct a factorization diagram for f by gluing a copy of B_q to X along A_q for every commutative *diagram in S. Namely, construct the pushout*



where p is induced from the universal property of pushouts. Set $i_1 = i$, $p = p_1$ and $Z = Z_1$. Repeat this construction inductively to obtain an object Z_∞ morphisms $i_\infty : X \to Z_\infty$ and $p_\infty : Z_\infty \to Y$ such that $p = p_\infty i_\infty$. We will now state our main lemma for proving the factorization . We will not state the small objects theorem in its full generality, instead only taking what is neccessary for our purposes.

Definition 18. ($\mathbb{Z}^+$-small) Let C be a category with all small colimits and let $F : \mathbb{Z}^+ \to C$ be a functor and A an object in C. We will then have maps $F(n) \to \varinjlim F$ and they induce for each n a map $Hom(A, F(n)) \to Hom(A, \varinjlim F)$ which combine to give a map $\varinjlim Hom(A, F(n)) \to Hom(A, \varinjlim F)$ which is canonical. If this map is a bijection for every functor $\mathbb{Z}^+$, we say that A is $\mathbb{Z}$-small.

Trying not to delve further into set theory or notational issues, let us quickly remark that a set is $\mathbb{Z}^+$-small iff it is finite, and for $Ch(R)$, a chain complex $C_\bullet$ is $\mathbb{Z}^+$ small iff only C_n is nonzero for finitely many n and for each such n, C_n is finitely presented. This is not all too hard to prove, but quite messy and we omit it. Now, with this:

Lemma 19. *(Quillen's small object argument) Let C be a category with all small colimits and let $\mathcal{L}$ be some set of morphisms in C, with a given well-ordering ω.For each $q \in \omega$, assume that A_q is $\mathbb{Z}^+$ small. Then there exists a weak factorization system $C(\mathcal{L}), \mathcal{R})$ where $C(\mathcal{L})$ is the set of of morphisms which are obtained by transfinite composition of pushouts of morphisms in $\mathcal{L}$ (as in our construction above) and $\mathcal{R}$ is the set of morphisms with the right lifting property with respect to $\mathcal{L}$.*

Proof. See, for example [9] p. 297 □

With this done, it can be shown (see for example, 2.3.13, Hovey) that the category of chain complexes (with increasing differential and 0 in negative degrees) has a set of maps $\mathcal{C}$ and $\mathcal{CW}$ which we call generating cofibrations and acyclic cofibrations respectively. In our case, they will have the property to be $\mathbb{Z}^+$-small, and further, a map is a fibration iff it has the right lifting property with respect to $\mathcal{CW}$ and an acyclic fibration iff it has the right lifting property with respect to maps in $\mathcal{C}$. Then for a map $f : C_\bullet \to D_\bullet$ the small *objects argument gives rise to a factorisation (taking $\mathcal{L} = \mathcal{C}$) $i_\infty : C_\bullet \to Z$, $p_\infty : Z \to D_\bullet$ such that p_∞ is an acyclic fibration. It is easily checked that i_∞ is a cofibration and the similar case for $\mathcal{L} = \mathcal{CW}$ gives the other factorization. We have thus finally shown that this is a model category.*

We now form the homotopy category of C, $\mathscr{H}C$ by localizing the weak equivalences and get a functor $Q : C \to \mathscr{H}C$. We should thus intuitively consider the homotopy category as the category where all the morphisms that induce homotopy equivalences, turns into isomorphisms. Further, it can be proved that we can think of the hom set between two objects as homotopy classes of maps between cofibrant and fibrant objects.

Category theory has taught us that one of the most important aspects when it comes to studying the structure of a certain kind, we must understand how it interacts with other structures. We would like to be able to compare different model categorical structures in some way. The right tool for this turns out to be a certain kind of adjunction, called a Quillen adjunction. First, we'll define a homotopy derived functor. Let C and D be any categories with weak equivalences. We define a functor $F : C \to D$ to be a (weakly) homotopical functor if it takes weak equivalences to weak equivalences. We define any functor $F : C \to E$ to be a homotopical functor if every weak equivalence is mapped onto an isomorphism in E. If this is the case, we have that the universal property of localization gives us a functor $\hat{F} : \mathscr{H}C \to E$ such that $\hat{F} \circ Q \cong F$, by a natural isomorphism γ. We call the functor $\hat{F}$ a derived functor. This is however, too much of a restriction, there are many functors that do not map weak equivalences to isomorphisms. We want to extend our notion to be able to create more derived functors.

Definition 20. A left derived functor of $F : C \to D$, where C is a model category, consists of a pair $(\mathbb{L}F, \gamma)$, where $\mathbb{L}F : \mathscr{H}C \to D$ is a functor and $\gamma : \mathbb{L}F \circ Q \to F$ is a natural transformation universal with the following property: For any pair (K, α) $K : \mathscr{H}C \to D$, $\alpha : K \circ Q \to F$ there is a unique $\beta : K \to \mathbb{L}F$ such that $\gamma \circ (\beta \circ Q) = \alpha$.

We also have a notion of a certain derived functor when both C and D has the structure of a model category. If $F : C \to D$, then a total left derived functor $\mathbb{L}F : \mathscr{H}C \to \mathscr{H}D$ is a left derived functor of $Q' \circ F$, where $Q' : D \to \mathscr{H}D$ is the localization functor. Let us remember that an object X of a model category is called cofibrant if the unique map from the initial object to X is a cofibration. Now, let us remember that we call a map that is both a weak equivalence and a fibration an acyclic fibration, and likewise for cofibration. In some favourable cases, a total left derived functor exists.

Theorem 21. *If $F : C \to D$ is a functor such that it maps acyclic cofibrations $c : X \to Y$, X and Y cofibrant, to weak equivalences, then the total left derived functor $(\mathbb{L}F, \alpha)$ exists.*

A similar theorem holds for the total right derived functor, just replace cofibration, cofibrant with fibrant. Now, we are finally ready to define a Quillen adjunction .

Definition 22. For C and D model categories, an adjoint pair of functors (L, R), $L : C \to D$, $R : D \to C$ is called a Quillen adjunction if the following equivalent conditions are satisfied:
(i) L preserves cofibrations and acyclic cofibrations
(ii) R preserves fibrations and acyclic fibrations
(iii) L preserves cofibrations and R fibrations.
(iv) L preserves acyclic cofibrations and R acyclic fibrations.

Example 23. *As above, let us consider the case where $C = Ch(R)$, complexes of R-modules and $D = C$. We have a model categorical structure on it, and I claim that the two functors $L = K \otimes_R -$ and $R = Hom_R(K,_)$ form a Quillen adjunction if K is a complex consisting of projective modules. Clearly, L and R are adjoint. We will verify that (L, R) satisfies (iii). L is an exact functor, so if $f : X \to Y$ is a cofibration, we have that for each n, $f_n : X_n \to Y_n$ is a monomorphism. Now, if f as before is a cofibration I claim that $L(f) : L(X) = K \otimes X \to L(Y) = K \otimes Y$ is a cofibration. We have that $L(f)_n : (K \otimes X)_n = \oplus_{l+m=n} K_l \otimes X_m \to \oplus_{l+m=n} K_l \otimes Y_m = (K \otimes Y)_n$ is a monomorphism by exactness of L. More than that, R preserves degreewise surjections as can easily be checked. Further, L preserves monomorphisms and as such R must preserve injectives. This gives that R preserves fibrations.*

From this definition it is immediate that the left adjoint preserves weak equivalences between cofibrant objects and right adjoint preserves weak equivalences between fibrant objects. Now, for a Quillen adjunction, we see that the left adjoint has a total left derived functor $\mathbb{L}L$ and $\mathbb{R}R$ a total right derived functor. They will form an adjoint pair. Now, it is natural to ask when these two derived functor determine an equivalence of categories. We first note that since both categories have a model categorical structure, we have a full subcategory of cofibrant objects, and then we invert the weak equivalences, and now, L will preserve the weak equivalences between cofibrant objects and R the same with weak equivalences between fibrant objects. So, it is in some sense natural, given the factorization of maps in C and D to ask for some relation between weak equivalences between fibrant and cofibrant objects. This is made precise by the following theorem.

