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Hopf algebras and Feynman graphs

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Abstract

Our main purpose in this project is to study several Hopf algebras of Feynman graphs, and do some calculations of the values of an antipode on concrete graphs. These Feynman graph Hopf algebras originated in the quantum field theory, more precisely in a relatively new approach to the renormalization of diverging Feynman integrals. In that approach to renormalization the antipode map plays a key role.

We give a comprehensive introduction into the theory of graded Hopf algebras. We describe in detail all the main definitions and theorems necessary to understand Hopf algebras of Feynman graphs, and consider many concrete graphs.

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Introduction

This text aims to first give a self-contained presentation of the fundamentals necessary to understand Hopf algebras, and then to present the interesting class of Hopf algebras associated to Feynman graphs. The reader will be introduced to multilinear maps of modules and the universal problem of multilinear maps, to which the tensor product is the solution. Relevant properties of the tensor product are proven and examples given. The theory is then extended from modules to associative algebras, modules with a structure of a ring, and an important example of such, the notion of tensor algebra, is explored. After the notion of algebra has been presented, its dual, the notion of coalgebra, is defined and studied. These two notions are then combined to a bialgebra, a module with structure of both an algebra and coalgebra in a compatible way. A Hopf algebra is a bialgebra with an anti-linear mapping called the antipode.

The text has two goals, one is of course to present Hopf algebras of Feynman graphs. Each aspect of the theory needed to understand Hopf algebras of Feynman graphs is in itself very important. Therefore a second goal is to make the treatment of those elements a good introduction, and some material not directly related to the final application but of general interest are presented. Hopf algebras and its applications to Feynman graphs should be seen as one of many interesting applications of the theory presented.

Chapter 1

Introduction to the tensor product

1.1 Modules

The concept a module is a generalization of the notion of a vector space. The main difference is that the scalars of a module instead of residing in a field come from a commutative ring R. It is also possible to define a module over a non-commutative ring, this will however not be the case in this text.

Definition. *An R-module M over a commutative ring R consist of an abelian group $(M, +)$ and an operation $R \times M \to M$, which have the following properties for $r, s \in R$ and $x, y \in M$*

$$r(x+y) = rx + ry \tag{1.1}$$
$$(r+s)x = rx + sx \tag{1.2}$$
$$(rs)x = r(sx) \tag{1.3}$$
$$1_R x = x \quad \textit{If } R \textit{ has an identity element } 1_R \tag{1.4}$$

It is important to note that a module does not necessarily have a basis, something that makes it different from a vector space. A module which is generated by a finite number of elements in M is called a **finitely generated module**. This is not the same as having a basis in the sense of vector spaces. Such a basis consists of a finite number of linearly independent elements. A module having such a basis is called a **free module**.

Definition. *If a R-module M is generated, using the defined operations, by a finite number of elements* $e_1, \ldots, e_n \in M$ *such that for* $r_1, \ldots, r_n \in R$

$$r_1 e_1, \ldots, r_n e_n = 0 \quad \Leftrightarrow \quad r_1 = r_2 = \ldots = r_n = 0$$

Then M is called a ***free module*** *and* $B = \{e_1, \ldots, e_n\}$ *is called* ***the basis of M*** *and the number of elements in B is called the* ***rank of B****.*

If the ring of scalars is in fact a field, the module is a vector space.

1.2 Multilinear maps and the universal problem

We will now describe two important concepts which together are used to describe the tensor product. These are the definition of multilinear maps, and the universal problem to which the tensor product pose a solution.

Definition. *Assume* $M_1, M_2, \ldots, M_n$ *and M are R-modules. Then a mapping*

$$\phi : M_1 \times M_2 \times \ldots. \times M_n \to M$$

is called multilinear if it is linear in each of its components. That is, if $m_1, m_2, \ldots, m_n$ *are elements in their respective R-module and* $r \in R$ *then the following equalities are satisfied:*

$$\begin{aligned} \phi(m_1, \ldots, m_i + m_i', \ldots, m_n) &= \phi(m_1, \ldots, m_i, \ldots, m_n) \\ &+ \phi(m_1, \ldots, m_i', \ldots, m_n) \qquad (1.5) \\ \phi(m_1, \ldots, r m_i, \ldots, m_n) &= r\phi(m_1, \ldots, m_i, \ldots, m_n) \qquad (1.6) \end{aligned}$$

for any $1 \leqslant i \leqslant n$ *and any* $r \in R$.

Examples of multilinear maps are the determinant with respect to each of its rows or columns, and the cross-product of vectors in $\mathbb{R}^3$.

When dealing with modules a linear map is called a homomorphism. Now assume that besides the mapping ϕ there is also a homomorphism $h : M \to$

N. It is easily verified that the composition $h \circ \phi$ is also a multilinear map.

$$\begin{aligned}
&h(\phi(m_1, \ldots, m_i + m_i', \ldots, m_n)) \\
=&h(\phi(m_1, \ldots, m_i, \ldots, m_n) + \phi(m_1, \ldots, m_i', \ldots, m_n)) \\
=&h(\phi(m_1, \ldots, m_i, \ldots, m_n)) + h(\phi(m_1, \ldots, m_i', \ldots, m_n)
\end{aligned}$$

By the multilinearity of ϕ and the linearity of h. Similarily we have:

$$\begin{aligned}
h(\phi(m_1, \ldots, rm_i, \ldots, m_n)) &= h(r\phi(m_1, \ldots, m_i, \ldots, m_n)) \\
&= rh(\phi(m_1, \ldots, m_i, \ldots, m_n))
\end{aligned}$$

We have now proven that the composition of ϕ and h is a multilinear map. The knowledge that a composition of a linear and multilinear mapping is a multilinear map gives rise to a more general question.

The universal problem of multilinear maps. *Find a pair of an R-module M and a multilinear mapping $\phi : M_1 \times \ldots \times M_n \to M$ such that for any multilinear mapping $\psi : M_1 \times \ldots \times M_n \to N$ there is exactly one homomorphism $h : M \to N$ such that $h \circ \phi = \psi$. A solution, (M, ϕ) to this problem is said to have the* ***universal property of multilinear maps****.*

Before we continue to find a solution to the universal problem we will conclude the following.

Theorem 1. *The solution to the universal problem is essentially unique in the sense that if two solutions (M, ϕ) and (M', ϕ') exist then there will always be inverse isomorphisms $\lambda : M \to M'$ and $\lambda' : M' \to M$.*

Proof. First of we conclude that if a pair (M, ϕ) is a solution, then whenever there are two homomorphisms $g : M \to N$ and $g' : M \to N$ such that $g \circ \phi = g' \circ \phi$. Then $g = g'$ by the uniqueness condition stated in the definition.

By the universal property there are homomorphisms λ and λ' such that $\lambda \circ \phi = \phi'$ and $\lambda' \circ \phi' = \phi$. From this it follows directly that

$$id \circ \phi = \phi = \lambda' \circ \phi' = \lambda' \circ \lambda \circ \phi$$

where id is the identity mapping of M. Now by what was stated at the beginning of the proof this implies that $\lambda' \circ \lambda = id$ and similarly $\lambda \circ \lambda' = id$. $\square$

As a consequence of this, the solutions to the universal problem is unique upto inverse isomorphisms. As a consequence one can often neglect the messy details of the construction as soon as its possibility has been proven and instead focus on this universal property to provide proofs of further properties.

1.3 The construction of the tensor product

We will now turn to finding a solution to the universal problem. First we construct a module called the free module generated by $M_1 \times M_2 \times \ldots \times M_n$. Then we quotient out by a submodule to impose an equivalence. We then show that this is a solution to the universal problem and name it the tensor product.

The **free module of a set** E is created seeing the whole set as a basis for an R-module. Any element in this R-module is a linear polynomial with the elements of E as indeterminates and with coefficients from R. This free module of E has the universal property that for any arbitrary mapping $\varphi : E \to N$, N being an R-module, there is an unique extension to a homomorphism from the free module of E to N. This is since we can, and must, define the mapping, where U(E) is the free module of E,

$$h : U(E) \to N$$

as

$$h(r_1 e_1 + \ldots + r_n e_n) = r_1 \varphi(e_1) + \ldots + r_n \varphi(e_n)$$

for $e_i \in E$ and $r \in R$.

If we now let $U(M_1, \ldots, M_n)$ be the free module generated by $M_1 \times \ldots \times M_n$, then the basis will consist of sequences $(m_1, \ldots, m_n)$. We define the mapping

$$\pi : M_1 \times \ldots \times M_n \to U(M_1, \ldots, M_n)$$

as the that maps $(m_1, \ldots, m_n)$ to the corresponding base element in $U(M_1, \ldots, M_n)$. By what we have just concluded, for any map $\varphi : M_1 \times \ldots \times M_n \to N$ there will be a unique extension to a homomorphism $h : U(M_1, \ldots, M_n) \to N$ such that $h \circ \pi = \varphi$. However, this does not pose a solution since the map π is not necessarily multilinear.

To solve this consider the submodule $V(M_1, \ldots, M_n)$ generated by elements on one of the forms:

$$\begin{aligned} (m_1, \ldots, m_i + m_i', \ldots, m_n) &- (m_1, \ldots, m_i, \ldots, m_n) \\ &- (m_1, \ldots, m_i', \ldots, m_n) \qquad (1.7) \\ (m_1, \ldots, rm_i, \ldots, m_n) &- r(m_1, \ldots, m_i, \ldots, m_n) \qquad (1.8) \end{aligned}$$

Define M as:

$$M = U(M_1, M_2, \ldots, M_n)/V(M_1, M_2, \ldots, M_n)$$

and define a mapping

$$\otimes : M_1 \times \ldots \times M_n \to M$$

so that $\otimes(m_1, m_2, \ldots, m_n)$ is the natural image of $(m_1, m_2, \ldots, m_n)$, considered as an element of $U(M_1, M_2, \ldots, M_n)$ in M.

Now reconsider the definition of a multilinear map, equation (1.5) and (1.6). Any of the elements on the form (1.7) or (1.8) will be mapped to zero by $\otimes$ as we now have defined it, since these elements will by definition belong to the submodule $V(M_1, \ldots, M_n)$. From this it follows that $\otimes$ satisfies the conditions (1.5) and (1.6) and thereby it is a multilinear map.

Theorem 2. $(\otimes, M)$ *is a solution to the universal problem.*

Proof. To prove this theorem suppose we have an arbitrary multilinear mapping

$$\psi : M_1 \times M_2 \times \ldots \times M_n \to N.$$

We know there is an R-homomorphism

$$h : U(M_1, M_2, \ldots, M_n) \to N.$$

Since ψ is multilinear any element on the form (1.7) or (1.8) will be mapped to zero in ψ and therefore ψ will vanish in $V(M_1, M_2, \ldots, M_n)$. As a consequence there is an induced map $h : M \to N$ such that $h \circ \otimes = \psi$. The uniqueness of the solution is already given by the construction of the homomorphism h from the free module. □

From now on we will be using an infix notation for the tensor product. If $(\otimes, M)$ is a solution to the universal problem then it is customary to write $M = M_1 \otimes_R M_2 \otimes_R \ldots \otimes_R M_n$ and to denote the element $\otimes(m_1, m_2, \ldots, m_n)$ by $m_1 \otimes_R m_2 \otimes_R \ldots \otimes_R m_n$, also the elements $m_1 \otimes_R m_2 \otimes_R \ldots \otimes_R m_n$ will be called **monomial tensors**. For ease of notation the suffix denoting the ring will often be omitted if it is obvious which ring is considered.

Theorem 3. *Each element of M, as defined earlier, can be expressed as a finite sum of elements of the form $m_1 \otimes \ldots \otimes m_n$. Or in other words, the tensor product is generated by the monomial tensors.*

Proof. Suppose that M' is the R-submodule of M generated by elements of the form $m_1 \otimes \ldots \otimes m_n$. Let $h_1 : M \to M/M'$ be the natural homomorphism and $h_2 : M \to M/M'$ be the null homomorphism.

For any element $m_1 \times \ldots \times m_n \in M_1 \times \ldots \times M_n$ it is obviously true that

$$h_1 \circ \otimes : M_1 \times \ldots \times M_n \to M/M'$$

and

$$h_2 \circ \otimes : M_1 \times \ldots \times M_n \to M/M'$$

will map the element to 0. But then $h_1 \circ \otimes = h_2 \circ \otimes$ which in turn implies that $h_1 = h_2$ so $M = M'$.

For any $x \in M = M'$, x can be written as

$$x = r(m_1 \otimes \ldots \otimes m_n) + r'(m'_1 \otimes \ldots \otimes m'_n) \ldots,$$

However, since $\otimes$ is multilinear

$$r(m_1 \otimes \ldots \otimes m_n) = (rm_1 \otimes \ldots \otimes m_n)$$

and similarly for all the other terms □

Many times the modules of interest are going to be free modules or vector spaces. Therefore the next theorem is of great interest.

Theorem 4. *Let M_i be a free R-module for $(i = 1, 2 \ldots, n)$ and B_i be its basis. Then $M_1 \otimes M_2 \otimes \ldots \otimes M_n$ is also a free R-module and its basis consists of the elements $b_1 \otimes b_2 \otimes \ldots \otimes b_n$ where $b_i \in B_i$. That is $B_1 \otimes B_2 \otimes \ldots \otimes B_n$ is a basis.*

Proof. Since we now that M is generated by elements on the form $m_1 \otimes \ldots \otimes m_n$ it will suffice to show that any such element is a linear combination of elements $b_1 \otimes b_2 \otimes \ldots \otimes b_n$. Since B_i is a basis for M_i we know that any element $m_i \in M_i$ can be written as $\sum_{i_\alpha} r_{i_\alpha} b_{i_\alpha} = m_i$. As a consequence:

$$\begin{aligned} m_1 \otimes \ldots \otimes m_n &= \sum_{1_\alpha} r_{1_\alpha} b_{1_\alpha} \otimes \ldots \otimes \sum_{n_\alpha} r_{n_\alpha} b_{n_\alpha} \\ &= \sum_{1_\alpha,\ldots,n_\alpha} r_{1_\alpha} \ldots r_{n_\alpha} b_{1_\alpha} \otimes \ldots \otimes b_{n_\alpha} \end{aligned}$$

by the multilinearity of $\otimes$. The linear independence of these elements follows directly from the linear independence of the respective B_i. This proves that the elements $b_{1_\alpha} \otimes \ldots \otimes b_{n_\alpha}$ generates the whole of M.

□

1.4 Examples of tensor products

Example 1. *Calculate* $\mathbb{Z}/5 \otimes_{\mathbb{Z}} \mathbb{Z}/7$.

To calculate $\mathbb{Z}/5 \otimes_{\mathbb{Z}} \mathbb{Z}/7$ we will make use multilinear properties of $\otimes_{\mathbb{Z}}$, the suffix will be left out for the rest of this example. First of it is clear that 5 annihilates the left factor and that 7 annihilates the right factor. But then it also follows that if $m \in \mathbb{Z}/5$ and $n \in \mathbb{Z}/7$

$$0 = 0 \cdot (m \otimes_{\mathbb{Z}} n) = (0 \cdot m) \otimes n = (5 \cdot m) \otimes n = m \otimes (5 \cdot n)$$

in other words, 5 also annihilates the right factor, similarly

$$0 = 0 \cdot (m \otimes n) = m \otimes (0 \cdot n) = m \otimes (7 \cdot n) = (7 \cdot m) \otimes n = (2 \cdot m) \otimes n.$$

Then we can also write

$$(5m \otimes n) - 2(2 \cdot m \otimes n) = (5 - 2 \cdot 2)m \otimes n = m \otimes n$$

But also

$$(5m \otimes n) - 2(2 \cdot m \otimes n) = 0 - 2 \cdot 0 = 0$$

Thereby any element $m \otimes n$ in $\mathbb{Z}/5 \otimes \mathbb{Z}/7$ is zero. But since any element of $\mathbb{Z}/5 \otimes \mathbb{Z}/7$ can be written as a sum of terms $m \otimes n$ it follows that any element in $\mathbb{Z}/5 \otimes \mathbb{Z}/7$ is 0. So

$$\mathbb{Z}/5 \otimes \mathbb{Z}/7 = 0$$

.

Example 2. $\mathbb{R}^3 \otimes_{\mathbb{R}} \mathbb{R}^3$ *over the field of real scalars.*

Now suppose that we have a basis $B = \{e_1, e_2, e_3\}$. Then by Theorem 3 we know that $\mathbb{R}^3 \otimes \mathbb{R}^3$ has a basis $B \otimes B$ such that the elements in this basis will be of the form $b \otimes b'$. That is elements on the form e_1e_2 or e_3e_1. The number of elements in this basis is $3^2 = 9$ since this is the number of ways you can create a two element sequence where each element is on of three elements. If we denote the basis

$$e_1' = e_1e_1 \quad e_2' = e_1e_2 \quad e_3' = e_1e_3 \quad e_4' = e_2e_1 \quad e_5' = e_2e_2,$$

$$e_6' = e_2e_3 \quad e_7' = e_3e_1 \quad e_8' = e_3e_2 \quad e_9' = e_3e_3$$

an element $x \in \mathbb{R}^3 \otimes \mathbb{R}^3$ is given by $x = a_1e_1' + a_2e_2' + \ldots + a_9e_9'$ where $a_i \in \mathbb{R}$. An example of a monomial tensor with usual vector notation would be

$$(1, 2, 3) \otimes (4, 5, 6) = (4, 5, 6, 8, 10, 12, 12, 15, 18)$$

We now have a new 9-dimensional vector space $\mathbb{R}^3 \otimes \mathbb{R}^3$ such that any multilinear mapping $\mathbb{R}^3 \times \mathbb{R}^3 \to V$, where V is any vector space, there is a linear map $h : \mathbb{R}^3 \otimes \mathbb{R}^3 \to V$. Since h is a linear map from one vector space to another it can be written as a transformation matrix A.

The usual scalar product is a multilinear map and it is in $\mathbb{R}^3$ defined by

$$(a_1, a_2, a_3) \cdot (a_1', a_2', a_3') = a_1a_1' + a_2a_2' + a_3a_3'$$

if we instead look at the induced linear map h from $\mathbb{R}^3 \otimes \mathbb{R}^3$ to $\mathbb{R}$ it would be $h(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9) = a_1 + a_5 + a_9$ and this linear map h would have the transformation matrix

$$A = (1,0,0,0,1,0,0,0,1)$$

To sum it up we have the following equality for $v, u \in \mathbb{R}^3$

$$v \cdot u = A(v \otimes u)$$

Similarly we can write the transformation matrix for the linear map induced by the cross product

$$v \times u = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} (v \otimes u)$$

Chapter 2

Some elementary properties of the tensor product

In this part of the text we will focus on some fundamental properties of the tensor product. Among those are isomorphisms that prove that the tensor product is in a sense both associative and commutative. Also we will study the tensor product of homomorphisms and the the tensor product of a direct sum.

