# SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

### MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

## The Four Functions Theorem

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## Sarah Alsaadi

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# The Four Functions Theorem

Sarah Alsaadi

# Contents

## Abstract

Let $\mathcal{P}(n)$ be the poset consisting of the subsets of $[n]$ ordered by inclusion and for polynomials $p(x), q(x) \in \mathbb{R}[x_1, x_2, \ldots, x_n]$, we put $q(x) \leq p(x)$ if the coefficients of $p(x) - q(x)$ are non-negative. Next we consider functions $\alpha, \beta, \gamma$ and $\delta : \mathcal{P}(n) \to \mathbb{R}_{\geq 0}$ satisfying

$$\alpha(A)\beta(B) \leq \gamma(A \cup B)\delta(A \cap B)$$

for every pair $A, B \in \mathcal{P}(n)$. In this thesis we will study several versions of the four functions theorem [1] and a new one, first noted by P. Brändén (private communication):

For each quadruple $\alpha, \beta, \gamma, \delta$ as above and every $A \in \mathcal{P}(n)$ one has

$$\left( \sum_{A \in \mathcal{P}(n)} \alpha(A) \prod_{a \in A} x_a \right) \left( \sum_{A \in \mathcal{P}(n)} \beta(A) \prod_{a \in A} x_a \right) \leq$$

$$\leq \left( \sum_{A \in \mathcal{P}(n)} \gamma(A) \prod_{a \in A} x_a \right) \left( \sum_{A \in \mathcal{P}(n)} \delta(A) \prod_{a \in A} x_a \right)$$

Notice that, since every finite distributive lattice is isomorphic to a sublattice of $\mathcal{P}(n)$ for some $n$, our main result holds for each finite distributive lattice. In the special case when $x_a = q$ for all $a$ we obtain the $q$-analogue of the four functions theorem due to Christofides [5] and when $x_a = 1$ for all $a$ we obtain the four functions theorem of Ahlswede and Daykin.

In Chapter 3, we consider applications to random graphs, linear extensions and to a correlation inequality for certain series weighted by Young tableaux.

# Chapter 1

# Introduction

The four functions theorem, proved by Ahlswede and Daykin in 1978 [1], is a correlation inequality for four functions defined on a finite distributive lattice. The four functions theorem is an extension of many other results such as the FKG inequality, Holley inequality, Kleitman's inequality etc, which has been frequently used in different fields such as statistical mechanics and probabilistic combinatorics. In the case of random graph it implies that the probability for a graph being planar given that it is Hamiltonian is less than the probability of it being planar. It is natural to expect that since being Hamiltonian indicates that the graph has many edges which makes it less likely to be planar. In this thesis we study the four functions theorem and give a generalization of it based on previous works of Björner [3] and Christofides [5]. Soon after Björner conjectured a $q$-analogue of the four functions theorem, Christofides proved it and based on that, it is natural to ask if we one could generalize it even further. In this work we prove a version of the four functions theorem for polynomials in several variables which automatically gives a polynomial version of the FKG inequality. In the special case where all the variables are equal we get the $q$-analogues of the four functions theorem and the FKG inequality.

Chapter 2 is expository. We define partial orders and show some results which will be needed in the proof of the four functions theorem. A major theorem in Chapter 2 is Birkhoff's representation theorem which states that each finite distributive lattice is isomorphic to some sublattice of $\mathcal{P}(n)$ for some $n$. Here $\mathcal{P}(n)$ is the poset consisting of the subsets of $[n]$ ordered by inclusion. In Chapter 3 we state and prove the four functions theorem and its generalization and in Chapter 4 we expose some applications.

# Chapter 2

# Preliminaries

In this chapter we introduce some basic definitions and theorems. Since the four functions theorem is defined for partially ordered set we need to define properties and show some basic results concerning these. In the end of this chapter, we will prove Birkhoff's representation theorem, the main theorem used when proving the four functions theorem. For further reading, see [6, 13].

**Definition 2.1.** *A partially ordered set* $P$ (or poset for short) is a set together with a binary relation denoted $\leq_P$, satisfying the following axioms:

    **Reflexivity:** $x \in P$, $x \leq_P x$.

    **Antisymmetry:** If $x \leq_P y$ and $y \leq_P x$, then $x = y$.

    **Transitivity:** If $x \leq_P y$ and $y \leq_P z$, then $x \leq_P z$.

    We will also use the notation $x \geq_P y$ to denote $y \leq_P x$, $x <_P y$ to denote $x \leq_P y$ and $x \neq y$, and $x >_P y$ to denote $y <_P x$. We write $x \not\leq_P y$ to denote that $x \leq_P y$ is false.

**Definition 2.2.** Two elements $x$ and $y$ of a poset $P$ are *comparable* if $x \leq_P y$ or $y \leq_P x$; otherwise $x$ and $y$ are *incomparable*.

**Example 2.1.** Let $n \in \mathbb{N} = \{0, 1, 2, \dots\}$. We can turn the set $2^{[n]}$ of all subsets of $[n] = \{1, 2, \dots, n\}$ into a poset $\mathcal{P}(n)$ by defining $S \leq_{\mathcal{P}(n)} T$ in $\mathcal{P}(n)$ if $S \subseteq T$ as sets, that is; $\mathcal{P}(n)$ consists of the subsets of $[n]$ ordered by inclusion.

**Definition 2.3.** Let $P$ be a poset and let $x, y \in P$, then we say $y$ *covers* $x$ if $x <_P y$ and if no element $z \in P$ satisfies $x <_P z <_P y$.

**Definition 2.4.** The *Hasse diagram* of a finite poset $P$ is the graph whose vertices are the elements of $P$ and whose edges are the cover relations, such that if $x <_P y$ then $y$ is drawn "above" $x$ (i.e., with a higher horizontal coordinate).

**Example 2.2.** The poset $\mathcal{P}(n)$ with $n = 3$ have the following Hasse diagram:
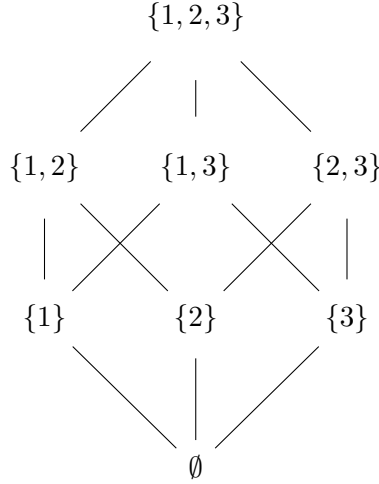


**Definition 2.5.** Let $P$ be a poset. A subset $S$ of $P$ together with a partial ordering of $S$ is a *subposet* of $P$ if for $x, y \in S$ we have $x \leq_S y$ in $S$ if and only if $x \leq_P y$ in $P$.

**Definition 2.6.** Let $P$ and $Q$ be two posets. A map $\varphi : P \to Q$ is said to be:

1. *order-preserving* if $x \leq_P y$ in $P$ implies $\varphi(x) \leq_Q \varphi(y)$ in $Q$;

2. an *order-embedding* if $x \leq_P y$ in $P$ if and only if $\varphi(x) \leq_Q \varphi(y)$ in $Q$;

3. an *order-isomorphism* if it is an order-embedding mapping $P$ onto $Q$, if there exists such a mapping we then write $P \cong Q$.

*Remark* 2.1. An order-embedding is automatically a one-to-one map since if $\varphi(x) = \varphi(y)$ in $Q$, then $\varphi(x) \leq_Q \varphi(y)$ and $\varphi(y) \leq_Q \varphi(x)$ in $Q$. This means that $x \leq_P y$ and $y \leq_P x$ in $P$ since $\varphi$ is an order-embedding, which gives that $x = y$.

**Definition 2.7.** Let $P$ be a poset. An *interval* $[x, y] = \{z \in P : x \leq_P z \leq_P y\}$ is a subposet of $P$ defined whenever $x \leq_P y$.

**Example 2.3.** In $P = \mathcal{P}(3)$, $[\emptyset, \{1, 3\}] = \{\emptyset, \{1\}, \{3\}, \{1, 3\}\}$ is an interval.

**Definition 2.8.** A poset $P$ is *locally finite* or *finitary* if every interval of $P$ is finite.

**Definition 2.9.** Let $P$ be a poset. We say that $P$ *has a* $\hat{0}$ if there exists an element $\hat{0} \in P$ such that $\hat{0} \leq_P x$ for all $x \in P$.

**Definition 2.10.** Let $P$ be a poset. We say that $P$ *has a* $\hat{1}$ if there exists an element $\hat{1} \in P$ such that $x \leq_P \hat{1}$ for all $x \in P$.

**Definition 2.11.** A *chain* $C$ is a poset in which any two elements are comparable.

**Definition 2.12.** The *length* of a finite chain $C$, denoted $l(C)$, is defined by $l(C) = |C| - 1$, where $|C|$ is the number of elements in $C$ .

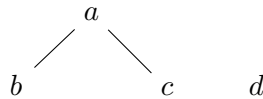**Definition 2.13.** An element $x$ of a poset $P$ is a *maximal element* if $y \geq_P x$ imply $x = y$ for each $y$ in $P$.

**Definition 2.14.** An element $x$ of a poset $P$ is a *minimal element* if $x \geq_P y$ imply $x = y$ for each $y$ in $P$.

**Definition 2.15.** Let $P$ be a finite poset and $x$ an element of $P$, the *rank* of $x$ is the length of the longest chain having $x$ as a maximal element and is denoted $r(x)$.

**Definition 2.16.** An *order ideal* (or semi-ideal, down-set or decreasing subset) of a poset $P$ is a subset $I$ of $P$ such that $x \in I$ and $y \leq_P x$ imply $y \in I$. The set of all order ideals of $P$, ordered by inclusion, forms a poset denoted by $J(P)$.

**Definition 2.17.** A *dual order ideal* (or filter, up-set or increasing subset) of a poset $P$ is a subset $I$ of $P$ such that $x \in I$ and $x \leq_P y$ imply $y \in I$.

**Example 2.4.** Let $P$ be the poset given by the following Hasse diagram:



The set of order ideals $J(P)$ are given by the following Hasse diagram:

$$\{a,b,c,d\}$$

$$\{a,b,c\} \qquad \{b,c,d\}$$

$$\{b,c\} \qquad \{b,d\} \qquad \{c,d\}$$

$$\{b\} \qquad \{c\} \qquad \{d\}$$

$$\emptyset$$
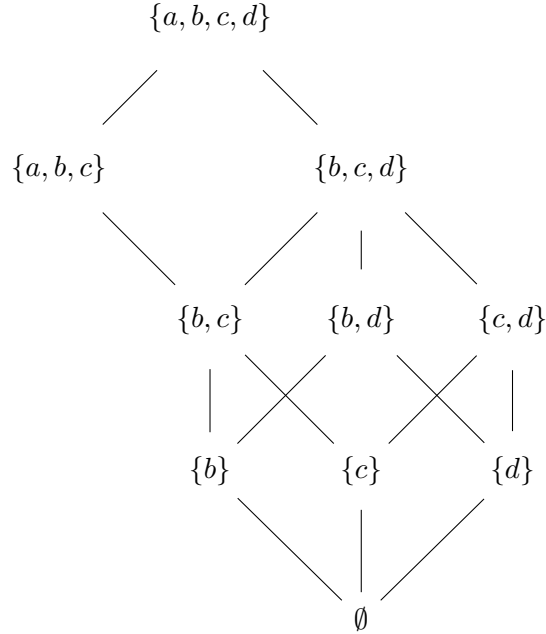
**Definition 2.18.** An order ideal of a poset $P$ is a *principal order ideal* if it is of the form $I = \{y \in P : y \leq_P x\}$, for some element $x \in P$.

**Example 2.5.** The principal order ideals of the poset given in the previous example are $\{\emptyset, \{b\}, \{c\}, \{d\}, \{a,b,c\}\}$.

**Definition 2.19.** If $x$ and $y$ belong to a poset $P$, then an upper bound of $x$ and $y$ is an element $z \in P$ satisfying $x \leq_P z$ and $y \leq_P z$. *A least upper bound* or *join* of $x$ and $y$ is an upper bound $z$ such that every upper bound $w$ of $x$ and $y$ satisfies $z \leq_P w$. If a least upper bound exists, then it is denoted by $x \vee y$.

**Definition 2.20.** If $x$ and $y$ belong to a poset $P$, then a lower bound of $x$ and $y$ is an element $z \in P$ satisfying $z \leq_P x$ and $z \leq_P y$. *A greatest lower bound* or *meet* of $x$ and $y$ is a lower bound $z$ such that every lower bound $w$ of $x$ and $y$ satisfies $w \leq_P z$. If a greatest lower bound of $x$ and $y$ exists, then it is denoted by $x \wedge y$.