Theorem 24. *A Quillen adjunction (L,R) is a Quillen Equivalence if the following equivalent conditions are satisfied:*
(i) For any map $f : L(X) \to Y$ with an adjoint map $g : X \to R(Y)$, X cofibrant and Y fibrant, the first map is a weak equivalence iff the latter is.
(ii) The total left derived functor $\mathbb{L}L$ is an equivalence of categories.
(iii) The total right derived functor $\mathbb{R}R$ is an equivalence of categories.

Remark. This is to me one of the most remarkable examples in mathematics of where quite simple objects can capture a lot of inherent structure of seemingly complex spaces. Namely, it is true that the category of simplicial sets is Quillen equivalent to the category of compactly generated Hausdorff spaces! We have two natural functors, one taking a simplicial set to its geometric realization, and another one taking a topological space to its singular complex.

2.4 Simplicial objects

Let Δ be the category consisting of objects $[n] = \{0, 1, \dots, n\}$, one for each non-negative integer n, and morphisms order-preserving maps. We call this category the simplicial category. The set of morphisms in Δ are generated by two classes of morphisms, face maps, $\delta_i^n : [n-1] \to [n]$ which is an injection that misses $i \in [n]$, and degeneracy maps $\rho_i^n : [n+1] \to [n]$ the surjection that repeats i, that is, $\rho_i^n(i) = \rho_i^n(i+1) = i$. If C is any category, a simplicial object with values in C is simply a functor $\Delta^{opp} \to C$. We have a category of simplicial objects with values in C, with morphisms being natural transformations. For a simplicial object X, $X(\rho_i^n) = \rho_n : X_n \to X_{n+1}$ and $X(\delta_i^n) = d_i^n : X_{n-1} \to X_n$. If $C = Set$, we call the category of simplicial objects simply simplicial sets, and the elements of $A([n]) \in Set$ for $A \in \Delta^{opp} \to Set$ for n-simplices. A certain kind of simplicial set will turn out to be very important later, for our study of simplicial homotopy. Let $\Delta[n]$ be the simplicial set, given by,for any $[m] \in \Delta$, $\Delta[n](m) = Hom_\Delta([m], [n])$. We will sometimes denote this by $\Delta^n(m)$. We say that a map of simplicial sets $f : A \to B$, is homotopic to a map $g : A \to B$ if there is a map of simplicial sets $F : A \times \Delta[1] \to B$, such that $F_0 = f$ and $F_1 = g$. We have a functor $|\cdot|$ from simplicial sets to the category of topological spaces, called the realization functor. It has a somewhat obtruse definition, but we shall try to elucidate this by giving an easy example.

Definition 25. Let $A : \Delta^{opp} \to Set$. The realization of A is

$$|A| = \varinjlim_{\Delta[n] \to A} |\Delta^n|$$

where each $|\Delta^n|$ the standard n-simplex in euclidean space. The colimit is taken over the category $\Delta[n] \to A$ consisting of maps $\Delta[n] \to A$ and morphisms between $f : \Delta[n] \to A$ and $g : \Delta[m] \to A$ are maps $h : \Delta[n] \to \Delta[n]$ such that $f = hg$.

It is then easy to see that for example, practically by definition, $|\Delta[n]| = |\Delta^n|$, the standard n-simplex.

We will now define the coskeleton of a simplicial object with values in C. Let C^Δ be simplicial objects with values in C. Then we have a k-th truncation functor $Tr_k : C^\Delta \to C^{\Delta_{n \le k}}$. Here $C^{\Delta_{n \le k}}$ denotes the full subcategory of C^Δ, where we simply "truncate" each simplicial object at the k-th simplex. If C is a category that has all finite inductive limits, this functor will have a right adjoint $cosk_k : C^{\Delta_{n \le k}} \to C^\Delta$, that is, we have $Hom_{C^{\Delta \le k}}(Tr_k(X), Y) \cong Hom_{C^\Delta}(X, cosk_n(Y))$. This functor can be constructed as the (right) Kan Extension of Tr_k. We have a left adjoint, provided C has all finite projective limits, constructed as the left Kan Extension of Tr_k called the skeleton, sk_k.

Let us spell out what this adjunction means in more concrete terms. It means that we have

$$Hom_{C^{\Delta}}(X, cosk_n Y) \cong Hom_{C^{\Delta^{\leq n}}}(sk_n X, Y).$$

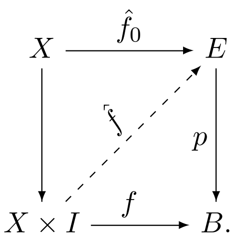
Let us assume that $C = Set$ for now. Then, the n-th skeleton, $sk_n(Y)$ of $Y \in Set^{\Delta^{\leq n}}$ is the simplicial set with no nondegenerate simplices of degree greater than n. By the Yoneda lemma, the n-simplices of a simplicial set X is given by $Hom_{Set^{\Delta}}(\Delta[n], X)$. We will compose the skeleton and coskeleton with the truncation functor, so that these both are morphisms from simplicial objects in C to simplicial objects in C. So, we see that since

$$Hom_{Set^{\Delta}}(\Delta^n, cosk_n X) \cong Hom_{Set^{\Delta^{\leq n}}}(Tr_n\Delta^n, Tr_n X) \cong Hom_{Set^{\Delta}}(sk_n\Delta^n, X)$$

, where the second isomorphism comes from the fact that $sk_n tr_n \Delta^n = sk_n \Delta^n$, the n-simplices of $cosk_n X$ are given by the maps $sk\Delta^n \to X$. We should think of the n-simplicies as follows: Every time we have a map from the skeleton of a n-simplex to X, there is an unique way to extend it to a map of all of the n-simplex to $cosk_n X$. Further, for $m > n$, it can be shown that the m-simplices are determined by their boundary.

2.4.1 Kan Fibration

Let us recall that for the category of topological spaces, a map $p : E \to B$ is called a fibration if for any homotopy $F : X \times I \to B$, and lift $\hat{f}_0 : X \to E$ of f_0, there exists a lifted homotopy $\hat{f} : X \times I \to E$ with $\hat{f}_0 = \hat{f}_{X\times 0}$. I.e, we can always find a dashed arrow in the diagram below:

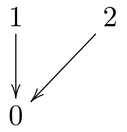


It turns out that to get a reasonable definition of homotopy in the category of simplicial sets we must restrict ourself to a certain kind of simplicial sets, called Kan complexes. If we do not restrict ourself to this subcategory, we cannot get a proper definition of homotopy groups. However, if we first define homotopy in the category of Kan complexes, there is a way to extend this to the whole of the category of simplicial sets. We will return to that soon. First, remember that by the Yoneda lemma, that maps $\Delta[n] \to X$ for X a simplicial set is in bijection with the n-simplices of X. We will write $\tau_{m,x}$ for the map representing the m-simplex labelled x.

Definition 26. Let $k \in [n]$. The k-th horn of $\Delta[n]$, $\wedge_k[n]$ is the smallest simplicial subset (i.e subfunctor) of $\Delta[n]$ containing all $d_i(id_n)$ for each $0 \leq i \leq n$, except $i = k$ where $id_n : \Delta[n] \to \Delta[n]$ is the identity map.

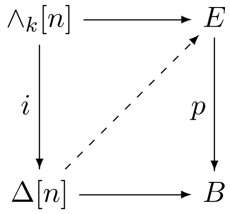
Note that $d_i(id_n)$ is indeed a n-1 simplex, since it is a map $\Delta[n] \to \Delta[n-1]$.