2.1 Basic isomorphisms

As has been discussed earlier one does often not need to make an actual construction of the tensor product for the use of its properties to solve problems. Instead it is sufficient to conclude that a construction is possible and thereafter make use of proven properties that follow directly. Two solutions to the universal problem will from this perspective be essentially the same since there is an unique isomorphism between them.

Theorem 5. *There is an isomorphism*

$$M_1 \otimes \ldots \otimes M_n \otimes N_1 \otimes \ldots \otimes N_p \simeq (M_1 \otimes \ldots \otimes M_n) \otimes (N_1 \otimes \ldots \otimes N_p)$$

in which $m_1 \otimes \ldots \otimes m_n \otimes n_1 \otimes \ldots \otimes n_p$ *is matched with* $(m_1 \otimes \ldots \otimes m_n) \otimes (n_1 \otimes \ldots \otimes n_p)$.

Proof. By the universal property of the tensor product there is a homomorphism

$$f : M_1 \otimes \ldots \otimes M_n \otimes N_1 \otimes \ldots \otimes N_p \to (M_1 \otimes \ldots \otimes M_n) \otimes (N_1 \otimes \ldots \otimes N_p)$$

induced by the multilinear mapping

$$M_1 \times \ldots M_n \times N_1 \times \ldots \times N_p \to (M_1 \otimes \ldots \otimes M_n) \otimes (N_1 \otimes \ldots \otimes N_p)$$

in which $(m_1, \ldots, m_n, n_1, \ldots, n_p)$ is mapped to $(m_1 \otimes \ldots \otimes m_n) \otimes (n_1 \otimes \ldots \otimes n_p)$. To prove that this is an isomorphism one has to reverse this homomorphism f.

Suppose that we hold $n_1, n_2, \ldots, n_p$ fixed. Again, by the universal property it is obvious that there is an homomorphism $M_1 \otimes \ldots \otimes M_n \to M_1 \otimes \ldots \otimes M_n \otimes N_1 \otimes \ldots \otimes N_p$ where $m_1 \otimes \ldots \otimes m_n$ is mapped with $m_1 \otimes \ldots \otimes m_n \otimes n_1 \otimes \ldots \otimes n_p$. Consequently, since this is a homomorphism we have that if the following relation is given

$$m_1 \otimes m_2 \otimes \ldots \otimes m_n + m'_1 \otimes m'_2 \otimes \ldots \otimes m'_n +$$

$$+ m''_1 \otimes m''_2 \otimes \ldots \otimes m''_n = 0$$

then

$$m_1 \otimes \ldots \otimes m_n \otimes n_1 \otimes \ldots \otimes n_p + m'_1 \otimes \ldots \otimes m'_n \otimes n_1 \otimes \ldots \otimes n_p +$$

$$+ m''_1 \otimes \ldots \otimes m''_n \otimes n_1 \otimes \ldots \otimes n_p = 0$$

Where 0 of course denotes the zero element in $M_1 \otimes \ldots \otimes M_n \otimes N_1 \otimes \ldots \otimes N_p$. We get similar results if the roles M_i and N_i are interchanged.

Now by Theorem 3 we know that any element $\xi \in M_1 \otimes M_2 \otimes \ldots \otimes M_n$ and $\eta \in N_1 \otimes N_2 \otimes \ldots \otimes N_n$ can be expressed using their respective monomial tensors. Now let

$$\xi = \sum m_1 \otimes \ldots \otimes m_n = \sum \mu_1 \otimes \ldots \otimes \mu_n$$

and

$$\eta = \sum n_1 \otimes \ldots \otimes n_p = \sum \nu_1 \otimes \ldots \otimes \nu_n$$

be two such representations for ξ and η each. By what we stated in the last paragraph we get the following equalities.

$$\sum \sum m_1 \otimes \ldots \otimes m_n \otimes n_1 \otimes \ldots \otimes n_p =$$

$$= \sum \sum \mu_1 \otimes \ldots \otimes \mu_n \otimes n_1 \otimes \ldots \otimes n_p =$$

$$= \sum \sum \mu_1 \otimes \ldots \otimes \mu_n \otimes \nu_1 \otimes \ldots \otimes \nu_p$$

Consequently $\xi \otimes \eta = m_1 \otimes \ldots \otimes m_n \otimes n_1 \otimes \ldots \otimes n_p$ depends only on ξ and η and are independent of the chosen representation. It follows that there is a mapping

$$(M_1 \otimes \ldots \otimes M_n) \times (N_1 \otimes \ldots \otimes N_p) \rightarrow M_1 \otimes \ldots \otimes M_n \otimes N_1 \otimes \ldots \otimes N_p$$

that takes (ξ, η) into the element $\sum \sum m_1 \otimes \ldots \otimes m_n \otimes n_1 \otimes \ldots \otimes n_p$. This mapping is obviously bilinear, and as a consequence there is an homomorphism

$$g : (M_1 \otimes \ldots \otimes M_n) \times (N_1 \otimes \ldots \otimes N_p) \rightarrow M_1 \otimes \ldots \otimes M_n \otimes N_1 \otimes \ldots \otimes N_p.$$

Now if the definitions of f and g are considered it is clear that $f \circ g$ and $g \circ f$ are both the identity mapping for their respective elements and the proof is done. □

Corollary 1. *There is an isomorphism*

$$(M_1 \otimes M_2) \otimes M_3 \simeq M_1 \otimes (M_2 \otimes M_3)$$

in which $(m_1 \otimes m_2) \otimes m_3$ *is mapped to* $m_1 \otimes (m_2 \otimes m_3)$

Proof. Theorem 4 provides us with isomorphisms

$$(M_1 \otimes M_2) \otimes M_3 \simeq M_1 \otimes M_2 \otimes M_3 \simeq M_1 \otimes (M_2 \otimes M_3).$$

□

As discussed in the beginning of this part of the text this implies that the tensor product is associative. Next we are going to prove that the tensor product is also commutative.

Theorem 6. *Let* $i_1, i_2, \ldots, i_n$ *be a permutation of* $(1, 2, \ldots, n)$*, then there is an isomorphism*

$$M_1 \otimes \ldots \otimes M_n \simeq M_{i_1} \otimes \ldots \otimes M_{i_n}$$

which associates $m_1 \otimes \ldots \otimes m_n$ *with* $m_{i_1} \otimes \ldots \otimes m_{i_n}$.

Proof. The multilinear mapping

$$M_1 \times \ldots \times M_n \to M_{i_1} \otimes \ldots \otimes M_{i_n}$$

mapping $(m_1, \ldots, m_n)$ to $m_{i_1} \otimes \ldots \otimes m_{i_n}$ induces a homomorphism

$$h : M_1 \otimes \ldots \otimes M_n \to M_{i_1} \otimes \ldots \otimes M_{i_n}.$$

Where $h(m_1, \ldots, m_n) = m_{i_1} \otimes \ldots \otimes m_{i_n}$. Similarly there is an induced homomorphism

$$g : M_{i_1} \otimes \ldots \otimes M_{i_n} \to M_1 \otimes \ldots \otimes M_n$$

with $g(m_{i_1} \otimes \ldots \otimes m_{i_n}) = m_1 \otimes \ldots \otimes m_n$.

It is now obvious that both $f \circ g$ and $g \circ f$ are identity mappings and the isomorphism is thereby proved. □

Looking at the criteria for an R-module in $(1-4)$ it is obvious that R itself can be looked at as an R-module. This is an important property which is often used.

Theorem 7. *Considering R as an R-module there are isomorphisms*

$$R \otimes M \simeq M$$

such that $r \otimes m$ is mapped to rm and

$$M \otimes R \simeq M$$

such that $m \otimes r$ is mapped to mr.

Proof. The mapping

$$\psi : R \times M \to M$$

in which $r \times m$ is mapped to rm is bilinear and therefore by the universal property induces an isomorphism

$$h : R \otimes M \to M$$

such that $h(r \otimes m) = \psi(r \times m) = rm$. Now we consider a mapping

$$g : M \to R \otimes M$$

such that $g(m) = 1 \otimes m$ which is obviously a homomorphism. But also

$$g(rm) = 1 \otimes rm = r(1 \otimes m) = r \otimes m$$

and thereby $f \circ g$ and $g \circ f$ are identity mappings and the isomorphism is proved. The case $M \otimes R \simeq M$ is proved analogous. □

2.2 Tensor product of homomorphisms

The tensor product is not a construction in any way confined to multilinear maps of modules. On the opposite the tensor product can be defined and studied for many different algebraic structures for which multilinearity is of interest. Later on in this text we will define the tensor product of algebras, coalgebras and also Hopf algebras. Now we are going to define the tensor product of homomorphisms.

Suppose we have n modules $M_1, \dots M_n$ and another set of n modules $M'_1, \dots, M'_n$ and that we have homomorphisms f_i taking $M_i \to M'_i$ for every $1 \leq i \leq n$. Then we can define a mapping

$$f_1 \otimes \dots \otimes f_n : M_1 \times \dots \times M_n \to M'_1 \otimes \dots \otimes M'_n$$

in which $(m_1, \dots, m_n)$ is mapped to $f(m_1) \otimes \dots \otimes f(m_n)$. This mapping can easily be shown to be multilinear. If f_i, g_i are homomorphisms of M into M' and r is in R we can form new homomorphisms $f_i + g_i$ and rf_i. Looking at the criteria for a multilinear mapping in (1.5) and (1.6) and this defined mapping we see that

$$\begin{aligned} r(f_1(m_1) \otimes \dots \otimes f_i(m_i) \otimes \dots \otimes f_n(m_n)) &= f_1(m_1) \otimes \dots \otimes rf_i(m_i) \otimes \dots \otimes f_n(m_n) \\ &= f_1(m_1) \otimes \dots \otimes f_i(rm_i) \otimes \dots \otimes f_n(m_n) \end{aligned}$$

By the multilinearity of the tensor product and the linearity of a homomorphism. Similarly condition (1.5) can be proven. Since this mapping is multilinear it induces a homomorphism

$$f_1 \times \dots \times f_n : M_1 \otimes \dots \otimes M_n \to M'_1 \otimes \dots \otimes M'_n.$$

This homomorphism does, by the universal property, satisfy

$$(f_1 \otimes \dots \otimes f_n)(m_1 \otimes \dots \otimes m_n) = f_1(m_1) \otimes \dots \otimes f_n(m_n)$$

Definition. *Let $M_1, \dots, M_n$ and $M'_1, \dots, M'_n$ be modules, and let*

$$f_i : M_i \to M'_i \quad (i = 1, 2, \dots, n)$$

be homomorphisms. The ***tensor product of the homomorphisms*** *f_i denoted $f_1 \otimes \dots \otimes f_n$ is the homomorphism*

$$M_1 \otimes \dots \otimes M_n \to M'_1 \otimes \dots \otimes M'_n$$

induced by the multilinear map

$$M_1 \times \ldots \times M_n \to M_1' \otimes \ldots \otimes M_n'$$

mapping $(m_1, \ldots, m_n)$ *with* $f_1(m_1) \otimes \ldots \otimes f_n(m_n)$.

Before we proceed we note two things. If each f_i is surjective then $f_1 \otimes \ldots \otimes f_n$ is surjective as well. If $M_i = M_i'$ for all i and f_i is the identity mapping of M_i then $f_1 \otimes \ldots \otimes f_n$ is the identity mapping of $M_1 \otimes \ldots \otimes M_n$.

Now to proceed suppose that in addition to the mappings f_i there are homomorphisms $g_i : M_i'' \to M$ for $i = 1, 2, \ldots, n$. From the definition it follows that

$$(f_1 \otimes \ldots \otimes f_n) \circ (g_1 \otimes \ldots \otimes g_n) = (f_1 \circ g_1) \otimes \ldots \otimes (f_n \circ g_n) \tag{2.1}$$

From this it follows that if each f_i is an isomorphism so is $f_1 \otimes f_2 \otimes \ldots \otimes f_n$. Because, if f_i is an isomorphism there is an inverse homomorphism

$$f_i^{-1} : M_i' \to M$$

such that $f_i \circ f_i^{-1} = f_i^{-1} \circ f_i$ is the identity mapping. Now by (2.1) we have

$$(f_1 \otimes \ldots \otimes f_n) \circ (f_1^{-1} \otimes \ldots \otimes f_n^{-1}) = (f_1 \circ f_1^{-1}) \otimes \ldots \otimes (f_n \circ f_n^{-1})$$

and since each $f_i \circ f_i^{-1}$ is the identity mapping this proves that $f_1 \otimes \ldots \otimes f_n$ is an isomorphism.

2.3 Tensor product of direct sum of modules

It is of interest to study modules which have representations as direct sums and to show that the tensor product of such modules does in itself have a representation as a direct sum.

Definition. *If a module N has a family of submodules $\{N_i\}_{i \in I}$ such that any element $n \in N$ has an unique representation of the form*

$$n = \sum_{i \in I} n_i \tag{2.2}$$

where $n_i \in N_i$ *and only finitely many summands are non-zero then* N *is called the direct sum of* $\{N_i\}_{i \in I}$*. When this is the case we will write*

$$N = \sum_{i \in I} N_i \tag{2.3}$$

or if we know that the family of submodules is finite we might write

$$N = N_1 \oplus N_2 \oplus \ldots \oplus N_n \tag{2.4}$$

instead.

Now this is the usual way of defining direct sums to which you might be accustomed. To complete the proofs we are interested in, we will instead use another, slightly more general, definition as well.

Suppose that N can be described as in (2.3). Then for each $i \in I$ we can define two homomorphisms called the inclusion mapping and the projection mapping. The inclusion mapping $\sigma_i : N_i \to N$ as the mapping which maps $n_i \in N_i$ to the corresponding element $n_i \in N$. We define the projection mapping $\pi_i : N \to N_i$ as the mapping that from the representation (2.2) of an element $n \in N$ picks out the summand from the submodule N_i. Now these two mappings have the following properties:

(i) $\pi_i \circ \sigma_j$ is a null homomorphism, mapping every element to the zero element, if $i \neq j$ and it is the identity mapping of N_i if $i = j$.

(ii) For each $n \in N$, $\pi_i(n)$ is non-zero for only finitely many values of i.

(iii) $\sum_{i \in I} \sigma_i \pi_i(n) = n$, for each $n \in N$,

We are going to base the slightly more general definition of a direct sum on these properties.

Suppose that N is a R-module, and that $\{N_i\}_{i \in I}$ is a family of R-modules. This family is though no longer assumed to consist of submodules of N. Suppose that for each $i \in I$ there are mappings $\sigma_i : N_i \to N$ and $\pi_i : N \to N_i$ such that the conditions $(i), (ii)$, and (iii) are satisfied. This is enough to supply us with a construction compatible with the definition given earlier of a direct sum.

Theorem 8. *Suppose N is an R-module, $\{N_i\}_{i\in I}$ is a family of R-modules and there are homomorphisms $\sigma_i : N_i \to N$ and $\pi_i : N \to N_i$ such that the conditions $(i), (ii)$, and (iii) are satisfied. Then N is a direct sum of the submodules $\{\sigma_i(N_i)\}_{i\in I}$.*

Proof. By (i), $\pi_i \circ \sigma_i$ is the identity mapping of N_i. Therefore σ_i must be an injection and π_i an surjection, otherwise $\pi \circ \sigma$ couldn't be the identity mapping of N_i. In particular the inclusion mapping σ_i maps N_i isomorphically onto $\sigma_i(N_i)$. Since N_i is a module, and σ_i is a homomorphism it follows that $\sigma_i(N_i)$ is a sub-module of N. From (ii) and (iii) it now follows that

$$N = \sum_{i\in I} \sigma_i(N_i)$$

All the necessary conditions from the definition has now been satisfied and N is a direct sum of the submodules $\{\sigma_i(N_i)\}_{i\in I}$. □

We have now introduced a more generalized notion of a direct sum where the $N_i's$ do not have to be submodules themselves but instead it is sufficient for the $\sigma_i(N_i)$ to be submodules of N. This is very important. Also the system formed by N, the N_i, and the homomorphisms σ_i and π_i is called a **complete representation of N as a direct sum**. The notation (2.2) and (2.3) will continuously be used.

We will turn to the real point of interest. Suppose that $M_1, M_2, \ldots, M_n$ are R-modules and that each has a complete representation as a direct sum on the form

$$M_\mu = \sum_{i\in I_\mu} M_i^\mu$$

with the homomorphisms

$$\sigma_i^\mu : M_i^\mu \to M_\mu \quad \text{and} \quad \pi_i^\mu : M_\mu \to M_i^\mu$$

.

Theorem 9. *Suppose we have R-modules $M_1, M_2, \ldots, M_n$ each with a complete representations as a direct sum. Then $M_1 \otimes M_2 \otimes \ldots \otimes M_n$ also has a complete representation as a direct sum*

$$M_1 \otimes M_2 \otimes \ldots \otimes M_n = \sum_{(i\in I)} M_i^1 \otimes M_i^2 \otimes \ldots \otimes M_i^n$$

where the inclusion and injection mappings are

$$\sigma_i^1 \otimes \ldots \otimes \sigma_i^n : M_i^1 \otimes \ldots \otimes M_i^n \to M_1 \otimes \ldots \otimes M_n$$

and

$$\pi_i^1 \otimes \ldots \otimes \pi_i^n : M_1 \otimes \ldots \otimes M_n \to M_i^1 \otimes \ldots \otimes M_i^n$$

Proof. Set $I = I_1 \times I_2 \times \ldots \times I_n$, $N = M_1 \otimes M_2 \otimes \ldots \otimes M_n$ and for i in I, set

$$N_i = M_i^1 \otimes M_i^2 \otimes \ldots \otimes M_i^n$$

$$\sigma_i = \sigma_i^1 \otimes \sigma_i^2 \otimes \ldots \otimes \sigma_i^n$$
$$\pi_i = \pi_i^1 \otimes \pi_i^2 \otimes \ldots \otimes \pi_i^n$$

To prove the theorem it will by Theorem 7 be sufficient to prove that conditions $(i), (ii)$, and (iii) holds. From what we know about the tensor product of homomorphisms and in particular from (2.1) it follows that condition (i) holds.