*Remark* 2.2. Note that a least upper bound and a greatest lower bound is unique.

**Definition 2.21.** A *lattice* is a poset $L$ for which every pair of elements has a least upper bound and a greatest lower bound.

*Remark* 2.3. Every finite lattice has a $\hat{0}$ and a $\hat{1}$.

**Definition 2.22.** Let $L$ be a lattice. A non-empty subset $S$ of $L$ is a *sublattice* of $L$ if $a, b \in S$ implies $a \vee b \in S$ and $a \wedge b \in S$.

**Definition 2.23.** A lattice $L$ which satisfies the following conditions is a *distributive lattice*. For $x, y, z \in L$:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

**Example 2.6.** The set $J(P)$ of order ideals of the poset $P$ is a distributive lattice where the lattice operations $\vee$ and $\wedge$ on order ideals are just ordinary union and intersection. Since the unions and intersections of order ideals are again an order ideal, it follows from the well-known distributivity of set union and intersection over one another that $J(P)$ is indeed a distributive lattice.

**Definition 2.24.** An element $x$ of a lattice $L$ is *join-irreducible* if $x$ is not the join of a finite set of other elements. The poset consisting of all join-irreducibles of $L$ is denoted $\mathcal{K}(L)$.

**Definition 2.25.** An element $x$ of a lattice $L$ is *meet-irreducible* if $x$ is not the meet of a finite set of other elements.

**Example 2.7.** Let $L$ be the following lattice:
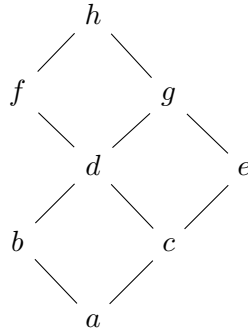


The join-irreducible elements of $L$ are $\{b, c, e, f\}$ and the meet-irreducible are $\{b, e, f, g\}$.

**Definition 2.26.** A poset $P$ satisfies the *descending chain condition* (DCC) if given any sequence $x_1 \geq_P x_2 \geq_P \cdots \geq_P x_n \geq_P \cdots$ of elements in $P$, there exists a $k \in \mathbb{N}$ such that $x_k = x_n$ whenever $k < n$ .

*Remark* 2.4. Every finite lattice satisfies DCC.

**Lemma 2.1.** *An ordered set $P$ satisfies DCC if and only if every non-empty subset $A$ of $P$ has a minimal element.*

*Proof.* We shall prove the contrapositive in both directions. We prove that $P$ has an infinite strictly descending chain if and only if there is a non-empty subset $A$ of $P$ which has no minimal element.

Assume that $x_1 >_P x_2 >_P \cdots >_P x_n >_P \cdots$ is an infinite descending chain in $P$; then $A := \{x_n : n \in \mathbb{N}\}$ has no minimal element.

Conversely, assume that $A$ is a non-empty subset $P$ which has no minimal element. Let $x_1 \in A$. Since $x_1$ is not minimal, there exists $x_2 \in A$ with $x_1 >_P x_2$. Similarly, there exists $x_3 \in A$ with $x_2 >_P x_3$. Continuing in this way, we obtain an infinite descending chain in $P$. $\qquad\square$

**Lemma 2.2.** *Let $L$ be a lattice satisfying DCC, and let $\mathcal{K}(L)$ denote the poset of join-irreducibles of $L$.*

   *i) Suppose $a, b \in L$ and $a \not\leq_L b$. Then there exists $x \in \mathcal{K}(L)$ such that $x \leq_L a$ and $x \not\leq_L b$.*

   *ii) $a = \bigvee\{x \in \mathcal{K}(L) : x \leq_L a\}$ for all $a \in L$.*

*Proof.*

   i) Let $a \not\leq_L b$ and let $S = \{x \in L : x \leq_L a, x \not\leq_L b\}$. The set $S$ is non-empty since it contains $a$. Hence, since $L$ satisfies DCC, there exists a minimal element $x$ in $S$. We claim that $x$ is join-irreducible. Suppose that $x$ is not join-irreducible, then $x = c \vee d$ with $c <_L x$ and $d <_L x$. By the minimality of $x$, neither $c$ nor $d$ lies in $S$. We thus see that $c <_L x \leq_L a$, so $c \leq_L a$, and similarly $d \leq_L a$. Therefore $c, d \notin S$ implies $c \leq_L b$ and $d \leq_L b$. This in turn imply $x = c \vee d \leq_L b$, which contradicts the assumption that $x$ is not join-irreducible, and we conclude that $x \in \mathcal{K}(L) \cap S$.

   ii) For each $a \in L$, let $\eta(a) = \{x \in \mathcal{K}(L) : x \leq_L a\}$. Clearly $a$ is an upper bound of $\eta(a)$. Let $c$ be another upper bound of $\eta(a)$. We claim that $a \leq_L c$, that is, $a$ is a least upper bound of $\eta(a)$.

   Suppose that $a \not\leq_L c$; then $a \wedge c <_L a$ and hence $a \not\leq_L a \wedge c$. By (i) there exists $x \in \eta(a)$ with $x \not\leq_L a \wedge c$. But $x \in \eta(a)$ implies $x \leq_L a$ (by definition) and $x \leq_L c$ since $c$ is an upper bound of $\eta(a)$. Thus $x$ is a lower bound of $\{a, c\}$ and consequently $x \leq_L a \wedge c$, a contradiction. $a$ is hence the least upper bound. This proves that $a = \bigvee \eta(a)$ in $L$.

$\qquad\square$

**Proposition 2.1.** *Let $P$ and $Q$ be finite posets. Then $J(P) \cong J(Q)$ if and only if $P \cong Q$.*

*Proof.* An order ideal $I$ of $P$ is join-irreducible in $J(P)$ if and only if it is a principal order ideal of $P$. The proposition follows from $P \cong \mathcal{K}(J(P))$, so we will prove that. Let $\phi : P \to \mathcal{K}(J(P))$ be the map defined by $\phi(x) =$

$\{y \in P : y \leq_P x\}$. The map is an order-embedding since $\phi(x) \subseteq \phi(y)$ if and only if $x \leq_P y$ and it is clear by the definition of a principal order ideal that the map is onto. $\square$

**Theorem 2.1** (Birkhoff's representation theorem)**.** *Let $L$ be a finite distributive lattice. There is a unique (up to isomorphism) finite poset $P$ for which $L \cong J(P)$.*

*Proof.* Because of the previous lemma which showed that $P \cong Q$ if and only if $J(P) \cong J(Q)$ we have showed that P is unique, so we only have to show that for a finite poset $P$, $L \cong J(P)$.

Let $P = \mathcal{K}(L)$, the subposet of join-irreducibles of $L$ and let $\eta : L \to J(\mathcal{K}(L))$ be the map defined by $a \to \eta(a) = \{x \in \mathcal{K}(L) : x \leq_L a\}$. For any element $a \in L$, $\eta(a) \in J(\mathcal{K}(L))$ since given $x \in \eta(a)$ and $y \in J(\mathcal{K}(L))$ with $y \leq_L x$, by transitivity $y \leq_L a$ which implies that $y \in \eta(a)$, so $\eta(a)$ is an ideal .

It remains to show that $\eta$ is a surjective order-embedding. We first prove that $\eta$ is an order-embedding.

We have that if $a \leq_L b$, then $\eta(a) \subseteq \eta(b)$. Assume that $\eta(a) \subseteq \eta(b)$, by the previous lemma we know that $a = \bigvee \eta(a) \leq_L \bigvee \eta(b) = b$.

To prove that $\eta$ is onto, let $U \in J(\mathcal{K}(L))$ and write $U = \{a_1, a_2, \ldots, a_k\}$. Define $a$ to be $\bigvee\{a_1, \ldots, a_k\}$. We will show that $U = \eta(a)$. Let $y \in U$, so $y = a_i$ for some $i$. Then $y$ is join-irreducible and $y \leq_L a$, hence $y \in \eta(a)$. Suppose that $y \in \eta(a)$. Then

$$\bigvee\{a_1, \ldots, a_k\} = a = \bigvee \eta(a) \ .$$

Apply $\wedge y$ to both sides. We get, by distributivity,

$$\bigvee\{a_i \wedge y : a_i \in U\} = \bigvee\{x \wedge y : x \in \eta(a)\} \ .$$

The right-hand side is just $y$, since one term is $y$ and all others are $\leq_L y$. So we get

$$\bigvee\{a_i \wedge y : a_i \in U\} = y \ .$$

Since $y$ is join-irreducible, it follows that some $a_i \in U$ satisfies $a_i \wedge y = y$, that is $y \leq_L a_i$. Since $U$ is an order ideal, $y \in U$, which imply $U = \eta(a)$. $\square$

**Lemma 2.3.** *Let $L$ be a finite distributive lattice and $\eta$ be the function defined above. Then $|\eta(a)| = r(a)$.*

*Proof.* Let $a \in L$ and $\eta(a) = \{a_1, a_2, \ldots, a_k\}$. Since $0 < a_1 < a_1 \vee a_2 < \cdots < a_1 \vee \cdots \vee a_k \leq a$, it follows that $|\eta(a)| = k \leq r(a)$. On the other hand, if $0 < y_1 < y_2 < \cdots < y_m = a$ is the longest chain having $a$ as a maximal element, then, since $\eta$ is injective, we have $\emptyset \subsetneq \eta(y_1) \subsetneq \eta(y_2) \subsetneq \cdots \subsetneq \eta(y_m)$ showing that $|\eta(a)| \geq m = r(a)$. Thus $|\eta(a)| = r(a)$. $\square$

Theorem 2.1 states that every finite distributive lattice $L$ is isomorphic to a sublattice of $\mathcal{P}(n)$ for some $n$. More specific, $n$ is the the number of join-irreducible elements of $L$.
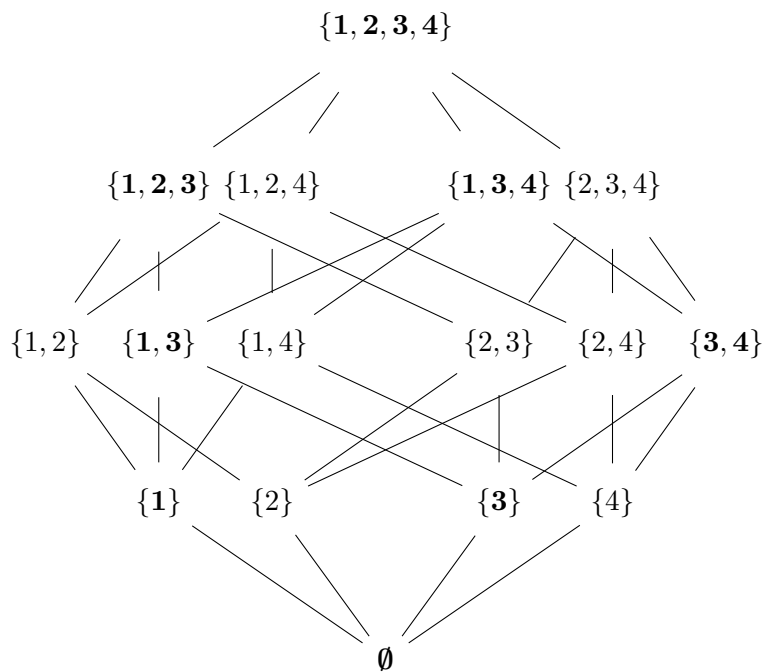
**Example 2.8.** Let $L$ be the lattice given by the following Hasse diagram:

$$
\begin{array}{ccc}
 & h & \\
\mathbf{f} & & g \\
 & d & \mathbf{e} \\
\mathbf{b} & & \mathbf{c} \\
 & a &
\end{array}
$$
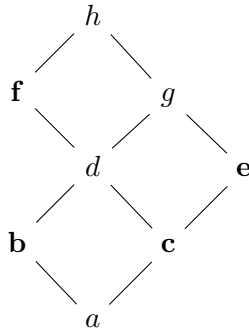
Elements shown in bold type are the join-irreducibles. The Hasse diagram of the subposet of join-irreducibles is given by:

$$
\begin{array}{cc}
f & e \\
| \quad \diagdown & | \\
b & c
\end{array}
$$
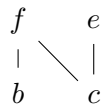
Since the number of join-irreducibles is 4, we get, by Theorem 2.1, that $L$ is isomorphic to a subposet of $\mathcal{P}(4)$. If we set $b = 1$, $f = 2$, $c = 3$, $e = 4$, one can easily see how $\mathcal{P}(\mathcal{K}(L))$ correspond to $\mathcal{P}(4)$. The Hasse diagrams are given by:

