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On Reflectionless Isoscattering Matrices

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Amar Rauf

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MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

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Amar Rauf

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Handledare: Pavel Kurasov

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ON REFLECTIONLESS ISOSCATTERING MATRICES

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By

Amar Rauf

raufamar@yahoo.com

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Supervisor at

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Sweden

Pavel Kurasov

Abstract

Hermitian Reflectionless Isoscattering matrices (RI-matrices) are studied. These are $n \times n$ unitary matrices with zero diagonal and all non-diagonal elements having the same absolute value:

$$S = S^* = S^{-1},$$

$$s_{jj} = 0, j = 1, 2, \cdots, n,$$

$$|s_{jk}| = |s_{lm}|, j \neq k, l \neq m, j, k, l, m = 1, 2, \cdots, n.$$

It is proven that such matrices are absent if the dimension n is odd. A complete discription of all such matrices in dimension 2, 4 and 6 is given.

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Contents

1 Introduction

A quantum graph is a metric graph with a differential operator acting on functions defined on the edges of the graph where each edge is viewed as interval of positive finite length. To make the operator self adjoint we use matching conditions imposed at the vertices. The study of quantum graphs started in 1980's, when P. Exner and P. Seba [2] investigated the free motion of quantum particles on a branching graph. The model goes back to 1930's when Linus Pauling examined quantum graph-like structures in physical chemistry to describe motion of free electron in organic molecules. Quantum graphs have applications as simplified models in mathematics, engineering, in mesoscopic physics and nanotechnology. One can find information and references for literature in articles [6], [10], [5], [7], [11] and [1].

The most commonly used differential operator in quantum graphs is the standard magnetic Schrödinger operator. Note that the stardard magnetic Schrödinger operator does not determine a unique self-adjoint operator. The operator is self-adjoint and defined on the domain consisting of functions from the Sobolev space W_2^2 satisfying matching/boundary conditions of the vertices, which in turn can be parametrized as:

$$i(S-I)\vec{u} = (S+I)\partial_n \vec{u},$$

where $\vec{u}$ and $\partial_n \vec{u}$ are the vectors formed by the limit values of function and their normal derivatives at a vertex and S is a unitary matrix. If the graph is finite and compact then the spectrum of the Schrödinger operator is pure discrete. The spectrum is formed by an infinite sequence of eigenvalues and has a unique accumulation point $+\infty$.

One of the important problems in this research area is the justification of quantum graphs as approximations for more relalistic models of waves in complex structures. If a graph Γ, the electric and magnetic potentials and the boundary-matching conditions are given then one can be asked to determine corresponding properties of such operator, say, spectral peoperties. The operator is normally denoted by $L^s_{q,a}(\Gamma)$. Such problems are called direct problems. In the inverse problems case one has to reconstruct the graph, determine the potentials of the differential operator and determine the appropriate boundary conditions if the spectrum of a differential operator is given. Another important area involves the relationship between the spectral properties of quantum and combinatorial graphs.

Our research work is to find all Reflectionless Isoscattering (RI-matrices) in even dimensions as no such matrices exist in odd dimensions. These are unitary matrices in which the entries in the main diagonal are zero while non-diagonal elements have the same absolute value. We shall study the case of Hermitian RI-matrices, since such matrices lead to energy-independent vertex scattering matrices.

In section 2 we talk about Hermitian matrices in general and their spectral properties. We also discuss the spectral properties of unitary Hermitian matrices and their orthogonal spectral decomposition in $\mathbb{C}^n$. Section 3 contains some basic definitions related to quantum graphs. We give general description of the Laplace operator and its domain, matching conditions via vertex scattering matrix and the energy resonant of the vertex scattering matrix. Motivation of this research work is presented in section 4. In section 5 we come up with general formula of 4×4 RI-matrices. All possible cases in dimension 6 are shown in section 6. We give the description of few typical cases in this section. The details of all possible cases can be found in the Appendix.

2 On spectral theory of Hermitian matrices

One uses different kinds of matrices in the theory of Quantum graphs. We will make use of unitary Hermitian matrices in our research work. A unitary matrix is an $n \times n$ complex matrix U satisfying the condition $U^*U = UU^* = I$, where I is the identity matrix and U^* is the conjugate transpose of U. A matrix A is called Hermitian, if it is equal to its conjugate transpose i-e $A = A^* = \overline{A}^t$. A real matrix is Hermitian if and only if it is symmetric. Hermitian matrices form one of the most studied classes of square matrices and many of their characteristics can be often calculated explicitly. Hermitian matrices are named after the French mathematician Charles Hermite, 1822-1901. There are several very powerful facts about Hermitian matrices that have found universal applications. First the spectrum of Hermitian matrices is real. Second, Hermitian matrices have a complete set of orthogonal eigenvectors, which make them diagonalizable. Third, these facts give a spectral representation for Hermitian matrices and corresponding method to approximate them by matrices of less rank.

An $n \times n$ Hermitian matrix A can be considered as a linear transformation in $\mathbb{C}^n$. Let us denote the inner product by $\langle \cdot, \cdot \rangle$.

Lemma 1. *Let A be Hermitian. Then the spectrum of A, $\sigma(A)$, is real.*

Proof. Let A be Hermitian matrix. Let λ be an eigenvalue of A. Let ψ be an eigenvector corresponding to the eigenvalue λ of A.

$$
\begin{aligned}
\lambda \langle \psi, \psi \rangle &= \langle \lambda\psi, \psi \rangle && \text{linearity of the complex inner product} \\
&= \langle A\psi, \psi \rangle && \text{since } \psi \text{ is an eigenvector} \\
&= \langle \psi, A^*\psi \rangle && \text{properties of conjugate matrix} \\
&= \langle \psi, A\psi \rangle && A \text{ is Hermitian so } A^* = A \\
&= \langle \psi, \lambda\psi \rangle && \text{definition of eigenvector: } \lambda\psi = A\psi \\
&= \overline{\lambda} \langle \psi, \psi \rangle && \text{anti-linearity of complex inner product}
\end{aligned}
$$

We have that $\psi \neq 0$, and because of the positive definiteness, it must be that $\langle \psi, \psi \rangle \neq 0$. It follows that $\lambda = \overline{\lambda} \Rightarrow \lambda \in \mathbb{R}$. □

Lemma 2. *Let A be a Hermitian matrix then the eigenvectors corresponding to distinct eigenvalues are orthogonal.*

Proof. Let λ and μ be distinct eigenvalues, with associated eigenvectors ψ and ϕ respectively. We have

$$
\begin{aligned}
\lambda \langle \phi, \psi \rangle &= \langle \lambda\phi, \psi \rangle && \text{linearity of the complex inner product} \\
&= \langle A\phi, \psi \rangle && \text{since } \phi \text{ is an eigenvector} \\
&= \langle \phi, A^*\psi \rangle && \text{properties of conjugate matrix} \\
&= \langle \phi, A\psi \rangle && A \text{ is Hermitian, so } A^* = A \\
&= \langle \phi, \mu\psi \rangle && \text{definition of eigen vector: } \mu\psi = S\psi \\
&= \mu \langle \phi, \psi \rangle && \text{anti-linearity of complex inner product}
\end{aligned}
$$

But $\lambda \neq \mu \Rightarrow \langle \phi, \psi \rangle = 0$. □

Lemma 3. *If A is unitary matrix then all of the eigenvalues of A have modulus equal to one.*

Proof. Let λ be an eigenvalue of A with associated eigenvector $\psi \neq 0$. Then:

$$
\begin{aligned}
\langle \psi, \psi \rangle &= \langle A^*A\psi, \psi \rangle && A \text{ is unitary matrix} \\
&= \langle A\psi, A\psi \rangle && \text{properties of Adjugate} \\
&= \langle \lambda\psi, \lambda\psi \rangle && \text{definition of eigenvector: } A\psi = \lambda\psi \\
&= \lambda\overline{\lambda} \langle \psi, \psi \rangle && \text{properties of complex inner product}
\end{aligned}
$$

Hence $(1 - \lambda\overline{\lambda}) \langle \psi, \psi \rangle = 0$. Since $\langle \psi, \psi \rangle \neq 0$, $\lambda\overline{\lambda} = 1 \Rightarrow |\lambda|^2 = 1 \Rightarrow |\lambda| = 1$. □

Lemma 4. *The spectrum of unitary Hermitian matrices consists of* ± 1.

Proof. Every Hermitian matrix has real eigenvalues while every unitary matrix has eigenvalues with absolute value 1. It follows that the spectrum of a unitary Hermitian matrix can contain only values ± 1. □

For example: $I, -I$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are unitary Hermitian matrices. We have: $\sigma(I) = \{1\},\ \sigma(-I) = \{-1\},\ \sigma\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \{1, -1\}$.

Lemma 5. *If P is an orthogonal projection in $\mathbb{C}^n$ then $1 - 2P$ is unitary.*

Proof. Consider

$$
\begin{aligned}
(I - 2P)^*(I - 2P) &= (I^* - 2P^*)(I - 2P) \\
&= (I^*I - 2P^*I - 2I^*P + 4P^*P) && \text{projection is orthogonal} \\
&= (I - 2P^* - 2P + 4PP) && \text{projection is Idempotent} \\
&= (I - 2P - 2P + 4P) \\
&= I
\end{aligned}
$$

□

Lemma 6. *Let A be a real symmetric $n \times n$ matrix with eigenvalues $\lambda_i, i = 1, 2, ..., n$ and corresponding eigenvectors $\psi_i, i = 1, 2, ..., n$. Then*

1)

$$A = \begin{pmatrix} \uparrow & \uparrow & \cdots & \uparrow \\ \psi_1 & \psi_2 & \cdots & \psi_n \\ \downarrow & \downarrow & \cdots & \downarrow \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \begin{pmatrix} \leftarrow & \psi_1 & \rightarrow \\ \leftarrow & \psi_2 & \rightarrow \\ \vdots & \vdots & \vdots \\ \leftarrow & \psi_n & \rightarrow \end{pmatrix}.$$

2)

$$A = \sum_{i=1}^{n} \lambda_i \psi_i \psi_i^t, i = 1, 2, ..., n.$$

Proof. 1) For any i, we have

$$Q\text{diag}\{\lambda_i\}Q^T\psi_i = Q\text{diag}\{\lambda_i\}e_i = Q\lambda_i e_i = \lambda_i Q e_i = \lambda_i \psi_i = A\psi_i,$$

where $Q = \begin{pmatrix} \uparrow & \uparrow & \cdots & \uparrow \\ \psi_1 & \psi_2 & \cdots & \psi_n \\ \downarrow & \downarrow & \cdots & \downarrow \end{pmatrix}$

2) This can be proven by multiplying both sides by ψ_j of the equation $A = \sum_{i=1}^{n} \lambda_i \psi_i \psi_i^t$. For any j, we have

$$\left(\sum_{i=1}^{n} \lambda_i \psi_i \psi_i^t \right) \psi_j = \lambda_j \psi_j = A\psi_j.$$

3 Quantum Graphs

A quantum graph is a graph equiped with a differential operator acting on the functions defined on the edges of the graph and accompanied by appropriate matching-boundary conditions.