Any element $n \in M_1 \otimes \ldots \otimes M_n$ has the form $n = m_1 \otimes m_2 \otimes \ldots \otimes m_n$. Condition (ii) and (iii) are in light of this representation of the elements n easily proven. By its multilinearity if any of the m_i in a monomial tensor i zero, then the whole monomial tensor is zero. Then if each $\pi_i^\mu(m_\mu)$ is nonzero for a finite number of values of i_μ then of course so is also $\pi_i(n)$ and thereby condition (ii) holds.

Similarly if each element m_μ can be written as $\displaystyle\sum_{i \in I_\mu} \sigma_i^\mu(m_\mu)$ then

$$n = m_1 \otimes \ldots \otimes m_n = \sum_{i \in I_1} \sigma_i^1(m_1) \otimes \ldots \otimes \sum_{i \in I_n} \sigma_i^n(m_n)$$

and by the multilinearity of the tensor product we can expand this tensor product of sums into a sum of monomial tensors.

$$\sum_{i \in I_1} \sigma_i^1(m_1) \otimes \ldots \otimes \sum_{i \in I_n} \sigma_i^n(m_n) = \sum_{i \in I} \sigma_i^1(m_1) \otimes \ldots \otimes \sigma_i^n(m_n)$$

□

Chapter 3

Associative algebras

Before proceeding to the study of the particular algebras of interest to us, we will define and get familiar with the concept of an associative algebra. We will only study algebras which possess an identity element. R and S will denote commutative rings with identity elements. Ring homomorphisms, and algebra homomorphisms, will be required to preserve identity elements.

3.1 Definition of an associative algebra

Associative algebras are modules that also have a compatible structure as a ring. The sum of two elements in an associative algebra A has to be the same whether the ring or the module structure is used. Also multiplication with elements from the underlying ring R must be commutative in the sense that

$$r(a_1a_2) = (ra_1)a_2 = a_1(ra_2) \tag{3.1}$$

where $a_1, a_2 \in A$ and $r \in R$. Note that this criteria is equivalent to

$$(r_1a_1)(r_2a_2) = (r_1r_2)(a_1a_2)$$

Definition. *Let A be a R-module. If A has an associative bilinear mapping mapping $A \times A \to A$, or in other words for $a_1, a_2, a_3 \in A$*

$$(a_1a_2)a_3 = a_1(a_2a_3)$$

such that it has an identity $\mathbf{1}_A$ *element for this operation, and if multiplication with elements from the underlying ring satisfies*

$$r(a_1a_2) = (ra_1)a_2 = a_1(ra_2)$$

then A *is called an* ***associative R-algebra****.*

There is also another way to look at algebras. Consider the mapping

$$\phi : R \to A$$

defined by $\phi(r) = r\mathbf{1}_A$. This mapping is both a ring-homomorphism and a homomorphism of R-modules. Also

$$\phi(r)a = r\mathbf{1}_A a = ra = ra\mathbf{1}_A = ar\mathbf{1}_A = a\phi(r) \tag{3.2}$$

so $\phi(R)$ is contained in the center of A. The mapping ϕ is called the **structural homomorphism** of the R-algebra A. This provides us with another way of looking at R-algebras. Suppose that A is a ring with an identity element. Assume we are given a ring-homomorphism $\phi : R \to A$ which maps R into the center of A. If we define $ra = \phi(r)a$ it is obvious that A with this mapping satisfies the conditions for a module $(1.1 - 1.4)$. Now A is an R-algebra with ϕ as its structural homomorphism. For example R with the identity mapping as ϕ is an algebra.

Definition. *Let* A *and* B *be* R*-algebras. A mapping*

$$f : A \to B$$

is called an ***algebra homomorphism*** *if it is both a homomorphism of rings and a homomorphism of* R*-modules.*

Note that if $\phi : R \to A$ and $\psi : R \to B$ are the structural homomorphisms of A and B then a mapping $f : A \to B$ is a algebra homomorphism if and only if $f \circ \phi = \psi$ and f is an ring homomorphism.

Definition. *If* C *is a subring of the* R*-algebra* A *(with* $\mathbf{1}_C = \mathbf{1}_A$*) as well as an* R*-submodule of* A*. Then* C *itself is an* R*-algebra and is called a subalgebra of* A*.*

3.2 Examples of associative algebras

Example 3. *The set of square $n \times n$ matrices with entries from a ring R form an associative algebra over R.*

Take the identity element, multiplication and addition mappings to be the usual ones for matrices and it is obvious that they satisfy the criteria for an R-algebra.

Example 4. *The complex numbers forms an associative algebra.*

Any complex number can be described as a vector in $\mathbb{R}^2$ where addition is the usual vector addition. If we define the the bilinear mapping $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ to be the normal multiplication of complex numbers they form an algebra.

Example 5. *The polynomials with real coefficients form an associative $\mathbb{R}$-algebra over the reals.*

The polynomials with real coefficients, $\mathbb{R}[X]$, are obviously compatible with the conditions $(1.1 - 1.4)$ for modules. Also if we define the multiplication mapping to be the usual one for multiplication of polynomials this bilinear map does comply with the criteria for an associative algebra.

Example 6. *The endomorphisms of a R-module M form a algebra.*

Homomorphisms of M into any R-module N can be added and be multiplied by elements of R, in fact they form a R-module often denoted $\text{Hom}_R(M, N)$. Now if $N = M$ these homomorphisms are in fact endomorphisms and we use the notation $\text{End}_R(M)$ instead. Now if f, g belong to $\text{End}_R(M)$ then so does $f \circ g$. If we now take the multiplication mapping to be defined as $\circ$ then $\text{End}_R(M)$ becomes a ring with identity. Also for $r \in R$ we have that

$$(rf) \circ g = r(f \circ g) = f \circ (rg)$$

which satisfies the condition (3.1) and we have that $\text{End}_R(M)$ is an R-algebra. The identity mapping is the identity element and the structural homomorphism $R \to \text{End}_R(M)$ sends r to the corresponding homothety, that is the mapping $M \to M$ in which $m \in M$ goes into rm.

Example 7. *Similarly to the free module of a set X it is possible to construct the free R-algebra or the free commutative R-algebra from a set X.*

The product of $X_1, X_2 \in X$ is simply written as the concatenation $X_1 \cdot X_2$. Depending on whether or not $X_1 \cdot X_2 = X_2 \cdot X_1$ we get the free algebra or the commutative free algebra. The free commutative algebra is in essence the same thing as the polynomial ring over R where the elements in X are taken as the indeterminates. The free (non-commutative) algebra can be seen as the noncommutative analouge of a polynomial ring, in other words $aX_1X_2 \neq aX_2X_1$.

To give an example we will do the calculations $(a_1X_1X_2 + a_2X_2X_1) \cdot X_1X_2$ first as an free algebra and then as an free commutative algebra.

$$\begin{aligned}(a_1X_1X_2 + a_2X_2X_1) \cdot X_1X_2 &= a_1X_1X_2X_1X_2 + a_2X_2X_1^2X_2 \\ (a_1X_1X_2 + a_2X_2X_1) \cdot X_1X_2 &= a_1X_1^2X_2^2 + a_2X_1^2X_2^2 = (a_1 + a_2)X_1^2X_2^2\end{aligned}$$

3.3 The tensor product of algebras

Suppose $A_1, A_2, \ldots, A_n$ are R-algebras. Then $A_1 \otimes A_2 \otimes \ldots \otimes A_n$ is obviously a R-module because of the module property of an algebra. We will show that in fact it does also have a natural structure as an R-algebra.

Theorem 10. *Let $A_1, A_2, \ldots, A_n$ be R-algebras. Then*

$$A_1 \otimes A_2 \otimes \ldots \otimes A_n$$

is an R-algebra where the R-module structure is the usual and the product of two elements $a_1 \otimes a_2 \otimes \ldots \otimes a_n$ and $a_1' \otimes a_2' \otimes \ldots \otimes a_n'$ is $a_1a_1' \otimes a_2a_2' \otimes \ldots \otimes a_na_n'$.

Proof. To prove that $A_1 \otimes A_2 \otimes \ldots \otimes A_n$ is an R-algebra we need to provide it with an associative multiplication mapping with unity such that it is commutative with respect to multiplication with scalars from the underlying ring R in the sense described earlier.

Now consider the multilinear mapping

$$A_1 \times A_2 \times \ldots. \times A_n \times A_1 \times A_2 \times \ldots \times A_n \to A_1 \otimes A_2 \otimes \ldots \otimes A_n$$

in which $(a_1, \ldots, a_n, a'_1, \ldots, a'_n)$ is mapped into $a_1a'_1 \otimes a_2a'_2 \otimes \ldots \otimes a_na'_n$. By the universal property this mapping induces a homomorphism

$$A_1 \otimes A_2 \otimes \ldots \otimes A_n \otimes A_1 \otimes A_2 \otimes \ldots \otimes A_n \to A_1 \otimes A_2 \otimes \ldots \otimes A_n$$

of R-modules. Theorem 5 states there is a R-module isomorphism

$$A_1 \otimes \ldots \otimes A_n \otimes A_1 \otimes \ldots \otimes A_n \simeq (A_1 \otimes \ldots \otimes A_n) \otimes (A_1 \otimes \ldots \otimes A_n)$$

We now combine the induced homomorphism with this isomorphism to form a homomorphism

$$(A_1 \otimes \ldots \otimes A_n) \otimes (A_1 \otimes \ldots \otimes A_n) \to A_1 \otimes \ldots \otimes A_n \tag{3.3}$$

in which $(a_1 \otimes \ldots \otimes a_n) \otimes (a_1 \otimes \ldots \otimes a_n)$ is mapped with $a_1a'_1 \otimes a_2a'_2 \otimes \ldots \otimes a_na'_n$.

We can now define the multiplication mapping μ to be the mapping

$$\mu : (A_1 \otimes \ldots \otimes A_n) \times (A_1 \otimes \ldots \otimes A_n)$$

where $\mu(a_1 \otimes \ldots \otimes a_n, a'_1 \otimes \ldots \otimes a'_n)$ is mapped to the image $(a_1 \otimes \ldots \otimes a_n) \otimes (a'_1 \otimes \ldots \otimes a'_n)$ under the mapping (3.3). Obviously, μ is a bilinear mapping. It follows from this definition that for any $x, x', x'' \in A_1 \otimes \ldots \otimes A_n$

$$\begin{aligned}
\mu(\mu(x, x'), x'') &= \mu(\mu(a_1 \otimes \ldots \otimes a_n, a'_1 \otimes \ldots \otimes a'_n), a''_1 \otimes \ldots \otimes a''_n) \\
&= \mu(a_1a'_1 \otimes \ldots \otimes a_na'_n, a''_1 \otimes \ldots \otimes a''_n) \\
&= a_1a'_1a''_1 \otimes \ldots \otimes a_na'_na''_n \\
&= \mu(a_1 \otimes \ldots \otimes a_n, a'_1a''_1 \otimes \ldots \otimes a'_na''_n) \\
&= \mu(x, \mu(x', x''))
\end{aligned}$$

This proves the associativity of μ. Also because of the bilinearity of μ

$$\mu(rx, x') = r\mu(x, x') = \mu(x, rx')$$

and if $e_1, \ldots, e_n$ are the respective identity elements of the algebras A_i then

$$\mu(e_1 \otimes \ldots \otimes e_n, a_1 \otimes \ldots \otimes a_n) = a_1 \otimes \ldots \otimes a_n = \mu(a_1 \otimes \ldots \otimes a_n, e_1 \otimes \ldots \otimes e_n)$$

so the mapping μ is commutative with respect to multiplication with $r \in R$ and has an identity element $e_1 \otimes \ldots \otimes e_n$.

□

3.4 Some basic properties of the tensor product of algebras

In this section some of the results proven for tensor products of modules will be proven to hold true also for tensor product of algebras. First we will prove an extension of Theorem 4.

Theorem 11. *Let $A_1, A_2, \ldots . A_n$ and $B_1, B_2, \ldots B_p$ be R-algebras. Then there is an isomorphism*

$$A_1 \otimes \ldots \otimes A_n \otimes B_1 \otimes \ldots \otimes B_p \simeq (A_1 \otimes \ldots \otimes A_n) \otimes (B_1 \otimes \ldots \otimes B_p)$$

of R-algebras in which $a_1 \otimes \ldots \otimes a_n \otimes b_1 \otimes \ldots \otimes b_p$ is associated with $(a_1 \otimes \ldots \otimes a_n) \otimes (b_1 \otimes \ldots \otimes b_p)$.

Proof. By Theorem 5 there is an isomorphism f of R-modules which satisfies the module conditions. All that is needed to prove this theorem is to show that f also is an isomorphism with respect to the multiplication mapping.

The isomorphism f satisfies the following

$$f(a_1 \otimes \ldots \otimes a_n \otimes b_1 \otimes \ldots \otimes b_p) = (a_1 \otimes \ldots \otimes a_n) \otimes (b_1 \otimes \ldots \otimes b_p)$$

Now let

$$x = a_1 \otimes \ldots \otimes a_n \otimes b_1 \otimes \ldots \otimes b_p$$

and

$$x' = a'_1 \otimes \ldots \otimes a'_n \otimes b'_1 \otimes \ldots \otimes b'_p.$$

By Theorem 10

$$xx' = a_1a'_1 \otimes \ldots \otimes a_na'_n \otimes b_1b'_1 \otimes \ldots \otimes b_pb'_p$$

and as an immediate consequence

$$f(xx') = (a_1a'_1 \otimes \ldots \otimes a_na'_n) \otimes (b_1b'_1 \otimes \ldots \otimes b_pb'_p)$$

but also

$$\begin{aligned} f(x)f(x') &= ((a_1 \otimes \ldots \otimes a_n) \otimes (b_1 \otimes \ldots \otimes b_p))((a'_1 \otimes \ldots \otimes a'_n) \otimes (b'_1 \otimes \ldots \otimes b'_p)) \\ &= ((a_1 \otimes \ldots \otimes a_n)(a'_1 \otimes \ldots \otimes a'_n)) \otimes ((b_1 \otimes \ldots \otimes b_p)(b'_1 \otimes \ldots \otimes b'_p)) \\ &= (a_1a'_1 \otimes \ldots \otimes a_na'_n) \otimes (b_1b'_1 \otimes \ldots \otimes b_pb'_p) \\ &= f(xx') \end{aligned}$$

Recall that by Theorem 3 any element of $A_1 \otimes \ldots \otimes A_n \otimes B_1 \otimes \ldots \otimes B_p$ can be expressed as a sum of monomial tensors. Since f is an isomorphism and thereby also R-linear it follows directly that if y and y' are any two elements of $A_1 \otimes \ldots \otimes A_n \otimes B_1 \otimes \ldots \otimes B_p$ expressed as sums of monomial tensors then $f(yy') = f(y)f(y')$. The theorem follows from the bijective property of f. □

Note that by an identical argument as in Corollary 2, there is an algebra-isomorphism of A_1, A_2, A_3

$$(A_1 \otimes A_2) \otimes A_3 \simeq A_1 \otimes (A_2 \otimes A_3)$$

In a similar manner, a theorem extending Theorem 6 can be proved.

Theorem 12. *Let $i_1, i_2, \ldots, i_n$ be a permutation of $1, 2, \ldots, n$. Then there is an isomorphism of algebras*

$$A_1 \otimes A_2 \otimes \ldots \otimes A_n \simeq A_{i_1} \otimes A_{i_2} \otimes \ldots \otimes A_{i_n}$$

which associates $a_1 \otimes \ldots \otimes a_n$ with $a_{i_1} \otimes \ldots \otimes a_{i_n}$

Proof. Theorem 6 provides us with an R-module isomorphism which we will denote f. As in the proof of Theorem 11 all we need to do is to prove that f is an isomorphism also with respect to the multiplication mapping. Let $x = a_1 \otimes \ldots \otimes a_n$ and $x' = a'_1 \otimes \ldots \otimes a'_n$

$$f(xx') = f(a_1a'_1 \otimes \ldots \otimes a_na'_n) = a_{i_1}a'_{i_1} \otimes \ldots \otimes a_{i_n}a'_{i_n} =$$

$$(a_{i_1} \otimes \ldots \otimes a_{i_n})(a'_{i_1} \otimes \ldots \otimes a'_{i_n}) = f(x)f(x')$$

which together with the same reasoning as in the last proof is enough. □

Now for the next proof recall that R is in itself a R-algebra. This theorem is an extension of Theorem 7.

Theorem 13. *Let A be an R-algebra, considering R as an R-algebra, there is an isomorphisms*

$$R \otimes A \simeq A$$

such that $r \otimes a$ is mapped with ra. There is a similar isomorphism for $A \otimes R \simeq ar$.

Proof. By Theorem 7 there is an isomorphism of R-modules f. Now

$$f((r \otimes a)(r' \otimes a')) = f(rr' \otimes aa') =$$

$$= (rr')(aa') = (ra)(r'a') = f(r \otimes a)f(r' \otimes a')$$

and the rest is obvious from previous proofs. $\square$

The tensor product of algebra homomorphisms will prove to have the expected properties.

Theorem 14. *Suppose $A_1, A_2, \ldots A_n$ and $A'_1, A'_2, \ldots, A'_n$ are R-algebras and that there are homomorphisms $f_i : A_i \to A'_1$. Then the tensor product of homomorphisms is an algebra homomorphism*

$$f_1 \otimes \ldots \otimes f_n : A_1 \otimes \ldots \otimes A_n \to A'_1 \otimes \ldots \otimes A'_n$$

Proof. In 2.2 the concept of tensor products of module homomorphisms was defined. It is obvious that the same mapping $f = f_1 \otimes \ldots \otimes f_n$ maps the identity element to the identity element if each f_i is an algebra homomorphism. By the previous proofs in this section it is enough to prove that for $x, x' \in A_1 \otimes \ldots \otimes A_n$ $f(xx') = f(x)f(x')$. We can write $x = a_1 \otimes \ldots a_n$ and $x' = a'_1 \otimes \ldots \otimes a'_n$ so

$$\begin{aligned} f(xx') &= f(a_1a'_1 \otimes \ldots \otimes a_na'_n) \\ &= f_1(a_1a'_1) \otimes \ldots \otimes f_n(a_na'_n). \end{aligned}$$

Each f_i is a homomorphism so

$$\begin{aligned} f(a_1a'_1) \otimes \ldots \otimes f(a_na'_n) &= f(a_1)f(a'_1) \otimes \ldots \otimes f(a_n)f(a'_n) \\ &= (f(a_1) \otimes \ldots \otimes f(a_n))(f(a'_1) \otimes \ldots \otimes f(a'_n)) \\ &= f(x)f(x') \end{aligned}$$

which completes the proof. $\square$

3.5 Graded algebras

Recall the associative algebra of polynomials over the real numbers, $\mathbb{R}[X]$. This algebra has a natural structure as a direct sum of the submodules

$\{\mathbb{R}[X]_n\}_{n\in\mathbb{Z}}$ where each submodule consists of the polynomials of degree n. This representation as a direct sum of submodules also has some extra structure. The product of two elements $a_kX^k \in \{\mathbb{R}[X]_k\}$ and $a_pX^p \in \{\mathbb{R}[X]_p\}$ is

$$a_kX^ka_pX^p = a_ka_pX^{k+p}$$

which is an element in $\{\mathbb{R}[X]_{k+p}\}$. This is true for any two elements of $\mathbb{R}[X]$. This extra structure on an algebra with a representation as a direct sum is called **a grading** on that algebra.