Example 27. *Let us take the k-th horn of* $\Delta[2]$. *Imagine that we have labelled the vertices (0-simplices) 0,1,2 in some order, and that we call* 01 *the 1-simplex such that* $d_0(01) = 1$ *and* $d_1(01) = 0$. *Then,* $\wedge_k[2]$ *can be seen to consist of all 0-simplices and 1-simplices of* $\Delta[2]$, *except the 1-simplex* $d_k(012)$, $0 \leq k \geq 2$. *The realization of for example* $\wedge_0[2]$ *can be visualized as follows*



i.e as a triangle but with the side 12 *removed. This clearly deformation retracts onto a point.*

Definition 28. A map of simplicial sets $p : E \to B$ is a Kan Fibration if for any $n \geq 1$ and commutative diagram

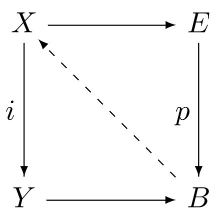


we have a dashed map of simplicial sets making each triangle commutative. We say that p has the right lifting property with respect to all inclusions $\wedge_k[n] \subset \Delta[n]$.

In some sense, every horn has a filler, meaning, that given a map onthe horn, we can extend it to the whole of $\Delta[n]$. We define a simplicial set S to be a Kan Complex if the map to the terminal object is a Kan fibration. A Kan Complex should be thought of as something analogous to a singular chain complex of a topological space.

Example 29. *Let* X *be a topological space, and define a singular n-simplex to be a continous map* $f : |\Delta^n| \to X$ *(where* $|\Delta^n|$ *is the realization of the standard n-simplex), and let* S_n *be all singular n-simplices, and set* $S = \amalg_n S_n(X)$, *with the usual face maps and degeneracy maps. We have that any map defined on* $|\wedge^k[n]|$ *extends to a map* $|\Delta^n|$, *since* $|\wedge^k[n]|$ *is a deformation retract of* $|\Delta^n|$. *Thus, singular chain complexes are Kan complexes.*

For further reference, we introduce the notion of a contractible Kan Object. Recall that we say that a map $p : E \to B$ has the right lifting property with respect to maps in $\mathscr{M}$ if for any commutative diagram



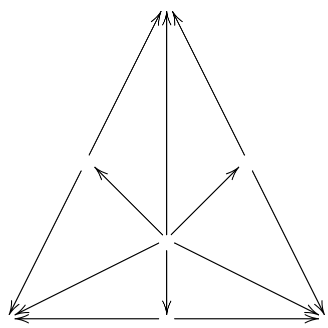
where $i \in \mathscr{M}$ there exists a dashed arrow making each triangle commutative.

Definition 30. We say that a map of simplicial sets $p : E \to B$ is an acyclic Kan Fibration if it has the right lifting property with respect to all boundary inclusions $\partial\Delta[n] \to \Delta[n]$. If B is the final object, we say that E is an contractible Kan complex.

It can be shown [Goerss-Jardine, I.7.10] that these induce isomorphisms on all simplicial homotopy groups and also are Kan Fibrations, justifying the terminology. To provide a good homotopy theory for the category of simplicial sets, Kan constructed a functor Ex^{∞}. Given **any** simplicial set X, $Ex^{\infty}(X)$ is a Kan complex. It will be a fibrant replacement functor, that is, a functor that takes an object and replaces it with a fibrant simplicial set, that is, so that it is Kan. To construct this functor, we need to introduce some concepts.

Definition 31. Let C be any locally small category. The nerve of C, $\mathscr{N}C$, is a simplicial set with $(\mathscr{N}C)_0 = Ob(C)$ and $(\mathscr{N}C)_1 = Mor(C)$,
$(\mathscr{N}C)_2 = \{$Pairs of composable morphisms $f : C_1 \to C_2, g : C_2 \to C_3\}$,
$(\mathscr{N}C)_k = \{$strings of length k consisting of composable morphisms$\}$.
The face maps $d_i : (\mathscr{N}C)_k \to (\mathscr{N}C)_{k-1}$ takes a string $C_0 \to C_1 \to \cdots \to C_k$ and composes the i:th and i+1th morphism, except for when $i = 0$ or
$i = n$, then it simply leaves out that arrow. The degeneracy maps
$s_i : (\mathscr{N}C)_k \to (\mathscr{N}C)_{k+1}$ adds the identity map to the i-th morphism, and we thus obtain a string of length k+1.

Example 32. *Let Δ^n be the standard n-simplex, and let us consider the non-degenerate simplices. They correspond to injective maps $[m] \to [n]$. We see that each choice of $m + 1$ elements of $[n]$ gives a non-degenerate simplex, and thus the non-degenerate simplices forms a poset $P\Delta^n$ ordered by inclusion. View this poset as a category, and form the nerve. We call the resulting category, $sd\Delta^n = \mathscr{N}P\Delta^n$.Let us explore the situation further for n = 2. Then the poset $P\Delta^2$ can be identified with all non-empty subsets of $\{0, 1, 2\}$. If we form the nerve, and draw it we can see that it resembles the barycentric subdivision of the triangle:*



In fact, the following is true:

Theorem 33. *The realization of $sd\Delta^n$ is homeomorphic to $|\Delta^n|$, the standard realization of the n-simplex by a homeomorphism taking $\{v_0, \ldots, v_k\} \in sd\Delta^n$ to the barycentre of the vertices.*

Proof. [4] 4.1. □

We now define subdivision for a general simplicial set X as

$$sdX = \varinjlim_{\Delta^n \to X} sd\Delta^n$$

indexed over the category $\Delta \to X$ defined previously in the context of realization. Now define $Ex(X)$ to be the simplicial set with n-simplices given by the set $Hom(sd\Delta^n, X)$. We have that Ex is a right adjoint to sd. We can see that it hold for standard simplices of the form Δ^m, since then we have that $Hom_{Sset}(\Delta^m, X) \cong X(m)$. So,

$$Hom_{Sset}(sd\Delta^m, X) \cong Ex(X)_m = Hom_{Sset}(\Delta^m, Ex(X))$$

as claimed, and this is clearly a natural isomorphism. Thus, for general X and Y

$$\begin{aligned} Hom_{Sset}(sdX, Y) = Hom_{Sset}(\varinjlim_{\Delta^n \to X} sd\Delta^n, Y) &\cong \varprojlim_{\Delta^n \to X} Hom_{Sset}(sd\Delta^n, Y) \\ \cong \varprojlim_{\Delta^n \to X} Ex(Y)(n) &= \varprojlim_{\Delta^n \to X} Hom_{Sset}(\Delta^n, ExY) \\ = Hom_{Sset}(\varinjlim_{\Delta^n \to X} \Delta^n, ExY) &= Hom_{Sset}(X, ExY). \end{aligned}$$

We have here used the fact that $\varinjlim_{\Delta^n \to X} \Delta^n \cong X$, and this is standard, since any presheaf defined on a small category is the colimit of representable functors. That aside, we have a last vertex map $lv : sd\Delta^n \to \Delta^n$ that is induced by the map of posets $s : P\Delta^n \to [n]$, $s(v_0, \ldots, v_n) = v_n$. By going to the colimit, we get $lv : sdX \to X$ for any simplicial set, and by adjointness, $e_X : X \to Ex(X)$. We have a functor F from the directed category $\mathbb{N}$ associated to the poset $(\mathbb{N}, \leq)$, where $F(n) = Ex^n(X)$, and $F(n \to n+1) = e_{Ex^n(X)} : Ex^n(X) \to Ex^{n+1}(X)$, and

$$F(m \to n) = e_{Ex_{n-1}(X)} \circ e_{Ex_{n-2}(X)} \circ \cdots \circ e_{Ex_m(X)}.$$

We define $Ex^\infty(X)$ as the colimit of this functor, and we get a functor $Ex^\infty : Sset \to Sset$.

Theorem 34. *For any simplicial set X, $Ex^\infty(X)$ is a Kan Complex and it preserves Kan Fibrations.*

Proof. [4] 4.8 □

Definition 35. Let $f, g : X \to Y$ be maps of simplicial sets. We say that f is simplicially homotopic to g if there is a map $F : X \otimes \Delta[1] \to Y$ such that the two restrictions of F to $X \otimes \Delta[0]$ is f resp. g.