3.1 Metric Graphs

Consider N closed or semi-infinite intervals E_n, which are subsets of $\mathbb{R}$, as

$$E_n = \begin{cases} [x_{2n-1}, x_{2n}], & n = 1, 2, \ldots, N_c \\ [x_{2n-1}, \infty), & n = N_c + 1, \ldots, N_c + N_i = N \end{cases},$$

where N_c and N_i denotes the number of compact and semi-infinite intervals respectively. The intervals E_n are called edges.

Consider

$$V = \{x_{2n-1}, x_{2n}\}_{n=1}^{N_c} \cup \{x_{2n-1}\}_{n=N_c+1}^{N}$$

the set of all end points and its arbitrary partition into M equivalence classes $V_m, m = 1, 2, \ldots, M$ called vertices.

A metric graph Γ is the union of edges:

$$\Gamma = \cup_{n=1}^{N} E_n/_{x \sim y}$$

where the equivalence relation $x \sim y$ is defined as follows:

$$x \sim y \Leftrightarrow \begin{cases} x, y \in V_m, x \neq y \\ x, y \in E_n, x = y \end{cases}.$$

The number v_m of elements in the class V_m is called the valence or degree of V_m. Thus

$$\#V = \sum_{m=1}^{M} v_m = 2N_c + N_i$$

where $\#V$ is the total number of end points. A function u defined on a metric graph is N-tuple of functions u_n defined on the corresponding intervals E_n. Thus a metric graph determines:

$$L_2(\Gamma) = \oplus \sum_{n=1}^{N} L_2(E_n)$$

where $L_2(\Gamma)$ on Γ consists of functions that are measurable and are square integrable on each edge E_n and such that:

$$\|u\|^2_{L_2(\Gamma)} = \sum_{n=1}^{N} \|u\|^2_{L_2(E_n)} < \infty.$$

The inner product on the Hilbert space is:

$$\langle u, v\rangle = \sum_{n=1}^{N} \int_{E_n} u(x)\overline{v(x)}dx$$

The values of the functions at the end points are given as:

$$u(x_j) = \lim_{x \to x_j} u(x)$$

and their normal derivatives are:

$$\partial_n u(x) = \begin{cases} \lim_{x \to x_j} \frac{d}{dx} u(x), & x_j \text{ is the left end point} \\ -\lim_{x \to x_j} \frac{d}{dx} u(x), & x_j \text{ is the right end point} \end{cases}$$

The limits are taken from inside the corresponding intervals.

3.2 Star Graph

Suppose we start with n edges, choose one vertex and then draw edges away from this vertes. The graph we would obtain is called the star graph having n semi-infinite edges denoted by S_n. Figure shows then star graph with 4 edges, S_4 :

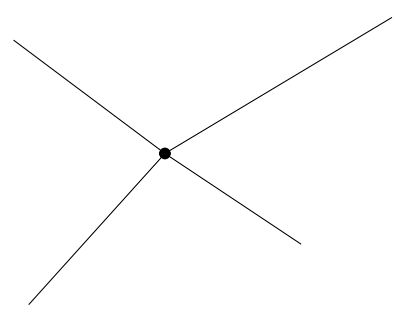

FIGURE 1. Star graph

This means the star graph of order n consists of a tree with one vertex of vertex degree n and n edges. One can see more information in literature [1], [7] and [9].

3.3 Laplace Operator

We can associate different differential operators on metric graphs. The differential operator describes the motion of the particles along the edges. Let us consider the following differential operators:

- the Laplace operator: $L = -\frac{d^2}{dx^2}$;
- the Schrödinger operator: $L_q = -\frac{d^2}{dx^2} + q(x)$;
- the magnetic Schrödinger operator: $L_{q,a} = (i\frac{d}{dx} + a(x))^2 + q(x)$.

Where q stands for electrical potential and a stands for magnetic potential. We have the following assumptions on the potentials:

(1) the potentials are real
$$q(x), a(x) \in \mathbb{R};$$
(2) the electric potential is square integrable and decays on infinite edges
$$q \in L_2(\Gamma),$$
$$\int_\Gamma (1+|x|)|q(x)|dx < \infty;$$
(3) the magnetic potential a is continuously differentiable
$$a \in C^1(\Gamma).$$

The most commonly used differential operator is Laplace operator. So, we introduce the Laplace operator to implement the dynamics of waves or particles travelling along the edges of the graph i.e. $L = -\frac{d^2}{dx^2}$. One can associate the maximal and minimal operator corresponding to the Laplace operator. The maximal operator $\oplus \sum L^{max}$ defined on the domain $Dom(L^{max}) = W_2^2(E_n)$, where W_2^2 is the Sobolev space of all square integrable functions with first and second derivatives. The domain can be written as the orthogonal sum of Sobolev spaces on the intervals E_n :

$$Dom(L^{max}) = \oplus \sum_{n=1}^{N} W_2^2(E_n)$$

The operator L^{max} can also be written as:

$$L^{max} = \oplus \sum_{n=1}^{N} L^n,$$

where L^n is given by the Laplacian on the domain $\oplus \sum_{n=1}^{N} W_2^2(E_n)$.

Similar relations can be obtained for the minimal operator on smooth functions $C_0^\infty(E_n)$. The Laplace operator $L = -\frac{d^2}{dx^2}$ on the domain $\sum_{n=1}^{N} W_2^2(E_n)$ satisfy in addition so called standard matching conditions at the vertices.

3.4 Standard matching-boundary conditions

Matching and boundary conditions play a very important role to make the operator self-adjoint. We can distinguish the vertices into internal and boundary vertices. The internal vertices have valence greater than one, so it is obvious that there will be at least two edges incident to an internal vertex. On the other hand the boundary vertices have valence one. The conditions defined on the internal vertices are called matching conditions. The standard matching conditions at all vertices can be define as:

$$\begin{cases} \text{u is continuous at the vertex } V_m, \\ \sum_{x_j \in V_m} \partial u(x_j) = 0. \end{cases}$$

If there are two edges incident the same vertex, the standard matching conditions imply nothing but the continuity of the function and of its first derivative. The two edges than can be identified with one edge, which has length equal to the sum of the two edges. For boundary vertices the standard conditions reduces to Neumann condition:

$$\partial u(x_j) = 0, x_j \epsilon V_m \in \partial\Gamma,$$

where $\partial\Gamma = \{V_m : v_m = 1\}$.

3.5 General matching conditions

As we know that matching-boundary conditions are localized to a single vertex, it is sufficient to discuss the problem of self-adjointness for a star graph. Let $\vec{u}^1 = \vec{u}(0)$ and $\partial\vec{u}^1 = \partial\vec{u}(0)$ denote the v-dimensional vectors of boundary values at the vertex V_1. On Γ, we define the operator L^S with

$$Dom(L^S) \subset W_2^2([0,\infty), \mathbb{C}^v),$$

satisfying the matching conditions:

$$i(S-I)\vec{u}^1 = (S+I)\partial\vec{u}^1$$

where S is a unitary $v \times v$ matrix.

3.6 Matching conditions and the Vertex Scattering matrix

The self-adjoint extension L^1 of L^{min} can be described by Neumann condition $\partial\vec{u} = 0$. It is the orthogonal sum of v identical Neumann Laplacians on $[0,\infty)$. The spectrum is pure absolutely continuous, has multiplicity v and fills the interval $[0,\infty)$. Hence all operators L^S have the same absolutely continuous spectrum. The corresponding generalized eigenfunction $\vec{\psi}$ are solutions to the differential equation:

$$-\frac{d^2}{dx^2}\vec{\psi} = \lambda\vec{\psi},$$

The solution of the above equation can be written as:

$$\psi\vec{(x)} = \vec{a}e^{-ikx} + \vec{b}e^{ikx}, \vec{a}, \vec{b} \in \mathbb{C}^v, k = -\lambda^2$$

Inserting this function into the matching conditions, we get:

$$\vec{a} = S_v(k)\vec{b},$$

where $\vec{a}$ and $\vec{b}$ are the amplitues of incoming and outgoing waves and $S_v(k)$ is the vertex scattering matrix corresponding to the energy $E = k^2$.

The boundary values of the function $\vec{\psi}$ are:

$$\vec{\psi}^1 = \vec{b} + S_v(k)\vec{b},$$

$$\partial\vec{\psi}^1 = -ik\vec{b} + ikS_v(k)\vec{b}.$$

Substitution into the matching conditions gives the relation:

$$i(S-I)(I+S_v(k))\vec{b} = (S+I)ik(-I+S_v(k))$$

and the following formula for the vertex scattering matrix:

$$S_v(k) = \frac{(k+1)S+(k-1)I}{(k-1)S+(k+1)I}.$$

We see that $S_v(1) = S$. It should be noted that the matching/boundary conditions at a vertex can be parametrized using the equation $A\vec{u} = B\partial\vec{u}$. However this parametrization is not unique, hence the significance of S. Also as mentioned above the elements of S have a bearing on the wave dynamics because its entries are the amplitudes of the incoming and outgoing waves.

3.7 Energy-resonant vertex S-matrices

Now we are going to show the energy dependance on vertex scattering matrix. Since the matrix S is unitary, we can write

$$S = \sum_{n=1}^{v} e^{i\theta_n} \langle \cdot, \vec{e_n} \rangle_{\mathbb{C}_v} \vec{e_n},$$

where $\theta_n \in [0, 2\pi), \vec{e_n} \in \mathbb{C}_v, S\vec{e_n} = e^{i\theta_n}\vec{e_n}$. Substituting this spectral representation into $S_v(k) = \frac{(k+1)S+(k-1)I}{(k-1)S+(k+1)I}$, we get

$$\begin{aligned} S(k) &= \sum_{n=1}^{v} \frac{(k+1)e^{i\theta_n} + (k-1)}{(k-1)e^{i\theta_n} + (k+1)} \langle \cdot, \vec{e_n} \rangle_{\mathbb{C}_v} \vec{e_n} \\ &= \sum_{n=1}^{v} \frac{k(e^{i\theta_n}+1) + (e^{i\theta_n}-1)}{k(e^{i\theta_n}+1) - (e^{i\theta_n}-1)} \langle \cdot, \vec{e_n} \rangle_{\mathbb{C}_v} \vec{e_n} \end{aligned}$$

The unitary matrix $S(k)$ has the same eigenvectors as the matrix S, but the corresponding eigenvalues in general depend on the energy. The eigenvalues ± 1 are stable, all eigenvalues different from ± 1 tend to 1 as $k \to \infty$. It follows that the high energy limit of $S(k)$ always exists and is given by

$$S(\infty) = \lim_{k\to\infty} S(k) = -P_{-1} + (I - P_{-1}),$$

where P_{-1} is the eigen projector onto the subspace corresponding to the eigenvalue -1,

$$P_{-1} = \sum_{\theta_n = \pi} \langle \cdot, \vec{e_n} \rangle_{\mathbb{C}_v} \vec{e_n}.$$

One can get more information in the research article [8].