Definition. *Let $\{A_n\}_{n\in\mathbb{Z}}$ be a family of submodules such that the algebra A is the direct sum of these modules. If the multiplication satisfies that*

$$a_k \in A_k, a_p \in A_p \quad \Rightarrow \quad a_ka_p \in A_{k+p}$$

*then $\{A_n\}_{n\in\mathbb{Z}}$ is said to constitute a **grading** of A. The elements of A_n is said to be **homogenous of degree n** and an algebra with such a grading is called a **graded algebra**. If $A_n = 0$ for $n < 0$ the grading is called a **non-negative grading**. We let $|x|$ denote the degree of x.*

We will prove some properties of a graded algebra.

Theorem 15. *The identity element $\mathbf{1}_A$ belongs to A_0*

Proof. Assume $\mathbf{1}_A \notin A_0$ and instead was of degree $q \neq 0$. Then the equality

$$\mathbf{1}_Aa_k = a_k = a_k\mathbf{1}_A$$

implies that $a_k \in A_k$ and $a_k \in A_{k+q}$ which contradicts that A is a direct sum of $\{A_n\}_{n\in\mathbb{Z}}$. □

Theorem 16. *A_0 is a R-subalgebra of A.*

Proof. Since we already know that A_0 is an submodule and that $\mathbf{1}_A \in A_0$, all we need to prove is closure for the multiplication mapping. For $a_0 \in A_0$ and $a_0' \in A_0$ we have

$$a_0a_0' \in A_{0+0}$$

and the proof is done. □

Theorem 17. *Assume A is a graded algebra and that A is generated as an R-algebra by A_1. Then the grading is non-negative and for $p > 0$, each element of A_p is a sum of products of p elements in A_1. Furthermore A_0 is generated, as an R-module, by the identity element $\mathbf{1}_A$ and therefore A_0 is contained in the center of A.*

Proof. A is generated as an R-module by products of elements of A_1, including the empty product and such a product will always be homogenous of degree p where p is the number of factors, which of course always will be ≥ 0. Also the empty product of A_1 is the identity element $\mathbf{1}_A$, and since the whole of A is generated as a R-module by products of A_1 and $\mathbf{1}_A$ is the only element in A_0 which can be written as a product of elements in A_1, A_0 must be generated by $\mathbf{1}_A$. $\square$

Theorem 18. *Let $A^{(1)}, A^{(2)}, \ldots, A^{(n)}$ be graded algebras then*

$$A = A^{(1)} \otimes A^{(2)} \otimes \ldots \otimes A^{(n)}$$

is also a graded algebra. The family of submodules constituting the grading on A are the submodules $\{A_n\}_{n\in\mathbb{Z}}$

$$A_n = \sum_{|I|=n} A^{(1)}_{i_1} \otimes A^{(2)}_{i_2} \otimes \ldots \otimes A^{(n)}_{i_n}$$

where $I = i_1, i_2, \ldots, i_n$ is any sequence of n integers and $|I| = i_1+i_2+\ldots+i_n$.

Proof. By Theorem 9 we know that

$$A^{(1)} \otimes A^{(2)} \otimes \ldots \otimes A^{(n)} = \sum_{I} A^{(1)}_{i_1} \otimes A^{(2)}_{i_2} \otimes \ldots \otimes A^{(n)}_{i_n}$$

and if we set

$$\sum_I A_I = \sum_I A^{(1)}_{i_1} \otimes A^{(2)}_{i_2} \otimes \ldots \otimes A^{(n)}_{i_n}$$

then $A = \sum_I A_I$. Let $J = (j_1, j_2, \ldots, j_n)$ be a second sequence of n integers. Then for $a_I \in A_I$ and $a_J \in A_J$ we have

$$a_I a_J = a_{i_1} a_{j_1} \otimes a_{i_2} a_{j_2} \otimes \ldots \otimes a_{i_n} a_{j_n}$$

and since each A^μ is a graded algebra $a_{i_\mu} a_{j_\mu} \in A^\mu_{i_\mu + j_\mu}$ for $1 \leq \mu \leq n$ and in addition

$$a_I a_J \in A^{(1)}_{i_1+j_1} \otimes A^{(2)}_{i_2+j_2} \otimes \ldots \otimes A^{(n)}_{i_n+j_n} = A_{I+J}$$

Now recall that

$$A_n = \sum_{|I|=n} A^{(1)}_{i_1} \otimes A^{(2)}_{i_2} \otimes \ldots \otimes A^{(n)}_{i_n}$$

so it follows easily from what we just have proven that for $x \in A_n$ and $y \in A_k$, $xy \in A_{n+k}$. $\square$

This grading of the tensor product of graded algebras will be called the **total grading** of the tensor product. Evidently, if each A^μ is non-negative, then so is the grading of A.

It is worth noting that because of the way Theorem 11 and 12 are formulated it is true that these isomorphisms are also isomorphisms of graded algebras. Before we can conclude the same for Theorem 13 we note the following. For an algebra B one can obtain a grading by defining $B_0 = B$ and $B_n = 0$ for $n \neq 0$. If we endow R with this trivial grading as an R-algebra it follows in the same way that Theorem 13 provides us with isomorphisms of graded algebras. Also if the homomorphisms f_i are assumed to be homomorphisms of graded algebras then Theorem 14 provides us with an homomorphism of graded algebras.

Chapter 4

The tensor algebra

In this chapter we construct an associative algebra out of the tensor powers of a module. Also, we show that the tensor algebra of a module is a solution to a universal problem.

4.1 Definition of the tensor algebra

The n:th tensor power of a module M, denoted $T_n(M)$, is simply

$$T_n(M) = \underbrace{M \otimes M \otimes \ldots \otimes M}_{\text{n times}}$$

and we denote $T(M) = \sum T_n(M)$

Theorem 19. *The set $T(M)$ of tensor powers of a module form an associative graded R-algebra where the multiplication mapping is the tensor product. The identity element is the identity element of R, 1_R, and the grading consists of $\{T_n\}_{n\in\mathbb{N}}$ where*

$$T_n = \underbrace{M \otimes M \otimes \ldots \otimes M}_{n\ times}$$

and this grading is non-negative. This algebra is generated (as an algebra) by $T_1(M)$.

Proof. To prove this theorem we have to show that the tensor product satisfies the conditions for a multiplicative mapping of an associative algebra.

Theorem 11 supplies us with an isomorphism

$$T_p \otimes T_q \simeq T_{p+q}$$

then Theorem 12 gives the isomorphisms $T_0 \otimes T_p \simeq T_p$ and $T_p \otimes T_0 \simeq T_p$ (recall that $R \simeq T_0$). Thus for $p \geq 0$ and $q \geq 0$ we have explicit isomorphisms $T_p \otimes T_q \simeq T_{p+q}$. As a direct consequence there is a bilinear map

$$\mu_{pq} : T_p \times T_q \to T_{p+q}$$

in which, for $x_p \in T_p$ and $y_q \in T_q$, $\mu_{pq}(x_p, y_q)$ is the image of $x_p \otimes y_p$ under the relevant isomorphism. To summarize, if $m_1, m_2, \ldots, m_p$ and $m'_1, m'_2, \ldots, m'_q$ belong to M then

$$\begin{aligned} &\mu_{pq}(m_1 \otimes m_2 \otimes \ldots \otimes m_p, m'_1 \otimes m'_2 \otimes \ldots \otimes m'_q) = \\ = \quad &m_1 \otimes m_2 \otimes \ldots \otimes m_p \otimes m'_1 \otimes m'_2 \otimes \ldots \otimes m'_q \end{aligned} \tag{4.1}$$

if $r \in T_0 = R$ we instead have

$$\mu_{0q}(r, y_q) = r y_q \tag{4.2}$$

$$\mu_{p0}(x_p, r) = x_p r = r x_p. \tag{4.3}$$

Now suppose $p, q \geq 0$ and let x_p, y_q, z_t belong to T_p, T_q, T_t respectively. Now it follows from the $(4.1 - 4.3)$ that

$$\mu_{p+q,t}(\mu_{pq}(x_p, y_q), z_t) = \mu_{p,q+t}(x_p, \mu_{qt}(y_p, z_t)$$

and we are now ready to define multiplication on A. Let $x, y \in A$, these elements have unique representations $x = x_0 + x_1 + x_2 + \ldots$ and $y = y_0 + y_1 + y_2 + \ldots$, where of course x_n, y_n belong to T_n. Now the required product may be defined as

$$\mu(x, y) = \sum_{p \geq 0, q \geq 0} \mu_{pq}(x_p, y_q).$$

It follows directly from the multilinearity of the tensor product and the bilinearity of μ that

$$\mu(rx, y) = r\mu(x, y) = \mu(x, ry),$$

and that multiplication is distributive with respect to addition. Also 1_R belongs to T_0 and $\mu(1_r, y) = y$, $\mu(x, 1_r) = x$ by $(4.2 - 3)4$. By what we have just done it is also clear that this algbra is graded and that the grading is the stated.

We will denote this tensor algebra of a module M with $T(M)$ and the grading with $\{T_n(M)\}_{n\in\mathbb{Z}}$. It is helpful to define a mapping

$$\phi : M \to T(M) \tag{4.4}$$

where M simply is mapped isomorphicly to T_1. With this notation one can conclude that $T(M)$ is generated as an algebra by $\phi(M) = T_1(M)$ since

$$\phi(M)\phi(M)\ldots\phi(M) = M \otimes M \otimes \ldots \otimes M.$$

□

4.2 The universal property of the tensor algebra

As stated earlier, the tensor algebra provides a solution to a universal problem. Let M be an R-module. Now suppose A is an R-algebra and that $\phi : M \to A$ is a homomorphism of R-modules. Let $h : A \to B$ be a homomorphism of R-algebras, then $h \circ \phi : M \to B$ is an R-module homomorphism of M into B. We are now ready to formulate the problem, which is similar to the universal problem of multilinear mappings.

The universal problem of algebras containing *M*. *Let M be an R-module, the problem is to choose A and $\phi : M \to A$ so that given any R-module homomorphism $\psi : M \to B$, where B is an R-algebra, there exists a unique homomorphism $h : A \to B$, of R-algebras such that $h \circ \phi = \psi$.*

It is obvious that if there are two solutions to this problem, (A, ϕ) and (A', ϕ'), then there are inverse algebra-isomorphisms $\lambda : A \to A'$ and $\lambda' : A' \to A$ such that $\lambda \circ \phi = \phi'$ and $\lambda' \circ \phi' = \phi$. So, much like the universal problem of multilinear mappings, this problem has essentially at most one solution.

Theorem 20. *The tensor algebra of M, $T(M)$ together with the natural mapping $\phi : M \to T(M)$ provides a solution of the universal problem. That is, for any R-module homomorphism $\psi : M \to B$, where B is an algebra, there is a unique algebra-homomorphism $h : A \to B$ such that $h \circ \phi = \psi$.*

Proof. Let $\psi : M \to B$ be a module homomorphism. We can construct a multilinear mapping

$$M \times M \times \ldots \times M \to B,$$

where $(m_1, m_2, \ldots, m_n)$ is mapped to $\psi(m_1)\psi(m_2)\ldots\psi(m_n)$. That this mapping is multilinear follows from the conditions for a multiplication map of the algebra B. This induces a mapping $h_p : T_p(M) \to B$ where

$$h_p(m_1 \otimes m_2 \otimes \ldots \otimes m_n) = \psi(m_1)\psi(m_2)\ldots\psi(m_n).$$

For $T_0(M) = R$ we can define $h_0 : T_0(M) \to B$ to be the structural homomorphism of $R \to B$. Now $T(M)$ is a direct sum of $\{T_n(M)\}_{n\geq 0}$, consequently we can define a mapping

$$h : A \to B$$

which agrees with h_n on $T_n(M)$. If x and y are homogenous elements it is easily verified, using the appropriate of $(4.1 - 4.3)$,that $h(xy) = h(x)h(y)$. It follows that $h : T(M) \to B$ is a homomorphism of R-algebras by the multilinear and linear properties the tensor product and the homomorphism as any element of $T(M)$ is a direct sum of homogenous elements.

The only thing left to prove is that $h \circ \phi = \psi$ and that h is the only homomorphism with this property. For $x \in M = T_1(M)$ we have $h(x) = \psi(x)$ and since $\phi(M) = M = T_1(M)$, $h \circ \phi = \psi$. But $M = T_1(M)$ generates $T(M)$, in the sense of an algebra, and as a consequence h must be the only algebra homomorphism which combined with ϕ gives ψ. □

Because of the great generality of the tensor algebra many other algebras of interest are created by imposing an equality on some of the elements by quotioning out an ideal. For example, it is possible to construct the symmetric algebra $S(M)$ of a module M by imposing a symmetric equivalence. If we define an ideal $I(M)$ as the ideal generated by all elements of the form $x \otimes y - y \otimes x$ we define the symmetric algebra as

$$S(M) = T(M)/I(M)$$

or in other words we impose the equivalence $x \otimes y = y \otimes x$. Other examples of interesting algebras constructed in this manner are the exterior algebra, universal enveloping algebras and Clifford algebras.

Chapter 5

Coalgebras

Coalgebras are the dual to algebras. For any mapping used to define an algebra there is an opposite mapping in the definition of a coalgebra. If we describe an algebra by commuting diagrams, which we will do soon, we obtain an coalgebra by reversing all the arrows. For example, in an algebra one has a mapping

$$A \otimes A \xrightarrow{\mu} A$$

while in a coalgebra one has a mapping

$$A \otimes A \xleftarrow{\Delta} A$$

going in the opposite direction.

Before the concept of a coalgebra will be studied closer and better defined, the concept of an algebra will be redefined. We will redefine it in a way which more naturally allows us to move on to the coalgebra and see similarities and differences. Also we will need this new description when we later define the concept of a Hopf algebra.

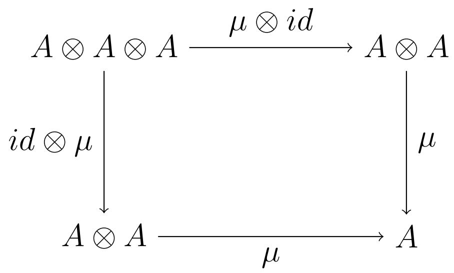
5.1 A new view of associative algebras

Since commutatative diagrams will be extensively used for this part we will start with a definition of such.

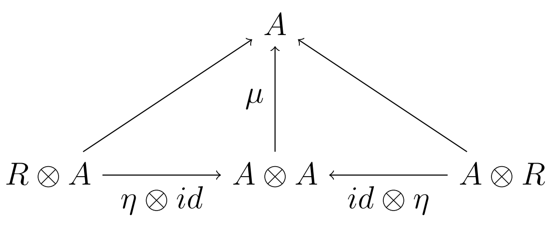
Definition. *A* ***commutative diagram*** *is a diagram where each vertex is an object and each arrow is a morphism such that all directed paths with the same initial and final vertex lead to the same result by composition.*

One can now make a new definition of an R-algebra by making use of these commutative diagrams.

Definition. *Let A be an R-module. Suppose there is an R-linear mapping μ such that the following diagram commutes:*



Which corresponds to μ being associative. Furthermore suppose there is a R-linear mapping $\eta : R \to A$ of A such that the diagram



commutes. $R \otimes A \to A$ and $A \otimes R \to A$ are the isomorphisms provided by Theorem 7. The triple (A, μ, η) will be said to constitute an associative R-algebra; μ is called the ***multiplication mapping*** *and η the* ***unit mapping****.*

For $x, y \in A$ we define $xy = \mu(x \otimes y)$. It is easily verified that this definition of an associative algebra is compatible with the one earlier given, where $\eta : R \to A$ is the structural homomorphism.

We are now going look at some of the important aspects of the theory of algebras from this new definition.

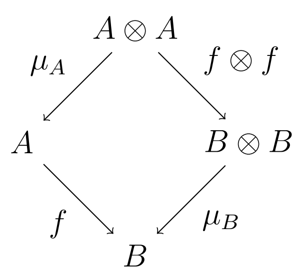
Theorem 21. *Let (A, μ_A, η_A) and (B, μ_B, η_B) be two associative R-algebras. If f is a mapping from A to B then f is an algebra-homomorphism if and only if the following conditions are satisfied*

(i) f *is R-linear*

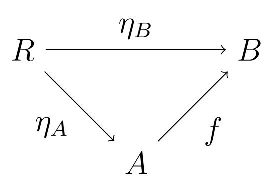
(ii) $f \circ \mu_a = \mu_b \circ (f \otimes f)$

(iii) $f \circ \eta_A = \eta_B$

Proof. The first condition is an obvious property since it is a property we already now an algebra-homomorphism must posess. The second condition is equivalent to that the following diagram



commutes. Similarly the third condition be described by the following commuting diagram.



Together, these conditions are equivalent to those given in 3.1. □

To describe the algebra created by the tensor product of n algebras we will need to define some new mappings. Let (A_i, μ_i, η_i) be algebras for each $1 \leq i \leq n$, and let

$$\Lambda_n : (A_1 \otimes \ldots \otimes A_n) \otimes (A_1 \otimes \ldots \otimes A_n) \to (A_1 \otimes A_1) \otimes \ldots \otimes (A_n \otimes A_n)$$

be an isomorphism matching $(a_1 \otimes \ldots \otimes a_n) \otimes (a'_1 \otimes \ldots a'_n)$ with $(a_1 \otimes a'_1) \otimes \ldots \otimes (a_n \otimes a'_n)$. Also let

$$\Delta_R^{(n)} : R \to \underbrace{R \otimes R \otimes \ldots \otimes R}_{\text{n factors}}$$

be the R-linear mapping in which $1 \to 1 \otimes 1 \otimes \ldots \otimes 1$.