We will now define a model categorical structure on the category of simplicial sets. The weak equivalences are the one that turns into weak equivalences in the category of topological spaces when we pass to the geometrical realization. The cofibrations are monomorphisms $f : X \to Y$ such that for each n $f : X_n \to Y_n$ is injective, and the fibrations are the Kan Fibrations. Quillen proved that this defines a model category (Homotopical Algebra, Quillen) where the tricky part is not really showing that we can turn Sset into a model category, but that the fibrations really are the Kan fibrations.

Definition 36. $\mathscr{H}$S, the extended homotopy category of simplicial sets have objects simplicial sets, and

$$Hom_{\mathscr{H}S}(X,Y) = [Hom_{Sset}(Ex^{\infty}X, Ex^{\infty}Y)]$$

where we by [] mean simplicial homotopy classes of maps.

3 Étale homotopy

We will describe and define Artin-Mazur's Étale Homotopy type of a locally noetherian scheme. The étale homotopy type contains a fantastic amount of detail, amongst other things, it contains all the information needed to compute its étale cohomology with certain restriction on coefficients. If X is a locally noetherian scheme, every scheme Y étale over X is a finite disjoint union of connected schemes. Associated to each such Y, we have a set $\pi_0(Y)$, consisting of the set of connected schemes making up Y. To get the étale homotopy type we will apply this functor π_0 to a certain class of coverings of X, called hypercoverings and from it derive a pro-object. The topological realization of this pro-object is the étale homotopy type of X.

3.1 Grothendieck topologies and sites

Grothendieck generalized the notion of a topology to categories. His generalization is as elegant as it is simple. Instead of focusing on the individual open sets, what is important is when something is covered or not. Let c be an object of the category C. A subfunctor $S \subset Hom_C(-,c)$ is a sieve on c. Each sieve on c can also be given as a collection of morphisms with codomain c such that this collection is closed under precomposing with morphisms in C. Much of the material here is my attempt to shorten the material in [8] and we refer the reader to it for more details.

Definition 37. A Grothendieck topology on a category C is a collection of sieves for each object, called covering sieves, one set of covering sieves for each $c \in C$, and we denote the covering sieves of c by $J(c)$. We require the covering sieves to satisfy the following properties:
(i) The maximal sieve $Hom_C(-,c)$ is a covering sieve of X for any object c..
(ii) If $S \in J(c)$ and $h : d \to c$ is a morphism, then the pullback $h^*(S) \in J(d)$.

(iii) Let $S \in J(c)$ and R be any sieve on C. If for each $h : d \to c$ in S, $h^*(R) \in J(d)$, then $R \in J(c)$.

Example 38. *Let us consider a topological space X, and let $C = Op(X)$, the category of open sets where morphisms are inclusions. Then a sieve on $U \in Op(X)$ is a family of open sets T such that if $T'' \subset T'$ and $T' \in T$, then $T'' \in T$. We say that a sieve is a covering sieve of U exactly when the union of all open sets in the sieve covers U, in the usual topological sense. Let us check that this definition satisfy the above axioms.*
(i) The maximal sieve corresponds to the set of all open subsets in U, since $Hom(-, U)$ can be identified with all open subsets contained in U. These clearly cover U. So the maximal sieve is a covering sieve.
(ii) The pullback is simply intersection here, so if T is any family of open sets covering U, that is, $U \subset \cup_{W \in T} W$ and $V \subset U$, then clearly $V \subset \cup_{W \in T} W \cap V$.
(iii) Let S be a covering sieve of U and R any sieve on U. Let us assume that for all $V \in S$, $V \subset \cup_{W \in R} W \cap V$. It is then clear that since the union of all open V in S covers U, and R covers all open in U, R must also cover U.

If a category C has fibered products there is a very convenient way of describing a basis that generates a Grothendieck topology.

Definition 39. Let $f : X \Rightarrow Z$ and $g : Y \to Z$ be morphisms in a category C. Then, the fiber product $X \times_Z Y$ (if it exists) is an object fitting into a commutative diagram

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{p_2} & Y \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array}$$

such that for any other object Q fitting into a commutative diagram

$$\begin{array}{ccc} Q & \xrightarrow{q_2} & Y \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array}$$

there is a unique morphism $s : Q \to X \times_Z Y$ such that $p_2 \circ s = q_2$ and $p_1 \circ s = q_1$.

Definition 40. A basis for a Grothendieck topology on a category C with fiber products is a collection of families of morphisms, one family of morphisms for each $c \in C$ denoted $K(c)$ called covering families with the property that all morphisms in $K(c)$ has codomain c. These collections are required to satisfy the following axioms:

(i) $K(c)$ contains all isomorphisms, more precisely, if $f : c' \to c$ is any isomorphism, then $\{f\} \in K(c)$.
(ii) If $\{f_i : c_i \to c | i \in I\} \in K(c)$, then for any $g : d \to c$, $\{c_i \times d \to d\} \in K(d)$ where $c_i \times d \to d$ is the canonical projection coming from the fiber product.
(iii) If $\{f_i : c_i \to c | i \in I\} \in K(c)$, and if for each i we have a covering family $\{g_{ij} : d_{ij} \to c_i | j \in I_i\} \in K(c_i)$ then the family of composites $\{f_i \circ g_{ij} : d_{ij} \to c | i \in I, j \in I_i\} \in K(c)$.

We call a category C with a Grothendieck topology J or a basis for a Grothendieck topology K for a site. Associated to each category C with a Grothendieck topology we have a category of sheaves on C, $Sh(C)$. It would be fair to say that the use of sheaves have been central throughout 20th century mathematics and continue to be to this day.

Definition 41. Let C be a small category, J a Grothendieck Topology on C and $P : C^{op} \to Set$ a presheaf (i.e a contravariant functor from C to Set) on C. Then, we say that P is a sheaf (of sets) on C if for every $X \in Ob(C)$ the diagram

$$P(X) \xrightarrow{i} \Pi_{(f:U_i \to X) \in S} P(U_i) \rightrightarrows \Pi_{(f:U_i \to X) \times (g:V \to U_i), f \in S, g \in Mor(C)} P(V)$$

is an equalizer of sets. The product in the second is thus taken over all pairs satisfying the condition. The first map i takes $x \in P(X)$ to $(P(f)(x))_f \in \Pi_{(f:U_i \to X) \in S} P(U_i)$. The upper of the stacked arrows takes $x_f \in \Pi_{(f:U_i \to X) \in S} P(U_i)$ to

$$(x_{fg}) \in \Pi_{(f:U_i \to X) \times (g:V \to U_i), f \in S, g \in Mor(C)} P(V).$$

This is sensible since $x_{fg} \in S$. The lower of the stacked arrows takes x_f to

$$P(g)(x_f) \in \Pi_{(f:U_i \to X) \times (g:V \to U_i), f \in S, g \in Mor(C)} P(V)$$

where g varies over all maps $g : V \to U_i$.

We denote by $Sh(C)$ the category of sheaves on C, where morphisms are natural transformations. We call any category of the form $Sh(C)$ for C any site a Grothendieck topos.

Example 42. *If X and Y are topological spaces, the category of sheaves $Sh(X)$ and $Sh(Y)$ are Grothendieck topoi. Further, if we have a continuous map $f : X \to Y$ it gives us a morphism $f : Op(X) \to Op(Y)$. We have two associated functors $f_* : Sh(X) \to Sh(Y)$ and $f^* : Sh(Y) \to Sh(X)$, called the direct image functor and inverse image functor. f_* right adjoint to f^*. Further, f^* is left exact.*

With this example as motivation, it would of course be natural to consider how we should capture a notion of a morphism of sites. We want to model it on the case for topological spaces, and we see that it has the property that for any sheaf $S \in Sh(X)$, $f_*(S)$, the presheaf on Y defined by, for $V \in O(Y)$, $f_*(S)(V) = S(f^{-1}(V))$ actually is a sheaf. We call a functor $F : C \to D$ between sites a continuous functor if for any $S \in Sh(D)$, the presheaf $F(S)$ on C given by $F(S)(c) = S(F(c))$ is a sheaf. We thus have that F induces a functor $\hat{F} : Sh(D) \to Sh(C)$ and it will have a left adjoint, $F^* : Sh(C) \to Sh(D)$ given by, for $X \in Sh(C)$ the left Kan Extension of X along $\hat{F}$. The adjunction is thus

$$Hom_{Sh(C)}(X, F_*(Y)) \cong Hom_{Sh(D)}(F^*(X), Y).$$

Definition 43. A continuous functor $F : C \to D$ is a morphism of sites from D to C if $F^* : Sh(C) \to Sh(D)$ preserves all finite limits.