3.8 Standard matching conditions

Let us discuss how to describe the standard matching conditions using the scattering matrix. To this end we calculate the vertex scattering matrix. One may simply substitute the Ansazts $\psi(\vec{x}) = \exp^{-ikx}\vec{b} + \exp^{ikx} S_v(k)\vec{b}$ into standard matching-boundary conditions, but it is wise to take into account that all edges in these matching conditions are equivalent and therefore the $v \times v$ matrix S is of the form

$$S_{ij}(k) = \begin{cases} T, & i \neq j, \\ R, & i = j, \end{cases} \quad \Rightarrow S(k) = \begin{pmatrix} R & T & T & \dots \\ T & R & T & \dots \\ T & T & R & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then the first and second conditions (continuity of the function and zero sum of normal derivatives) in standard matching-boundary conditions imply that:

$$\begin{array}{lcl} 1 + R & = & T \\ ik(-1 + R + (v-1)T) & = & 0. \end{array}$$

The transition and reflection coefficients are:

$$\begin{cases} T = 2/v, \\ R = -1 + 2/v. \end{cases}$$

The matrix S corresponding to standard matching conditions is then given by:

$$S = \begin{pmatrix} -1 + 2/v & 2/v & 2/v & \dots \\ 2/v & -1 + 2/v & 2/v & \dots \\ 2/v & 2/v & -1 + 2/v & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which allows to write the standard matching conditions in the form $i(S - I)\vec{u} = (S + I)\partial\vec{u}$:

$$i \begin{pmatrix} -2 + 2/v & 2/v & 2/v & \dots \\ 2/v & -2 + 2/v & 2/v & \dots \\ 2/v & 2/v & -2 + 2/v & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \vec{u} = \begin{pmatrix} 2/v & 2/v & 2/v & \dots \\ 2/v & 2/v & 2/v & \dots \\ 2/v & 2/v & 2/v & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \partial\vec{u}.$$

4 Problem discription

4.1 Equi-transmitting matrix

An $n \times n$ unitary matrix σ is equi-tranmitting if $\sigma_{ii} = 0$ for all i and non-diagonal elements have amplitudes: $|\sigma_{ij}| = (v-1)^{-\frac{1}{2}}$ for $i \neq j$, where v is the valency of the vertex.

If we consider a Star graph then the probability that a particle is scattered from edge j to the edge i is: $|\sigma_{ij}|^2$ and $|\sigma_{11}|^2 + |\sigma_{21}|^2 + |\sigma_{31}|^2 + \ldots + |\sigma_{n1}|^2 = 1$.

J. M. Harrison, U. Smilansky and B. Winn [3] gave examples of scattering matrices in which back-scattering is prohibited. They showed that the set of equi-transmitting matrices are neither empty nor trivial. The example of 2×2 equi-transmitting matrix is

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In dimension 3 no equi-transmitting matrices exist. J M Harrison, U Smilansky and B Winn [3] used skew-Hadamard matrices [13], [12] and Dirichlet characters [4] for the construction of examples of such matrices. Let us consider one such example in dimension 5 :

$$\sigma = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega & \omega^2 \\ 1 & 1 & 0 & \omega^2 & \omega \\ 1 & \omega & \omega^2 & 0 & 1 \\ 1 & \omega^2 & \omega & 1 & 0 \end{pmatrix}, \qquad \text{where } \omega = \exp^{\frac{2\pi i}{3}}.$$

One can see that the above matrix is symmetric but not Hermitian. If ω would be a real number, then the matrix σ would be Hermitian. But ω is not real and therefore σ is not Hermitian. That's why the spectrum is not contained in $\{1, -1\}$. One possible question: Suppose the matrix depends on the energy would it still be possible that nothing is reflecting back. We know that for vertex scattering matrix the high energy limit of $S(k)$ always exists and is given by:

$$S(\infty) = \lim_{k \to \infty} S(k) = -P_{-1} + (I - P_{-1}),$$

where P_{-1} is the eigen projector onto the subspace corresponding to the eigenvalue -1. Let us calculate the eigenvectors of the matrix σ corresponding to the eigenvalue -1. Consider

$$\frac{1}{2} \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega & \omega^2 \\ 1 & 1 & 0 & \omega^2 & \omega \\ 1 & \omega & \omega^2 & 0 & 1 \\ 1 & \omega^2 & \omega & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ s \\ t \end{pmatrix} = -1 \begin{pmatrix} x \\ y \\ z \\ s \\ t \end{pmatrix}, \qquad \text{where } \omega = \exp^{\frac{2\pi i}{3}}$$

implies that

$$\begin{aligned} 2x + y + z + s + t &= 0 \\ x + 2y + z + \omega s + \omega^2 t &= 0 \\ x + y + 2z + \omega^2 s + \omega t &= 0 \\ x + \omega y + \omega^2 z + 2s + t &= 0 \\ x + \omega^2 y + \omega z + s + 2t &= 0 \end{aligned}$$

We get the following eigenvetor $\{-2, 1, 1, 1, 1\}$ corresponding to eigenvalue -1. The eigen projector onto the subspace corresponding to the eigenvalue -1 is given by:

$$P_{-1}\begin{pmatrix} x \\ y \\ z \\ s \\ t \end{pmatrix} = \frac{-2x+y+z+s+t}{8}\begin{pmatrix} -2 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{8} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{8} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{8} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{8} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ s \\ t \end{pmatrix}.$$

Since $\sigma(\infty) = \lim_{k\to\infty}\sigma(k) = -P_{-1} + (I - P_{-1}) = I - 2P_{-1}$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

Now one can notice that the first entry in the main diagonal is 0 and remaining entries are equal to $\frac{3}{4}$ in the above matrix which depends on the energy i.e $\sigma(k)$, where $k \to \infty$. It means that the waves we are sending into the star graph are reflecting back almost with the same amplitudes. This is in reality non-physical because the authors gave examples of the matrices which are symmetric. One needs here Hermitian matrices in order to stop the back-scattering.

Our goal is to describe all unitary Hermitian $n \times n, n = 2, 4, 6$ matrices with zero diagonal and all non diagonal elements having the same absolute value

$$\begin{gathered} S = S^* = S^{-1}, \\ s_{jj} = 0,\ j = 1, \ldots, n, \\ |s_{jk}| = |s_{lm}|,\ j \neq k,\ l \neq m,\ j, k, l, m = 1, \ldots, n. \end{gathered}$$

Such matrices will be called **Reflectionless Isoscattering** matrices, **RI-matrices**.

5 Reflectionless Isoscattering Matrices

Theorem 1. *No RI-matrices exist if the dimension n is odd.*

Proof. Every RI-matrix is diagonalizable and therefore has precisely n eigenvalues. Possible eigenvalues are ± 1. Their sum cannot be equal to zero if the number of eigen values is odd. $\square$

It remains to describe all RI-matrices are even dimensions.

5.1 Dimension Two

It is clear that all RI-matrices in dimension two are of the form:

$$S = \begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix}, \quad \theta \in [0, 2\pi).$$

Every such matrix can be written as:

$$S = \mathrm{diag}\{1, e^{-i\theta_1}\} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathrm{diag}\{1, e^{i\theta_1}\}$$

5.2 Dimension four

RI-matrices in dimension four are of the form:

$$S = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & e^{i\theta_1} & e^{i\theta_2} & e^{i\theta_3} \\ e^{-i\theta_1} & 0 & a & b \\ e^{-i\theta_2} & \overline{a} & 0 & c \\ e^{-i\theta_3} & \overline{b} & \overline{c} & 0 \end{pmatrix}.$$

Every RI-matrix possesses the following represention:

$$S = \mathrm{diag}\{1, e^{-i\theta_1}, e^{-i\theta_2}, e^{-i\theta_3}\} \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & a & b \\ 1 & \overline{a} & 0 & c \\ 1 & \overline{b} & \overline{c} & 0 \end{pmatrix} \mathrm{diag}\{1, e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}\},$$

where $\theta_j \in [0, 2\pi)$ are arbitrary. The numbers $a, b, c \in \mathbb{C}$ have unit absolute value and should be chosen so that the rows (and hence the columns as well) in the matrix $C = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & a & b \\ 1 & \overline{a} & 0 & c \\ 1 & \overline{b} & \overline{c} & 0 \end{pmatrix}$ are orthogonal. The normalization condition is satisfied automatically, since $|a| = |b| = |c| = 1$. Let us write down the corresponding orthogonality conditions:

$$\begin{cases} a + b = 0 \\ \overline{a} + c = 0 \\ \overline{b} + \overline{c} = 0 \end{cases} \qquad \begin{cases} 1 + b\overline{c} = 0 \\ 1 + ac = 0 \end{cases} \qquad \begin{cases} 1 + \overline{a}b = 0 \end{cases}$$

It follows that $b = -a$ and $c = -\overline{a}$ implying

$$a^2 = -1 \Rightarrow a = \pm i.$$

We get precisely two possible matrices C^1 and C^2 :

$$C^1 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & i & -i \\ 1 & -i & 0 & i \\ 1 & i & -i & 0 \end{pmatrix} \text{ and } C^2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -i & i \\ 1 & i & 0 & -i \\ 1 & -i & i & 0 \end{pmatrix}$$

There are two non-intersecting 3-parameter families of 4×4 Hermitian RI-matrices:

$$S^1 = \operatorname{diag}\{1, e^{-i\theta_1}, e^{-i\theta_2}, e^{-i\theta_3}\}\frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & i & -i \\ 1 & -i & 0 & i \\ 1 & i & -i & 0 \end{pmatrix}\operatorname{diag}\{1, e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}\},$$

$$S^2 = \operatorname{diag}\{1, e^{-i\theta_1}, e^{-i\theta_2}, e^{-i\theta_3}\}\frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -i & i \\ 1 & i & 0 & -i \\ 1 & -i & i & 0 \end{pmatrix}\operatorname{diag}\{1, e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}\},$$

where $\theta_j \in [0, 2\pi)$ are arbitrary.

Matrix diagonalization is the process of converting a square matrix into a diagonal matrix. Diagonal matrices are easy to handle. Let us diagonalize these matrices. It is clear that it is enough to diagonalize the matrix $\frac{1}{\sqrt{3}}C$. The eigenvalues of $\frac{1}{\sqrt{3}}C$ are $\lambda = \pm 1$. For the first matrix and eigenvalue $\lambda = 1$ we have:

$$\frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & i & -i \\ 1 & -i & 0 & i \\ 1 & i & -i & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

which implies

$$\begin{pmatrix} -\sqrt{3} & 1 & 1 & 1 \\ 1 & -\sqrt{3} & i & -i \\ 1 & -i & -\sqrt{3} & i \\ 1 & i & -i & -\sqrt{3} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Choosing $z = 2$ and $w = 0$, we get the following system of equations:

$$\begin{cases} -\sqrt{3}x + y &= -2 \\ x - \sqrt{3}y &= -2i \end{cases}$$

Solving the above system of equations, we get

$$x = \sqrt{3} + i$$

$$y = 1 + i\sqrt{3}.$$

Summing up we have:

$$\begin{cases} x = \sqrt{3} + i \\ y = 1 + \sqrt{3}i \\ z = 2 \\ w = 0 \end{cases}$$

By choosing $w = \sqrt{3}$, we have

$$\begin{pmatrix} -\sqrt{3} & 1 & 1 \\ 1 & -\sqrt{3} & i \\ \sqrt{3}-i & 1-i\sqrt{3} & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ i \\ 0 \end{pmatrix}$$

Determinent of the abover matrix is 12. So by Cramer's rule, we get

$$\begin{cases} x = 1 \\ y = -i \\ z = i \\ w = \sqrt{3} \end{cases}$$

Similarly for eigen value $\lambda = -1$, Choose $z = 2$ and $w = 0$, we get

$$\begin{cases} x = i - \sqrt{3} \\ y = 1 - \sqrt{3}i \\ z = 2 \\ w = 0 \end{cases}$$