Now we can describe the algebra $A_1 \otimes A_2 \otimes ... \otimes A_n$ from Section 3.3 as having the mappings

$$\mu_{A_1 \otimes \ldots \otimes A_n} = (\mu_1 \otimes \mu_2 \otimes \ldots \otimes \mu_n) \circ \Lambda_n \tag{5.1}$$

and

$$\eta_{A_1 \otimes \ldots \otimes A_n} = (\eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n) \circ \Delta_R^{(n)}. \tag{5.2}$$

Next in turn are graded algebras. Assume (A, μ, η) is an algebra and assume $\{A_n\}_{n \in \mathbb{Z}}$ is a family of submodules of A such that

$$A = \sum_{n \in \mathbb{Z}} A_n,$$

which means that we suppose that $\{A_n\}_{n \in \mathbb{Z}}$ grades A *as an R-module*. From this we also have a grading $\{(A \otimes A)_n\}_{n \in \mathbb{Z}}$ of the module $A \otimes A$, where

$$(A \otimes A)_n = \sum_{p+q=n} A_p \otimes A_q,$$

this is of course the usual total grading on $A \otimes A$. Now the statement that $\{A_n\}_{n \in \mathbb{Z}}$ is an algebra-grading is equivalent to that the mappings $\mu : A \otimes A \to A$ and $\eta : R \to A$ preserves degrees, with R being granted the trivial grading.

Suppose now that $(A^{(1)}, \mu_1, \eta_1), (A^{(2)}, \mu_2, \eta_2), \ldots, (A^{(n)}, \mu_n, \eta_n)$ are graded R-algebras. For each two sequences of integers $I = (i_1, i_2, \ldots, i_n)$ and $J = (j_1, j_2, \ldots, j_n)$ there is an isomorphism between

$$(A_{i_1}^{(1)} \otimes A_{i_2}^{(2)} \otimes \ldots \otimes A_{i_n}^{(n)}) \otimes (A_{j_1}^{(1)} \otimes A_{j_2}^{(2)} \otimes \ldots \otimes A_{j_n}^{(n)}) \tag{5.3}$$

and

$$(A^{(1)}_{i_1} \otimes A^{(1)}_{j_1}) \otimes (A^{(2)}_{i_2} \otimes A^{(2)}_{j_2}) \otimes \ldots \otimes (A^{(n)}_{i_n} \otimes A^{(n)}_{j_n}). \qquad (5.4)$$

Now since $(A^{(1)} \otimes A^{(2)} \otimes \ldots \otimes A^{(n)}) \otimes (A^{(1)} \otimes A^{(2)} \otimes \ldots \otimes A^{(n)})$ is the direct sum of the modules (5.3) while $(A^{(1)} \otimes A^{(1)}) \otimes (A^{(2)} \otimes A^{(2)}) \otimes \ldots \otimes (A^{(n)} \otimes A^{(n)})$ is the direct sum of modules (5.4) we can combine the various isomorphisms and define an isomorphism

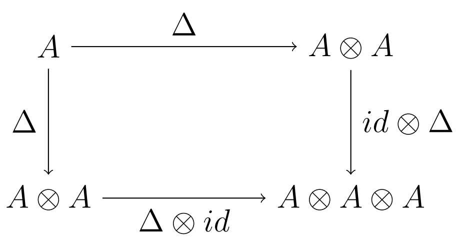
$$\Lambda_n : (A^{(1)} \otimes \ldots \otimes A^{(n)}) \otimes (A^{(1)} \otimes \ldots \otimes A^{(n)}) \to (A^{(1)} \otimes A^{(1)}) \otimes \ldots \otimes (A^{(n)} \otimes A^{(n)})$$

such that we can now describe $A^{(1)} \otimes A^{(2)} \otimes \ldots \otimes A^{(n)}$ as a *graded algebra* with this new Λ_n in (5.1).

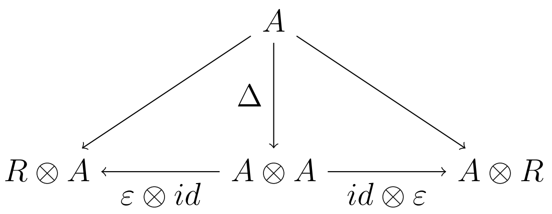
5.2 Definition of a coalgebra

We have now done the necessary preperations to describe a coalgebra as a concept dual to that of an algebra. That is, for each commutative diagram we have used to define an algebra, we obtain the corresponding diagram for coalgebras by reversing the arrows.

Definition. *Let A be an R-module and suppose that we are given R-linear mappings $\Delta : A \to A \otimes A$ and $\varepsilon : A \to R$. If the following two diagrams*



and



are commutative ($A \to A \otimes R$ maps a into $a \otimes 1$). We will call the triple (A, Δ, ε) a ***coalgebra****. The mapping $\Delta : A \to A \otimes A$ is called the* ***comultiplication*** *mapping of the coalgebra and the commutativity of the first diagram is described by saying that comultiplication is* ***coassociative****. The mapping $\varepsilon : A \to R$ is known as the* ***counit****.*

R itself becomes a coalgebra if we define the two mappings to be $\Delta : R \to R \otimes R$ to be the R-linear mapping which carries 1 into $1 \otimes 1$ and $\varepsilon : R \to R$ to be the identity mapping. This allows us to speak of R as an coalgebra, and whenever we do, this will be the structure considered. We will now make two relevant definitions.

Definition. *Suppose $(A, \Delta_A, \varepsilon_A)$ and $(B, \Delta_B, \varepsilon_B)$ are coalgebras. A mapping $f : A \to B$ is called a* ***homomorphism of coalgebras*** *if the following conditions are satisfied*

(i) f *is R-linear*

(ii) $\Delta_B \circ f = (f \otimes f) \circ \Delta_A$

(iii) $\varepsilon_B \circ f = \varepsilon_A$

if the mapping f is a bijection it is called an ***isomorphism of coalgebras****.*

Definition. *Let (A, Δ, ε) be an R-coalgebra and let $\{A_n\}_{n \in \mathbb{Z}}$ be a grading of A considered as a module. Now let $A \otimes A$ have the usual induced total grading $\{(A \otimes A)_n\}$ and let R have the trivial grading.*

We say that A is a ***graded coalgebra*** *with the grading $\{A_n\}_{n \in \mathbb{Z}}$ if Δ and ε preserves the degrees of homogenous elements. Also, a* ***homomorphism of graded coalgebras*** *is coalgebra homomorphism which preserves degrees.*

Note that this definition implies that if (A, Δ, ε) is a graded coalgebra, then ε maps any homogeneous element of a degree not equal to zero into the zero element of R.

5.3 The tensor product of coalgebras

Before we move on to define the tensor product of coalgebras we will define some concepts needed.

Suppose $A_1, A_2, \ldots, A_p$ are R-modules. We define the isomorphism

$$V_n : (A_1 \otimes A_1) \otimes \ldots \otimes (A_n \otimes A_n) \to (A_1 \otimes \ldots \otimes A_n) \otimes (A_1 \otimes \ldots \otimes A_n) \quad (5.5)$$

in which $(a_1 \otimes a_1') \otimes \ldots \otimes (a_n \otimes a_n')$ is matched with the element $(a_1 \otimes \ldots \otimes a_n) \otimes (a_1' \otimes \ldots \otimes a_n')$. Also in a similar manner we define the isomorphism

$$\begin{aligned} W_n : (A_1 \otimes A_1 \otimes A_1) \otimes \ldots \otimes (A_n \otimes A_n \otimes A_n) \\ \to (A_1 \otimes \ldots \otimes A_n) \otimes (A_1 \otimes \ldots \otimes A_n) \otimes (A_1 \otimes \ldots \otimes A_n). \end{aligned} \quad (5.6)$$

That these two module isomorphisms exist is a consequence of Theorem 5 and 6.

Suppose that for $1 \leq i \leq n$, $(A_i, \Delta_i, \varepsilon_i)$ is an R-coalgebra. And set

$$A = A_1 \otimes A_2 \otimes \ldots \otimes A_n$$

at the moment only considered as a module. Next we define the R-linear mapping $\Delta : A \to A \otimes A$ to be the composition

$$\Delta = V_n \circ (\Delta_1 \otimes \Delta_2 \otimes \ldots \otimes \Delta_n).$$

Also we define a mapping

$$\mu_R^n : \underbrace{R \otimes R \otimes \ldots \otimes R}_{\text{n times}} \to R$$

in which $\mu_R^n(r_1 \otimes r_2 \otimes \ldots \otimes r_n) = r_1 r_2 \ldots r_n$ with which we can define the mapping $\varepsilon : A \to R$ to be

$$\varepsilon = \mu_R^n \circ (\varepsilon_1 \otimes \varepsilon_2 \otimes \ldots \otimes \varepsilon_n)$$

which is R-linear.

Theorem 22. *Let $(A_i, \Delta_i, \varepsilon_i)$ be R-coalgebras. Then, with the notation just described, (A, Δ, ε) is also an R-coalgebra*

Proof. To establish our claim we will have to prove that the conditions described in the definition of a coalgebra are satisfied.

The mapping

$$(id \otimes \Delta_1) \circ \Delta_1 \otimes (id \otimes \Delta_2) \circ \Delta_2 \otimes \ldots \otimes (id \otimes \Delta_n) \circ \Delta_n$$

maps $A_1 \otimes A_2 \otimes \ldots \otimes A_n$ into $(A_1 \otimes A_1 \otimes A_1) \otimes \ldots \otimes (A_n \otimes A_n \otimes A_n)$ and consequently

$$W_n \circ ((id \otimes \Delta_1) \circ \Delta_1 \otimes \ldots \otimes (id \otimes \Delta_n) \circ \Delta_n) \tag{5.7}$$

maps A into $A \otimes A \otimes A$. To show the coassociativity of Δ as defined we will start by with proving that $(id \otimes \Delta) \circ \Delta$ is equal to the mapping (5.7).

Since any element in $A \otimes A$ can be written as a finite sum of monomial tensors we know that

$$\Delta_i(a_i) = \sum_{\lambda} a_i'(\lambda) \otimes a_i''(\lambda)$$

where λ is some labelling. To simplify the notation we will instead write $\Delta_i(a_i) = \sum a_i' \otimes a_i''$. This notation will be called the **summative notation** of the coproduct. Because of this we can write

$$(\Delta_1 \otimes \ldots \otimes \Delta_n)(a_1 \otimes \ldots \otimes a_n) = (\sum a_1' \otimes a_1'') \otimes \ldots \otimes (\sum a_n' \otimes a_n'')$$

and as a consequence

$$\Delta(a_1 \otimes \ldots \otimes a_n) = \sum (a_1' \otimes \ldots a_n') \otimes (a_1'' \otimes \ldots \otimes a_n'')$$

where the summation is taken over all the different labellings. This rewriting is possible because of the multilinearity of the tensor product. It follows that

$$(id \otimes \Delta)(\Delta(a_1 \otimes \ldots \otimes a_n)) = \sum (a_1' \otimes \ldots \otimes a_n') \otimes \Delta(a_1'' \otimes \ldots \otimes a_n'').$$

We also have that,

$$\begin{aligned}((id \otimes \Delta_1) \circ \Delta_1 \otimes \ldots \otimes (id \otimes \Delta_n) \circ \Delta_n)(a_1 \otimes \ldots \otimes a_n) \\ = (\sum a_1' \otimes \Delta_1(a_1'')) \otimes \ldots \otimes (\sum a_n' \otimes \Delta_n(a_n''))\end{aligned}$$

and from this it follows that

$$W_n \circ ((id \otimes \Delta_1) \circ \Delta_1 \otimes \ldots \otimes (id \otimes \Delta_n) \circ \Delta_n)(a_1 \otimes \ldots \otimes a_n)$$

is mapped to

$$\sum (a_1' \otimes \ldots \otimes a_n') \otimes V_n(\Delta_1(a_1'') \otimes \ldots \otimes \Delta_n(a_n'')) =$$

$$= \sum (a_1' \otimes \ldots \otimes a_n') \otimes \Delta(a_1'' \otimes \ldots \otimes a_n'')$$

and we have now proven the claim that

$$(id \otimes \Delta) \circ \Delta = W_n \circ ((id \otimes \Delta_1) \circ \Delta_1 \otimes \ldots \otimes (id \otimes \Delta_n) \circ \Delta_n).$$

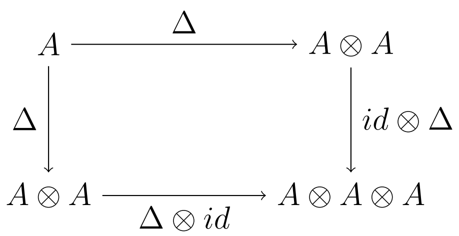
In the same manner it can be proved that

$$(\Delta \otimes id) \circ \Delta = W_n \circ ((\Delta_1 \otimes id) \circ \Delta_1 \otimes \ldots \otimes (\Delta_n \otimes id) \circ \Delta_n).$$

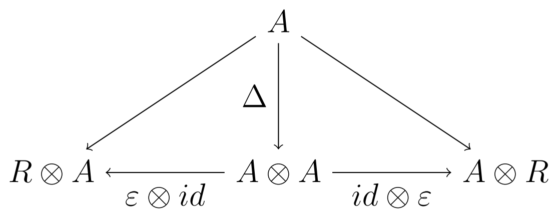
For each Δ_i we have by its coassociativity that $(id \otimes \Delta_i) \circ \Delta_i = (\Delta_i \otimes id) \circ \Delta_i$. And as a consequence the following equality holds

$$\begin{aligned} (id \otimes \Delta) \circ \Delta &= W_n \circ ((id \otimes \Delta_1) \circ \Delta_1 \otimes \ldots \otimes (id \otimes \Delta_n) \circ \Delta_n) \\ &= W_n \circ ((\Delta_1 \otimes id) \circ \Delta_1 \otimes \ldots \otimes (\Delta_n \otimes id) \circ \Delta_n) \\ &= (\Delta \otimes A) \circ \Delta \end{aligned}$$

and the commutativity of the diagram



has been proven. Now to complete the proof we have to prove the commutativity of the diagram



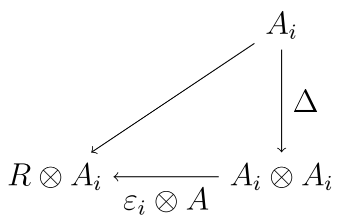
as well. We consider only the left triangle since the proof of the right is almost identical. It is enough to prove that the effect of applying the composition of homomorphisms

$$A \xrightarrow{\quad \Delta \quad} A \otimes A \xrightarrow{\varepsilon \otimes id} R \otimes A \xrightarrow{\quad \sim \quad} A$$

to an element $a_1 \otimes a_2 \otimes \ldots \otimes a_n$ is to leave the element unchanged, when the last isomorphism is the one given by theorem 13. Now when these homomorphisms are applied the image is

$$(\sum \varepsilon_1(a_1')a_1'') \otimes \ldots \otimes (\sum \varepsilon_n(a_n')a_n'')$$

but we already know that for each i that $\sum \varepsilon_i(a_i')a_i'' = a_i$ since the diagram



commutes. Accordingly the first diagram commutes and the proof is done. □

Definition. *The coalgebra (A, Δ, ε) as described in Theorem 21 is called the tensor product of coalgebras and will be denoted $A_1 \otimes A_2 \otimes \ldots \otimes A_n$.*

5.4 Some properties of the tensor product of coalgebras

As noted in the beginning of this chapter the coalgebra is a concept dual to that of an algebra, and as a consequence for most of the results of an algebra there is a corresponding result for coalgebras.

Theorem 23. *Let $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ be coalgebras and for each $1 \leq i \leq n$ let*

$$f_i : A_i \to B_i$$

be a homomorphism of coalgebras. Then

$$f_1 \otimes \ldots \otimes f_n : A_1 \otimes \ldots \otimes A_n \to B_1 \otimes \ldots \otimes B_n$$

is also a coalgebra homomorphism.

Proof. We know that $f_1 \otimes \ldots \otimes f_n$ is a module-homomorphism, i.e. R-linear. Now put $f = f_1 \otimes \ldots \otimes f_n$, $A = A_1 \otimes \ldots \otimes A_n$ and $B = B_1 \otimes \ldots \otimes B_n$. Using the summative notation of the coproduct we have

$$\Delta_A(a_1 \otimes \ldots \otimes a_n) = \sum (a_1' \otimes \ldots \otimes a_n') \otimes (a_1'' \otimes \ldots \otimes a_n'')$$

and therefore

$$(f \otimes f)(\Delta_A(a_1 \otimes \ldots a_n)) = \sum (f_1 a_1' \otimes \ldots \otimes f_n a_n') \otimes (f_1 a_1'' \otimes \ldots \otimes f_n a_n'').$$

Because f_i is an coalgebra homomorphism $\Delta_{B_i}(f_i a_i) = \sum f_i a_i' \otimes f_i a_i''$ and as a consequence

$$\Delta_B(f_1 a_1 \otimes \ldots \otimes f_n a_n) = \sum (f_1 a_1' \otimes \ldots \otimes f_n a_n') \otimes (f_1 a_1'' \otimes \ldots \otimes f_n a_n'').$$

This shows that $\Delta_B \circ f = (f \otimes f) \circ \Delta_A$. Now for counits, for each i we have the equality $\varepsilon_{B_i} \circ f = \varepsilon_{A_i}$ and therefore

$$\begin{aligned}(\varepsilon_B \circ f)(a_1 \otimes \ldots \otimes a_n) &= \varepsilon_{B_1}(f_1 a_1)\varepsilon_{B_2}(f_2 a_2) \ldots \varepsilon_{B_n}(f_n a_n) \\ &= \varepsilon_{A_1}(a_1)\varepsilon_{A_2}(a_2) \ldots \varepsilon_{A_n}(a_n)\end{aligned}$$

so $\varepsilon_B \circ f = \varepsilon_A$ and the proof is done. □

Theorem 24. *Let $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_p$ be coalgebras. Then there is an isomorphism*

$$A_1 \otimes \ldots \otimes A_n \otimes B_1 \otimes \ldots \otimes B_p \simeq (A_1 \otimes \ldots \otimes A_n) \otimes (B_1 \otimes \ldots \otimes B_p)$$

.