$$\{\mathbf{b}, \mathbf{f}, \mathbf{c}, \mathbf{e}\}$$

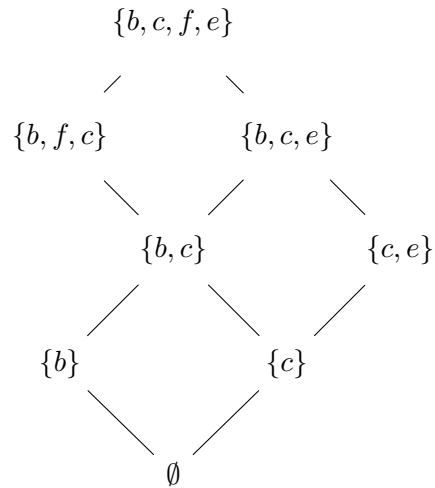$$\{\mathbf{b}, \mathbf{f}, \mathbf{c}\} \quad \{b, f, e\} \qquad \{\mathbf{b}, \mathbf{c}, \mathbf{e}\} \quad \{f, c, e\}$$

$$\{b, f\} \quad \{\mathbf{b}, \mathbf{c}\} \quad \{b, e\} \qquad \{f, c\} \quad \{f, e\} \quad \{\mathbf{c}, \mathbf{e}\}$$

$$\{\mathbf{b}\} \quad \{f\} \qquad \{\mathbf{c}\} \quad \{e\}$$

$$\emptyset$$

$$\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\}$$

$$\{\mathbf{1}, \mathbf{2}, \mathbf{3}\} \quad \{1, 2, 4\} \qquad \{\mathbf{1}, \mathbf{3}, \mathbf{4}\} \quad \{2, 3, 4\}$$

$$\{1, 2\} \quad \{\mathbf{1}, \mathbf{3}\} \quad \{1, 4\} \qquad \{2, 3\} \quad \{2, 4\} \quad \{\mathbf{3}, \mathbf{4}\}$$

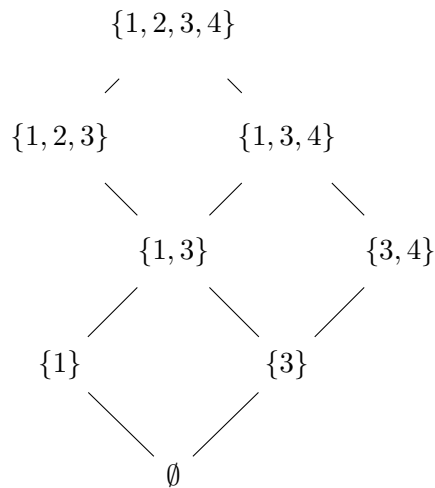$$\{\mathbf{1}\} \quad \{2\} \qquad \{\mathbf{3}\} \quad \{4\}$$

$$\emptyset$$
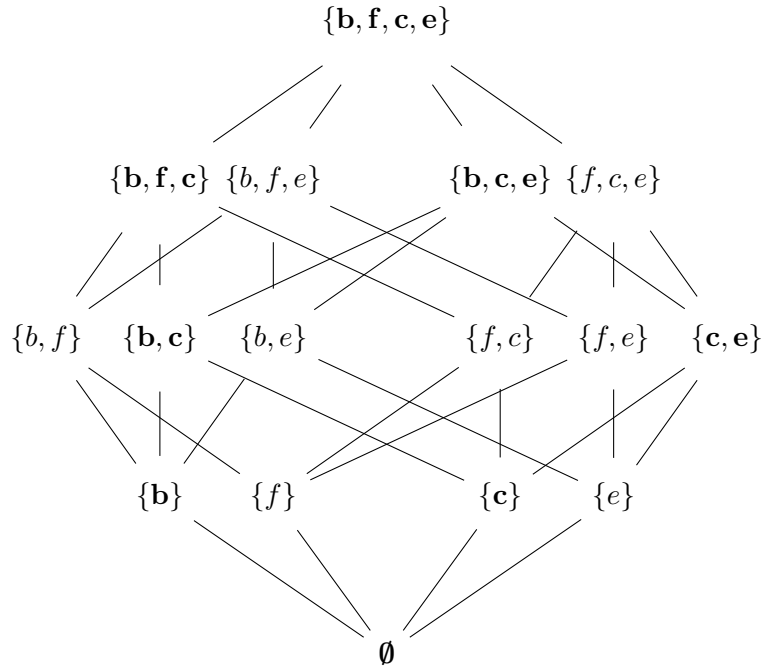
The elements in bold type are the elements of $J(\mathcal{K}(L))$ and we get the

following Hasse diagram for $J(\mathcal{K}(L))$:

$$\{b,c,f,e\}$$

$$\{b,f,c\} \qquad \{b,c,e\}$$

$$\{b,c\} \qquad \{c,e\}$$

$$\{b\} \qquad \{c\}$$

$$\emptyset$$

This is the subposet of $P(4)$ which correspond to $J(\mathcal{K}(L))$.

$$\{1,2,3,4\}$$

$$\{1,2,3\} \qquad \{1,3,4\}$$

$$\{1,3\} \qquad \{3,4\}$$

$$\{1\} \qquad \{3\}$$

$$\emptyset$$

Not surprisingly $L$, $J(\mathcal{K}(L))$ and the set $V = \{\emptyset, \{1\}, \{3\}, \{1,3\}, \{3,4\},$ $\{1,2,3\}, \{1,3,4\}, \{1,2,3,4\}\}$ have the same Hasse diagram. This is just the statement of Theorem 2.1.

# Chapter 3

# The four functions theorem

In this chapter we state and prove the four functions theorem which is the main result of this thesis. We will also discuss the FKG inequality which follows as a special case of the four functions theorem.

Given families $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(n)$ we write $\mathcal{A} \vee \mathcal{B} = \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\}$ and $\mathcal{A} \wedge \mathcal{B} = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}$. Given $f : \mathcal{P}(n) \to \mathbb{R}$ and $\mathcal{A} \subseteq \mathcal{P}(n)$ we let $\tilde{f}(\mathcal{A}) = \sum_{A \in \mathcal{A}} f(A)$. The following theorem is appears in [1].

**Theorem 3.1** (The four functions theorem of Ahlswede and Daykin)**.** *Let* $n \in \mathbb{N}$ *and* $\alpha, \beta, \gamma$ *and* $\delta : \mathcal{P}(n) \to \mathbb{R}_{\geq 0}$*. If*

$$\alpha(A)\beta(B) \leq \gamma(A \cup B)\delta(A \cap B) \tag{3.1}$$

*for every pair* $A, B \in \mathcal{P}(n)$*, then*

$$\tilde{\alpha}(\mathcal{A})\tilde{\beta}(\mathcal{B}) \leq \tilde{\gamma}(\mathcal{A} \vee \mathcal{B})\tilde{\delta}(\mathcal{A} \wedge \mathcal{B}) \tag{3.2}$$

*for every pair* $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(n)$*.*

*Proof.* The proof is by induction on $n$. Consider the case $n = 1$, in which $\mathcal{P}(n) = \mathcal{P}(1) = \{\emptyset, \{1\}\}$. Then 3.1 becomes

$$\begin{aligned} \alpha(\emptyset)\beta(\emptyset) &\leq \gamma(\emptyset)\delta(\emptyset) \\ \alpha(\emptyset)\beta(\{1\}) &\leq \gamma(\{1\})\delta(\emptyset) \\ \alpha(\{1\})\beta(\emptyset) &\leq \gamma(\{1\})\delta(\emptyset) \\ \alpha(\{1\})\beta(\{1\}) &\leq \gamma(\{1\})\delta(\{1\}) \end{aligned} \tag{3.3}$$

Now, if $\mathcal{A}$ and $\mathcal{B}$ consists of a single element, (3.2) will follow immediately from (3.1). E.g., if $\mathcal{A} = \{\emptyset\}$ and $\mathcal{B} = \{\{1\}\}$, (3.2) becomes $\alpha(\emptyset)\beta(\{1\}) \leq \gamma(\{1\})\delta(\emptyset)$, and this is just the second inequality of (3.3).

That (3.2) holds for the case when one of $\mathcal{A}$ or $\mathcal{B}$ consists of one element and the other of two is easily seen by direct inspection. For example, if $\mathcal{A} = \{\emptyset\}$ and $\mathcal{B} = \{\emptyset, \{1\}\}$, then (3.2) becomes

$$\alpha(\emptyset)(\beta(\emptyset) + \beta(\{1\})) \leq (\gamma(\emptyset) + \gamma(\{1\}))\delta(\emptyset)$$

and this follows from the first and second inequality of (3.3).

If $\mathcal{A} = \{\emptyset, \{1\}\} = \mathcal{B}$, then (3.2) becomes

$$(\alpha(\emptyset) + \alpha(\{1\}))(\beta(\emptyset) + \beta(\{1\})) \leq (\gamma(\emptyset) + \gamma(\{1\}))(\delta(\emptyset) + \delta(\{1\}))$$

Since, by (3.3), $\alpha(\emptyset)\beta(\emptyset) \leq \gamma(\emptyset)\delta(\emptyset)$ and $\alpha(\{1\})\beta(\{1\}) \leq \gamma(\{1\})\delta(\{1\})$ we only need to show

$$\alpha(\emptyset)\beta(\{1\}) + \alpha(\{1\})\beta(\emptyset) \leq \gamma(\emptyset)\delta(\{1\}) + \gamma(\{1\})\delta(\emptyset) \qquad (3.4)$$

Now, we have two cases to consider. Case one: $\gamma(\{1\})\delta(\emptyset) = 0$. Case two: $\gamma(\{1\}\delta(\emptyset) \neq 0$.

If $\gamma(\{1\})\delta(\emptyset) = 0$, then (3.4) becomes $0 \leq \gamma(\emptyset)\delta(\{1\})$, since by (3.3), $\alpha(\emptyset)\beta(\{1\})$ and $\alpha(\{1\})\beta(\emptyset)$ will be 0. Now the requested inequality follows since $\gamma$ and $\delta$ are both greater or equal to zero.

If $\gamma(\{1\}\delta(\emptyset) \neq 0$, by the first inequality of (3.3) we have $\gamma(\emptyset) \geq \frac{\alpha(\emptyset)\beta(\emptyset)}{\delta(\emptyset)}$ and by the fourth we have $\delta(\{1\}) \geq \frac{\alpha(\{1\})\beta(\{1\})}{\gamma(\{1\})}$. Hence, we are done if we can show that

$$\alpha(\emptyset)\beta(\{1\}) + \alpha(\{1\})\beta(\emptyset) \leq \left(\frac{\alpha(\emptyset)\beta(\emptyset)}{\delta(\emptyset)}\right)\left(\frac{\alpha(\{1\})\beta(\{1\})}{\gamma(\{1\})}\right) + \gamma(\{1\})\delta(\emptyset)$$
$$(3.5)$$

since

$$\left(\frac{\alpha(\emptyset)\beta(\emptyset)}{\delta(\emptyset)}\right)\left(\frac{\alpha(\{1\})\beta(\{1\})}{\gamma(\{1\})}\right) + \gamma(\{1\})\delta(\emptyset) \leq \gamma(\emptyset)\delta(\{1\}) + \gamma(\{1\})\delta(\emptyset)$$

It is not hard to see that (3.5) is equivalent to

$$(\gamma(\{1\})\delta(\emptyset) - \alpha(\emptyset)\beta(\{1\}))(\gamma(\{1\})\delta(\emptyset) - \alpha(\{1\})\beta(\emptyset)) \geq 0$$

The last inequality is true since $\gamma(\{1\})\delta(\emptyset) \geq \alpha(\emptyset)\beta(\{1\}))$ and $\gamma(\{1\})\delta(\emptyset) \geq \alpha(\{1\})\beta(\emptyset)$. We conclude that the Theorem is true for $n = 1$.