Now for $w = \sqrt{3}$, we have

$$\begin{cases} x = 1 \\ y = i \\ z = -i \\ w = \sqrt{3} \end{cases}$$

Since its easier to work with unitary matrices, we normalize the eigenvectors and obtain the matrix below:

$$\begin{pmatrix} \frac{\sqrt{3}+i}{\sqrt{12}} & \frac{1}{\sqrt{6}} & \frac{i-\sqrt{3}}{\sqrt{12}} & \frac{-1}{\sqrt{6}} \\ \frac{1+i\sqrt{3}}{\sqrt{12}} & \frac{-i}{\sqrt{6}} & \frac{1-i\sqrt{3}}{\sqrt{12}} & \frac{i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{i}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{-i}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

In the same way the second matrix can be written in normalized form as,

$$\begin{pmatrix} \frac{\sqrt{3}-i}{\sqrt{12}} & \frac{1}{\sqrt{6}} & \frac{-i-\sqrt{3}}{\sqrt{12}} & \frac{-1}{\sqrt{6}} \\ \frac{1-i\sqrt{3}}{\sqrt{12}} & \frac{i}{\sqrt{6}} & \frac{1+i\sqrt{3}}{\sqrt{12}} & \frac{-i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-i}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{i}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Theorem 2. *The set of all 4-dimensional RI-matrices form two three-paramete families,*

$$S^1 = \mathrm{diag}\{1, e^{-i\theta_1}, e^{-i\theta_2}, e^{-i\theta_3}\} \begin{pmatrix} \frac{\sqrt{3}+i}{\sqrt{12}} & \frac{1}{\sqrt{6}} & \frac{i-\sqrt{3}}{\sqrt{12}} & \frac{-1}{\sqrt{6}} \\ \frac{1+i\sqrt{3}}{\sqrt{12}} & \frac{-i}{\sqrt{6}} & \frac{1-i\sqrt{3}}{\sqrt{12}} & \frac{i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{i}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{-i}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\sqrt{3}-i}{\sqrt{12}} & \frac{1-i\sqrt{3}}{\sqrt{12}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{i}{\sqrt{6}} & \frac{-i}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{-i-\sqrt{3}}{\sqrt{12}} & \frac{1+i\sqrt{3}}{\sqrt{12}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{-1}{\sqrt{6}} & \frac{-i}{\sqrt{6}} & \frac{i}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix} \mathrm{diag}\{1, e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}\},$$

and

$$S^2 = \mathrm{diag}\{1, e^{-i\theta_1}, e^{-i\theta_2}, e^{-i\theta_3}\} \begin{pmatrix} \frac{\sqrt{3}-i}{\sqrt{12}} & \frac{1}{\sqrt{6}} & \frac{-i-\sqrt{3}}{\sqrt{12}} & \frac{-1}{\sqrt{6}} \\ \frac{1-i\sqrt{3}}{\sqrt{12}} & \frac{i}{\sqrt{6}} & \frac{1+i\sqrt{3}}{\sqrt{12}} & \frac{-i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-i}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{i}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\sqrt{3}+i}{\sqrt{12}} & \frac{1+i\sqrt{3}}{\sqrt{12}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{-i}{\sqrt{6}} & \frac{i}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{i-\sqrt{3}}{\sqrt{12}} & \frac{1-i\sqrt{3}}{\sqrt{12}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{-1}{\sqrt{6}} & \frac{i}{\sqrt{6}} & \frac{-i}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix} \mathrm{diag}\{1, e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}\},$$

where $\theta_j \in [0, 2\pi)$.

5.3 Dimension Six

Dimension six

Let us study RI-matrices of dimension 6. Similar to case $n = 4$, let us introduce the diagonal matrices

$$D = \mathrm{diag}\{1, e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4}, e^{i\theta_5}\}$$

with $\theta_j \in [0, 2\pi]$ arbitrary and matrix

$$C = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & a & b & c & d \\ 1 & \overline{a} & 0 & e & f & g \\ 1 & \overline{b} & \overline{e} & 0 & h & j \\ 1 & \overline{c} & \overline{f} & \overline{h} & 0 & k \\ 1 & \overline{d} & \overline{g} & \overline{j} & \overline{k} & 0 \end{pmatrix},$$

Then every 6×6 Hermitian RI-matrix can be written in the form

$$S = D^{-1} \frac{1}{\sqrt{5}} \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & a & b & c & d \\ 1 & \overline{a} & 0 & e & f & g \\ 1 & \overline{b} & \overline{e} & 0 & h & j \\ 1 & \overline{c} & \overline{f} & \overline{h} & 0 & k \\ 1 & \overline{d} & \overline{g} & \overline{j} & \overline{k} & 0 \end{pmatrix} D,$$

where parameter $a, b, c, d, e, f, g, h, j, k \in \mathbb{C}$ have unit absolute value, should be chosen so that all row vectors are orthogonal. Let us write down all 15 orthogonality conditions:

$$\begin{cases} a + b + c + d = 0 & (1) \\ \overline{a} + e + f + g = 0 & (2) \\ \overline{b} + \overline{e} + h + j = 0 & (3) \\ \overline{c} + \overline{f} + \overline{h} + k = 0 & (4) \\ \overline{d} + \overline{g} + \overline{j} + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 + b\overline{e} + c\overline{f} + d\overline{g} = 0 & (6) \\ 1 + ae + c\overline{h} + d\overline{j} = 0 & (7) \\ 1 + af + bh + d\overline{k} = 0 & (8) \\ 1 + ag + bj + ck = 0 & (9) \end{cases}$$

(5.1)

$$\begin{cases} 1+\overline{a}b+f\overline{h}+g\overline{j}=0 & (10)\\ 1+\overline{a}c+eh+g\overline{k}=0 & (11)\\ 1+\overline{a}d+ej+fk=0 & (12)\end{cases} \qquad \begin{cases} 1+\overline{b}c+\overline{e}f+jk=0 & (13)\\ 1+\overline{b}d+\overline{e}g+hk=0 & (14)\end{cases}$$

$$\begin{cases} 1+\overline{c}d+\overline{f}g+\overline{h}j=0 & (15)\end{cases}$$

We are going to use the following elementary fact:

Lemma 7. *The sum of four complex numbers z_1, z_2, z_3, z_4 having the same absolute value $|z_1| = |z_2| = |z_3| = |z_4|$ is equal to zero*

$$z_1 + z_2 + z_3 + z_4 = 0$$

if and only if at least one of the following equalities hold:

$$z_1 = -z_2,\ z_3 = -z_4;$$
$$or$$
$$z_1 = -z_3,\ z_2 = -z_4;$$
$$or$$
$$z_1 = -z_4,\ z_2 = -z_3.$$

To prove this fact one takes into account that the vectors z_j form a romb.

The 15 equations are of similar form. Here we discuss five typical families of RI-matrices. All other families have also been computed and can be found in the Appendix. The different casses are indicated as $n_1 \cdot n_2 \cdot n_3 \cdot n_4$. For example case $1 \cdot 3 \cdot 2 \cdot 2$ is given by:

$$b = -a, d = -c$$
$$g = -\overline{a}, f = -e$$
$$h = \overline{a}, g = -\overline{e}$$
$$k = \overline{e}$$

To solve the system (5.1) of 15 equations, we are going to apply lemma 7 to the first few equations. We first look at equation (1) in (5.1). We see that lemma 7 gives us three possibilities enumerated by index $n_1 = 1, 2, 3$. We proceed than to equation (2) in resulting system. This gives us another three possibilities enumerated as $n_2 = 1, 2, 3$ for each of the above mentioned three possibilities. Proceeding similarly we end up with the table given below.

n_1		n_2		n_3		n_4	
1	$b=-a, d=-c$	1	$e=-\bar{a}, g=-f$				
		2	$f=-\bar{a}, g=-e$	1	$e=a, j=-h$		
				2	$h=\bar{a}, j=-\bar{e}$		
				3	$j=\bar{a}, h=-\bar{e}$	1	$k=e$
						2	$k=\bar{e}$
		3	$g=-\bar{a}, f=-e$	1	$e=\bar{a}, j=-h$		
				2	$h=\bar{a}, j=-\bar{e}$	1	$k=e$
						2	$k=\bar{e}$
				3	$j=\bar{a}, h=-\bar{e}$		
2	$c=-a, d=-b$	1	$e=-\bar{a}, g=-f$	1	$a=\bar{b}, j=-h$		
				2	$h=-\bar{b}, j=a$	1	$k=b$
						2	$k=\bar{b}$
				3	$j=-\bar{b}, h=a$		
		2	$f=-\bar{a}, g=-e$				
		3	$g=-\bar{a}, f=-e$	1	$e=-b, j=-h$	1	$k=a$
						2	$k=\bar{a}$
				2	$h=-\bar{b}, j=-\bar{e}$	1	$e=-a, k=b$
						2	$b=-\bar{a}, k=\bar{e}$
						3	$e=-\bar{b}, k=\bar{a}$
				3	$j=-\bar{b}, h=-\bar{e}$		
3	$d=-a, c=-b$	1	$e=-\bar{a}, g=-f$	1	$a=\bar{b}, j=-h$		
				2	$h=-\bar{b}, j=a$		
				3	$j=-\bar{b}, h=a$	1	$k=b$
						2	$k=\bar{b}$
		2	$f=-\bar{a}, g=-e$	1	$e=-b, j=-h$	1	$k=a$
						2	$k=\bar{a}$
				2	$h=-\bar{b}, j=-e$		
				3	$j=-\bar{b}, h=-\bar{e}$	1	$a=-\bar{b}, k=e$
						2	$e=-\bar{b}, k=a$
						3	$k=\bar{b}, e=-a$
		3	$g=-\bar{a}, f=-e$				

Table 1

We observe from the table that there are different cases as depending on how many times lemma 7 is applied:

- Two applications of lemma 7 for example case 1.1.
- Three applications of lemma 7 for example case 1.2.1.
- Four applications of lemma 7 for example case 1.2.3.1.

Let us discuss few typical casses.