Proof. We know from Theorem 5 that there is a corresponding isomorphism of modules, we now prove that this isomorphism is also an isomorphism of coalgebras if the modules are in fact coalgebras. Denote $A = A_1 \otimes \ldots \otimes A_n$, $B = B_1 \otimes \ldots \otimes B_p$ and $C = A_1 \otimes \ldots \otimes A_n \otimes B_1 \otimes \ldots \otimes B_p$. We denote the module-isomorphism

$$\phi : A_1 \otimes \ldots \otimes A_n \otimes B_1 \otimes \ldots \otimes B_p \xrightarrow{\sim} (A_1 \otimes \ldots \otimes A_n) \otimes (B_1 \otimes \ldots \otimes B_p).$$

For this proof to be complete it suffices to show that ϕ is a coalgebra-homomorphism. For $(\phi \otimes \phi) \circ \Delta_C$ we have

$$((\phi \otimes \phi) \circ \Delta_C)(a_1 \otimes \ldots \otimes a_n \otimes b_1 \otimes \ldots \otimes b_p)$$
$$= \sum((a'_1 \otimes \ldots \otimes a'_n) \otimes (b'_1 \otimes \ldots \otimes b'_p)) \otimes ((a''_1 \otimes \ldots \otimes a''_n) \otimes (b''_1 \otimes \ldots \otimes b''_p)).$$

But for $\Delta_{A \otimes B} \circ \phi$ we have

$$(\Delta_{A \otimes B} \circ \phi)(a_1 \otimes \ldots \otimes a_n \otimes b_1 \otimes \ldots \otimes b_p) = \Delta_{A \otimes B}((a_1 \otimes \ldots \otimes a_n) \otimes (b_1 \otimes \ldots \otimes b_p))$$
$$= \sum((a'_1 \otimes \ldots \otimes a'_n) \otimes (b'_1 \otimes \ldots \otimes b'_p)) \otimes ((a''_1 \otimes \ldots \otimes a''_n) \otimes (b''_1 \otimes \ldots \otimes b''_p)),$$

so $(\phi \otimes \phi) \circ \Delta_C = \Delta_{A \otimes B} \circ \phi$. The result of applying both ε_C and $\varepsilon_{A \otimes B} \circ \phi$ to $a_1 \otimes \ldots \otimes a_n \otimes b_1 \otimes \ldots \otimes b_p$ is

$$\varepsilon_{A_1}(a_1) \ldots \varepsilon_{A_n}(a_n) \varepsilon_{B_1}(b_1) \ldots \varepsilon_{B_p}(b_p)4$$

so $\varepsilon_{A \otimes B} \circ \phi = \varepsilon_C$ and the proof is done □

Theorem 25. *Let $A_1, A_2, \ldots, A_n$ be coalgebras and let $I = i_1, i_2, \ldots, i_n$ be a permutation of $(1, 2, \ldots, n)$. Then there is an isomorphism*

$$A_1 \otimes A_2 \otimes \ldots \otimes A_n \simeq A_{i_1} \otimes A_{i_2} \otimes \ldots \otimes A_{i_n}.$$

Proof. We denote $A = A_1, \ldots, A_n$, $B = A_{i_1}, \ldots A_{i_n}$ and the corresponding module isomorphism with

$$\phi : A_1 \otimes \ldots \otimes A_n \xrightarrow{\sim} A_{i_1} \otimes \ldots \otimes A_{i_n}.$$

Similar to the previous two proofs, if one apply $(\phi \otimes \phi) \circ \Delta_A$ and $\Delta_B \circ \phi$ the result is

$$\sum(a'_{i_1} \otimes \ldots \otimes a'_{i_n}) \otimes (a''_{i_1} \otimes \ldots \otimes a''_{i_n})$$

so $(\phi \otimes \phi) \circ \Delta_A = \Delta_B \circ \phi$. Also the result of applying both $\varepsilon_B \circ \phi$ and ε_A is

$$\varepsilon_{A_1}(a_1) \ldots \varepsilon_{A_n}(a_n).$$

□

Theorem 26. *Let A be an R-coalgebra. Regarding R as an R-coalgebra there are isomorphisms*

$$R \otimes A \simeq A \quad \text{and } A \otimes R \simeq A$$

.

Proof. Let $\phi : A \otimes R \to A$ be the corresponding module-isomorphism in which $\phi(a \otimes r) = ar$. Now

$$(\phi \otimes \phi)(\Delta_{A\otimes R}(a \otimes r) = r \sum a' \otimes a'' = \Delta_A(\phi(a \otimes r)$$

so $(\phi \otimes \phi) \circ \Delta_{A\otimes R} = \Delta_A \circ \phi$. Also

$$\varepsilon_A(\phi(a \otimes r)) = r\varepsilon_A(a) = \varepsilon_{A\otimes R}(a \otimes r)$$

and therefore $\varepsilon_A \circ \phi = \varepsilon_{A\otimes R}$. Similarly for $R \otimes A$. □

Theorem 27. *Let $A^{(1)}, A^{(2)}, \ldots, A^{(n)}$ be graded coalgebras, then the usual total grading on the module*

$$A^{(1)} \otimes A^{(2)} \otimes \ldots \otimes A^{(n)}$$

is a coalgebra grading.

Proof. From Theorem 21 we have a natural structure of $A^{(1)} \otimes A^{(2)} \otimes \ldots \otimes A^{(n)}$ as an coalgebra. Also if each $\Delta^{(i)}$ and $\varepsilon^{(i)}$ preserves grading under the usual total grading then so does Δ and ε, as these mappings are described in (5.3). □

5.5 The convolution product

Let A be an R-algebra and C be an R-coalgebra. Then there is an associative product on the R-module $Hom_R(C, A)$, the R-module of module homomorphisms from C to A, called the convolution product denoted $*$. The convolution product is for $\varphi, \psi \in Hom_R(C, A)$ given by:

$$\varphi * \psi = \mu_A \circ (\varphi \otimes \psi) \circ \Delta_C \tag{5.8}$$

which reads in summative notation

$$(\varphi * \psi)(x) = \sum \varphi(x')\psi(x''). \tag{5.9}$$

The associativity of this product is a direct consequence of the associativity of μ and the coassociativity of Δ. The identity of the convolution product is the mapping $\eta \circ \varepsilon$. To see this consider for $f \in Hom_R(C, A)$ and $x \in C$

$$(f * (\eta \circ \varepsilon))(x) = \sum f(x')(\eta \circ \varepsilon)(x'') = \sum f(x')\varepsilon(x'')\mathbf{1} = f(x)$$

by the definition of the unit and counit mapping.

Chapter 6

Hopf algebras

6.1 Bialgebras

A bialgebra is a structure that is both an algebra and a coalgebra at the same time, in such a way that the two structures are compatible.

Definition. *Let A be an R-module. Suppose there are R-linear mappings*

$$\mu : A \otimes A \to A \tag{6.1}$$
$$\eta : R \to A \tag{6.2}$$
$$\Delta : A \to A \otimes A \tag{6.3}$$
$$\varepsilon : A \to R \tag{6.4}$$

*then $(A, \mu, \eta, \Delta, \varepsilon)$ is called a **bialgebra** if the following four conditions are satisfied*

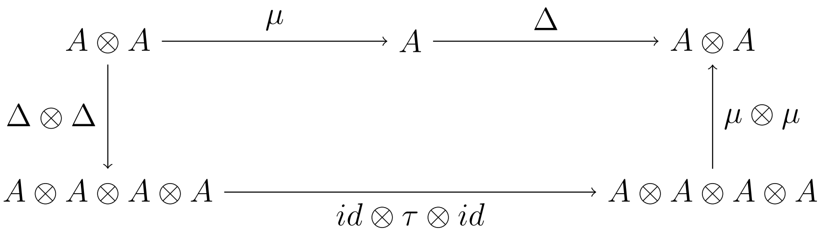
(i) (A, μ, η) *is an R-algebra*

(ii) (A, Δ, ε) *is a R-coalgebra*

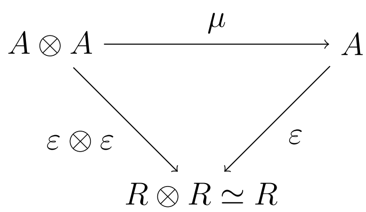
(iii) $\Delta : A \to A \otimes A$ *and* $\varepsilon : A \to R$ *are homomorphisms of algebras*

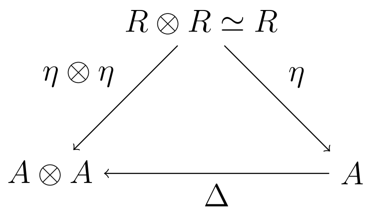
(iv) $\mu : A \otimes A \to A$ *and* $\eta : R \to A$ *are homomorphisms of coalgebras*

We can also say that the following diagrams commute.

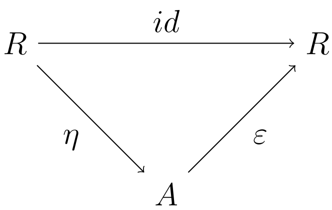


where $\tau : A \otimes A \to A \otimes A$ *maps* $a' \otimes a''$ *to* $a'' \otimes a'$*.*





and



These conditions of a bialgebra are however not independent.

Theorem 28. *Suppose* (A, μ, η) *is an algebra and* (A, Δ, ε) *is a coalgebra. Then the following statements are true.*

(i) $\mu : A \otimes A \to A$ *is compatible with comultiplication if and only if* $\Delta : A \to A \otimes A$ *is compatible with multiplication*

(ii) $\mu : A \otimes A \to A$ *preserves counits if and only if* $\varepsilon : A \to R$ *is compatible with multiplication*

(iii) $\eta : R \to A$ *is compatible with comultiplication if and only if* $\Delta : A \to A \otimes A$ *preserves identity elements*

(iv) $\eta : R \to A$ *preserves counits if and only if* $\varepsilon : A \to R$ *preserves identity elements*

Proof. (i) The mapping $\mu : A \otimes A \to A$ is compatible with comultiplication if and only if $\Delta \circ \mu = (\mu \otimes \mu) \circ \Delta_{A \otimes A}$. We know that

$$\Delta_{A \otimes A} = (id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta)$$

from the description of the tensor product of coalgebras, τ is the mapping in which $\tau(a \otimes a') = a' \otimes a$. Consequently we can write the condition as

$$\Delta \circ \mu = (\mu \otimes \mu) \circ (id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta). \tag{6.5}$$

But $\Delta : A \to A \otimes A$ is compatible with multiplication precisely when $\Delta \circ \mu = \mu_{A \otimes A} \circ (\Delta \otimes \Delta)$ and since

$$\mu_{A \otimes A} = (\mu \otimes \mu) \circ (id \otimes \tau \otimes id)$$

this condition is equivalent to (6.5).

(ii) The mapping $\mu : A \otimes A \to A$ preserves counits if and only if $\varepsilon \circ \mu = \varepsilon_{A \otimes A}(A \otimes A)$, which in turn by definition of $\varepsilon_{A \otimes A}$ can be written as $\varepsilon \circ \mu = \mu_R \circ (\varepsilon \otimes \varepsilon)$. But this is the condition for $\varepsilon : A \to R$ to be compatible with multiplication.

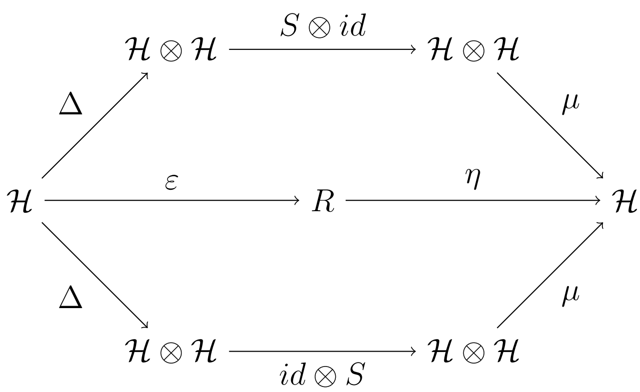
(iii) The structural homomorphism $\eta : R \to A$ is compatible comultiplication if and only if $(\eta \otimes \eta) \circ \Delta_R = \Delta \circ \eta$. And for this to occur it is necessary and sufficient that $\Delta(1_A) = 1_A \otimes 1_A$.

(iv) For $\eta : R \to A$ to preserve counits we require that $\varepsilon_R = \varepsilon \circ \eta$ and this occurs precisely when $\varepsilon(1_A) = 1_R$. □

Corollary 2. *Suppose* (A, μ, η) *is an algebra and* (A, Δ, ε) *is a coalgebra. Then* $\mu : A \otimes A \to A$ *and* $\eta : R \to A$ *are homomorphisms of coalgebras if and only if* $\Delta : A \to A \otimes A$ *and* $\varepsilon : A \to R$ *are homomorphisms of algebras.*

6.2 Hopfalgebras

Definition. *A **Hopf algebra** is a bialgebra $\mathcal{H}$ with a linear map $S : \mathcal{H} \to \mathcal{H}$ called the antipode, such that the following diagram commutes:*



Note that commuting diagram can be translated to the following equality using the summative notation for a coproduct:

$$\sum S(x')x'' = \sum x'S(x'') = (\eta \circ \varepsilon)(x).$$

Since $(\eta \circ \varepsilon)$ is the unit element of the convolution product this statement is the same as the antipode $S : \mathcal{H} \to \mathcal{H}$ is the inverse of the identity mapping $id : \mathcal{H} \to \mathcal{H}$ under the convolution product as defined in 5.5. Since the inverse of a function is unique this also means that the antipode is uniquely defined if it exists. The convolution from $\mathcal{H} \to \mathcal{H}$ is always defined since $\mathcal{H}$ has a structure as both an algebra and a coalgebra. From this follows an important property of the antipode mapping.

Theorem 29. *Let $\mathcal{H}$ be a Hopf algebra with an antipode S. Then for any $g, h \in \mathcal{H}$*

$$S(hg) = S(g)S(h)$$

or in other words, the antipode is an antihomomorphism.

Proof. Consider the three functions from $\mathcal{H} \otimes \mathcal{H}$ to $\mathcal{H}$ defined by

$$\begin{aligned} F(h \otimes g) &= S(g)S(h) \\ G(h \otimes g) &= S(hg) \\ M(h \otimes g) &= hg, \end{aligned}$$

for any $h, g \in \mathcal{H}$. We want to show that, with respect to the convolution product from $(\mathcal{H}\otimes\mathcal{H})$ to $\mathcal{H}$, that both F and G are inverses to M under the convolution. Let h, g be any elements in $\mathcal{H}$.

$$\begin{aligned}
(M * F)(h \otimes g) &= \sum_{(g),(h)} M(h' \otimes g')F(h'' \otimes g'') \\
&= \sum_{(g),(h)} h'g'S(g'')S(h'') \\
&= \sum_{(h)} h'(\eta \circ \varepsilon)(g)S(h'') \\
&= \sum_{(h)} h'\varepsilon(g)\mathbf{1}S(h'') \\
&= \varepsilon(h)\varepsilon(g)\mathbf{1} \\
&= \varepsilon(h \otimes g)\mathbf{1} \\
&= (\eta \circ \varepsilon)(h \otimes g)
\end{aligned}$$

similarly

$$\begin{aligned}
(G * M)(h \otimes g) &= \sum G(h' \otimes g')M(h'' \otimes g'') \\
&= \sum S(h'g')h''g'' \\
&= \varepsilon(hg)\mathbf{1} \\
&= (\eta \circ \varepsilon)(h \otimes g)
\end{aligned}$$

since inverses are unique the theorem follows. □

Definition. *A Hopfalgebra $\mathcal{H}$ is called a* ***graded Hopf algebra*** *if it is graded as an algebra, as a coalgebra and if $x \in \mathcal{H}_n$ implies $S(x) \in \mathcal{H}_n$ i.e. for $h_i \in \mathcal{H}_n$:*

$$\begin{aligned}
\mathcal{H} &= \sum_{i \in I} \mathcal{H}_i \\
h_p \cdot h_q &\in \mathcal{H}_{p+q} \\
\Delta(h_n) &\in \sum_{p+q=n} \mathcal{H}_p \otimes \mathcal{H}_q \\
S(h_n) &\in \mathcal{H}_n
\end{aligned}$$

If the antipode criterium is omitted this is the definition of a ***graded bialgebra****.*

A graded Hopf algebra, or bialgebra, is called **connected** if $\mathcal{H}_0$ is generated by the identity element. Notice that if $\mathcal{H}$ is a vector space or a free module this is the same as $\mathcal{H}_0$ being one-dimensional.

Note that from the definition of graded coalgebras the kernel of ε is all elements of degree 1 or higher.

Theorem 30. *Let $\mathcal{H}$ be a connected graded Hopf algebra. For any $x \in \mathcal{H}_n$ we can write:*

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \tilde{\Delta}(x), \qquad \tilde{\Delta}(x) \in \bigoplus_{p+q=n, p\neq 0, q\neq 0} \mathcal{H}_p \otimes \mathcal{H}_q.$$

The map $\tilde{\Delta}$ is coassociative and $\tilde{\Delta}_k = (id^{\otimes k-1} \otimes \tilde{\Delta}) \circ (id^{\otimes k-2} \otimes \tilde{\Delta}) \ldots \tilde{\Delta}$ sends $\mathcal{H}_n$ into $(\mathcal{H}_{n-k})^{(\otimes k+1)}$.

Proof. It is always to split up the sum given by the coproduct, using the summative notation, in two parts. One with all terms where either $|x'| = 0$ or $|x''| = 0$ and the one where $|x'|, |x''| \geq 1$. Now if we group all terms where $|x'| = 0$ together it follows from the connectedness of $\mathcal{H}$ that we can write:

$$\Delta(x) = a(x \otimes 1) + b(1 \otimes x) + \tilde{\Delta}(x)$$

where $a, b \in R$ and $\tilde{\Delta}(x) \in \ker \varepsilon \otimes \ker \varepsilon$, that is:

$$\tilde{\Delta}(x) = \sum x' \otimes x'', \quad \text{such that } |x'|, |x''| \geq 1$$

The co-unit property now tells us, together with the isomorphism $R \otimes \mathcal{H} \simeq \mathcal{H} \otimes R \simeq \mathcal{H}$, that:

$$\begin{aligned} x &= (\varepsilon \otimes id)(\Delta(x)) = & bx \\ x &= (id \otimes \varepsilon)(\Delta(x)) = & ax \end{aligned}$$

which gives $a = b = 1$. To prove the coassociativity we now compute:

$$\begin{aligned} (\Delta \otimes id)\Delta(x) &= (\Delta \otimes id)(x \otimes 1 + 1 \otimes x + \tilde{\Delta}(x)) = \\ &= x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x \\ &+ \sum (x' \otimes x'' \otimes 1 + x' \otimes 1 \otimes x'' + 1 \otimes x' \otimes x'') \\ &+ (\tilde{\Delta} \otimes id)(\tilde{\Delta}(x)) \end{aligned}$$

and

$$\begin{aligned}
(id \otimes \Delta)\Delta(x) &= (id \otimes \Delta)(x \otimes 1 + 1 \otimes x + \tilde{\Delta}(x)) = \\
&= x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x \\
&+ \sum (x' \otimes x'' \otimes 1 + x' \otimes 1 \otimes x'' + 1 \otimes x' \otimes x'') \\
&+ (id \otimes \tilde{\Delta})(\tilde{\Delta}(x))
\end{aligned}$$

since Δ is co-associative, i.e. $(id \otimes \Delta)\Delta(x) = (\Delta \otimes id)\Delta(x)$ it follows that $\tilde{\Delta}$ is co-associative.