Definition 44. A geometric morphism $f : Sh(C) \to Sh(D)$ between Grothendieck topoi is an adjoint par (f^*, f_*) such that f^* is left exact. We call f_* the direct image of f and f^* the inverse image of f.

Let X denote a fixed scheme. We have the usual Zariski topology on X, but this is for most cases much too coarse to actually capture subtle geometric properties of the scheme, since the open sets are so "large". The small étale site remedies this by replacing them with étale open sets, which should be thought of as somewhat akin to local isomorphisms in the complex analytic case. Grothendieck first used the (small) étale site to define étale cohomology.

Definition 45. With X as above, the site $X_{\text{ét}}$ consists of a category, with objects schemes with an étale morphism $f : U \to X$ and morphisms the obvious one.We have a basis for a Grothendieck topology, given by families of étale morphisms $\{f_i : U_i \to X\}$ such that $\cup_i im f_i = X$.

Given a presheaf $F : C^{op} \to Set$, it is natural to ask if we can make this into a sheaf in some natural manner. The answer is yes:

Theorem 46. *Let C be a Grothendieck site and F a presheaf on C. Then there is a sheafication functor $Sh : Presh(C) \to Sh(C)$ which is left adjoint to the natural inclusion functor $i : Sh(C) \to Presh(C)$.*

Proof. [8] III.5. □

3.2 Etale morphisms

We will here briefly remind the reader on some classes of morphisms from algebraic geometry of particular importance for our purposes. We will assume that the reader is familiar with the basic notions of scheme theory.

Definition 47. Let $f : X \to Y$ be a morphism of schemes. We say that f is unramified if it is locally of finite presentation and if for all x in X, the map $\mathcal{O}_{Y,f(x)}/\eta \to \mathcal{O}_{X,x}/m$, where m and η are the maximal ideals of $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,f(x)}$ respectively, is a finite and separable field extension.

Definition 48. Let $f : X \to Y$ be a morphism of schemes. We say that f is an étale morphism if it is unramified and flat.

Example 49. *If K and L are fields, and L is a finite field extension of K, the morphism* $f : SpecL \to SpecK$ *is étale iff L is a separable field extension of K.*

Example 50. *Let us consider one of the easiest examples of an étale morphism. Let k be a field and* $f(x)$ *a monic irreducible polynmial and set* $X = speck[x]/(f(x))$*. The canonical ring homomorphism* $k \to k[x]/(f(x))$ *induces a morphism of schemes* $X \to speck$ *which is étale iff f is a separable polynomial. Generalizing the argument, it is easy to see that if* $f()$ *is a monic polynomial with coefficients in k, the induced maps* $X \to speck$ *is étale iff its irreducible factors only occur with multiplicity one and all are separable polynomials.*

We will stop here, and refer the reader to [6] for a more detailed treatment.

3.3 Profinite completion of spaces

To later be able to fully understand Artin-Mazur's comparison theorem for étale homotopy, we will have to develop the basics of Profinite homotopy theory. The reader is referred to [2] for proofs.

Definition 51. (A class of groups) A class $\mathcal{C}$ of groups is a full subcategory of *Grp* such that:
(i) The trivial group is in $\mathcal{C}$.
(ii)A If $G \in \mathcal{C}$ and $H \subset G$ is a subgroup, then $H \in \mathcal{C}$. Further, if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence of groups, $B \in \mathcal{C}$ if and only if $A, C \in \mathcal{C}$.

Example 52. *Consider the* $\mathcal{C}$ *to be the full subcategory consisting of finite groups. Then clearly the trivial groups in* $\mathcal{C}$ *and any subgroup of a finite group is finite. The condition on short exact sequences follows from for example the first isomorphism theorem.*

Example 53. *Let* $\mathcal{C}$ *consist of finite groups with cardinality a power of p for p a prime. I claim that this is a class of groups. Indeed, the trivial group is in* $\mathcal{C}$ *and by Lagrange's theorem, the order of every subgroup H of a group of cardinality a power of p also has cardinality a power of p. Lastly, if we have a short exact sequence as above, then, denoting by # cardinality,* $\#C = \#B/\#A$ *so (ii) clearly holds.*

We want a condition so that if $G \in \mathcal{C}$ and A an abelian G-module such that $A \in \mathcal{C}$, then the cohomology groups $H^q(G, A)$ stay in $\mathcal{C}$. The following ensures this:

Definition 54. Let $\mathcal{C}$ be a class of groups. We say that $\mathcal{C}$ is a complete class of groups if for $A, B \in \mathcal{C}$ the product $\Pi_{b \in B} A_b \in \mathcal{C}$ where A_b means a copy of A indexed by b.

Example 55. *Take finitely generated abelian groups to be* $\mathcal{C}$*. This is not a complete class, since* $\Pi_{a \in \mathbb{Z}} \mathbb{Z}_a$ *is not a finitely generated abelian group. However, it is clear that the previous two examples are complete.*

Now, I will briefly explain why if a class of groups is complete, the cohomology groups as mentioned above stays in the class. This follows instantly from that the cochain groups C_q given in the group cohomology case are $C_q = A^{G\times\cdots\times G}$ which will be in $\mathcal{C}$, since we assumed it to be complete. Now, we have that Z_q, the class of cycles is a subgroup of C_q and further, the class of B_q is also a subgroup of C_q so by (ii) it is in $\mathcal{C}$ and from the obvious short exact sequence

$$0 \longrightarrow B_q \longrightarrow Z_q \longrightarrow H^q(G,A) \longrightarrow 0$$

(ii) gives that $H^q(G,A)$ is in $\mathcal{C}$.

Definition 56. Let $\{G_i\}$ be an inverse system of groups, that is, a functor $F : I \to Grp$ such that I is small and cofiltered. Set $G = \varprojlim_{i\in I} G_i$. A group is called a profinite group if it is isomorphic to a group arising as the inverse limit of a system of groups in this way.

Example 57. *Let us consider the inverse system $G_m = \mathbb{Z}/p^m\mathbb{Z}$, $m \in \mathbb{N}$, and if $n > m$ homomorphism $G_n \to G_m$ given by the canonical quotient map.Taking the inverse limit we obtain the p-adic integers.*

Now, let $\mathcal{C}$ be a class of groups and G a profinite group. Consider now the category consisting of maps $G \to H$ where $H \in \mathcal{C}$ and H varies through $\mathcal{C}$, and morphisms are commutative triangles.We can by (ii) check that the opposite category is cofiltering and thus, we can take the inverse limit. We call this object $\hat{G}$ the profinite completion of G in $\mathcal{C}$. This gives us a pro-object in $\mathcal{C}$-groups with a canonical map $G \to \hat{G}$ universal with respect to maps from G into pro-$\mathcal{C}$-groups.