<u>Case 1.1</u>

$$b = -a, d = -c$$

$$e = \overline{a}, g = -f$$

Assume equation (1) is satisfied that is $b = -a, d = -c$. The system (5.1) is reduced to 14 equations:

$$\begin{cases} \overline{a} + e + f + g = 0 & (2) \\ -\overline{a} + \overline{e} + h + j = 0 & (3) \\ \overline{c} + \overline{f} + \overline{h} + k = 0 & (4) \\ -\overline{c} + \overline{g} + \overline{j} + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 - a\overline{e} + c\overline{f} - c\overline{g} = 0 & (6) \\ 1 + ae + c\overline{h} - c\overline{j} = 0 & (7) \\ 1 + af - ah - c\overline{k} = 0 & (8) \\ 1 + ag - aj + ck = 0 & (9) \end{cases}$$

$$\begin{cases} \underline{1 - |a|^2} + f\overline{h} + g\overline{j} = 0 & (10) \\ 1 + \overline{a}c + eh + g\overline{k} = 0 & (11) \\ 1 - \overline{a}c + ej + fk = 0 & (12) \end{cases} \qquad \begin{cases} 1 - \overline{a}c + \overline{e}f + j\overline{k} = 0 & (13) \\ 1 + \overline{a}c + \overline{e}g + hk = 0 & (14) \end{cases}$$

$$\begin{cases} \underline{1 - |c|^2} + \overline{f}g + \overline{h}j = 0 & (15), \end{cases}$$

where the underlined expressions are zero. Equation (15) $\overline{f}g + \overline{h}j = 0$ can be deleted, since it can be obtained by multiplying equation (10) $f\overline{h} + g\overline{j} = 0$ by nonzero factor $\overline{f}j$. 13 equations remain.

$$\begin{cases} \overline{a} + e + f + g = 0 & (2) \\ -\overline{a} + \overline{e} + h + j = 0 & (3) \\ \overline{c} + \overline{f} + \overline{h} + k = 0 & (4) \\ -\overline{c} + \overline{g} + \overline{j} + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 - a\overline{e} + c\overline{f} - c\overline{g} = 0 & (6) \\ 1 + ae + c\overline{h} - c\overline{j} = 0 & (7) \\ 1 + af - ah - c\overline{k} = 0 & (8) \\ 1 + ag - aj + ck = 0 & (9) \end{cases}$$

(5.2)

$$\begin{cases} f\overline{h} + g\overline{j} = 0 & (10) \\ 1 + \overline{a}c + eh + g\overline{k} = 0 & (11) \\ 1 - \overline{a}c + ej + fk = 0 & (12) \end{cases} \qquad \begin{cases} 1 - \overline{a}c + \overline{e}f + j\overline{k} = 0 & (13) \\ 1 + \overline{a}c + \overline{e}g + hk = 0 & (14) \end{cases}$$

Let us have a look at equation (2). Again it is satisfied if and only if one of the following three equations are satisfied: $e = -\overline{a}, g = -f$or $f = -\overline{a}, g = -e$ or $g = -\overline{a}, f = -e$. Consider $e = -\overline{a}, g = -f$ is satisfied. 12 equations remain.

$$\begin{cases} -\overline{a} - a + h + j = 0 & (3) \\ \overline{c} + \overline{f} + \overline{h} + k = 0 & (4) \\ -\overline{c} - \overline{f} + \overline{j} + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 + a^2 + c\overline{f} + c\overline{f} = 0 & (6) \\ \underline{1 - |a|^2} + c\overline{h} - c\overline{j} = 0 & (7) \\ 1 + af - ah - c\overline{k} = 0 & (8) \\ 1 - af - aj + ck = 0 & (9) \end{cases}$$

$$\begin{cases} f\overline{h} - f\overline{j} = 0 & (10) \\ 1 + \overline{a}c - \overline{a}h - f\overline{k} = 0 & (11) \\ 1 - \overline{a}c - \overline{a}j + fk = 0 & (12) \end{cases} \qquad \begin{cases} 1 - \overline{a}c - af + j\overline{k} = 0 & (13) \\ 1 + \overline{a}c + af + hk = 0 & (14) \end{cases}$$

From equation (10) we see that $h = j$. The equation (3) implies $h \in \mathbb{R}$. Remembering that $|h| = 1$ we conclude that $h = \pm 1 (= j)$ which implies $a = \pm 1$. Summing equations (4) and (5) we see that $k = \mp 1$. So the system reduces to:

$$\begin{cases} c + f = 0 \\ 1 + c\overline{f} = 0 \end{cases}$$

implying that $f = -c$. Finally we get all entries of C :

$$C = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \mp 1 & c & -c \\ 1 & \pm 1 & 0 & \mp 1 & -c & c \\ 1 & \mp 1 & \mp 1 & 0 & \pm 1 & \pm 1 \\ 1 & \overline{c} & -\overline{c} & \pm 1 & 0 & \mp 1 \\ 1 & -\overline{c} & \overline{c} & \pm 1 & \mp 1 & 0 \end{pmatrix}$$

Now consider the case 1.3.2.1.

Case 1.3.2.1

$$b = -a, d = -c$$

$$g = -\overline{a}, f = -e$$

$$h = \overline{a}, j = -\overline{e}$$

$$k = e$$

The equation (2) in system (5.2) is satisfied if $g = -\overline{a}, f = -e$. The system of 13 equations reduces to the system of 12 equations:

$$\begin{cases} -\overline{a} + \overline{e} + h + j = 0 & (3) \\ \overline{c} - \overline{e} + \overline{h} + k = 0 & (4) \\ -\overline{c} - a + \overline{j} + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 - a\overline{e} - c\overline{e} + ca = 0 & (6) \\ 1 + ae + c\overline{h} - c\overline{j} = 0 & (7) \\ 1 - ae - ah - c\overline{k} = 0 & (8) \\ \underline{1 - |a|^2} - aj + ck = 0 & (9) \end{cases}$$

$$\begin{cases} e\overline{h} + \overline{a}\overline{j} = 0 & (10) \\ 1 + \overline{a}c + eh - \overline{a}\overline{k} = 0 & (11) \\ 1 - \overline{a}c + ej - ek = 0 & (12) \end{cases} \qquad \begin{cases} \underline{1} - \overline{a}c - \underline{|e|^2} + j\overline{k} = 0 & (13) \\ 1 + \overline{a}c - \overline{ea} + hk = 0 & (14) \end{cases}$$

The equatoin (13) $-\overline{a}c + j\overline{k} = 0$ can be deleted because it can be obtaind by multiplying equation (9) by nonzero factor $\overline{a}\overline{k}$. We see that equation (3) is satisfied if and if one of the following three equalities holds: $e = a, j = -h$ or $h = \overline{a}, j = -\overline{e}$ or $j = \overline{a}, h = -\overline{e}$. Looking at the equality $h = \overline{a}, j = -\overline{e}$, leading to the system of 10 equations:

$$\begin{cases} \overline{c} - \overline{e} + a + k = 0 & (4) \\ -\overline{c} - a - e + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 - a\overline{e} - c\overline{e} + ca = 0 & (6) \\ 1 + ae + ca + ce = 0 & (7) \\ \underline{1} - ae - \underline{|a|^2} - c\overline{k} = 0 & (8) \\ a\overline{e} + ck = 0 & (9) \end{cases}$$

$$\begin{cases} ea - \overline{a}e = 0 & (10) \\ 1 + \overline{a}c + e\overline{a} - \overline{a}\overline{k} = 0 & (11) \\ \underline{1} - \overline{a}c - \underline{|e|^2} - ek = 0 & (12) \end{cases} \qquad \begin{cases} 1 + \overline{a}c - \overline{ea} + \overline{a}k = 0 & (14) \end{cases}$$

From equation (10) we see that $a \in \mathbb{R}$, therefore $a = \pm 1$. By adding equations (4) and (5), we get $Re(k) = Re(e) \Rightarrow k = e$ or $k = \overline{e}$. Consider the case when $k = e$. Subtitute $k = e$ in the above system of equations, we have

$$\begin{cases} \overline{c} - \overline{e} + a + e = 0 & (4) \\ -\overline{c} - a - e + \overline{e} = 0 & (5) \end{cases} \qquad \begin{cases} 1 - a\overline{e} - c\overline{e} + ca = 0 & (6) \\ 1 + ae + ca + ce = 0 & (7) \\ -ae - c\overline{e} = 0 & (8) \\ a\overline{e} + ce = 0 & (9) \end{cases}$$

(5.3)

$$\begin{cases} ea - \overline{a}e = 0 & (10) \\ 1 + \overline{a}c + e\overline{a} - \overline{ae} = 0 & (11) \\ -\overline{a}c - e^2 = 0 & (12) \end{cases} \qquad \begin{cases} 1 + \overline{a}c - \overline{ea} + \overline{a}e = 0 & (14) \end{cases}$$

Put $a = 1$ in the above system (5.3), we get

$$\begin{cases} \overline{c} - \overline{e} + 1 + e = 0 & (4) \\ -\overline{c} - 1 - e + \overline{e} = 0 & (5) \end{cases} \qquad \begin{cases} 1 - \overline{e} - c\overline{e} + c = 0 & (6) \\ 1 + e + c + ce = 0 & (7) \\ -e - c\overline{e} = 0 & (8) \\ \overline{e} + ce = 0 & (9) \end{cases}$$

$$\begin{cases} e - e = 0 & (10) \\ 1 + c + e - \overline{e} = 0 & (11) \\ -c - e^2 = 0 & (12) \end{cases} \qquad \begin{cases} 1 + c - \overline{e} + e = 0 & (14) \end{cases}$$

Subtracting equation (7) from equation (6), we get $c = -1$. Using the value of c in equation (4), we get $e = \pm 1$. Now substitute $a = -1$ into the system (5.3) , we get the following system of equations:

$$\begin{cases} \overline{c} - \overline{e} - 1 + e = 0 & (4) \\ -\overline{c} + 1 - e + \overline{e} = 0 & (5) \end{cases} \qquad \begin{cases} 1 + \overline{e} - c\overline{e} - c = 0 & (6) \\ 1 - e - c + ce = 0 & (7) \\ e - c\overline{e} = 0 & (8) \\ -\overline{e} + ce = 0 & (9) \end{cases}$$

$$\begin{cases} -e + e = 0 & (10) \\ 1 - c - e + \overline{e} = 0 & (11) \\ c - e^2 = 0 & (12) \end{cases} \qquad \begin{cases} 1 - c + \overline{e} - e = 0 & (14) \end{cases}$$

Subtracting equation (7) from equation (6), we get $c = 1$. Equation (4) implies $e = \pm 1$. Summing up, we have if $a = \pm 1 \Rightarrow c = \mp 1$. We get the following two families:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \mp 1 & \mp 1 & \pm 1 \\ 1 & \pm 1 & 0 & \pm 1 & \mp 1 & \mp 1 \\ 1 & \mp 1 & \pm 1 & 0 & \pm 1 & \mp 1 \\ 1 & \mp 1 & \mp 1 & \pm 1 & 0 & \pm 1 \\ 1 & \pm 1 & \mp 1 & \mp 1 & \pm 1 & 0 \end{pmatrix}$$

Let us now discuss the case 2.1.1.