Assume now that the statement holds for $n = k - 1$ for some $k \geq 2$. Suppose the functions $\alpha, \beta, \gamma, \delta : \mathcal{P}(k) \to \mathbb{R}_{\geq 0}$ satisfy (3.1) with $n = k$ and let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(k)$ be given. Define new functions $\alpha', \beta', \gamma', \delta' : \mathcal{P}(k-1) \to \mathbb{R}_{\geq 0}$ as follows

$$\alpha'(A') = \sum_{\substack{A \in \mathcal{A} \\ A' = A \cap [k-1]}} \alpha(A) \qquad\qquad \gamma'(C') = \sum_{\substack{C \in \mathcal{A} \vee \mathcal{B} \\ C' = C \cap [k-1]}} \gamma(C)$$

$$\beta'(B') = \sum_{\substack{B \in \mathcal{B} \\ B' = B \cap [k-1]}} \beta(B) \qquad\qquad \delta'(D') = \sum_{\substack{D \in \mathcal{A} \wedge \mathcal{B} \\ D' = D \cap [k-1]}} \delta(D)$$

Thus, for $A' \in \mathcal{P}(k-1)$

$$\alpha'(A') = \begin{cases} \alpha(A') + \alpha(A' \cup \{k\}) & \text{if } A' \in \mathcal{A}, A' \cup \{k\} \in \mathcal{A} \\ \alpha(A') & \text{if } A' \in \mathcal{A}, A' \cup \{k\} \notin \mathcal{A} \\ \alpha(A' \cup \{k\}) & \text{if } A' \notin \mathcal{A}, A' \cup \{k\} \in \mathcal{A} \\ 0 & \text{if } A' \notin \mathcal{A}, A' \cup \{k\} \notin \mathcal{A} \end{cases}$$

With these definitions we have the following

$$\tilde{\alpha}(\mathcal{A}) = \sum_{A \in \mathcal{A}} \alpha(A) = \sum_{A' \in \mathcal{P}(k-1)} \Big( \sum_{\substack{A \in \mathcal{A} \\ A' = A \cap [k-1]}} \alpha(A) \Big) = \sum_{A' \in \mathcal{P}(k-1)} \alpha'(A') = \tilde{\alpha}'(\mathcal{P}(k-1))$$

$$\tilde{\beta}(\mathcal{B}) = \sum_{B \in \mathcal{B}} \beta(B) = \sum_{B' \in \mathcal{P}(k-1)} \Big( \sum_{\substack{B \in \mathcal{B} \\ B' = B \cap [k-1]}} \beta(B) \Big) = \sum_{B' \in \mathcal{P}(k-1)} \beta'(B') = \tilde{\beta}'(\mathcal{P}(k-1))$$

$$\tilde{\gamma}(\mathcal{A} \vee \mathcal{B}) = \sum_{C \in \mathcal{A} \vee \mathcal{B}} \gamma(C)$$
$$= \sum_{C' \in \mathcal{P}(k-1)} \Big( \sum_{\substack{C \in \mathcal{A} \vee \mathcal{B} \\ C' = C \cap [k-1]}} \gamma(C) \Big) = \sum_{C' \in \mathcal{P}(k-1)} \gamma'(C') = \tilde{\gamma}'(\mathcal{P}(k-1))$$

$$\tilde{\delta}(\mathcal{A} \wedge \mathcal{B}) = \sum_{D \in \mathcal{A} \wedge \mathcal{B}} \delta(D)$$
$$= \sum_{D' \in \mathcal{P}(k-1)} \Big( \sum_{\substack{D \in \mathcal{A} \wedge \mathcal{B} \\ D' = D \cap [k-1]}} \delta(D) \Big) = \sum_{D' \in \mathcal{P}(k-1)} \delta'(D') = \tilde{\delta}'(\mathcal{P}(k-1))$$

Therefore, if

$$\alpha'(A')\beta'(B') \leq \gamma'(A' \cup B')\delta'(A' \cap B') \tag{3.6}$$

for all $A', B' \in \mathcal{P}(k-1)$, then by the induction hypotheses we have

$$\tilde{\alpha}(\mathcal{A})\tilde{\beta}(\mathcal{B}) = \tilde{\alpha}'(\mathcal{P}(k-1))\tilde{\beta}'(\mathcal{P}(k-1)) \leq$$
$$\leq \tilde{\gamma}'(\mathcal{P}(k-1) \vee \mathcal{P}(k-1))\tilde{\delta}'(\mathcal{P}(k-1) \wedge \mathcal{P}(k-1)) = \tilde{\gamma}(\mathcal{A} \vee \mathcal{B})\tilde{\delta}(\mathcal{A} \wedge \mathcal{B})$$

and this is just (3.2). To prove (3.6), note that this is just case $n = 1$. To see this, define $\bar{\alpha}(\emptyset) = \alpha(A')$, $\bar{\alpha}(\{1\}) = \alpha(A' \cup \{k\})$, $\bar{\beta}(\emptyset) = \beta(A')$, $\bar{\beta}(\{1\}) = \beta(A' \cup \{k\})$, $\bar{\gamma}(\emptyset) = \gamma(A')$, $\bar{\gamma}(\{1\}) = \gamma(A' \cup \{k\})$, $\bar{\delta}(\emptyset) = \delta(A')$ and $\bar{\delta}(\{1\}) = \delta(A' \cup \{k\})$. For instance, for $\alpha'$ we would have

$$\alpha'(A') = \begin{cases} \bar{\alpha}(\emptyset) + \bar{\alpha}(\{1\}) & \text{if } A' \in \mathcal{A}, A' \cup \{k\} \in \mathcal{A} \\ \bar{\alpha}(\emptyset) & \text{if } A' \in \mathcal{A}, A' \cup \{k\} \notin \mathcal{A} \\ \bar{\alpha}(\{1\}) & \text{if } A' \notin \mathcal{A}, A' \cup \{k\} \in \mathcal{A} \\ 0 & \text{if } A' \notin \mathcal{A}, A' \cup \{k\} \notin \mathcal{A} \end{cases}$$

Since we have already treated this case, the proof is now completed. $\qquad \square$

Given a lattice $L$ and subsets $X, Y \subseteq L$ we write $X \vee Y = \{x \vee y : x \in X, y \in Y\}$ and $X \wedge Y = \{x \wedge y : x \in X, y \in Y\}$. Since every finite distributive lattice is isomorphic to some sublattice of $\mathcal{P}(n)$, we get the following corollary.

**Corollary 3.1.** Let $L$ be a finite distributive lattice and let $\alpha, \beta, \gamma$ and $\delta : L \to \mathbb{R}_{\geq 0}$ satisfy

$$\alpha(x)\beta(y) \leq \gamma(x \vee y)\delta(x \wedge y) \qquad (3.7)$$

for every pair $x, y \in L$. Then

$$\tilde{\alpha}(X)\tilde{\beta}(Y) \leq \tilde{\gamma}(X \vee Y)\tilde{\delta}(X \wedge Y)$$

for every pair $X, Y \subseteq L$.

*Proof.* Take any bijection $m : \mathcal{K}(L) \to \{1, 2, \ldots, |\mathcal{K}(L)|\}$ and define $\eta : L \to \mathcal{P}(|\mathcal{K}(L)|)$ by $\eta(x) = \{m(s) : s \in \mathcal{K}(L), s \leq x\}$. The map $\eta$ is an embedding of $L$ into $P(|\mathcal{K}(L)|)$. Extend now each of $\alpha$, $\beta$, $\gamma$ and $\delta$ to the whole of $P(|\mathcal{K}(L)|)$ by defining them to be 0 outside $L$. The result will follow from Theorem 3.1. $\qquad \square$

*Remark* 3.1. Note that above, we have used Birkhoff's representation theorem!

**Definition 3.1.** A function $f : L \to \mathbb{R}$ is called *increasing* if $x \leq_L y$ implies $f(x) \leq f(y)$ and *decreasing* if $x \geq_L y$ implies $f(x) \leq f(y)$.

**Definition 3.2.** A function $\mu : L \to \mathbb{R}_{\geq 0}$ is said to be *log-supermodular* if it satisfies

$$\mu(x)\mu(y) \leq \mu(x \vee y)\mu(x \wedge y) \qquad (3.8)$$

$\mu$ is said to be *log-modular* if it satisfies

$$\mu(x)\mu(y) = \mu(x \vee y)\mu(x \wedge y)$$

*Remark* 3.2. Any function $\mu : L \to \mathbb{R}_{\geq 0}$ defined on a totally ordered lattice is automatically log-modular since the meet and join is just the minimum respectively maximum of the elements.

The following theorem is due to Fortuin, Kasteleyn and Ginibre [7].

**Corollary 3.2** (The FKG inequality)**.** Let $L$ be a finite distributive lattice, $\mu : L \to \mathbb{R}_{\geq 0}$ a log-supermodular function and $f, g : L \to \mathbb{R}_{\geq 0}$ either both increasing or both decreasing. Then

$$\sum_{x \in L} f(x)\mu(x) \sum_{x \in L} g(x)\mu(x) \leq \sum_{x \in L} \mu(x) \sum_{x \in L} f(x)g(x)\mu(x) \qquad (3.9)$$

*Proof.* We prove the corollary by applying Corollary 3.1 with $X = Y = L$ and $\alpha = f\mu$, $\beta = g\mu$, $\gamma = \mu$ and $\delta = fg\mu$ when $f$ and $g$ are decreasing functions and with $\alpha = f\mu$, $\beta = g\mu$, $\gamma = fg\mu$ and $\delta = \mu$ when $f$ and $g$ are increasing.

For $f$ and $g$ decreasing functions, (3.7) becomes

$$f(x)\mu(x)g(y)\mu(y) \leq \mu(x \vee y)f(x \wedge y)g(x \wedge y)\mu(x \wedge y)$$

For $f$ and $g$ increasing functions, (3.7) becomes

$$f(x)\mu(x)g(y)\mu(y) \leq f(x \vee y)g(x \vee y)\mu(x \vee y)\mu(x \wedge y)$$

Since the function $\mu$ is log-supermodular we have

$$\mu(x)\mu(y) \leq \mu(x \vee y)\mu(x \wedge y)$$

When both $f$ and $g$ are decreasing functions, we have

$$f(x)g(y) \leq f(x \wedge y)g(x \wedge y)$$

and when both are increasing, we have

$$f(x)g(y) \leq f(x \vee y)g(x \vee y)$$

Hence (3.7) is satisfied in both cases, which indeed implies (3.9). □

*Remark* 3.3. When the functions $f$ and $g$ are counter monotone, the inequality is reversed.

*Remark* 3.4. If $\mu$ satisfies (3.8) and $L_0 \subset L$ is the support of $\mu$:

$$L_0 = \{x \in L : \mu(x) > 0\}$$

and if $x \in L_0$ and $y \in L_0$, then by (3.8) $\mu(x \vee y)\mu(x \wedge y) > 0$ which implies that both $x \vee y$ and $x \wedge y$ are in $L_0$. By definition, we get that $L_0$ is a sublattice of $L$. So, if (3.8) holds for all $x, y \in L_0$ it also holds for all $x, y \in L$. This means that the FKG inequality can be formulated for an infinite distributive lattice. We then require that either the sums converge or that the function $\mu$ has a finite support.

**Example 3.1.** Let $L = \mathbf{Z}_{>0}$, $\mu(n) = 1/2^n$, $f(n) = n$ and $g(n) = n^2$. The lattice $L$ is a finitary distributive lattice, the function $\mu$ is log-modular and $f, g$ are both increasing. From the FKG inequality it follows that:

$$\sum_{n \in \mathbf{Z}_{>0}} \frac{n}{2^n} \sum_{n \in \mathbf{Z}_{>0}} \frac{n^2}{2^n} \leq \sum_{n \in \mathbf{Z}_{>0}} \frac{n^3}{2^n}$$

*Remark* 3.5. R.L Graham [9] notes that, in some sense the FKG inequality has its roots in the old result of Chebyshev's sum inequality. It serves to extend the Chebyshev's sum inequality to the case where the underlying set is only supposed to be partially ordered as opposed to the totally ordered index set of integers occurring in the Chebyshev's sum inequality.

**Example 3.2.** Let $(a_k : 0 \leq k \leq n)$ be an arbitrary positive sequence and $(b_i : 0 \leq i \leq n)$, $(c_i : 0 \leq i \leq n)$ be both increasing or both decreasing sequences. Define functions $\mu, f, g : \{0, 1, \ldots, n\} \to \mathbb{R}_{\geq 0}$ by $\mu(k) = a_k$, $f(k) = b_k$ and $g(k) = c_k$. The function $\mu$ is log-modular since the domain is a totally ordered set, and the functions $f, g$ are both decreasing or increasing. From the FKG inequality it follows that:

$$\sum_{k=0}^{n} a_k b_k \sum_{k=0}^{n} a_k c_k \leq \sum_{k=0}^{n} a_k \sum_{k=0}^{n} a_k b_k c_k$$

By letting $a_k = 1$ for all $k$ we get Chebyshev's sum inequality:

$$\sum_{k=0}^{n} b_k \sum_{k=0}^{n} c_k \leq n \sum_{k=0}^{n} b_k c_k$$

**Definition 3.3.** For sets $A$ and $B$ in $\mathcal{P}(n)$, the *set difference* of $A$ in $B$ is the set of elements in $B$, but not in $A$. The set difference of $A$ in $B$ is denoted $B \setminus A$.