Example 58. *Let G be any group and $\mathcal{C}$ the class of finite groups. The above category will then consist of maps $G \to H$, where H is a finite group. It can easily be seen that for computing the inverse limit in the opposite category, it is enough to consider the maps where $\varphi : G \to H$ are surjective. This means that $G/ker\varphi \cong H$, and conversly, for every normal subgroup N such that G/N has finite index, we obtain a map. This gives us that that the profinite completion of G in the class of finite groups is the inverse limit of G/N, where N runs through all normal subgroups of finite index in G and the maps between are the canonical ones.*

Artin-Mazur constructs an analogue of this for CW-complexes, but we can equivalently use it for realizations of simplicial sets. Namely, let $\mathcal{C}$ be a class of groups and we call a simplicial set X pointed if there is a choice of basepoint in X, and a morphism between simplicial sets is pointed if it sends the distinguished basepoints to the distinguished basepoint. We have an obvious category of pointed simplicial sets, and we can restrict it to a full subcategory consisting of simplicial sets X such that the homotopy groups of the realizations of X is in $\mathcal{C}$. Call this category $\mathcal{C} - \mathcal{H}_\bullet$. They then show that if X is a pro-simplicial pointed set, meaning its an inverse limit of an inverse system of simplicial pointed sets, then there is an object $\hat{X} \in$ Pro-$\mathcal{C} - \mathcal{H}_\bullet$ with a map $X \to \hat{X}$ which is universal with respect to maps from X into objects of Pro-$\mathcal{C} - \mathcal{H}_\bullet$ .We call this pro-simplicial set the $\mathcal{C}$-completion of X. The construction is messy and we refer the reader to Artin-Mazur, 3.4.Intuitively we get $\hat{X}$ by ignoring all facts about X except those concerning maps into objects of $\mathcal{C} - \mathcal{H}_\bullet$.

Obviously, we have a notion of homotopy groups for the $\hat{X}$, the $\mathcal{C}$-completion of X. Let us assume that $\mathcal{C}$ is a complete class, now and $X = \{X_i\}$ a pro-simplicial set. We have a map $X \to \hat{X}$ and this induces a map $\pi_n \hat{(X)} \to \pi_n(\hat{X})$ for each n, where we by $\pi_n \hat{(X)}$ mean the $\mathcal{C}$ completion of the inverse system of groups $\pi_n(X) = \{\pi_n(X_i)\}$ .If $n = 1$ then it can be shown (see Artin-Mazur) that $\pi_1(\hat{X}) \cong \pi_1 \hat{(X)}$. More generally, they prove the following

Theorem 59. *Let X be a pro-simplicial space and suppose that $\pi_q(X) = 0$ for $1 \leq q < n$.Then $\pi_n \hat{(X)} \cong \pi_n(\hat{X})$.*

We will use this material in the section on étale homotopy, namely, it will play a crucial part in understanding Artin-Mazur's comparison theorem. The moral behind why profinite completions are important is that when analyzing the homotopy type, the information we are looking for can be quite hard to obtain. Going to the profinite completion may sometimes make this information more accessible.

3.4 Hypercoverings

Let C be a site with all finite products and coproducts, and fiber products, assume further that all representable presheaves are actually sheaves under the Grothendieck topology of C. This is the case for many different topologies, for example, the Zariski topology and étale topology. It can be shown that each category can be given a finest topology such that all representable presheaves are sheaves. We want to define the notion of a pointed site. A point of a site C is simply a morphism of sites $Set \to C$. We then say that a pointed site is a site C with a choice of a fixed morphism of sites $Set \to C$.

Definition 60. Let C be a (pointed) site. A hypercovering of C is a simplicial object K with values in C such that
(i) If e is the terminal object of C, then the map $K_0 \to e$ is a covering.
(ii] For every $n \geq 0$, the canonical morphism $K_{n+1} \to (Cosk_n K)_{n+1}$ is a covering.

We say that K covers an object X if K is a hypercovering of the category C/X which has as objects morphisms in C with codomain X and morphisms between objects of C/X are compatible maps. A morphism of sites clearly preserves hypercoverings. The definition of a hypercovering might at first seem rather obtruse, but in some sense we could see it as a generalization of a Cech complex in the following sense. For an open cover $U = \{U_i\}$ of a topological space X the Cech complex first takes as 0-simplicies all open sets in U, and 1-simplicies all non-empty intersections $U_i \cap U_j$, and 2-simplicies all

non-empty triple intersections and so on. We want to believe if that we chose a nice enough open covering of a topological space X, then the cohomology of the Cech complex should be the same as that of the original space. However, in many cases, say Cech étale cohomology, doesn't agree with the usual étale cohomology of a scheme. To remedy this we have hypercoverings. Let U be a hypercovering of X. Then:
(i) U has 0-simplicies given by the open sets forming an open cover of X, as before. So U_0 is an open cover of X.
(ii) U_1 is an open covering of the double intersections of open sets in U_0.
(ii) U_2 is an open covering of the non-empty triple intersections of open sets in U_1 .
(iii) U_3 is an open covering of the non-empty quadruple intersections of the opens in U_2, etc.
So we allow for a greater degree of freedom, and it is a theorem of Verdier [1] (2, 7, Appendix), that the étale cohomology of the hypercovering is the same as that of the original scheme.

Example 61. *Let $C = Set$ considered as a site with coverings as surjective families of maps. A hypercovering of $\{p\} \in Set$, a terminal object of Set, is is a simplicial set K such that $K_0 = \{f_i : k_i \to p\}$ covers the terminal object, i.e it's non-empty. The requirement that $K_{n+1} \to (cosk_n K.)_{n+1}$ is a covering is equivalent to the fact that for each map $g \in Hom_{Sset}(sk_n \Delta^{n+1}, K)$ we have that there is at least one $f \in Hom_{Sset}(\Delta^{n+1}, K)$ such that the image of it under the canonical map $sk_n \Delta^{n+1} \to \Delta^{n+1}$ is g. This since the condition that $K_{n+1} \to (Cosk_n K.)_{n+1}$ is a covering (i.e surjective), gives, from the adjunction properties that the map*

$$Hom(\Delta[n+1], K) \cong K_{n+1} \to Hom(sk\Delta[n+1], K) \cong (cosk_n K)_{n+1}$$

is surjective. This translates to the fact that every map

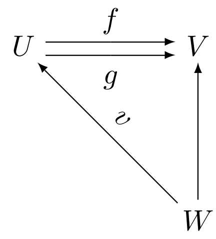
$$sk\Delta[n+1] = \partial\Delta[n+1] \to K$$

there is at least one map $\Delta[n+1] \to K$ inducing it. This shows that K is an acyclic Kan complex, since by what we just have said K has the right lifting property with respect to the boundary inclusions.

We have an associated category of hypercoverings, with objects hypercoverings and morphisms between them. Let X now be a locally noetherian scheme, then $\acute{\mathrm{E}}\mathrm{t}(X)$ is locally connected. We have a functor $\pi : \acute{\mathrm{E}}\mathrm{t}(X) \to Set$ which associates the set of connected components to an object of Ét(X). For each hypercovering U of X we have an object $\pi(U) \in Sset$. We call the category of étale hypercoverings of X for $HR(X_{\grave{\mathrm{E}}\mathrm{t}})$. If we applied the functor π to all hypercoverings of X, we would like to view them as a pro-objects, analogous to gluing them together. However, the category of étale hypercoverings of X is not cofiltering, a major problem to be overcome. Artin-Mazur overcomes this by passing to the homotopy category of étale hypercoverings, $HC(X_{\grave{\mathrm{E}}\mathrm{t}})$.

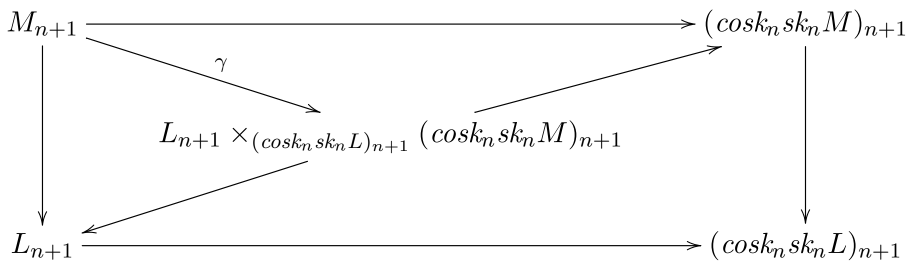
Theorem 62. *Let X be a locally noetherian scheme. Then the category $HC(X_{\grave{E}t})$ is cofiltering.*

Proof. Let $U \to X$ and $V \to X$ be hypercoverings. We form the hypercover $U \times_X V \to X$ which is for n, $U \times_X V_n \to X$ The fact that this is an étale hypercovering follows from that the property of being surjective, étale are stable under base change and composition. It is easy to see that this hypercovering dominates both U and V. Further, it is clearly non-empty. So we are only left to show that if $f, g : U \rightrightarrows V$ are morphisms of hypercoverings of X, there is a hypercovering W of X such there exists $v : W \to U$ such that



commutes up to simplicial homotopy, that is there is a homotopy between $f \circ v$ and $g \circ v$. To achieve this, we will need some auxillary lemmas.