<u>Case 2.1.1</u>

$$c = -a, d = -b$$

$$e = -\overline{a}, g = -f$$

$$a = \overline{b}, j = -h$$

For $c = -a, d = -b$, the 15 orthogonality conditions in system (5.1) reduces to 14 equations:

$$\begin{cases} \overline{a} + e + f + g = 0 & (2) \\ \overline{b} + \overline{e} + h + j = 0 & (3) \\ -\overline{a} + \overline{f} + \overline{h} + k = 0 & (4) \\ -\overline{b} + \overline{g} + \overline{j} + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 + b\overline{e} - a\overline{f} - b\overline{g} = 0 & (6) \\ 1 + ae - a\overline{h} - b\overline{j} = 0 & (7) \\ 1 + af + bh - b\overline{k} = 0 & (8) \\ 1 + ag + bj - ak = 0 & (9) \end{cases}$$

(5.4)

$$\begin{cases} 1 + \overline{a}b + f\overline{h} + g\overline{j} = 0 & (10) \\ \underline{1 - |a|^2} + eh + g\overline{k} = 0 & (11) \\ 1 - \overline{a}b + ej + fk = 0 & (12) \end{cases} \qquad \begin{cases} 1 - \overline{b}a + \overline{e}f + j\overline{k} = 0 & (13) \\ \underline{1 - |b|^2} + \overline{e}g + hk = 0 & (14) \end{cases}$$

$$\begin{cases} 1 + \overline{a}b + \overline{f}g + \overline{h}j = 0 & (15) \end{cases}$$

Equation (14) can be deleted, because it can be obtained from equation (11) by multiplying non-zero factor $\overline{e}k$. 13 equations left. Let us study the first case from equation (2) $e = -\overline{a}, g = -f$. The system (5.4) reduces to 12 equations:

$$\begin{cases} \overline{b} - a + h + j = 0 & (3) \\ -\overline{a} + \overline{f} + \overline{h} + k = 0 & (4) \\ -\overline{b} - \overline{f} + \overline{j} + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 - ba - a\overline{f} + b\overline{f} = 0 & (6) \\ \underline{1 - |a|^2} - a\overline{h} - b\overline{j} = 0 & (7) \\ 1 + af + bh - b\overline{k} = 0 & (8) \\ 1 - af + bj - ak = 0 & (9) \end{cases}$$

$$\begin{cases} 1 + \overline{a}b + f\overline{h} - f\overline{j} = 0 & (10) \\ -\overline{a}h - f\overline{k} = 0 & (11) \\ 1 - \overline{a}b - \overline{a}j + fk = 0 & (12) \end{cases} \qquad \begin{cases} 1 - \overline{b}a - af + j\overline{k} = 0 & (13) \end{cases}$$

$$\begin{cases} \underline{1} + \overline{a}b - \underline{|f|^2} + \overline{h}j = 0 & (15) \end{cases}$$

Equation (15) can be deleted, since it can be obtained from equation (7) by multiplying nonzero factor by $\overline{b}h$. Let us consider Equation (3). It is satisfied if and only if one of the following three equalities holds: $a = \overline{b}, j = -h$ or $h = -\overline{b}, j = a$

or $j=-\bar{b}, h=a$. Consider the first equality $a=\bar{b}, j=-h$, leading to the system of 10 equations:

$$\begin{cases} -b+\bar{f}+\bar{h}+k=0 & (4) \\ -\bar{b}-\bar{f}-\bar{h}+\bar{k}=0 & (5) \end{cases} \qquad \begin{cases} \underline{1-|b|^2}-\overline{bf}+b\bar{f}=0 & (6) \\ -\overline{bh}+b\bar{h}=0 & (7) \\ 1+\bar{b}f+bh-b\bar{k}=0 & (8) \\ 1-\bar{b}f-bh-\bar{b}k=0 & (9) \end{cases}$$

$$\begin{cases} 1+|b|^2+f\bar{h}+f\bar{h}=0 & (10) \\ -bh-f\bar{k}=0 & (11) \\ 1-b^2-bj+fk=0 & (12) \end{cases} \qquad \begin{cases} \underline{1-|b|^2}-\bar{b}f-h\bar{k}=0 & (13) \end{cases}$$

From equation (6), we get $b=\pm 1$. From equation (10) it follows that $h=-f$. Put $b=\pm 1$ and $h=-f$ in equation (9), we get $k=\pm 1$. We get two one-parameter of families:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \pm 1 & \mp 1 & \mp 1 \\ 1 & \pm 1 & 0 & \mp 1 & f & -f \\ 1 & \pm 1 & \mp 1 & 0 & -f & f \\ 1 & \mp 1 & \bar{f} & -\bar{f} & 0 & \pm 1 \\ 1 & \mp 1 & -\bar{f} & \bar{f} & \pm 1 & 0 \end{pmatrix}$$

Now we are going to discuss the case 2.3.2.3.

Case 2.3.2.3

$$c=-a, d=-b$$

$$g=-\bar{a}, f=-e$$

$$h=-\bar{b}, j=-\bar{e}$$

$$a=-\bar{b}, k=\bar{a}$$

Equation (2) of system (5.4) is satiesfied if we substitute $g=-\bar{a}, f=-e$. The system of 13 equations reduces to the system of 12 equations:

$$\begin{cases} \bar{b}+\bar{e}+h+j=0 & (3) \\ -\bar{a}-\bar{e}+\bar{h}+k=0 & (4) \\ -\bar{b}-a+\bar{j}+\bar{k}=0 & (5) \end{cases} \qquad \begin{cases} 1+b\bar{e}+a\bar{e}+ba=0 & (6) \\ 1+ae-a\bar{h}-b\bar{j}=0 & (7) \\ 1-ae+bh-b\bar{k}=0 & (8) \\ \underline{1-|a|^2}+bj-ak=0 & (9) \end{cases}$$

$$\begin{cases} 1+\bar{a}b-e\bar{h}-\bar{a}\bar{j}=0 & (10) \\ eh-\bar{a}\bar{k}=0 & (11) \\ 1-\bar{a}b+ej-ek=0 & (12) \end{cases} \qquad \begin{cases} \underline{1}-\bar{b}a-\underline{|e|^2}+j\bar{k}=0 & (13) \end{cases}$$

$$\begin{cases} 1+\bar{a}b+\overline{ea}+\bar{h}j=0 & (15) \end{cases}$$

The equation (13) $-\bar{b}a+j\bar{k}=0$ can be deleted, because it can be obtained from equation (9) $bj-ak=0$ by multiplying by the non-zero factor $\overline{bk}$. 11 equations remain. Now equation (3) is satisfied if and only if one of the following three equalities holds: $e=-b, j=-h$ or $h=-\bar{b}, j=-\bar{e}$ or $j=-\bar{b}, h=-\bar{e}$. Here we shall consider the second equality $h = -\bar{b}, j = -\bar{e}$. Hence the equation (3) of the above system is satisfied. We obtain the following system of 10 equations:

$$\begin{cases} -\overline{a} - \overline{e} - b + k = 0 & (4) \\ -\overline{b} - a - e + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 + b\overline{e} + a\overline{e} + ba = 0 & (6) \\ 1 + ae + ab + be = 0 & (7) \\ \underline{1} - ae - \underline{|b|}^2 - b\overline{k} = 0 & (8) \\ -b\overline{e} - ak = 0 & (9) \end{cases}$$

$$\begin{cases} 1 + \overline{a}b + eb + \overline{a}e = 0 & (10) \\ -e\overline{b} - \overline{a}\overline{k} = 0 & (11) \\ \underline{1} - \overline{a}b - \underline{|e|}^2 - ek = 0 & (12) \end{cases} \qquad \begin{cases} 1 + \overline{a}b + \overline{ea} + b\overline{e} = 0 & (15) \end{cases}$$

We can delete equation (5) because it is same as equation (4). Similarly we can delete equation (9) because it is same as equaion (11). 8 equations remain:

$$\begin{cases} -\overline{a} - \overline{e} - b + k = 0 & (4) \end{cases} \qquad \begin{cases} 1 + b\overline{e} + a\overline{e} + ba = 0 & (6) \\ 1 + ae + ab + be = 0 & (7) \\ \underline{1} - ae - \underline{|b|}^2 - b\overline{k} = 0 & (8) \end{cases}$$

$$\begin{cases} 1 + \overline{a}b + eb + \overline{a}e = 0 & (10) \\ -e\overline{b} - \overline{a}\overline{k} = 0 & (11) \\ \underline{1} - \overline{a}b - \underline{|e|}^2 - ek = 0 & (12) \end{cases} \qquad \begin{cases} 1 + \overline{a}b + \overline{ea} + b\overline{e} = 0 & (15) \end{cases}$$

Equation (4) is satisfied if and only if one of the following holds: $e = -a, k = b$ or $b = -\overline{a}, k = \overline{e}$ or $e = -\bar{b}, k = \overline{a}$. By considering the third equality $e = -\bar{b}, k = \overline{a}$, we are getting the following system of 8 equations:

$$\begin{cases} 1 - b^2 - ab + ab = 0 & (6) \\ 1 - a\overline{b} + ab - |b|^2 = 0 & (7) \\ a\overline{b} - ba = 0 & (8) \end{cases} \qquad \begin{cases} 1 + \overline{a}b - |b|^2 - \overline{a}b = 0 & (10) \\ -\overline{b}\overline{B} - |a|^2 = 0 & (11) \\ -\overline{a}b - \overline{a}e = 0 & (12) \end{cases}$$

$$\begin{cases} 1 + \overline{a}b - b\overline{a} - |b|^2 = 0 & (15) \end{cases}$$

From equation (6), it follows that $b = \pm 1$. We have two-one parameter faimilies

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & a & \pm 1 & -a & \mp 1 \\ 1 & \overline{a} & 0 & \mp 1 & \pm 1 & -\overline{a} \\ 1 & \pm 1 & \mp 1 & 0 & \mp 1 & \pm 1 \\ 1 & -\overline{a} & \pm 1 & \mp 1 & 0 & \overline{a} \\ 1 & \mp 1 & -a & \pm 1 & a & 0 \end{pmatrix}$$

The last case that we want to be consider is 3.2.2.

Case 3.2.2

$$d = -a, c = -b$$

$$f = -\overline{a}, g = -e$$

$$h = -\bar{b}, j = -e$$

Equation (1) of system (5.1) is satisfied by considering $d = -a, c = -b$. We get the system of 14 equations: The 15 orthogonality conditions reduces to 14 equations:

$$\begin{cases} \overline{a} + e + f + g = 0 & (2) \\ \overline{b} + \overline{e} + h + j = 0 & (3) \\ -\overline{b} + \overline{f} + \overline{h} + k = 0 & (4) \\ -\overline{a} + \overline{g} + \overline{j} + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 + b\overline{e} - b\overline{f} - a\overline{g} = 0 & (6) \\ 1 + ae - b\overline{h} - a\overline{j} = 0 & (7) \\ 1 + af + bh - a\overline{k} = 0 & (8) \\ 1 + ag + bj - bk = 0 & (9) \end{cases}$$

$$\begin{cases} 1 + \overline{a}b + f\overline{h} + g\overline{j} = 0 & (10) \\ 1 - \overline{a}b + eh + g\overline{k} = 0 & (11) \\ \underline{1 - |a|^2} + ej + fk = 0 & (12) \end{cases} \qquad \begin{cases} \underline{1 - |b|^2} + \overline{e}f + j\overline{k} = 0 & (13) \\ 1 - \overline{b}a + \overline{e}g + hk = 0 & (14) \end{cases}$$

$$\begin{cases} 1 + \overline{b}a + \overline{f}g + \overline{h}j = 0 & (15) \end{cases}$$