The following two lemmas are due to Christofides [5].

**Lemma 3.1.** *Let* $\alpha, \beta, \gamma$ *and* $\delta : \mathcal{P}(n) \to \mathbb{R}_{\geq 0}$ *satisfy*

$$\alpha(A)\beta(B) \leq \gamma(B \setminus A)\delta(A \setminus B)$$

*for every pair* $A, B \in \mathcal{P}(n)$. *Then*

$$\tilde{\alpha}(\mathcal{P}(n))\tilde{\beta}(\mathcal{P}(n)) \leq \tilde{\gamma}(\mathcal{P}(n))\tilde{\delta}(\mathcal{P}(n))$$

*Proof.* Apply the four functions theorem to the functions $\alpha, \beta', \gamma'$ and $\delta$, Where $\beta'(B) := \beta(B^c)$ and $\gamma'(C) := \gamma(C^c)$. We then have, for $A, B \in \mathcal{P}(n)$

$$\alpha(A)\beta'(B) = \alpha(A)\beta(B^c) \leq \gamma(A \cup B^c)\delta(A \cap B^c) = \gamma'(B \setminus A)\delta(A \setminus B)$$

and the conclusion follows. $\qquad\square$

**Theorem 3.2.** *Let* $\alpha, \beta, \gamma$ *and* $\delta : \mathcal{P}(n) \to \mathbb{R}_{\geq 0}$ *satisfy*

$$\alpha(A)\beta(B) \leq \gamma(A \cup B)\delta(A \cap B)$$

*for every pair* $A, B \in \mathcal{P}(n)$. *Then*

$$\sum_{A \in \mathcal{P}(n)} \alpha(A)\beta(A^c) \leq \sum_{C \in \mathcal{P}(n)} \gamma(C)\delta(C^c)$$

*Proof.* Define $f, g : \mathcal{P}(n) \to \mathbb{R}_{\geq 0}$ by $f(A) = \alpha(A)\beta(A^c)$ and $g(A) = \gamma(A^c)\delta(A)$. Observe that $\sum_{A \in \mathcal{P}(n)} f(A) = \sum_{A \in \mathcal{P}(n)} \alpha(A)\beta(A^c)$ and $\sum_{A \in \mathcal{P}(n)} g(A) = \sum_{A \in \mathcal{P}(n)} \gamma(A)\delta(A^c)$. Then,

$$f(A)f(B) = (\alpha(A)\beta(B^c))(\alpha(B)\beta(A^c)) \leq$$
$$\leq \gamma(A \cup B^c)\delta(A \cap B^c)\gamma(B \cup A^c)\delta(B \cap A^c) =$$
$$= \gamma(A \cup B^c)\delta(B \cap A^c)\gamma(B \cup A^c)\delta(A \cap B^c) =$$
$$= g(A^c \cap B)g(A \cap b^c) = g(B \setminus A)g(A \setminus B)$$

If we apply the previous Lemma with $\alpha = \beta = f$ and $\gamma = \delta = g$, we obtain

$$(f(\mathcal{P}(n)))^2 \leq (g(\mathcal{P}(n)))^2$$

The result follows since all functions used take only non-negative values.  $\square$

**Definition 3.4.** Let $L$ and $K$ be lattices. A map $f : L \to K$ is a *lattice homomorphism* if $f$ is join-preserving and meet-preserving, that is, for all $a, b \in L$, $f(a \vee b) = f(a) \vee f(b)$ and $f(a \wedge b) = f(a) \wedge f(b)$.

**Definition 3.5.** A *lattice embedding* is a one-to-one lattice homomorphism.

Consider a pair of polynomials $p(x), q(x) \in \mathbb{R}[x_1, x_2, \ldots, x_n]$, where $q(x) \leq p(x)$. This means that $p(x) - q(x) \in \mathbb{R}_{\geq 0}[x_1, x_2, \ldots, x_n]$. We now state the polynomial version of the four functions theorem. W.l.o.g. we may assume that $\mathcal{A} = \mathcal{B} = \mathcal{A} \vee \mathcal{B} = \mathcal{A} \wedge \mathcal{B} = \mathcal{P}(n)$ since for fixed $\mathcal{A}$ and $\mathcal{B}$ we can always modify the functions $\alpha, \beta, \gamma$ and $\delta$ by setting $\alpha(A) = 0$ if $A \notin \mathcal{A}$, $\beta(B) = 0$ if $B \notin \mathcal{B}$, $\gamma(C) = 0$ if $C \notin \mathcal{A} \vee \mathcal{B}$ and $\delta(D) = 0$ if $D \notin \mathcal{A} \wedge \mathcal{B}$. The following theorem was first noted by P. Brändén (private communication).

**Theorem 3.3.** *Let $\alpha, \beta, \gamma$ and $\delta : \mathcal{P}(n) \to \mathbb{R}_{\geq 0}$ satisfy*

$$\alpha(A)\beta(B) \leq \gamma(A \cup B)\delta(A \cap B)$$

*for every pair $A, B \in \mathcal{P}(n)$. Then*

$$\left( \sum_{A \in \mathcal{P}(n)} \alpha(A) \prod_{a \in A} x_a \right) \left( \sum_{A \in \mathcal{P}(n)} \beta(A) \prod_{a \in A} x_a \right) \leq$$

$$\leq \left( \sum_{A \in \mathcal{P}(n)} \gamma(A) \prod_{a \in A} x_a \right) \left( \sum_{A \in \mathcal{P}(n)} \delta(A) \prod_{a \in A} x_a \right) \quad (3.10)$$

*Proof.* We have to show that the coefficient of any monomial of the left-hand side of the inequality is less of equal than the coefficient of the same monomial of the right-hand side. For a monomial $x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n}$ let $R = \{i : s_i = 0\}$, $S = \{i : s_i = 1\}$, $T = \{i : s_i = 2\}$ and $\psi : \mathcal{P}(S) \to \mathcal{P}(n)$, defined by

$A \to \psi(A) = A \cup T$. Note that $\psi$ is a lattice embedding since $\psi(A \cup B) = A \cup B \cup T = (A \cup T) \cup (B \cup T) = \psi(A) \cup \psi(B)$, $\psi(A \cap B) = (A \cap B) \cup T = (A \cup T) \cap (B \cup T) = \psi(A) \cap \psi(B)$ and if $A \neq B$ then $A \cup T \neq B \cup T$ which implies that $\psi(A) \neq \psi(B)$.

Define new functions $\alpha', \beta', \gamma'$ and $\delta' : \mathcal{P}(S) \to \mathbb{R}_{\geq 0}$ by $\alpha' = \alpha \circ \psi$, $\beta' = \beta \circ \psi$, $\gamma' = \gamma \circ \psi$ and $\delta' = \delta \circ \psi$. The coefficient of the monomial is $\sum_{A \in \mathcal{P}(S)} \alpha'(A)\beta'(A^c)$ and by the previous theorem, this is less or equal to $\sum_{A \in \mathcal{P}(S)} \gamma'(A)\delta'(A^c)$ which is the coefficient of the same monomial on the right-hand side of (3.10). $\qquad\square$

Recall that $\eta(a)$ was defined as the set of join-irreducibles less than or equal to $a$.

**Corollary 3.3.** Let $L$ be a finite distributive lattice and let $\alpha, \beta, \gamma$ and $\delta : L \to \mathbb{R}_{\geq 0}$ satisfy

$$\alpha(w)\beta(z) \leq \gamma(w \vee z)\delta(w \wedge z)$$

for every pair $w, z \in L$. Then

$$\left( \sum_{t \in W} \alpha(t) \prod_{a \in \eta(t)} x_a \right)\left( \sum_{t \in Z} \beta(t) \prod_{a \in \eta(t)} x_a \right) \leq$$
$$\left( \sum_{t \in W \vee Z} \gamma(t) \prod_{a \in \eta(t)} x_a \right)\left( \sum_{t \in W \wedge Z} \delta(t) \prod_{a \in \eta(t)} x_a \right) \quad (3.11)$$

for every pair $W, Z \subseteq L$.

By letting $x_1 = x_2 = \cdots = x_n = q$ in Theorem 3.3 we get the following $q$-analogue due to Christofides [5].

**Theorem 3.4.** *Let $\alpha, \beta, \gamma$ and $\delta : \mathcal{P}(n) \to \mathbb{R}_{\geq 0}$ satisfy*

$$\alpha(A)\beta(B) \leq \gamma(A \cup B)\delta(A \cap B)$$

*for every pair $A, B \in \mathcal{P}(n)$. Then*

$$\sum_{A \in \mathcal{P}(n)} \alpha(A)q^{|A|} \sum_{A \in \mathcal{P}(n)} \beta(A)q^{|A|} \leq \sum_{A \in \mathcal{P}(n)} \gamma(A)q^{|A|} \sum_{A \in \mathcal{P}(n)} \delta(A)q^{|A|}$$

**Corollary 3.4.** Let $L$ be a finite distributive lattice and let $\alpha, \beta, \gamma$ and $\delta : L \to \mathbb{R}_{\geq 0}$ satisfy

$$\alpha(w)\beta(z) \leq \gamma(w \vee z)\delta(w \wedge z)$$

for every pair $w, z \in L$. Then

$$\sum_{t \in W} \alpha(t)q^{r(t)} \sum_{t \in Z} \beta(t)q^{r(t)} \leq \sum_{t \in W \vee Z} \gamma(t)q^{r(t)} \sum_{t \in W \wedge Z} \delta(t)q^{r(t)}$$

for every pair $W, Z \subseteq L$.

We now state the polynomial version of the FKG inequality.

**Corollary 3.5.** Let $L$ be a finite distributive lattice, $\mu : L \to \mathbb{R}_{\geq 0}$ a log-supermodular function and $f, g : L \to \mathbb{R}_{\geq 0}$ which are either both increasing or both decreasing. Then

$$\left( \sum_{y \in L} f(y)\mu(y) \prod_{a \in \eta(y)} x_a \right) \left( \sum_{y \in L} g(y)\mu(y) \prod_{a \in \eta(y)} x_a \right) \leq$$
$$\left( \sum_{y \in L} \mu(y) \prod_{a \in \eta(y)} x_a \right) \left( \sum_{y \in L} f(y)g(y)\mu(y) \prod_{a \in \eta(y)} x_a \right) \quad (3.12)$$

*Proof.* The proof is analogous to the one for Corollary 3.2. $\qquad \square$

By letting $x_1 = x_2 = \cdots = x_n = q$ in Theorem 3.5 we also obtain the following $q$-analogue due to Björner [3].

**Corollary 3.6.** Let $L$ be a finite distributive lattice, $\mu : L \to \mathbb{R}_{\geq 0}$ a log-supermodular function and $f, g : L \to \mathbb{R}_{\geq 0}$ which are either both increasing or both decreasing. Then

$$\left( \sum_{y \in L} f(y)\mu(y)q^{r(y)} \right) \left( \sum_{y \in L} g(y)\mu(y)q^{r(y)} \right) \leq$$
$$\left( \sum_{y \in L} \mu(y)q^{r(y)} \right) \left( \sum_{y \in L} f(y)g(y)\mu(y)q^{r(y)} \right) \quad (3.13)$$

# Chapter 4

# Applications

## 4.1 Random graphs

This chapter is devoted to applications of the four functions theorem. The theorem is used in many different areas and we will investigate a few of those; random graphs, Young's lattice and linear extensions.

**Definition 4.1.** A *graph* is a pair $G = (V, E)$ of sets such that $E \subseteq \binom{V}{2}$ ; thus, the elements of $E$ are 2-element subsets of $V$. The elements of $V$ are the vertices (or nodes) of the graph $G$, the elements of $E$ are its edges.

**Definition 4.2.** Let $X$ be a set. The subset $\mathcal{F} \subseteq \mathcal{P}(X)$ is called a $\sigma$-algebra if it satisfies the following

1. $\mathcal{F}$ is non-empty.

2. $\mathcal{F}$ is closed under complementation: if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.

3. $\mathcal{F}$ is closed under countable unions: if $A_1, A_2, \cdots$ are in $\mathcal{F}$, then $A = A_1 \cup A_2 \cup \cdots \in \mathcal{F}$.

Note that $\mathcal{F}$ contains both the empty set and $X$. This follows from the fact that, since $\mathcal{F}$ is non-empty, it contains at least one element $A \in \mathcal{F}$. By 2, $A^c \in \mathcal{F}$, and by 3, $X = A \cup A^c \in \mathcal{F}$. Using 2 again, we see that $\emptyset = X^c \in \mathcal{F}$.