Lemma 63. *Let C be any site with fibre products, $X \in Ob(C)$. For simplicial objects K,L,M with values in C, let us assume that:*
(i) K is a hypercovering of X
(ii) $M_0 \to L_0$ is a covering
(iii) For all n ≥ 0, the arrow γ in the diagram



is a covering.
Then the fiber product $K \times_L M$ is a hypercovering of X.

Proof. [13] 20.4.1 ☐

Lemma 64. *Let C be a site with fiber products, X an object of C, and L a simplicial object with values in C. Let $n \geq 0$ and consider the commutative diagram*

$$\begin{array}{ccc} Hom(\Delta[1], L)_{n+1} & \longrightarrow & (cosk_n sk_n Hom(\Delta[1], L))_{n+1} \\ \downarrow & & \downarrow g \\ (L \times L)_{n+1} & \xrightarrow{f} & (cosk_n sk_n (L \times L))_{n+1} \end{array} \qquad (65)$$

coming from the morphisms $e_i : \Delta^0 \to \Delta^1$, $i = 0, 1$. Then the fiber product of f and g is equal to $Hom(U, L)_0$ where $U \subset \Delta[1] \times \Delta[n+1]$ is the smallest simplicial subset such that both $\Delta[n+1] \amalg \Delta[n+1]$ and $\Delta[1] \times \partial\Delta[n+1]$ map into it.

Proof. [13] 20.6.1 □

With these two technical lemmas we can finally prove the last part. Consider the diagram

$$\begin{array}{ccc} U \times_{V\times V} \mathrm{Hom}(\Delta^1, V) & \longrightarrow & \mathrm{Hom}(\Delta^1, V) \\ \Big\downarrow v & & \Big\downarrow \\ U & \xrightarrow{\quad f,g \quad} & V \times V. \end{array}$$

The horisontal morphism $U \times_{V\times V} \mathrm{Hom}(\Delta^1, V) \to \mathrm{Hom}(\Delta^1, V)$ is equivalent to a morphism $(U \times_{V\times V} \mathrm{Hom}(\Delta^1, V))(\Delta^1) \to V$,which gives a simplicial homotopy between $f \circ h$ and $g \circ h$, so we are done if we can show that $U \times_{X\times X} \mathrm{Hom}(\Delta^1, X)$ is a hypercovering. We use the lemma 4.11 for this.
1) is satisfied (U is by hypothesis a hypercovering of X) .
2) is true since $Hom(\Delta[1], V)_0 = V_1$ and $(d_0^1, d_0^1) : L_1 \to L_0 \times L_0$ is a covering, since L is one (remember the condition on coskeleton). Finally, for 3) we use lemma 4.12, this gives that γ in the first lemma is the morphism $Hom(\Delta[1] \times \Delta[n+1], V)_0 \to Hom(U, L)_0$ with U as in the 4.12 . But this is a covering (see Stacks 5.2). □

As we have seen, to show that the homotopy category of hypercoverings is cofiltered is not by any means easy. We will later see another way to prove that the category is cofiltering.

Definition 66. We say that a category C is distributive if:
(i) C has finite fiber products.
(ii) C has an initial object.
(iii) For every family of objects X_α such that $\amalg X_\alpha$ exists and any family of morphisms $X_\alpha \to Y$ and for any $X \to Y$, the canonical map $\amalg X_\alpha \times_Y X \to X \times_Y (\amalg X_\alpha)$ induced by the universal properties is an isomorphism.

Definition 67. Let C be a distributive category. We say that an object $X \in C$ is connected if it is not the initial object and has no non-trivial coproduct decomposition, i.e if $X = X_1 \amalg X_2$ implies that $X_i = \emptyset$ for some i. We say that a distributive category is locally connected if every object can be decomposed into a coproduct of connected objects.

Definition 68. Let C be a locally connected category. The Verdier functor $\Pi : C \to Set$ takes an object $X \in C$ to the set of connected components of X, and maps between objects into maps between the connected components in the obvious way.

3.5 The étale homotopy type

Definition 69. Let X be a locally Noetherian scheme. The étale homotopy type of X, $\acute{\mathrm{E}}\mathrm{t}(X)$ is a pro-object in the homotopy category of simplicial sets, defined by Ét(X) = $\Pi HR(X_{\acute{e}t}) = \{\pi_0(U)\}_{U \in HR(X)}$.

Taking the geometrical realisation of the étale homotopy type, we can define homology groups and homotopy groups. This has the nice application that for any abelian group A, if we form the locally constant sheaf $\underline{\text{A}}$ and consider cohomology, $H^q(\underline{\text{A}}(K)) = H^q(\pi(K), A)$ for any hypercovering K. Combining this with that we have a canonical isomorphism for any pointed site between $H^q(C, \underline{\text{A}})$ and $\varinjlim_K H^q(\underline{\text{A}}(K)) = \varinjlim_K H^q(\pi(K.), A)$ (Verdier's hypercovering theorem) we can achieve some surprising results. The latter is isomorphic as a pro-object to ΠX, thus we see that we can compute any cohomology with locally constant coefficients through the geometrical realization of this homotopy type! How lovely this homotopy type though may seem, it is lacking in many regards. First and foremost, it is hard to find any suitable model categorical structure in this construction, thus not permitting us to use a variety of powerful tools. Also, the étale homotopy type is constructed in an ad-hoc fashion, and can be hard to compute.

Example 70. *Let $C = G - set$ be the site of left G-sets, coverings surjective maps. We have a morphism of sites $Set \to G - set$ given by the forgetful functor. There is a terminal hypercovering K, $K_n = G \times n\ timesG$. So in this case, $\Pi C \cong \pi(K)$. Remember that an object X of a category (with coproducts) is connected if it has no non-trivial decomposition as a coproduct and it is not the initial object. Every G-set decomposes (by standard group theory) as a coproduct of transitive G-sets which is the quotient of G by some subgroup. Following these ideas we see that $\pi(K)$ is isomorphic to the orbit space of K under the obvious action . This is exactly how Eilenberg-Maclane spaces $K(G, 1)$ are most commonly constructed [Milnor, Construction of Universal Bundles II] . We will in brief explain how to construct $K(G, 1)$, and how to see that it really is $K(G, 1)$. Remember that a space X is $K(G, 1)$ if its only non-zero homotopy group is $\pi_1(X) = G$. Now, if we consider the simplicial set EG such that $EG_n = G \times Gn\ times \times G$, with face and degeneracy maps $d_i(g_1, \ldots g_{n+1}) = (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1})$ and $s_i(g_1, \ldots, g_{n+1}) = (g_1, \ldots, g_{i-1}, e, g_i, \ldots, g_{n+1})$ for $1 \leq i \leq n$. There is an action of G on EG, taking $(g_1, \ldots, g_n)$ to $(gg_1, \ldots, gg_n)$ for $g \in G$, and it is easy to see that this action is free, and further, EG is contractible (note that it is isomorphic to $cosk_0(G)$ which we showed to be contractible and Kan).Let $BG = EG/G$ be the orbit space, with $\pi_1(BG) = G$ and since $EG \to BG$ is a fibration with fiber G with the discrete topology, there is a long exact sequence of homotopy groups which shows that $\pi_i(BG) = 0$ for $i > 1$.*

The étale homotopy type allows us to define étale homotopy groups. It is easy to see that the construction of the étale homotopy type of a locally noetherian scheme has an analogous definition for a locally noetherian scheme with a choice of a geometric point $x \in X$, so that $(X_{\grave{\text{Et}}}, x)$ becomes a pointed site. We have a category of pointed hypercoverings and the associated homotopy category is cofiltering, so we can form the pointed étale homotopy type for X, we call it $(\Pi X, x)$. Then, with a geometric point, we can define a pro-group, by $\pi_n((\Pi X, x)) = \{\pi_n(\pi_0(U)\}_{U \in HR(X,x)}$.