Equation (13) $\overline{e}f + j\overline{k} = 0$ can be deleted, since it can be obtained from equation (12) $ej + fk = 0$, by multiplying nonzero factor $\overline{e}\overline{k}$. From the second equation, we can choose $f = -\overline{a}, g = -e$. We obtain the system of 12 equations:

$$\begin{cases} \overline{b} + \overline{e} + h + j = 0 & (3) \\ -\overline{b} - a + \overline{h} + k = 0 & (4) \\ -\overline{a} - \overline{e} + \overline{j} + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 + b\overline{e} + ba + a\overline{e} = 0 & (6) \\ 1 + ae - b\overline{h} - a\overline{j} = 0 & (7) \\ \underline{1 - |a|^2} + bh - a\overline{k} = 0 & (8) \\ \underline{1} - ae + bj - bk = 0 & (9) \end{cases}$$

$$\begin{cases} 1 + \overline{a}b - \overline{a}\overline{h} - e\overline{j} = 0 & (10) \\ 1 - \overline{a}b + eh - e\overline{k} = 0 & (11) \\ ej - \overline{a}k = 0 & (12) \end{cases} \qquad \begin{cases} \underline{1} - \overline{b}a - \underline{|e|^2} + hk = 0 & (14) \end{cases}$$

$$\begin{cases} 1 + \overline{b}a + ae + \overline{h}j = 0 & (15) \end{cases}$$

Equation (14) can be deleted, because it can be obtaind from equation (8) by multiplying non-zero factor $\overline{b}k$. Equation (3) is satisfied if and if only if one of the following three equalities holds: $e = -b, j = -h$ or $h = -\overline{b}, j = -e$ or $j = -\overline{b}, h = -\overline{e}$. Considering the second equality leading the above system to the system of 10 equations:

$$\begin{cases} -\overline{b} - a - b + k = 0 & (4) \\ -\overline{a} - \overline{e} - e + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 + b\overline{e} + ba + a\overline{e} = 0 & (6) \\ 1 + ae + b^2 + ae = 0 & (7) \\ -b\overline{b} - a\overline{k} = 0 & (8) \\ 1 - ae - b\overline{e} - bk = 0 & (9) \end{cases}$$

$$\begin{cases} 1 + \overline{a}b + \overline{a}b + e^2 = 0 & (10) \\ 1 - \overline{a}b - e\overline{b} - e\overline{k} = 0 & (11) \\ -e\overline{e} - \overline{a}k = 0 & (12) \end{cases} \qquad \begin{cases} 1 + \overline{b}a + ae + b\overline{e} = 0 & (15) \end{cases}$$

From equation (8), we get $a = -k$. Using $a = -k$ in equation (4), we get $k = Re(b) \Rightarrow b = \pm 1, k = \pm 1 \Rightarrow a \mp 1$. Now put $a = -1, b = 1$ and $k = 1$ in equation (11), we get $e = 1$. Put $a = 1, b = -1$ and $k = -1$ in equation (11), we get $e = -1$.

Summing up, we have $a = \mp 1, b = \pm 1, k = \pm 1$ and $e = \pm 1$. So, we get the following two families:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \mp 1 & \pm 1 & \mp 1 & \pm 1 \\ 1 & \mp 1 & 0 & \pm 1 & \pm 1 & \mp 1 \\ 1 & \pm 1 & \pm 1 & 0 & \mp 1 & \mp 1 \\ 1 & \mp 1 & \pm 1 & \mp 1 & 0 & \pm 1 \\ 1 & \pm 1 & \mp 1 & \mp 1 & \pm 1 & 0 \end{pmatrix}$$

5.4 General discussion

By solving all the cases mentioned in the table, we noticed that there are one-parameter families of matrices. The first row and the first column in each matrix is fixed. So we are left with 5×5 matrices from which we are getting one-parameter families of matrices. We constructed these matrices in such a way that the entries on the main diagonal are zero. In each constructed family precisely one of the row and column with the same number do not contain a free parameter.

It is natural to divide constructed families into groups in accordance to which column is parameter free. Such column contains four numbers two time $+1$ and two time -1. There precisely 6 possibilities to arrange these numbers:

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} = 6.$$

It appears that if the parameter free column is fixed, then all the other entries are determined uniquely up to one arbitrary parameter. Therefore we ger $5 \times 6 = 30$ one-parameter families of RI-matrices.

Now we are going to mention all 30 different one-parameter family of matrices. First we write down the families of matrices in which second row and second column is fixed i-e $(3456)_2$.

$$A_2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \pm 1 & \mp 1 & \mp 1 \\ 1 & \pm 1 & 0 & \mp 1 & a & -a \\ 1 & \pm 1 & \mp 1 & 0 & -a & a \\ 1 & \mp 1 & \overline{a} & -\overline{a} & 0 & \pm 1 \\ 1 & \mp 1 & -\overline{a} & \overline{a} & \pm 1 & 0 \end{pmatrix}_{(3456)_2}$$

First we interchage third and fifth columns in the above families, we get

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \mp 1 & \pm 1 & \pm 1 & \mp 1 \\ 1 & \pm 1 & a & \mp 1 & 0 & -a \\ 1 & \pm 1 & -a & 0 & \mp 1 & a \\ 1 & \mp 1 & 0 & -\overline{a} & \overline{a} & \pm 1 \\ 1 & \mp 1 & \mp 1 & \overline{a} & -\overline{a} & 0 \end{pmatrix}$$

Now interchange third and fifth rows to get two new families $(3546)_2$.

$$B_2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \mp 1 & \pm 1 & \pm 1 & \mp 1 \\ 1 & \mp 1 & 0 & -\bar{b} & \bar{b} & \pm 1 \\ 1 & \pm 1 & -b & 0 & \mp 1 & b \\ 1 & \pm 1 & b & \mp 1 & 0 & -b \\ 1 & \mp 1 & \pm 1 & \bar{b} & -\bar{b} & 0 \end{pmatrix}_{(3546)_2}$$

The third and final possibility is to interchange third and sixth column and then third and sixth row to get $(3645)_2$, we have

$$C_2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \mp 1 & \pm 1 & \mp 1 & \pm 1 \\ 1 & \mp 1 & 0 & \bar{c} & \pm 1 & -\bar{c} \\ 1 & \pm 1 & c & 0 & -c & \mp 1 \\ 1 & \mp 1 & \pm 1 & -\bar{c} & 0 & \bar{c} \\ 1 & \pm 1 & -c & \mp 1 & c & 0 \end{pmatrix}_{(3645)_2}$$

Consider $(2456)_3$.

$$A_3 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \mp 1 & a & -a \\ 1 & \pm 1 & 0 & \pm 1 & \mp 1 & \mp 1 \\ 1 & \mp 1 & \pm 1 & 0 & -a & a \\ 1 & \bar{a} & \mp 1 & -\bar{a} & 0 & \pm 1 \\ 1 & -\bar{a} & \mp 1 & \bar{a} & \pm 1 & 0 \end{pmatrix}_{(2456)_3}$$

By interchanging second and fifth rows and columns, we get B_3

$$B_3 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \mp 1 & -\bar{b} & \bar{b} & \pm 1 \\ 1 & \mp 1 & 0 & \pm 1 & \pm 1 & \mp 1 \\ 1 & -b & \pm 1 & 0 & \mp 1 & b \\ 1 & b & \pm 1 & \mp 1 & 0 & -b \\ 1 & \pm 1 & \mp 1 & \bar{b} & -\bar{b} & 0 \end{pmatrix}_{(2546)_3}$$

Third possibility is to interchange first second and sixth columns and then intechange the second and sixth rows, we have

$$C_3 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \mp 1 & \bar{c} & \pm 1 & -\bar{c} \\ 1 & \mp 1 & 0 & \pm 1 & \mp 1 & \pm 1 \\ 1 & c & \pm 1 & 0 & -c & \mp 1 \\ 1 & \pm 1 & \mp 1 & -\bar{c} & 0 & \bar{c} \\ 1 & -c & \pm 1 & \mp 1 & c & 0 \end{pmatrix}_{(2654)_3}$$

The third choice is to fix forth column and row, we get A_4

$$A_4 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm1 & \mp1 & a & -a \\ 1 & \pm1 & 0 & \mp1 & -a & a \\ 1 & \mp1 & \mp1 & 0 & \pm1 & \pm1 \\ 1 & \overline{a} & -\overline{a} & \pm1 & 0 & \mp1 \\ 1 & -\overline{a} & \overline{a} & \pm1 & \mp1 & 0 \end{pmatrix}_{(2356)_4}$$

Interchange second and fifth columns and then the rows, we obtain

$$B_4 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -\overline{b} & \pm1 & \overline{b} & \mp1 \\ 1 & -b & 0 & \mp1 & \pm1 & b \\ 1 & \pm1 & \mp1 & 0 & \mp1 & \pm1 \\ 1 & b & \pm1 & \mp1 & 0 & -b \\ 1 & \mp1 & \overline{b} & \pm1 & -\overline{b} & 0 \end{pmatrix}_{(2536)_4}$$

For $(2635)_4$, we interchange the second and sixth columns and then rows.

$$C_4 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \overline{c} & \pm1 & \mp1 & -\overline{c} \\ 1 & c & 0 & \mp1 & -c & \pm1 \\ 1 & \pm1 & \mp1 & 0 & \pm1 & \mp1 \\ 1 & \mp1 & -\overline{c} & \pm1 & 0 & \overline{c} \\ 1 & -c & \pm1 & \mp1 & c & 0 \end{pmatrix}_{(2635)_4}$$

The forth option is to fix fifth column and fifth row, we have

$$A_5 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm1 & a & \mp1 & -a \\ 1 & \pm1 & 0 & -a & \mp1 & a \\ 1 & \overline{a} & -\overline{a} & 0 & \pm1 & \mp1 \\ 1 & \mp1 & \mp1 & \pm1 & 0 & \pm1 \\ 1 & -\overline{a} & \overline{a} & \mp1 & \pm1 & 0 \end{pmatrix}_{(2346)_5}$$

Now interchange second and forth columns and then rows to get B_5

$$B_5 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -\overline{b} & \overline{b} & \pm1 & \mp1 \\ 1 & -b & 0 & \pm1 & \mp1 & b \\ 1 & b & \pm1 & 0 & \mp1 & -b \\ 1 & \pm1 & \mp1 & \mp1 & 0 & \pm1 \\ 1 & \mp1 & \overline{b} & -\overline{b} & \pm1 & 0 \end{pmatrix}_{(2436)_5}$$

To get C_5, we interchange second and sixth columns and rows.

$$C_5 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \overline{c} & \mp1 & \pm1 & -\overline{c} \\ 1 & c & 0 & -c & \mp1 & \pm1 \\ 1 & \mp1 & -\overline{c} & 0 & \pm1 & \overline{c} \\ 1 & \pm1 & \mp1 & \pm1 & 0 & \mp1 \\ 1 & -c & \pm1 & c & \mp1 & 0 \end{pmatrix}_{(2634)_5}$$

In the last we are going to fix sixth column and the sixth row, we get

$$A_6 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & a & -a & \pm 1 & \mp 1 \\ 1 & \overline{a} & 0 & \mp 1 & -\overline{a} & \pm 1 \\ 1 & -\overline{a} & \mp 1 & 0 & \overline{a} & \pm 1 \\ 1 & \pm 1 & -a & a & 0 & \mp 1 \\ 1 & \mp 1 & \pm 1 & \pm 1 & \mp 1 & 0 \end{pmatrix}_{(2345)_6}$$

Interchange second and forth columns and then rows

$$B_6 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \mp 1 & -\overline{b} & \overline{b} & \pm 1 \\ 1 & \mp 1 & 0 & \overline{b} & -\overline{b} & \pm 1 \\ 1 & -b & b & 0 & \pm 1 & \mp 1 \\ 1 & b & -b & \pm 1 & 0 & \mp 1 \\ 1 & \pm 1 & \pm 1 & \mp 1 & \mp 1 & 0 \end{pmatrix}_{(2435)_6}$$

Finally the families $(2534)_6$ are obtained by interchanging the second and fifth columns and then second and fifth rows respectively

$$C_6 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & c & -c & \pm 1 & \mp 1 \\ 1 & \overline{c} & 0 & \mp 1 & -\overline{c} & \pm 1 \\ 1 & -\overline{c} & \mp 1 & 0 & \overline{c} & \pm 1 \\ 1 & \pm 1 & -c & c & 0 & \mp 1 \\ 1 & \mp 1 & \pm 1 & \pm 1 & \mp 1 & 0 \end{pmatrix}_{(2534)_6}$$