**Definition 4.3.** Let $\mathcal{F}$ be a $\sigma$-algebra over a set $X$. A function $\mu$ from $\mathcal{F}$ to the extended real number line is called a (positive) measure if it satisfies the following

1. $\mu(A) \geq 0$ for all $A \in \mathcal{F}$.

2. For all countable collections $(E_i)_{i \in I}$ of pairwise disjoint sets in $\mathcal{F}$: $\mu(\bigcup_{i \in I} E_i) = \sum_{i \in I} \mu(E_i)$.

3. $\mu(\emptyset) = 0$

A probability measure is a measure $\mu$ which satisfies $\mu(X) = 1$.

**Definition 4.4.** A *measure space* is a triple $(X, \mathcal{F}, \mu)$ where $X$ is a set, $\mathcal{F}$ is a $\sigma$-algebra over $X$ and $\mu$ is a measure with domain $\mathcal{F}$.

A special kind of measure space is a probability space defined as follows:

**Definition 4.5.** A *probability space* is a triple $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega$ is a set, $\mathcal{F}$ is a $\sigma$-algebra of subsets of $\Omega$, $\mathbb{P}$ is a measure on $\mathcal{F}$, and $\mathbb{P}(\Omega) = 1$.

In the simplest case $\Omega$ is a finite set and $\mathcal{F}$ is $\mathcal{P}(\Omega)$, the set of all subsets of $\Omega$. Then $\mathbb{P}$ is determined by a function $p : \Omega \to [0, 1]$ defined by $p(\omega) = \mathbb{P}(\{\omega\})$, namely

$$\mathbb{P}(A) = \sum_{\omega \in A} p(\omega), A \subset \Omega$$

**Definition 4.6.** Given a measure $\mu$, we say that a set $E$ is $\mu$-measurable if

$$\mu(A) = \mu(A \cap E) + \mu(A - E)$$

for any $A \subset X$. Where $A - E$ is defined as $\{x \in A : x \notin E\}$.

**Definition 4.7.** Let $f$ be a real-valued function defined on a measurable set $X_0$ of a measure space. We say that $f$ is a *measurable function* on $X_0$ if the inverse image of any open set in $\mathbb{R}$ is measurable.

**Definition 4.8.** A real valued *random variable $Z$* is a measurable real-valued function on a probability space, $Z : \Omega \to \mathbb{R}$.

*Binomial random graph* [4, 10], denoted $G(n, p_{ij}) = G(n, (p_{ij})_{i,j})$; where $n$ is the number of vertices and $0 \leq p_{ij} \leq 1$ for $1 \leq i < j \leq n$ is a random graph model. The sample space $\Omega$ is the set of all graphs with vertex set $V = \{1, 2, \ldots, n\}$ in which the edges are chosen independently, and for $1 \leq i < j \leq n$ the probability of $ij$ being an edge is $p_{ij}$. Denote this sample space by $\Gamma$. Finally

$$\mathbb{P}(A) = \sum_{G \in A} \mathbb{P}(\{G\}) = \sum_{G \in A} \prod_{ij \in E} p_{ij} \prod_{ij \in E^c} (1 - p_{ij}), A \subset \Omega$$

A binomial random graph $G$ is an element of $G(n, p_{ij})$ which can be considered as a graph generated by a random process, more precisely; it is a graph on $n$ vertices where each edge $ij$ is assigned in an independent way a probability $p_{ij}$.

**Definition 4.9.** Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. We call $G$ and $G'$ *isomorphic*, and write $G \simeq G'$, if there exist a bijection $\phi : V \to V'$ with $ij \in E \iff \phi(i)\phi(j) \in E'$ for all $i, j \in V$.

**Definition 4.10.** A graph $G'$ is a *subgraph* of a graph $G$ if $G'$ is isomorphic to a graph all of whose vertices and edges are in $G$ and we write $G' \subset G$ .

**Definition 4.11.** A *graph property* is a set of graphs $Q$ that is closed under isomorphism. A property $Q$ is *monotone increasing* if for $G$ and $H$ graphs on $n$ such that $G \in Q$ and $G \subset H$ implies $H \in Q$. A property is *monotone decreasing* if for $G$ and $H$ graphs on $n$ such that $G \in Q$ and $H \subset G$ implies $H \in Q$.

**Example 4.1.** Examples of increasing graph properties are:

1. $G$ is hamiltonian.

2. $G$ has chromatic number at least k.

3. $G$ is k-connected.

**Example 4.2.** Examples of decreasing graph properties are:

1. $G$ is bipartite.

2. $G$ is 3-colourable.

3. $G$ is planar.

Define now a partial order $\leq_\Gamma$ on $\Gamma$, the set of all graphs on $n$ vertices, where $G_1 \leq_\Gamma G_2$ if all the edges in $G_1$ is also in $G_2$. This makes $\Gamma$ a finite distributive lattice since it is of the form $\mathcal{P}(X)$, where $X$ is the set of all pairs of $n$. For $n = 3$, the Hasse diagram for the poset is the following:
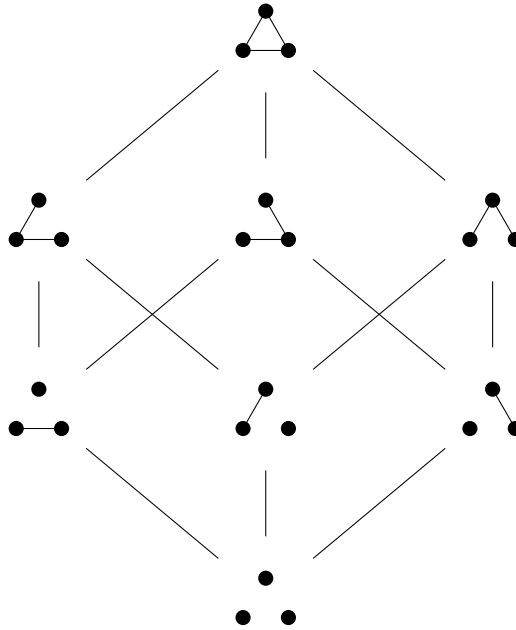
**Corollary 4.1.** Let the random variables $Z_1, Z_2 : \Gamma \to \mathbb{R}_{\geq 0}$ be both increasing or decreasing functions. Then

$$\sum_{G \in \Gamma} \mathbb{P}(\{G\}) \prod_{ij \in E} x_{ij} \sum_{G \in \Gamma} Z_1(G) Z_2(G) \mathbb{P}(\{G\}) \prod_{ij \in E} x_{ij} \geq$$

$$\sum_{G \in \Gamma} Z_1(G) \mathbb{P}(\{G\}) \prod_{ij \in E} x_{ij} \sum_{G \in \Gamma} Z_2(G) \mathbb{P}(\{G\}) \prod_{ij \in E} x_{ij}$$

If we set $x_{ij} = q$ for all $ij \in E$ we get the following corollary:

**Corollary 4.2.** Let the random variables $Z_1, Z_2 : \Gamma \to \mathbb{R}_{\geq 0}$ be both increasing or decreasing functions. Then

$$\sum_{G \in \Gamma} \mathbb{P}(\{G\}) q^{|E(G)|} \sum_{G \in \Gamma} Z_1(G) Z_2(G) \mathbb{P}(\{G\}) q^{|E(G)|} \geq$$

$$\sum_{G \in \Gamma} Z_1(G) \mathbb{P}(\{G\}) q^{|E(G)|} \sum_{G \in \Gamma} Z_2(G) \mathbb{P}(\{G\}) q^{|E(G)|}$$

If we set $x_{ij} = 1$ for all $ij \in E$ we get the following theorem, which appears in [10]:

**Theorem 4.1.** *Let the random variables $Z_1, Z_2 : \Gamma \to \mathbb{R}_{\geq 0}$ be both increasing or decreasing functions. Then*

$$\mathbb{E}(Z_1 Z_2) = \sum_{G \in \Gamma} Z_1(G) Z_2(G) \mathbb{P}(\{G\}) \geq$$

$$\sum_{G \in \Gamma} Z_1(G) \mathbb{P}(\{G\}) \sum_{G \in \Gamma} Z_2(G) \mathbb{P}(\{G\}) = \mathbb{E}(Z_1) \mathbb{E}(Z_2)$$

*In particular, if $Q_1$ and $Q_2$ are two increasing or decreasing graph properties, then*

$$\mathbb{P}(Q_1 \cap Q_2) \geq \mathbb{P}(Q_1) \mathbb{P}(Q_2)$$

*Proof.* We prove the theorem by applying the FKG-inequality. Recall that $\Gamma$ is a finite distributive lattice. We also have that $\mathbb{P}$ is a log-modular function since

$$\mathbb{P}(\{G_1\}) \mathbb{P}(\{G_2\}) = \prod_{ij \in E_1} p_{ij} \prod_{ij \in E_1^c} (1 - p_{ij}) \prod_{ij \in E_2} p_{ij} \prod_{ij \in E_2^c} (1 - p_{ij}) =$$

$$= \prod_{\substack{ij \in \\ E_1 \cup E_2}} p_{ij} \prod_{\substack{ij \in \\ (E_1 \cup E_2)^c}} (1 - p_{ij}) \prod_{\substack{ij \in \\ E_1 \cap E_2}} p_{ij} \prod_{\substack{ij \in \\ (E_1 \cap E_2)^c}} (1 - p_{ij}) =$$

$$= \mathbb{P}(\{G_1 \vee G_2\}) \mathbb{P}(\{G_1 \wedge G_2\})$$

Thus, the first result follows directly from corollary 2.2. To see how the second result follows from the FKG inequality let $Q_1$ and $Q_2$ be two increasing properties and let $I_{Q_1} : \Gamma \to \{0, 1\}$ be the *indicator function,* i.e

$$I_{Q_1}(G) = \begin{cases} 1 & \text{if } G \in Q_1 \\ 0 & \text{if } G \notin Q_1 \end{cases} .$$

$I_{Q_2}$ and $I_{Q_1 \cap Q_2}$ are defined in the same way. Then,

$$\mathbb{P}(Q) = \sum_{G \in \Gamma} I_Q(G)\mathbb{P}(\{G\}) = \mathbb{E}(I_Q)$$

and we get

$$\mathbb{P}(Q_1 \cap Q_2) = \mathbb{E}(I_{Q_1 \cap Q_2}) \geq \mathbb{E}(I_{Q_1})\mathbb{E}(I_{Q_2}) = \mathbb{P}(Q_1)\mathbb{P}(Q_1)$$

$\square$

## 4.2 Young's lattice

**Definition 4.12.** A *Young diagram*[8] is a collection of boxes arranged in left-justified row, with a (weakly) decreasing number of boxes in each row.

Listing the number of boxes in each row of a Young diagram gives a partition of the integer $n$ that is the total number of boxes. Conversely, every partition of $n$ corresponds to a Young diagram. We usually denote a partition by $\lambda$ or $\sigma$. It is given by a sequence of weakly decreasing positive integers and we write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, where $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_k$ and $|\lambda| = \sum_i \lambda_i = n$. We let $Par(n)$ denote the set of all partitions on $n$, with $Par(0)$ consisting of the empty set. We also let $Par := \bigcup_{n \geq 0} Par(n)$.