Using the definition of $\tilde{\Delta}_k$ we have for any $x \in \mathcal{H}_n$ that

$$\tilde{\Delta}_k(x) = \sum x^{(1)} \otimes x^{(2)} \otimes \ldots \otimes x^{(k+1)}$$

with a total of $k+1$ terms with every term $|x^{(j)}| \geq 1$, from the definition of $\tilde{\Delta}$. Also the filtration imposes that

$$\sum_{j=1}^{k+1} |x^{(j)}| \leq n$$

so the maximal degree of any term is $n - k$. □

The convolution product is a binary product on the set of homomorphisms from an R-coalgebra C to an R-algebra A, denoted $Hom_R(C, A)$. We are particularly interested in the case where the R-coalgebra C is in fact a connected graded bialgebra $\mathcal{H}$. We can then construct an interesting group.

Definition. *Let $\mathcal{H}$ be a connected graded bialgebra and A be any algebra. We define the subset $G_{\mathcal{L}(\mathcal{H},A)}$ of $Hom_R(\mathcal{H}, A)$ as the following set:*

$$G_{\mathcal{L}(\mathcal{H},A)} = \{\varphi \in Hom_R(\mathcal{H}, A), \quad \varphi(1) = \mathbf{1}_A\}.$$

Theorem 31. *Let $\mathcal{H}$ be a connected graded R-bialgebra and let A be any R-algebra. Then set $G_{\mathcal{L}(\mathcal{H},A)}$ together with the convolution product form a group. The identity element of this group is the function $e = \eta_A \circ \varepsilon$.*

Proof. When the convolution product was introduced the associativity and the unit mapping was proved and the closure property follows easily from definition. The only thing left to prove is the inverse property. To do this let us consider the formal power series:

$$\varphi^{*-1}(x) = (e - (e - \varphi))^{*-1}(x) = \sum_{k \geq 0} (e - \varphi)^{*k}(x)$$

where a^{*n} is defined as $a * a * \ldots * a$, the convolution taken n times. If this formal power series is well defined then it must be the inverse. Compare with the well known equality.

$$\frac{1}{1-x} = \sum x^n.$$

Now by the definition of the elements in the set G and of e it is evident that $(e - \varphi)(1) = 0$ and as an immediate consequence $(e - \varphi)^{*k}(1) = 0$.

For any $x \in \ker \varepsilon$ it is evident that $(e - \varphi)(x) = -\varphi(x)$. As a consequence

$$(e - \varphi)^{*k} = \mu^{k-1}(-\varphi \otimes \ldots \otimes -\varphi)\tilde{\Delta}^{k-1}(x)$$

When $x \in \mathcal{H}_n$ this expression vanishes for $k \geq n+1$ by Theorem 30. As a consequence the formal series ends up with a finite number of terms for any x, therefore φ^{*-1} is well defined and the theorem is proved. □

The next theorem is a direct consequence of what we now know about graded bialgebras, Hopf algebras and the convolution product. This is a theorem we will make heavy use of later.

Theorem 32. *Any connected graded bialgebra $\mathcal{H}$ is a graded Hopf algebra in the sense that an antipode always can be defined by:*

$$S(x) = \sum_{k \geq 0} (e - id)^{*k}(x)$$

It can be calculated by $S(1) = 1$ and alternatively by any of the two formulas for $x \in \ker \varepsilon$:

$$S(x) = -x - \sum S(x')x''$$
$$S(x) = -x - \sum x'S(x'')$$

which are recursive on the degree of x.

Proof. The antipode, when it exists, is the inverse of the identity mapping under the convolution product, thus the inverse of the identity mapping in $G_{\mathcal{L}(\mathcal{H},\mathcal{H})}$ as defined in Theorem 31. The first antipode formula is now a direct consequence of applying Theorem 31 to the identity mapping. It follows from Theorem 31 that the sum is finite in the sense that all but finitely many terms are zero.

Consider the antipode of an element x of degree n, then

$$\begin{aligned} S(x) &= \sum_{k\geq 0}^{n}(e-id)^{*k}(x) \\ &= \sum_{k\geq 0}^{n}\mu^{k-1}(-id\otimes\ldots\otimes -id)\tilde{\Delta}^{k-1}(x) \\ &= \mu^{n-1}(-id\otimes\ldots - id)\tilde{\Delta}^{n-1}(x)+\ldots+\mu^{2}(-id\otimes -id)\tilde{\Delta}^{2}(x)-id(x) \\ &= \mu((..((\tilde{\Delta}\ldots)\tilde{\Delta}+-id)\otimes -id)\tilde{\Delta}+-id)(x) \end{aligned}$$

where μ simply represents the multiplication mapping of the correct number of elements. This can in summative notation be written as

$$S(x) = -x - \sum S(x')x''$$

and the second formula can be derived similarly.

□

6.3 Examples of Hopf algebras

Example 8. *Let G be a group and let RG be the free module of G as described in 1.3. The product in G extends uniquely to a bilinear map $\mu : RG \times RG \to RG$ in which $(rg, r'g')$ is mapped to $rr'gg'$. The neutral element of G is the unit element for μ. It is possible to endow RG with the following co-unital coalgebra structure:*

$$\begin{aligned} \Delta(\sum r_i g_i) &= \sum r_i (g_i \otimes g_i) \\ \varepsilon(\sum r_i g_i) &= \sum r_i \end{aligned}$$

The antipode is given by:

$$S(g) = g^{-1}, \quad g \in G.$$

Proof. We check some of the conditions. We first prove that the coproduct and product are compatible. For any $g, h \in G$ we have:

$$\Delta(gh) = gh \otimes gh = (g \otimes g)(h \otimes h) = \Delta(g) \cdot \Delta(h).$$

For the antipode criteria we can conclude the following for any $g \in G$:

$$\begin{aligned} \mu((S \otimes id)\Delta(g)) = g^{-1}g &= e \\ \mu((id \otimes S)\Delta(g)) = gg^{-1} &= e \\ \eta \circ \varepsilon(g) &= e \end{aligned}$$

so the antipode defined satisfies the criteria. □

Example 9. *There is a natural structure as a cocommutative, the coproduct is commutative, Hopf algebra on the tensor algebra $T(M)$.*

It is possible to define different coalgebraic structures on the tensor algebra, but not all are such that they fulfill the conditions for a bialgebra and/or a Hopf algebra. One coalgebra structure that does fulfill these conditions are the shuffle coproduct, which is defined as

$$\Delta(x_1 \otimes x_2 \otimes \ldots \otimes x_n) = \sum_{p=0}^{n} \sum_{\sigma \in \mathrm{Sh}_{p,m-p}} (x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(p)}) \otimes (x_{\sigma(p+1)} \otimes \ldots \otimes x_{\sigma(n)})$$

the summation is taken over the $(p, n-p)$ shuffles of $(1, 2, ..., n)$. The $(p, n-p)$ shuffles are all the ways in which it is possible to partition $(1, 2, ..., n)$ in such a way that

$$\sigma(1) < \sigma(2) < \ldots < \sigma(p)$$
$$\sigma(p+1) < \sigma(p+2) < \ldots < \sigma(n).$$

Another way to define this coproduct is to define it as the algebra morphism from $T(M)$ into $T(M) \otimes T(M)$ such that

$$\Delta(1) = 1 \otimes 1$$

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

which extends to

$$\Delta(x_1 \otimes x_2 \otimes \ldots \otimes x_n) = \Delta(x_1)\Delta(x_2)\ldots\Delta(x_n)$$

so for example

$$\begin{aligned} \Delta(x_1 \otimes x_2) &= (x_1 \otimes 1 + 1 \otimes x_1)(x_2 \otimes 1 + 1 \otimes x_2) \\ &= (x_1 \otimes 1)(x_2 \otimes 1) + (x_1 \otimes 1)(1 \otimes x_2) \\ &\quad +(1 \otimes x_1)(x_2 \otimes 1) + (1 \otimes x_1)(1 \otimes x_2) \\ &= (x_1 \otimes x_2) \otimes (1 \otimes 1) + (x_1 \otimes 1) \otimes (1 \otimes x_2) \\ &\quad +(1 \otimes x_2) \otimes (x_1 \otimes 1) + (1 \otimes 1) \otimes (x_1 \otimes x_2) \\ &= (x_1 \otimes x_2) + (x_1) \otimes (x_2) + (x_2) \otimes (x_1) + (x_1 \otimes x_2). \end{aligned}$$

The counit is defined as $\varepsilon(1) = \mathbf{1}$ and $\varepsilon|_M = 0$. With these two mappings $T(M)$ is a cocommutative bialgebra. For these mappings $\eta \circ \varepsilon(1) = 1$, so $S(1) = 1$, and $\eta \circ \varepsilon(x) = 0$ for $x \in M = T_1(M)$. For $x \in T_1(M)$ the antipode can be defined as

$$S(x) = -x$$

for which

$$S * id(x) = S(x)1 + 1x = -x + x = 0$$

by the definition of multiplication with a scalar id $T(M)$, so $id * S(x) = S(x) * id = 0$. Since $T(M)$ is generated by $T_1(M)$ we can simply extend this to $T(M)$ with the use of Theorem 29 so that

$$S(x_1 \otimes \ldots \otimes x_n) = S(x_n)\ldots S(x_1) = (-1)^n(x_n \otimes \ldots \otimes x_1)$$

Example 10. *The universal enveloping algebra of a Lie algebra is also a cocommutative Hopf algebra.*

A Lie algebra can be constructed out of an algebra by defining the commutator

$$[x, y] = xy - yx$$

Then we can construct the universal enveloping algebra of the Lie algebra L from $T(L)$ by factoring out the ideal I generated by elements

$$[x, y] - x \otimes y + y \otimes x.$$

Similarly one can construct the symmetric algebra, equivalent to the commutative free algebra of M, by factoring $T(M)$ by the ideal generated by elements

$$x \otimes y - y \otimes x.$$

And also the exterior algebra of a module M by factoring by the ideal generated by elements

$$x \otimes x.$$

Chapter 7

Hopf algebras of graphs

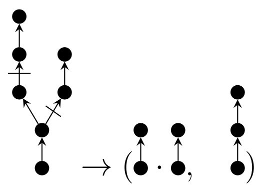
In this part we will take a closer look at particular kind of Hopf algebras. Hopf algebras created from different sets of graphs. The algebraic structure in these examples are the same, the free commutative R-algebra as described in example 7. We will see that depending on the additional properties of the graphs it is possible to create different gradings and coalgebraic structures. These sets of graphs will be shown to have a graded connected bialgebraic structure, and therefore by Theorem 32 also Hopf algebraic structures.

7.1 Hopf algebra of rooted trees

A **rooted tree** is a finite simply connected oriented graph such that each vertex has exactly one incoming edge except one distinguished vertex, called the **root**, which has no incoming edge. A vertex without any outgoing edges is called a **leaf**. A **rooted forest** is a disjoint union of rooted trees, $s = \{t_1, t_2, \ldots, t_n\}$ which we simply denote by the commutative product $t_1 \cdot t_2 \cdot \ldots \cdot t_n$. Two examples of rooted trees with 5 vertices are:

To construct a bialgebraic structure, let $\mathcal{H}$ be the free commutative R-algebra of rooted forests generated via the disjoint union by all rooted trees, where the empty tree is the identity element **1**. To define a bialgebra on this set we must define a counit and a comultiplication such that these structures together with the free commutatvie structure satisfies the axioms of a bialgebra. We let the co-unit ε be the map sending **1** to 1 and any non-empty forest to 0.

The coproduct is somewhat more complicated, to define it we will first need some other concepts. An **elementary cut** on a tree is a cut on some edge of that tree. An **admissable cut** is a collection of elementary cuts such that any path from the root to a leaf passes at most one edge with an elementary cut on it. The empty cut as well as the total cut, a cut under or above the whole tree, are considered to be elementary. A cut on a forest is said to be admissable if its restriction to any tree factor is admissable. Any admissable cut sends a forest to a couple $(P^c(F), R^c(F))$, called the **crown** and the **trunk**. The trunk is, as its name suggests, the graph containing the original root, and the crown is the other. The trunk of an admissable cut on a single tree is a tree itself, the crown on the other hand is a forest. We give an example of an admissable cut on a tree, and the pair of forests associated to the cut.

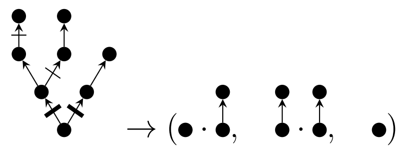


Note how the crown in this example is a forest while the trunk is a single tree. We let $\mathrm{Adm}(F)$ denote the set of admissable cuts of a forest F and are now ready to define the coproduct.

$$\Delta_{Tr}(F) = \sum_{c \in \mathrm{Adm} F} P^c(F) \otimes R^c(F)$$

The compatability of the product and coproduct is clear from the definition of admissable cuts on a forest. The grading defined by the number of vertices in a forest is obviously compatabile both with the coproduct and the product. The important thing left to prove is the coassociativity of the defined coproduct.

To prove the coassociativity we define the concept of **bi-admissable cuts**. We say that two cuts (c_1, c_2) are bi-admissable if each of the cuts is admissable, and if c_1 never bypasses c_2, or in other words if c_2 never cuts the trunk of c_1. Any bi-admissable cut $c = (c_1, c_2)$ of a tree creates not only a trunk and a crown, bu also a middle part $M^c(F)$. An example of a bi-admissable cut is:



Another way to understand a bi-admissable cut is to see it as a sequence of one admissable cut followed by a second cut on the trunk, or first one admissable cut and then a second on the crown. This description is possible since the two cuts (c_1, c_2), by definition do not cross each other and each one of them is an admissable cut in its own right.

Any two sequential admissable cuts yield the same result as some bi-admissable cut, and also any bi-admissable cut yields the same result as both some combination of one admissable cut and a second admissable cut of its crown or some admissable cut followed by an admissable cut of its trunk. We denote the set of all bi-admissable cuts by $\mathrm{Adm}_2 F$, and note that

$$(\Delta \otimes id)\Delta(F) = \sum_{c \in \mathrm{Adm}_2 F} P^c(F) \otimes M^c(F) \otimes R^c(F) =$$

$$= (id \otimes \Delta)\Delta(F)$$

so the defined coproduct is in fact coassociative and then we have defined a connected graded bialgebra, $\mathcal{H}_0$ is generated by the empty tree. By Theorem 32, this gives us a Hopf algebraic structure where the antipode is defined recursively on $\ker \varepsilon$ as

$$\begin{aligned} S(F) &= -F - \sum_{c \in \mathrm{Adm}^*(F)} S(P^c(F)) \cdot R^c(F) \\ &= -F - \sum_{c \in \mathrm{Adm}^*(F)} P^c(F) \cdot S(R^c(F)) \end{aligned}$$

and $S(\mathbf{1}) = 1$, $\mathrm{Adm}^*(F)$ is the set of non-trivial admissable cuts. In the end of this chapter we show some concrete examples of calculations of the antipode in this Hopf algebra.

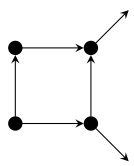
7.2 Hopf algebra of Feynman graphs

The notion of Feynman graphs originated in quantum field theory. They are used as a graphical interpretation of the complex mathematical expressions governing the behavior of subatomic particles. A Feynman graph consists of internal and external edges of different types, together with different types of vertices. Every external edge has a vector attached to it (called an exterior momenta), and the sum of all exterior momenta is always zero. The different types of edges and vertices available are given by the particles studied. For example Feynman graphs in quantum electrodynamcis have two types of edges and three types of vertices. For our purposes , however, it is not relevant exactly what kind of edges or vertices there are, instead we will a look at some more general results.

7.2.1 Some basic concepts

Three different Hopf algebraic structures on Feynman graphs will be considered. The first is more general while the following two make use of more structure when defining the coproduct. Before discussing these three cases some definitions and properties will be defined.

An **oriented Feynman graph** is an oriented graph with a finite number of vertices and edges. Orientation means that a choice of direction is given to each edge. An edge will be classified as **internal** if it is connected to a vertex in both ends or **external** if it is connected to a vertex in only one edge. Below follows an example of an oriented Feynman graph with two external and four internal edges.



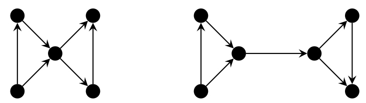
A **cycle** in an oriented Feynman graph is a finite collection $(e_1, \ldots, e_n)$ of oriented internal edges such that the target of e_k coincides with the source of e_{k+1} for any $k = 1, \ldots, n$ modulo n, i.e. the target of e_n is the source of e_1.

The **loop number**, $L(\Gamma)$ of a graph Γ is given by:

$$L(\Gamma) = |I(\Gamma)| - |V(\Gamma)| + 1$$

where $|I(\Gamma)|$ is the number of internal edges and $|V(\Gamma)|$ is the number of vertices.

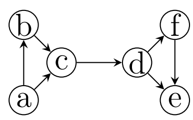
A **one-particle irreducible (1PI) graph** is a connected oriented Feynman graph which remains connected when any internal edge is cut. A disconnected graph is said to be locally 1PI if all of its connected components are 1PI. The left graph is an example of a 1PI graph and the right is a non-1PI graph.



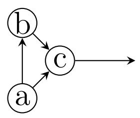
Let P be a non-empty subset of $V(\Gamma)$, where Γ is an oriented Feynman graph. We define the associated subgraph $\Gamma(P)$ in the following way. The internal edges of $\Gamma(P)$ are the internal edges of Γ with source and target in $\Gamma(P)$, and the external edges of $\Gamma(P)$ are all the edges of Γ with the source or the target in $\Gamma(P)$ but not both. The orientations of the edges in $\Gamma(P)$ are the same as the orientation when seen as an edge in Γ. We say that $\Gamma(P)$ is a **connected subgraph** if and only if for any $v, w \in P$ there is an undirected path from v to w consisting only of internal edges to $\Gamma(P)$. We set $\Gamma(\emptyset) = \mathbf{1}$ where $\mathbf{1}$ is the empty graph. An obvious result is that if $Q \subseteq P \subseteq V(\Gamma)$ then:

$$\Gamma(P)(Q) = \Gamma(Q)$$

since all internal edges of $\Gamma(Q)$ will be internal edges of $\Gamma(P)$ and any external edges of $\Gamma(Q)$ will either be an external or internal edge of $\Gamma(P)$.