Example 71. *Take as above C to be the site of left G-sets and the étale homotopy type, which we showed to be isomorophic to $K(G, 1)$. So here $\pi_1(C) = \pi_1(K(G, 1)) = G$ and $\pi_m(C) = 0$ for $m \geq 2$.*

Example 72. *Let k be a field and let G be the absolute galois group of k. We know that the étale site of k is equivalent to the category of left G-sets, and so our above computation gives that $\acute{E}tk \cong K(G,1)$, so that $\pi_1(K(G,1)) = G$ and $\pi_m(K(G,1)) = 0$ for $m > 1$.*

This étale homotopy type is in general, very hard to compute. However, there are some theorems that in some cases make the object tractable for computation. We need to introduce more terminology for this however.

Definition 73. Let X be a connected and locally connected topological space with a choice of a basepoint $x \in X$.Let C_{ord} be the site with objects coproducts of open sets in X, with the following Grothendieck topology : A covers B iff $B \subset A$ as open sets.

Definition 74. We see that this is a locally connected category, meaning that we can apply ΠX and we get by this, homotopy pro-groups for X (actually, for the category C_{ord}), we call $\pi_q(C_{ord})$ simply $\breve{\pi}_q(X)$. The homotopy pro-groups of C_{ord} will have a close connection with the étale homotopy type of X, if X is a scheme of finite type over the complex numbers, as we will see in a few paragraphs.

Let $S_\bullet X$ denote the simplicial set such that $S_q X$ consists of singular q-simplices, i.e continuous maps $\Delta^q \to X$ (where Δ^q is the standard q-simplex). Let X be a topological space. We say that X is paracompact if every open cover $\{U_\alpha\}$ has a refinement $\{V_\beta\}$ of open sets such that this refinement satisfies the following: For every point $x \in X$ there is a neighborhood W of x such that W only has finitely many non-empty intersections with the open sets in $\{V_\beta\}$.

Example 75. *Any compact space is paracompact, obviously, since a finite refinement $\{V_\beta\}$ only have finitely many open sets.*

Example 76. *Let X be any infinite set and let us define a topology called the particular point topology. Choose some $x \in X$ We say that $U \subset X$ is open iff either $U = \emptyset$ or $x \in U$. I claim that this is not paracompact. Indeed, let $\{U_\alpha\}$ be the open cover $\{x, p\}$, $p \in X$. Now, this open set has no open refinement except itself, and p lies in every open set of the covering. Thus, it is not paracompact.*

Theorem 77. *Let X be a connected pointed topological space,and assume that every open subset of X is paracompact and that X is locally contractible, meaning that every point contains arbitrarly small neighborhood that are contractible. Assume further that $U_\bullet$ is a hypercovering of the site C_{ord} as previously mentioned such that for every $q \in \mathbb{N}$, the connected components of U_q are contractible. Then, the simplicial set $\pi(U_\bullet)$ is canonically homotopic to $S_\bullet X$.With this, it follows that $\Pi X \cong S_\bullet X$ and that $\pi_q(X) \cong \breve{\pi}(X)$.*

It is proven in [7] that if X is a connected, pointed scheme of finite type over $Spec\mathbb{C}$, it is triangulable. This implies that the conditions of our above theorem are satisfied and as such, letting C_{ord} be as above where the open sets of X are in the analytical topology [for definition, see [12]], $\Pi C_{ord} = S_\bullet X$ as pro-objects in the homotopy category of pointed simplicial sets. This is **not** a statement about the étale homotopy type of X, but of the object we get by appling the connected components functor to the site C_{ord}.

Proof. This is a highly technical proof, involving properties of bismplicial sets which is out of scope for this thesis. The reader is referred to [2], theorem 12.1. □

Definition 78. Let X be a locally Noetherian scheme. We say that X is geometrically unibranch if for every $x \in X$, the integral closure of the stalk $\mathcal{O}_{X,x}$ is again local.

Theorem 79. *Let X be a noetherian, connected, geometrically connected and pointed scheme. The étale homotopy type Ét(X) is then a profinite-object, in the sense that Ét(X) $\in$ Pro-$\mathcal{C} - \mathcal{H}_\bullet$ where $\mathcal{C}$ is the complete class of finite groups as in 2.3 .*

With these preliminaries, Artin-Mazur helps us compute the étale homotopy type for schemes of finite type over $\mathbb{C}$ which are geometrically unibranch. Namely, we have the following theorem:

Theorem 80. *Let X be a connected,geometrically unibranch, pointed scheme of finite type over $\mathbb{C}$. Let C be the category consisting of coproducts of open sets in X with the analytical topology, with a Grothendieck Topology such that $\amalg U_i$ covers $\amalg V_i$ iff $\amalg V_i \subset \amalg U_i$. Set $X_{cl} = \Pi C$. Then the étale homotopy type of X is isomorphic to the profinite completion of X_{cl}.*

Now, we have that $X_{cl} = \Pi C$ is isomorphic to $S_\bullet X$, and as such, $\widehat{\pi_n(X)} \cong \pi_n(X_{\text{Ét}})$ where the left hand side is the profinite completion of $\pi_n(X)$, where we view X with the analytical topology.

Remark. In a letter to Faltings Grothendieck mentions an idea of anabelian geometry, which has gained much interest the last decade. Without going into definitions too much, Grothendieck asks how much information regarding a variety X that can be recovered from its étale fundamental group $\pi(X, x)$ for various choices of x.It is shown in [11] i showed that for certain cases such as hyperbolic curves over number fields, the isomorphism class of X can be recovered from its étale fundamental group. Grothendieck formulated a certain famous section conjecture in anabelian geometry which is still unsolved to this day. However, there is the natural extension of Grothendieck's ideas of anabelian geometry, which asks:
To what extent is the isomorphism type of a scheme / variety determined by the étale fundamental group together with all the higher étale homotopy groups?
It seems as if this should, from a topological viewpoint, capture a larger class of schemes than we get from simply considering the étale fundamental group. It should only be possible to determine the isomorphism class of the scheme X if the higher étale homotopy groups vanishes. I hope that I will be able to give the reader more details on anabelian geometry, the section conjecture and étale homotopy in a later paper.

Example 81. *So for example, take $X = P^1_{\mathbb{C}}$ projective space over the complex numbers. Then we have that $\pi_1(P^1_{\mathbb{C}})$ is trivial, and as such, the profinite completion is also trivial, for any choise of basepoint. This implies that $\pi_1(P^1_{\mathbb{C},\acute{E}t})$ is trivial.*

Example 82. *Let us consider $X = Spec\mathbb{F}_p$ a finite field. Then it is true that the absolute Galois group of X is $\hat{\mathbb{Z}}$, so that $\acute{E}t\mathbb{F}_p \cong K(\hat{\mathbb{Z}}, 1)$ .This suggests some very nice ideas. First of all, note that we have that $\pi_1(\acute{E}t\mathbb{F}_p) \cong \hat{\mathbb{Z}}$ and the fundamental groups vanishes for higher homotopy groups. The circle S^1 has just the same property, but with $\mathbb{Z}$ instead of $\hat{\mathbb{Z}}$ i.e it is $K(\mathbb{Z}, 1)$. We might consider $Spec\mathbb{F}_{|}$ as an arithmetical analogue of the topological circle. With this, we get a further analogy between prime ideals in the number ring $Spec\mathcal{O}_K$ and knots on a 3-manifold. Namely, note that a prime ideal in $q \in \mathcal{O}_K$ we have a natural map $\mathcal{O}_K/q = Spec\mathbb{F}_p \to Spec\mathcal{O}_K$ and I claim that we can view $Spec\mathcal{O}_K$ as an arithmetical analogue of a 3-manifold. This since the étale cohomological dimension is 3 ([10]) up to 2-torsion, and satisfies an arithmetic type Poincaré duality theorem [3].*

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