**Example 4.3.** Let $n = 3$, the partitions of 3 are $(1, 1, 1)$, $(2, 1)$ and $(3)$. The Young diagram of each of the partitions are:



$$(1, 1, 1) \quad (2, 1) \quad (3)$$

We define a partial order $\subseteq_{Par}$ on partitions by $\sigma \subseteq_{Par} \lambda$ for any $\lambda, \sigma \in Par$ if $\sigma_i \leq \lambda_i$ for all $i$. If we identify a partition by its Young diagram, then $\subseteq_{Par}$ is given by containment of diagrams.
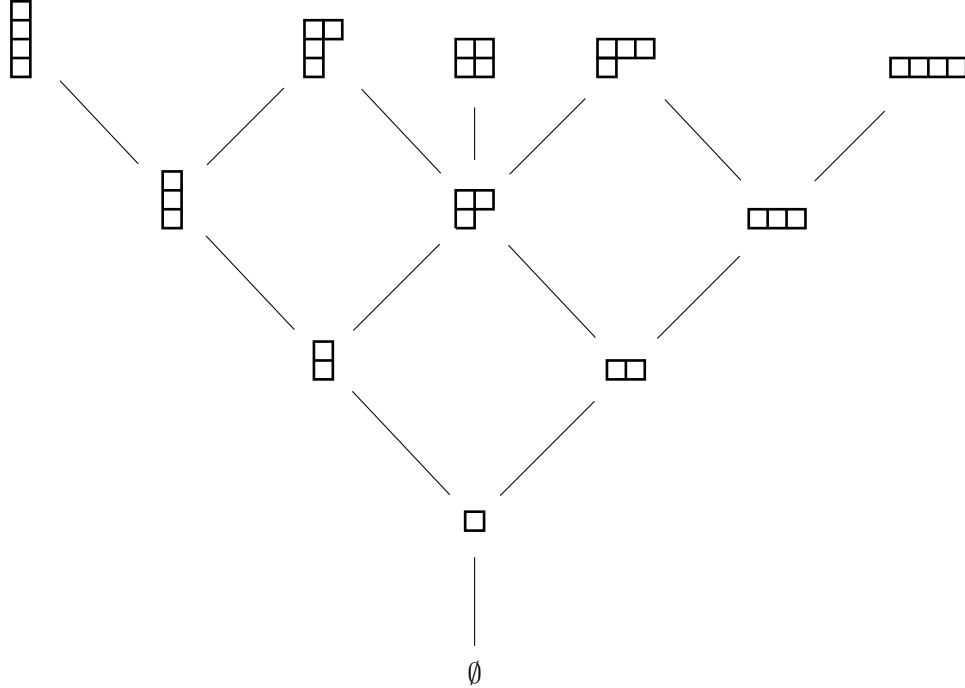
**Definition 4.13.** The Young's lattice $Y$ is the set of $Par$ together with the partial order $\subseteq_{Par}$. The lattice operations are

$$\lambda \vee \sigma = (\max(\lambda_1, \sigma_1) \geq \max(\lambda_2, \sigma_2) \geq \cdots \geq \max(\lambda_k, \sigma_k))$$

$$\lambda \wedge \sigma = (\min(\lambda_1, \sigma_1) \geq \min(\lambda_2, \sigma_2) \geq \cdots \geq \min(\lambda_k, \sigma_k))$$

The join-irreducibles of the lattice $Y$ are those of the form $(i, i, \ldots, i)$ where $i \geq 1$, that is, all diagrams with the form of a rectangle.

**Example 4.4.** The Hasse diagram of the Young's lattice up to $n = 4$ is the following:



**Definition 4.14.** A *Young tableaux* is a Young diagram filled with positive integers such that the filling is

1. weakly increasing across each row

2. strictly increasing down each row

We say that the tableaux is a tableaux on the diagram $\lambda$, or that $\lambda$ is the shape of the tableaux.

**Definition 4.15.** A *standard Young tableaux* is a tableaux in which the entries of the boxes are numbers from 1 to $n$, each occurring once.

**Definition 4.16.** The number of standard Young tableaux of shape $\lambda$ is denoted $f_\lambda$.

**Proposition 4.1.** Suppose that $0 \leq s \leq t$. Then the function $\psi : Y \to \mathbb{R}_{\geq 0}$ given by

$$\psi(\lambda) = \frac{f_\lambda^t}{(|\lambda|!)^s}$$

is log-supermodular.

*Proof.* For a proof, see [3].

$\square$

The Young's lattice is a locally finite distributive lattice; every interval is finite and the meet and join operations are represented by intersections and unions of the corresponding Young diagrams.

Recall that the join-irreducibles of the lattice $Y$ are those of the form $\mu = (i, i, \ldots, i)$ where $i \geq 1$. Let $\mu$ be a join-irreducible element and define $x_\mu = x_{ij}$ where $j$ is the number of $i$'th. Corollary 3.5 for the Young's lattice becomes:

**Theorem 4.2.** *Let $Y$ be the Young's lattice and $\epsilon : Y \to \mathbb{R}_{\geq 0}$ be a log-supermodular function. Let $0 \leq s \leq t$ and $g, h : Y \to \mathbb{R}_{\geq 0}$ be comonotone functions. Then*

$$\left( \sum_{\lambda \in Y} g(\lambda)\epsilon(\lambda)\psi(\lambda) \prod_{\mu \in \ \eta(\lambda)} x_\mu \right)\left( \sum_{\lambda \in Y} h(\lambda)\epsilon(\lambda)\psi(\lambda) \prod_{\mu \in \ \eta(\lambda)} x_\mu \right) \leq$$
$$\left( \sum_{\lambda \in Y} \epsilon(\lambda)\psi(\lambda) \prod_{\mu \in \ \eta(\lambda)} x_\mu \right)\left( \sum_{\lambda \in Y} g(\lambda)h(\lambda)\epsilon(\lambda)\psi(\lambda) \prod_{\mu \in \ \eta(\lambda)} x_\mu \right) \quad (4.1)$$

*Observe that since we do not require that $Y$ is finite, the sums are formal power series.*

*Proof.* The ideal $\eta(\lambda)$ is a subset of the interval $I = [\hat{0}, \lambda]$ and since $L$ is finitary, every interval is finite and $\eta(\lambda)$ is finite, thus, corollary 3.5 is valid for each $\eta(\lambda)$ and the conclusion follows. $\qquad\square$

If we set $x_\mu = q$ we get the following $q$-analogue due to Björner [3].

**Theorem 4.3.** *Let $Y$ be the Young's lattice and $\epsilon : Y \to \mathbb{R}_{\geq 0}$ be a log-supermodular function. Let $0 \leq s \leq t$ and $g, h : Y \to \mathbb{R}_{\geq 0}$ be comonotone functions. Then*

$$\left( \sum_{\lambda \in Y} g(\lambda)\epsilon(\lambda)\psi(\lambda)q^{|\lambda|} \right)\left( \sum_{\lambda \in Y} h(\lambda)\epsilon(\lambda)\psi(\lambda)q^{|\lambda|} \right. \leq$$
$$\left( \sum_{\lambda \in Y} \epsilon(\lambda)\psi(\lambda)q^{|\lambda|} \right)\left( \sum_{\lambda \in Y} g(\lambda)h(\lambda)\epsilon(\lambda)\psi(\lambda)q^{|\lambda|} \right) \quad (4.2)$$

## 4.3 Linear extensions

**Definition 4.17.** Let $(P, <)$ be a finite poset. For $n = |P|$, denote by $\Lambda$ the set of all one-to-one mappings of $P$ onto $[n]$.

**Definition 4.18.** Let $(P, <)$ be a finite poset. A map $\lambda \in \Lambda$ is said to be a *linear extension* of $P$ if $x <_P y$ implies $\lambda(x) < \lambda(y)$. The set of linear extension of $P$ is denoted by $\Lambda(P)$.

The following theorem is due to Shepp [11].

**Theorem 4.4.** *Let $(P, <)$ be a poset consisting of two chains, $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ and assign a uniform probability distribution to $\Lambda(P)$. Let also $Q$ and $Q'$ be two events both of which are sets of the form $\{a_{i_1} < b_{j_1}, a_{i_2} < b_{j_2}, \ldots\}$. Then*

$$\mathbb{P}(Q \cap Q'|P) \geq \mathbb{P}(Q|P)\mathbb{P}(Q'|P)$$

*Proof.* The proof is done by using the FKG inequality. A finite distributive lattice $\Gamma$ and functions $\mu$, $f$ and $g$ on $\Gamma$ are defined such that the assumption in the FKG inequality are fulfilled. The result then follows as a consequence of the FKG inequality.

Define a lattice $\Gamma$ with elements of the form $\bar{x} = (x_1, x_2, \ldots, x_m)$, $x_i \in \mathbb{N}$, where $1 \leq x_1 < x_2 < \ldots < x_m \leq m + n$ and $\bar{x} \leq \bar{x}'$ if $x_i \leq x_i'$ for $1 \leq i \leq m$. It is easily checked that the lattice $\Gamma$ is a finite distributive lattice.

The lattice operations are

$$\bar{x} \vee \bar{x}' = (\ldots, \max(x_i, x_i'), \ldots)$$
$$\bar{x} \wedge \bar{x}' = (\ldots, \min(x_i, x_i'), \ldots)$$

Let $(P, <)$ be a poset consisting of two chains, $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, relations of the form $a_i < b_j$ and $b_r < a_s$ are allowed. Associate a unique mapping $\lambda_{\bar{x}} \in \Lambda$ to each $\bar{x} \in \Gamma$ by setting:

$$\lambda_{\bar{x}}(a_i) = x_i, \lambda_{\bar{x}}(b_j) = y_j$$

where $[m + n] \setminus \{x_1 < \ldots < x_m\} = \{y_1 < \ldots < y_n\}$.

Finally, define

$$\mu(\bar{x}) = \begin{cases} 1 & \text{if } \lambda_{\bar{x}} \in \Lambda(P) \\ 0 & \text{otherwise} \end{cases}$$

$$f(\bar{x}) = \begin{cases} 1 & \text{if } \lambda_{\bar{x}} \in \Lambda(Q) \\ 0 & \text{otherwise} \end{cases}$$

$$g(\bar{x}) = \begin{cases} 1 & \text{if } \lambda_{\bar{x}} \in \Lambda(Q') \\ 0 & \text{otherwise} \end{cases}$$

where $Q$ and $Q'$ are sets of the form $\{a_{i_1} < b_{j_1}, a_{i_2} < b_{j_2}, \ldots\}$. We will show that the function $\mu$ is log-supermodular and that the functions $f$ and $g$ are decreasing, since if we do that, Corollary 3.2 holds for the lattice defined above.

First we show that $\mu$ is log-supermodular, i.e., that

$$\mu(\bar{x})\mu(\bar{x}') \leq \mu((\bar{x} \vee \bar{x}')\mu(\bar{x} \wedge \bar{x}')$$

for all $\bar{x}, \bar{x}' \in \Gamma$. Suppose that $\mu(\bar{x})\mu(\bar{x}') = 1$. Then $\lambda_{\bar{x}} \in \Lambda(P)$ and $\lambda_{\bar{x}'} \in \Lambda(P)$. If $a_i < a_j$ in $P$, where $i < j$, then

$$\lambda_{\bar{x}}(a_i) = x_i < x_j = \lambda_{\bar{x}}(a_j)$$

$$\lambda_{\bar{x}'}(a_i) = x_i' < x_j' = \lambda_{\bar{x}'}(a_j)$$

and so

$$\lambda_{\bar{x} \vee \bar{x}'}(a_i) = \max(x_i, x_i') < \max(x_j, x_j') = \lambda_{\bar{x} \vee \bar{x}'}(a_j)$$

$$\lambda_{\bar{x} \wedge \bar{x}'}(a_i) = \min(x_i, x_i') < \min(x_j, x_j') = \lambda_{\bar{x} \wedge \bar{x}'}(a_j)$$

Similarly, if $b_i < b_j$ in $P$ , where $i < j$, then

$$\lambda_{\bar{x}}(b_i) = y_i < y_j = \lambda_{\bar{x}}(b_j)$$

$$\lambda_{\bar{x}'}(b_i) = y_i' < y_j' = \lambda_{\bar{x}'}(b_j)$$

Since $y_j$ = the number of $x$:s which are less than $y_j$ and $y$:s which are less or equal to $y_j$, we can write $y_j$ and $y_j'$ as follows,

$$y_j = j + \#(i : x_i < y_j)$$

and

$$y_j' = j + \#(i : x_i' < y_j')$$

so,

$$\min(y_j, y_j') = \min(j + \#(i : x_i < y_j), j + \#(i : x_i' < y_j')) =$$
$$j + \min(\#(i : x_i < y_j), \#(i : x_i' < y_j')) =$$
$$j + \#(i : \max(x_i, x_i') < \min(y_j, y_j'))$$

since if $x_i < y_j$ and $x_i' < y_j'$ then

$$x_i \leq i + j - 1, x_i' \leq i + j - 1$$

and

$$y_j \geq i + j, y_j' \geq i + j$$

Thus,

$$\max(x_i, x_i') \leq i + j - 1 < i + j \leq \min(y_j, y_j').$$

This implies that,

$$\lambda_{\bar{x} \vee \bar{x}'}(b_i) = \min(y_i, y_i') < \min(y_j, y_j') = \lambda_{\bar{x} \vee \bar{x}'}(b_j)$$

analogously

$$\lambda_{\bar{x} \wedge \bar{x}'}(b_i) = \max(y_i, y_i') < \max(y_j, y_j') = \lambda_{\bar{x} \wedge \bar{x}'}(b_j)$$

In the case where $a_i < b_j$ in $P$ then

$$\lambda_{\bar{x}}(a_i) = x_i < y_j = \lambda_{\bar{x}}(b_j)$$

$$\lambda_{\bar{x}'}(a_i) = x_i' < y_j' = \lambda_{\bar{x}'}(b_j)$$

We have that $x_i =$ the number of $x$:s and $y$:s which are less or equal to $x_i$. There are exactly $i$ $x$:s and at most $j-1$ $y$:s less than or equal to $x_i$. This implies

$$x_i \leq i+j-1, x_i' \leq i+j-1$$

Similarly we get that

$$y_j \geq i+j, y_j' \geq i+j$$

Consequently,

$$\lambda_{\bar{x} \vee \bar{x}'}(a_i) = \max(x_i, x_i') \leq i+j-1$$

and

$$\lambda_{\bar{x} \vee \bar{x}'}(b_j) = \min(y_j, y_j') \geq i+j$$

i.e.,

$$\lambda_{\bar{x} \vee \bar{x}'}(a_i) < \lambda_{\bar{x} \vee \bar{x}'}(b_j)$$

The argument for $b_i < a_j$ is similar. This shows that $\lambda_{\bar{x} \vee \bar{x}'} \in \Lambda(P)$, i.e., $\mu(\bar{x} \vee \bar{x}') = 1$. In almost the same way it follows that $\mu(\bar{x} \wedge \bar{x}') = 1$. Therefore, we have shown that

$$\mu(\bar{x})\mu(\bar{x}') = 1 \Rightarrow \mu(\bar{x} \wedge \bar{x}')\mu(\bar{x} \vee \bar{x}') = 1$$

so $\mu$ is log-supermodular.