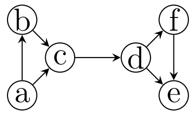


If we to the above graph, Γ, define $P = \{a, b, c\}$ then the subgraph $\Gamma(P)$ is

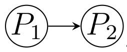


A **covering subgraph** of Γ is an oriented Feynman graph γ (in general disconnected), consisting of a collection $\{\Gamma(P_1), \ldots, \Gamma(P_n)\}$ of connected subgraphs such that $P_i \cap P_j = \emptyset$ when $i \neq j$, and such that any vertex of Γ

belongs to some P_j, $j \in \{1, \ldots, n\}$. For any covering subgraph γ we can define the **contracted graph** Γ/γ by shrinking all the connected components of γ onto a single vertex. Using the same graph Γ



we can define the covering subgraph $\gamma = \{\Gamma(P_1), \Gamma(P_2)\}$ with $P_1 = \{a, b, c\}$, $P_2 = \{d, e, f\}$. The contracted graph Γ/γ is



7.2.2 The full Hopf algebra of oriented Feynman graphs

We start with a more general Hopf algebraic structure, and continue later with similar constructions on more specific Feynman graphs. Let $\tilde{\mathcal{H}}$ be the free commutative algebra generated by connected oriented Feynman graphs. The unit $\mathbf{1}$ is identified with the empty graph. We define the coproduct as:

$$\Delta_F(\Gamma) = \sum_{\gamma \in \mathrm{Cov}(\Gamma)} \gamma \otimes \Gamma/\gamma$$

where $\mathrm{Cov}(\Gamma)$ is the set of all covering subgraphs of Γ. Since the product in $\tilde{\mathcal{H}}$ is just the concatenation, this is obviously an algebra homomorphism.

If we let γ be a covering subgraph of Γ and δ be a covering subgraph of γ, then looking at the definition of contraction one notes the following property:

$$\Gamma/\gamma = (\Gamma/\delta)/(\gamma/\delta),$$

this is clear since it does not matter if we first contract some of the subgraphs of the connected graphs in γ befor contracting them fully. By similar reasoning, if we let $\tilde{\gamma}$ be a covering graph of Γ/δ there is a bijection $\gamma \to \tilde{\gamma} = \gamma/\delta$ from covering subgraphs of Γ containing δ to covering subgraphs of Γ/δ, given by shrinking δ. As a consequence we have:

$$\begin{aligned}
(\Delta_F \otimes id)\Delta_F(\Gamma) &= \sum_{\delta \in \mathrm{Cov}(\gamma),\, \gamma \in \mathrm{Cov}(\Gamma)} \delta \otimes \gamma/\delta \otimes \Gamma/\gamma \\
(id \otimes \Delta_F)\Delta_F(\Gamma) &= \sum_{\delta \in \mathrm{Cov}(\Gamma),\, \tilde{\gamma} \in \mathrm{Cov}(\Gamma/\delta)} \delta \otimes \tilde{\gamma} \otimes (\Gamma/\delta)/\tilde{\gamma}
\end{aligned}$$

and since these two expressions coincide the coassociativity of the defined coproduct is proved. We let the counit ε be the mapping defined by $\varepsilon(\mathbf{1}) = 1$ and $\varepsilon(\Gamma) = 0$ for any non-empty graph Γ. The number of vertices does not make up a grading for this bigalgebra structure since the number of vertices is not preserved by the coproduct. Instead we define the grading to be number of internal edges, something which is preserved, since exactly those edges which are internal in the covering subgraph γ are contracted in Γ/γ. There are many different kinds of graphs without internal edges, so $\tilde{\mathcal{H}}_0$ is not one-dimensional and therefore $\tilde{\mathcal{H}}$ is not connected, something that prevents the application of Theorem 32.

To solve this problem let us define a new space by identifying all degree zero elements with $\mathbf{1}$, i.e.:

$$\mathcal{H} = \tilde{\mathcal{H}}/\mathcal{J}$$

where $\mathcal{J}$ is the bi-ideal generated by elements $\tilde{h}_0 - \mathbf{1}$, with $\tilde{h}_0$ in $\tilde{\mathcal{H}}_0$. Now this bialgebra $\mathcal{H}$ is obviously connected, and therefore a connected graded bialgebra. So we can define the Hopf algebraic structure as in Theorem 32:

$$S_F(\Gamma) = -\Gamma - \sum_{\gamma \in \mathrm{Cov}^*(\Gamma)} S(\gamma) \cdot \Gamma/\gamma$$

$$S_F(\Gamma) = -\Gamma - \sum_{\gamma \in \mathrm{Cov}^*(\Gamma)} \gamma \cdot S(\Gamma/\gamma)$$

where $\mathrm{Cov}^*(\Gamma)$ is the set of on non-trivial covering sub graphs of Γ and S($\mathbf{1}$) = 1.

7.2.3 The Hopf algebra of 1PI graphs

We can make a similar construction of locally 1PI graphs. Let $\tilde{\mathcal{H}}_{1PI}$ be the free algebra generated by 1PI graphs. Then we can define a coproduct, where 1PICov(Γ) is the set of 1PI covering subgraphs:

$$\Delta_{1PI}(\Gamma) = \sum_{\gamma \in 1\mathrm{PICov}(\Gamma)} \gamma \otimes \Gamma/\gamma$$

since the same argument about transitivity of subgraphs holds for the subset of the covering subgraphs which are 1PI the coassociativity is given. The grading consisting of the number of internal edges still makes sense, but there is also an alternative grading available in the form of the loop number.

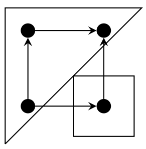
The coproduct Δ_{1PI} which only sums over 1PI covering subgraphs preserves the loop number, unlike the more general coproduct.

The elements of degree one is in both gradings the residues without any internal edges, since a 1PI graph with loop number one cannot have any internal edges. We can therefore, similarly to the general case, construct the Hopf algebra $\mathcal{H}_{1PI}$ from $\tilde{\mathcal{H}}_{1PI}$ by identifying any element of degree 0 with the identity element $\mathbf{1}$.

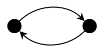
7.2.4 The Hopf Algebra of cycle free graphs

A cycle free graph is a graph that does not contain any cycle as described earlier. Such a graph has a natural poset structure given by, $v \leq w$ if and only if there exist a path from v to w, i.e. a sequence $(e_1, \ldots, e_n)$ such that the target of e_k is the source of e_{k+1} and the source of e_1 is v and the target of e_n is w. This structure would not be possible on any graph with cycles since the antisymmetry of the relation would not be satisfied.

Let $\tilde{\mathcal{H}}_{CF}$ be the free algebra generated by cycle-free graphs. Any covering subgraph will in itself be cycle free, but the same is not necessarily true for the contracted graph Γ/γ. For instance, consider the graph:



$\rightarrow \quad \Gamma/\gamma =$



for which Γ/γ is not a cycle free graph. We call a covering subgraph γ **poset-compatible** if the contracted graph Γ/γ is cycle-free. We can now define the coproduct if we denote the set of poset-compatabile covering subgraphs of Γ with $\mathrm{PoCov}(\Gamma)$:

$$\Delta_{CF}(\Gamma) = \sum_{\gamma \in \mathrm{PoCov}(\Gamma)} \gamma \otimes \Gamma/\gamma$$

this coproduct is also coassociative since the argument of transitive shrinking is valid also for this subset of covering subgraphs. The relevant grading is the one consisting of the number of internal edges. We can also construct an associated Hopf algebra $\mathcal{H}_{CF}$ by identifying the zero degree elements with $\mathbf{1}$.

Note that the set of rooted trees is a subset of the connected oriented cyclefree graphs, and therefore we can use this structure, as well as the one in 7.1, for rooted trees.

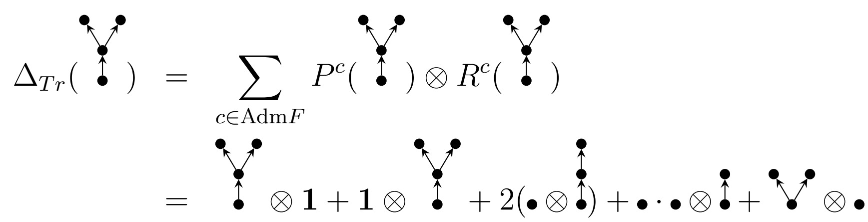
Another important remark is that in all of these examples it is possible to completely disregard any external edges and still obtain an Hopf algebraic structure, and in many cases it is convenient to do so.

7.3 Examples of Hopf algebraic calculations

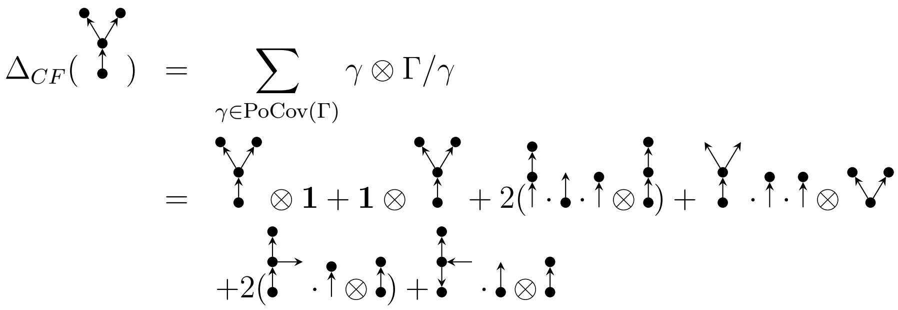
This part will mainly be concerned with calculations of the antipode of different graphs for the different Hopf algebraic structures discussed.

Example 11. *Two different Hopfalgebraic structures on trees.*

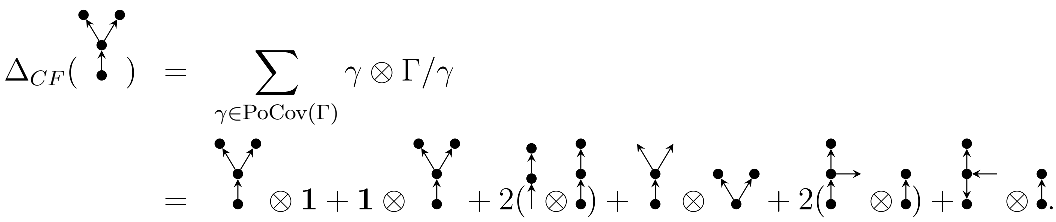
The essential difference between the different structures is the defined coproduct, and as a consequence of that the antipode is also different. In our first example we will look at a simple tree and see the difference between the first Hopf algebraic structure on trees, which we will denote Δ_{Tr} and then the cycle free structure. We start by calculating the coproducts for the different structures. Note how the number of vertices is conserved in the first and the number of inner edges in the second but not the other way around, this refelcts the different gradings. From the tree structure we get:



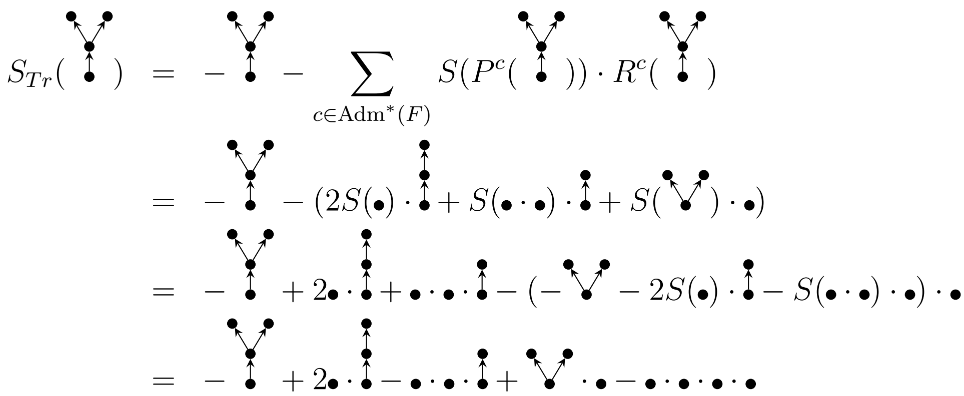
And for the cycle-free structure we have.



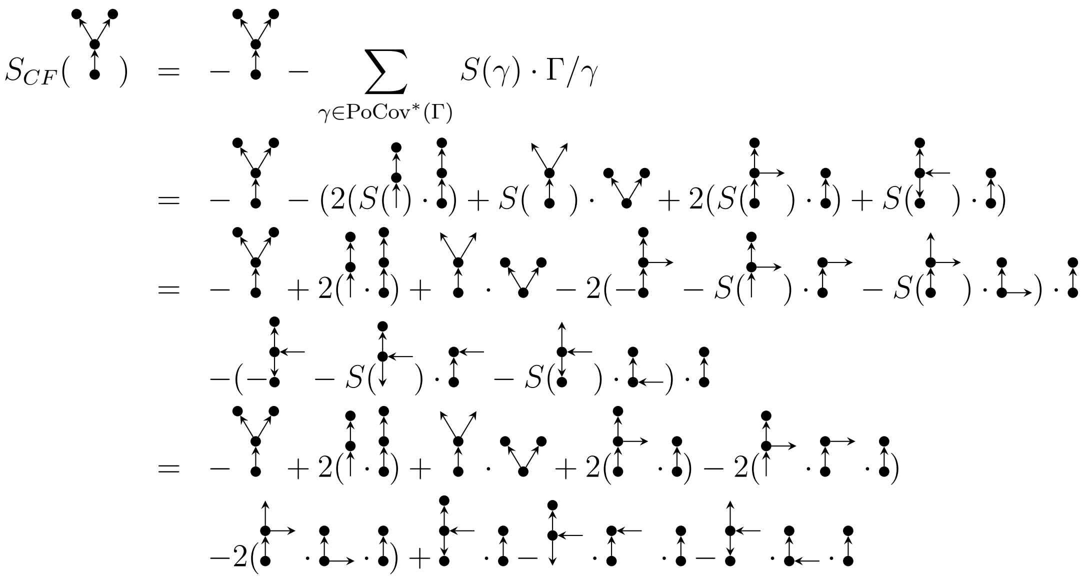
We can also write the coproduct in $\mathcal{H}$, where we identify any zero degree element with the identity, as



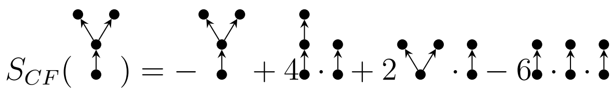
To compute the antipodes we use the formulas and get.



and



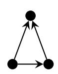
The exterior structure of the graphs are of limited interest and greatly increases the complexity of the antipode calculations, the latest computation disregarding exterior edges become.



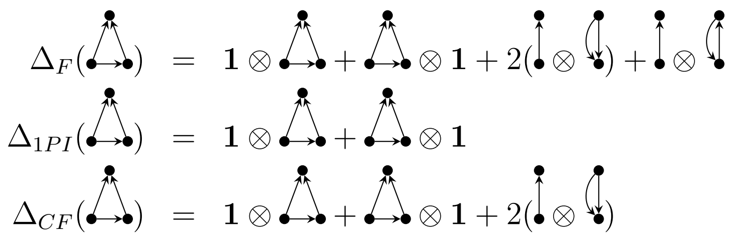
We will continue to do the calculations disregarding exterior edges since this better highlights the differences of interest.

Example 12. *A cycle free and locally 1PI graph.*

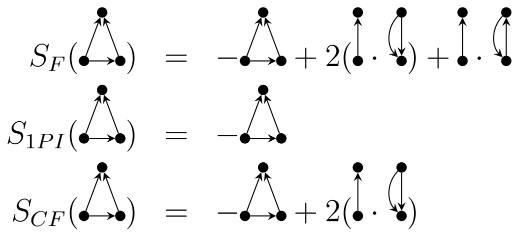
In this example we will directly consider the Hopfalgebraic structure where all zero degree elements equal the identity element. The graph under consideration is the simplest non-trivial example of a cycle free and 1PI graph namely.



The coproduct computations of this graph is

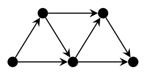


The antipodes are simple because of the low degree.

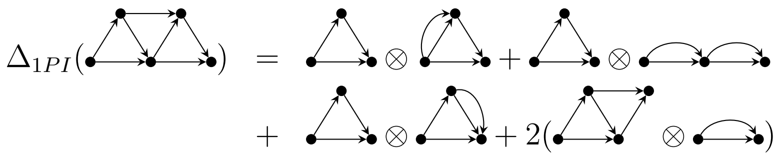


Example 13. *A somewhat more complicated 1PI graph.*

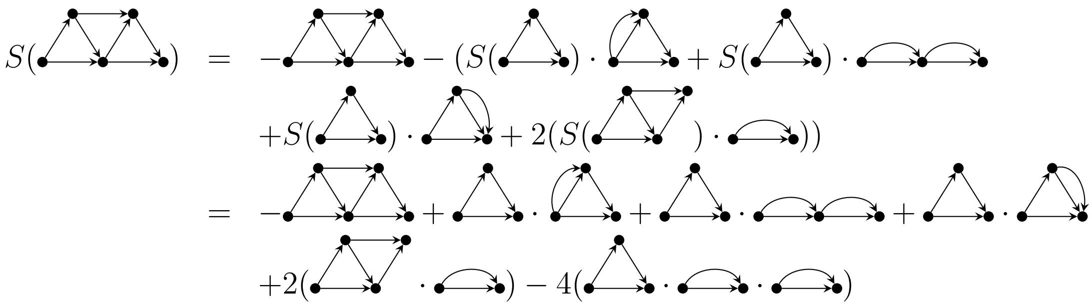
In this example we consider the graph



which have 5 non-trivial locally 1PI covering subgraphs, the three triangles and the two paralellograms. So the coproduct is, with any zero degree element identified with **1**,



With formula for calculating the antipode we get



7.4 Note on Feynman integrals

Feynman graphs are used to describe the interactions of subatomic particles. Depending on the theory being used, there are many different theories using Feynman graphs. A set of rules are used to generate integrals from the graph which describe the behavior of the particles. A problem that arises is that these integrals generally do not converge, something they would have to do to make sense. Therefore, various techniques created to tackle problems with infinity, called renormalization, have been used. One set of such techniques rely on a Hopf algebraic structure and this is why it is important to find and describe Hopf algebraic structures of Feynman graphs. See [4] for a thorough explanation of such techniques.

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