We now show that the functions $f$ and $g$ are both decreasing functions. Suppose $\bar{x} \leq \bar{x}'$ and $f(\bar{x}') = 1$. Then, by definition, $\lambda_{\bar{x}'} \in \Lambda(Q)$. If $a_i < b_j$ in $Q$ then

$$\lambda_{\bar{x}'}(a_i) = x_i' < y_j' = \lambda_{\bar{x}'}(b_j)$$

Which implies that,

$$\lambda_{\bar{x}}(a_i) = x_i \leq x_i' \leq i+j-1$$

Suppose now that $y_j < x_i$ this implies that $x_i \geq i+j > x_i'$, a contradiction. So $x_i < y_j$ and we get

$$\lambda_{\bar{x}}(a_i) = x_i < y_j = \lambda_{\bar{x}}(b_j)$$

Thus, $\lambda_{\bar{x}} \in \Lambda(Q)$ and $f(\bar{x}) = 1$, and consequently, $f$ is decreasing. Similar argument shows that $g$ is decreasing. The sums in Corollary 3.2 become,

$$|Q \cap Q' \cap P||P| \geq |Q \cap P||Q' \cap P|$$

i.e.,

$$\mathbb{P}(Q \cap Q'|P) \geq \mathbb{P}(Q|P)\mathbb{P}(Q'|P)$$

<div align="right">□</div>

The join-irreducibles of the lattice $\Gamma$ are those of the form $(1, 2, \ldots, i, j, j+1, \ldots)$ where $i \geq 1$ and $j \geq i + 2$ or of the form $(i, i+1, \ldots)$ where $i \geq 2$. Let $z_i$ denote the variable associated to the join-irreducibles of the form $\alpha_i = (i, i+1, \ldots)$ and $y_{ij}$ the variable associated to the join-irreducibles of the form $\beta_{ij} = (1, 2, \ldots, i, j, j+1, \ldots)$. We then have the following theorem:

**Corollary 4.3.** *Let* $\Gamma, \mu, f$ *and* $g$ *be defined as above. Then*

$$\Big( \sum_{\bar{x} \in \Gamma} f(\bar{x})\mu(\bar{x}) \prod_{\alpha_i, \beta_{ij} \in \eta(\bar{x})} z_i y_{ij} \Big) \Big( \sum_{\bar{x} \in \Gamma} g(\bar{x})\mu(\bar{x}) \prod_{\alpha_i, \beta_{ij} \in \eta(\bar{x})} z_i y_{ij} \Big) \leq$$
$$\Big( \sum_{\bar{x} \in \Gamma} \mu(\bar{x}) \prod_{\alpha_i, \beta_{ij} \in \eta(\bar{x})} z_i y_{ij} \Big) \Big( \sum_{\bar{x} \in \Gamma} f(\bar{x})g(\bar{x})\mu(\bar{x}) \prod_{\alpha_i, \beta_{ij} \in \eta(\bar{x})} z_i y_{ij} \Big)$$

When $z_i = y_{ij} = q$ for all $i$ and $j$ we get the following theorem:

**Corollary 4.4.** *Let* $\Gamma, \mu, f$ *and* $g$ *be defined as above. Then*

$$\Big( \sum_{\bar{x} \in \Gamma} f(\bar{x})\mu(\bar{x})q^{r(\bar{x})} \Big) \Big( \sum_{\bar{x} \in \Gamma} g(\bar{x})\mu(\bar{x})q^{r(\bar{x})} \Big) \leq$$
$$\Big( \sum_{\bar{x} \in \Gamma} \mu(\bar{x})q^{r(\bar{x})} \Big) \Big( \sum_{\bar{x} \in \Gamma} f(\bar{x})g(\bar{x})\mu(\bar{x})q^{r(\bar{x})} \Big)$$

*Remark* 4.1. When all the $z_i = y_{ij} = 1$ we get Theorem 4.4.

The following example shows that if the chain condition is weakened even slightly then the previous theorem will no longer be true.

**Example 4.5.** Let $P$ be the poset consisting of $A = \{a_1 < a_2\}$ and $B = \{b_1 < b_3, b_2 < b_3\}$, in addition we have $\{b_1 < a_2, a_1 < b_2\}$. Let also $Q = \{a_1 < b_1\}$ and $Q' = \{a_2 < b_3\}$. The Hasse diagram of $A$ and $B$ is the following:
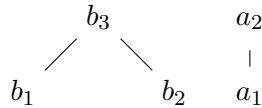


and the linear extensions consistent with $P$ are the following:

| $b_3$ | $a_2$ | $b_3$ | $b_3$ | $a_2$ | $b_3$ | $b_3$ | $a_2$ |
|-------|-------|-------|-------|-------|-------|-------|-------|
| \| | \| | \| | \| | \| | \| | \| | \| |
| $a_2$ | $b_3$ | $b_2$ | $a_2$ | $b_3$ | $b_2$ | $a_2$ | $b_3$ |
| \| | \| | \| | \| | \| | \| | \| | \| |
| $b_1$ | $b_1$ | $a_2$ | $b_2$ | $b_2$ | $a_2$ | $b_2$ | $b_2$ |
| \| | \| | \| | \| | \| | \| | \| | \| |
| $b_2$ | $b_2$ | $b_1$ | $b_1$ | $b_1$ | $a_1$ | $a_1$ | $a_1$ |
| \| | \| | \| | \| | \| | \| | \| | \| |
| $a_1$ | $a_1$ | $a_1$ | $a_1$ | $a_1$ | $b_1$ | $b_1$ | $b_1$ |

we then have that,

$$\mathbb{P}(Q|P \cap Q') = \frac{3}{5} < \frac{5}{8} = \mathbb{P}(Q|P)$$

The following theorem is due to Shepp [12].

**Theorem 4.5.** *For any partial order $(P, <)$ on the set $\{x_1, x_2, \ldots, x_n\}$ , the sets $Q = \{x_1 < x_2\}$ and $Q' = \{x_1 < x_3\}$ are positively correlated, i.e.;*

$$\mathbb{P}(Q \cap Q'|P) \geq \mathbb{P}(Q|P)\mathbb{P}(Q'|P)$$

*Proof.* Also here, the proof is done by using the FKG inequality. A finite distributive lattice $\Gamma$ and functions $\mu$, $f$ and $g$ on $\Gamma$ are defined such that the assumption in the FKG inequality are fulfilled. The result then follows as a consequence of the FKG inequality.

Define a lattice $\Gamma$ with elements of the form $\bar{x} = (x_1, x_2, \ldots, x_n)$ where each $x_i \in \{1, 2, \ldots, N\}$. We say $\bar{x} < \bar{x}'$ if $x_1 \geq x_1'$ and $x_i - x_1 \leq x_i' - x_1'$ for $i = 2, \ldots, n$.

The lattice operations are:

$$(\bar{x} \vee \bar{x}')_i = \max(x_i - x_1, x_i' - x_1') + \min(x_1, x_1'), i = 1, \ldots, n$$

$$(\bar{x} \wedge \bar{x}')_i = \min(x_i - x_1, x_i' - x_1') + \max(x_1, x_1'), i = 1, \ldots, n$$

Finally, define

$$\mu(\bar{x}) = \begin{cases} 1 & \text{if } \bar{x} \text{ satisfies the inequalities in } P \\ 0 & \text{otherwise} \end{cases}$$

$$f(\bar{x}) = \begin{cases} 1 & \text{if } x_1 \leq x_2 \\ 0 & \text{otherwise} \end{cases}$$

$$g(\bar{x}) = \begin{cases} 1 & \text{if } x_1 \leq x_3 \\ 0 & \text{otherwise} \end{cases}$$

It is shown in [12] that $\Gamma$ defines a distributive lattice, the functions $f$, $g$ are increasing and $\mu$ is a log-supermodular function. The sums in Corollary 3.2 become,

$$|Q \cap Q' \cap P||P| \geq |Q \cap P||Q' \cap P| \tag{4.3}$$

Letting $N \to \infty$, the probability that $x_i = x_j$ for $i \neq j$ tends to zero and so, it follows that 4.3 also holds for the permutations induced by $\{x_1, x_2, \ldots, x_n\}$. We then get after dividing by $\mathbb{P}(P)^2$:

$$\mathbb{P}(Q \cap Q'|P) \geq \mathbb{P}(Q|P)\mathbb{P}(Q'|P)$$

<div align="right">□</div>

The join-irreducibles of the lattice $\Gamma$ defined above are those of the form $\alpha_k = (k, 1, 1, \ldots)$ or of the form $\beta_{kl} = (k, 1, 1, \ldots, N - k, 1, 1, \ldots)$ where $k = 1, 2, \ldots, N - 1$ or of the form $\gamma_{kl} = (N, 1, 1, \ldots, k, 1, 1, \ldots)$ where $k = 2, 3, \ldots, N$.

Let $z_k$ denote the variable associated to the join-irreducibles of the form $(k, 1, 1, \ldots)$, $y_{kl}$ the variable associated to the join-irreducibles of the form $(k, 1, 1, \ldots, N - k, 1, 1, \ldots)$ where $l$ is the positions of $N - k$ and $w_{kl}$ the variable associated to the join-irreducibles of the form $(N, 1, 1, \ldots, k, 1, 1, \ldots)$ where $l$ is the position of $k$.

We then have the following theorem:

**Corollary 4.5.** *Let* $\Gamma, f, g, \mu$ *be defined as above. Then*

$$\left( \sum_{\bar{x} \in \Gamma} f(\bar{x})\mu(\bar{x}) \prod_{\alpha_k, \beta_{kl}, \gamma_{kl} \in \eta(\bar{x})} z_k y_{kl} w_{kl} \right) \left( \sum_{\bar{x} \in \Gamma} g(\bar{x})\mu(\bar{x}) \prod_{\alpha_k, \beta_{kl}, \gamma_{kl} \in \eta(\bar{x})} z_k y_{kl} w_{kl} \right) \leq$$
$$\left( \sum_{\bar{x} \in \Gamma} \mu(\bar{x}) \prod_{\alpha_k, \beta_{kl}, \gamma_{kl} \in \eta(\bar{x})} z_k y_{kl} w_{kl} \right) \left( \sum_{\bar{x} \in \Gamma} f(\bar{x})g(\bar{x})\mu(\bar{x}) \prod_{\alpha_k, \beta_{kl}, \gamma_{kl} \in \eta(\bar{x})} z_k y_{kl} w_{kl} \right)$$

When $z_k = y_{kl} = w_{kl} = q$ for all $k$ and $l$ we get the following theorem:

**Corollary 4.6.** *Let* $\Gamma, f, g, \mu$ *be defined as above. Then*

$$\left( \sum_{\bar{x} \in \Gamma} f(\bar{x})\mu(\bar{x})q^{r(\bar{x})} \right) \left( \sum_{\bar{x} \in \Gamma} g(\bar{x})\mu(\bar{x})q^{r(\bar{x})} \right) \leq$$
$$\left( \sum_{\bar{x} \in \Gamma} \mu(\bar{x})q^{r(\bar{x})} \right) \left( \sum_{\bar{x} \in \Gamma} f(\bar{x})g(\bar{x})\mu(\bar{x})q^{r(\bar{x})} \right)$$

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