In addition, we get 12 parameter free matrices like the case 2.3.1.1:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \pm 1 & \mp 1 & \mp 1 \\ 1 & \pm 1 & 0 & \mp 1 & \pm 1 & \mp 1 \\ 1 & \pm 1 & \mp 1 & 0 & \mp 1 & \pm 1 \\ 1 & \mp 1 & \pm 1 & \mp 1 & 0 & \pm 1 \\ 1 & \mp 1 & \mp 1 & \pm 1 & \pm 1 & 0 \end{pmatrix}.$$

We know that the absolute value of the parameter is 1 so, we observe that in each of the above 30 cases, the parameter can assume value $+1$ or -1. So in total we have 60 parameter free matrices. It should be noted that we obtained 12 parameter free matrices in the course of constructing the RI-matrices. Each of these 12 matrices turn out to be an intersection of 5 different one-parameter matrices. Let us group the above 30 cases into the following five families:

$$\{A_i, B_i, C_i\}_{i=2}^{6}.$$

The choice of one-parameter matrix in the intersection described above is done in such a way that we pick one and only one matrix in each of the five families. Let us introduce the notation $A_n^j(k)$, where j indicates whether we are choosing the upper or lower sign. The choice of the upper sign is indicated by u while that of the lower sign is indicated by l. n indicates the row and column index without parameter and it takes the values 2, 4 and 6. k is the value of the parameter and which can assume either $+1$ or -1. Similar descriptions apply for $B's$ and $C's$. For example

$A_2^u(1)$ stands for the upper matrix A_2 with parameter value $a = 1$. Let us consider the following matrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \pm 1 & \mp 1 & \mp 1 \\ 1 & \pm 1 & 0 & \mp 1 & \pm 1 & \mp 1 \\ 1 & \pm 1 & \mp 1 & 0 & \mp 1 & \pm 1 \\ 1 & \mp 1 & \pm 1 & \mp 1 & 0 & \pm 1 \\ 1 & \mp 1 & \mp 1 & \pm 1 & \pm 1 & 0 \end{pmatrix}.$$

This is the intersection point of the matrices $A_2^u(1)$, $C_3^l(1)$, $B_4^u(-1)$, $C_5^l(1)$ and $B_6^l(-1)$.

6 Appendix

As explained in subsection 5.3, the determination of the RI-matrices is by repeated application of Lemma 6 until one parameter or parameter free RI-matrix is obtained upto a sign. Detail discussion and computation follow below.

Case 1.0

Assume equation 1 is satisfied that is $b = -a$ and $d = -c$. The system is reduced to:

$$\begin{cases} \overline{a} + e + f + g = 0 & (2) \\ -\overline{a} + \overline{e} + h + j = 0 & (3) \\ \overline{c} + \overline{f} + \overline{h} + k = 0 & (4) \\ -\overline{c} + \overline{g} + \overline{j} + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 - a\overline{e} + c\overline{f} - c\overline{g} = 0 & (6) \\ 1 + ae + c\overline{h} - c\overline{j} = 0 & (7) \\ 1 + af - ah - c\overline{k} = 0 & (8) \\ 1 + ag - aj + ck = 0 & (9) \end{cases}$$

$$\begin{cases} \underline{1 - |a|^2} + f\overline{h} + g\overline{j} = 0 & (10) \\ 1 + \overline{a}c + eh + g\overline{k} = 0 & (11) \\ 1 - \overline{a}c + ej + fk = 0 & (12) \end{cases} \qquad \begin{cases} 1 - \overline{a}c + \overline{e}f + j\overline{k} = 0 & (13) \\ 1 + \overline{a}c + \overline{e}g + hk = 0 & (14) \end{cases}$$

$$\left\{ \underline{1 - |c|^2} + \overline{f}g + \overline{h}j = 0 \quad (15), \right.$$

where the underlined expressions are zero. Equation (15) $\overline{f}g + \overline{h}j = 0$ can be deleted, since it can be obtained by multiplying equation (10) $f\overline{h} + g\overline{j} = 0$ by nonzero factor $\overline{f}j$. 13 equations remain.

$$\begin{cases} \overline{a} + e + f + g = 0 & (2) \\ -\overline{a} + \overline{e} + h + j = 0 & (3) \\ \overline{c} + \overline{f} + \overline{h} + k = 0 & (4) \\ -\overline{c} + \overline{g} + \overline{j} + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 - a\overline{e} + c\overline{f} - c\overline{g} = 0 & (6) \\ 1 + ae + c\overline{h} - c\overline{j} = 0 & (7) \\ 1 + af - ah - c\overline{k} = 0 & (8) \\ 1 + ag - aj + ck = 0 & (9) \end{cases}$$

(6.1)

$$\begin{cases} f\overline{h} + g\overline{j} = 0 & (10) \\ 1 + \overline{a}c + eh + g\overline{k} = 0 & (11) \\ 1 - \overline{a}c + ej + fk = 0 & (12) \end{cases} \qquad \begin{cases} 1 - \overline{a}c + \overline{e}f + j\overline{k} = 0 & (13) \\ 1 + \overline{a}c + \overline{e}g + hk = 0 & (14) \end{cases}$$

Now we are going to study these cases separately. Let us have a look at equation (2). Again it is satisfied if and only if one of the following three equations are satisfied: $e = -\overline{a}, g = -f$or $f = -\overline{a}, g = -e$ or $g = -\overline{a}, f = -e$.

Case 1.1

$$b = -a, d = -c$$
$$e = -\overline{a}, g = -f$$

12 equations remain.

$$\begin{cases} -\overline{a} - a + h + j = 0 & (3) \\ \overline{c} + \overline{f} + \overline{h} + k = 0 & (4) \\ -\overline{c} - \overline{f} + \overline{j} + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 + a^2 + c\overline{f} + c\overline{f} = 0 & (6) \\ \underline{1 - |a|^2} + c\overline{h} - c\overline{j} = 0 & (7) \\ 1 + af - ah - c\overline{k} = 0 & (8) \\ 1 - af - aj + ck = 0 & (9) \end{cases}$$

$$\begin{cases} f\overline{h} - f\overline{j} = 0 & (10) \\ 1 + \overline{a}c - \overline{a}h - f\overline{k} = 0 & (11) \\ 1 - \overline{a}c - \overline{a}j + fk = 0 & (12) \end{cases} \qquad \begin{cases} 1 - \overline{a}c - af + j\overline{k} = 0 & (13) \\ 1 + \overline{a}c + af + hk = 0 & (14) \end{cases}$$

From equation (10) we see that $h = j$. The equation (3) implies $h \in \mathbb{R}$. Remembering that $|h| = 1$ we conclude that $h = \pm 1 (= j)$ which implies $a = \pm 1$. Summing equations (4) and (5) we see that $k = \mp 1$. So the system reduces to:

$$\begin{cases} c + f = 0 \\ 1 + c\overline{f} = 0 \end{cases}$$

implying that $f = -c$. Finally we get two one-parameter families:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \mp 1 & c & -c \\ 1 & \pm 1 & 0 & \mp 1 & -c & c \\ 1 & \mp 1 & \mp 1 & 0 & \pm 1 & \pm 1 \\ 1 & \overline{c} & -\overline{c} & \pm 1 & 0 & \mp 1 \\ 1 & -\overline{c} & \overline{c} & \pm 1 & \mp 1 & 0 \end{pmatrix}$$

<u>Case 1.2</u>

Let us make another choice:

$$b = -a, d = -c$$

$$f = -\overline{a}, g = -e$$

The system (6.2) of 13 equations reduces to 12 equations:

$$\begin{cases} -\overline{a} + \overline{e} + h + j = 0 & (3) \\ \overline{c} - a + \overline{h} + k = 0 & (4) \\ -\overline{c} - \overline{e} + \overline{j} + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 - a\overline{e} - ca + c\overline{a} = 0 & (6) \\ 1 + ae + c\overline{h} - c\overline{j} = 0 & (7) \\ \underline{1 - |a|^2} - ah - c\overline{k} = 0 & (8) \\ 1 - ae - aj + ck = 0 & (9) \end{cases}$$

(6.2)

$$\begin{cases} -\overline{a}\overline{h} - e\overline{j} = 0 & (10) \\ 1 + \overline{a}c + eh - e\overline{k} = 0 & (11) \\ 1 - \overline{a}c + ej - \overline{a}k = 0 & (12) \end{cases} \qquad \begin{cases} 1 - \overline{a}c - \overline{e}\overline{a} + j\overline{k} = 0 & (13) \\ \underline{1} + \overline{a}c - \underline{|e|^2} + hk = 0 & (14) \end{cases}$$

Equation (14) can be deleted because it can be obtained by multiplying equation (8) by the nonzero factor $\overline{a}k$. Looking at equation (2), we can make the following choices.

The first possible choice is:

<u>Case 1.2.1</u>

$$b = -a, d = -c$$

$$f = -\overline{a}, g = -e$$

$$e = a, j = -h$$

10 equations remaining:

$$\begin{cases} \overline{c} - a + \overline{h} + k = 0 & (4) \\ -\overline{c} - \overline{a} - \overline{h} - \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} \underline{1 - |a|^2} - ca + c\overline{a} = 0 & (6) \\ 1 + a^2 + c\overline{h} + c\overline{h} = 0 & (7) \\ -ah - c\overline{k} = 0 & (8) \\ 1 - a^2 + ah + ck = 0 & (9) \end{cases}$$

$$\begin{cases} \overline{a}\overline{h} - a\overline{h} = 0 & (10) \\ 1 + \overline{a}c + ah - a\overline{k} = 0 & (11) \\ 1 - \overline{a}c - ah - \overline{a}k = 0 & (12) \end{cases} \qquad \begin{cases} \underline{1} - \overline{a}c - \underline{|a|^2} - h\overline{k} = 0 & (13) \end{cases}$$

Equation (10) implies $a \in \mathbb{R}$ and therefore $a = \pm 1$. Using the value of a in equation (7), we get $c\overline{h} = -1$ implies $h = -c$. Put $h = -c$ in equation (8), we obtain $k = \pm 1$. It leads to the following two families of matrices:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \mp 1 & c & -c \\ 1 & \pm 1 & 0 & \pm 1 & \mp 1 & \mp 1 \\ 1 & \mp 1 & \pm 1 & 0 & -c & c \\ 1 & \overline{c} & \mp 1 & -\overline{c} & 0 & \pm 1 \\ 1 & -\overline{c} & \mp 1 & \overline{c} & \pm 1 & 0 \end{pmatrix}$$

Let us do another choice:

Case 1.2.2

$$b = -a, d = -c$$
$$f = -\overline{a}, g = -e$$
$$h = \overline{a}, j = -\overline{e}$$

leads to the system (6.3) to the system of 10 equations.

$$\begin{cases} \overline{c} - a + a + k = 0 & (4) \\ -\overline{c} - \overline{e} - e + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 - a\overline{e} - ca + c\overline{e} = 0 & (6) \\ 1 + ae + ca + ce = 0 & (7) \\ -|a|^2 - c\overline{k} = 0 & (8) \\ 1 - ae + a\overline{e} + ck = 0 & (9) \end{cases}$$

$$\begin{cases} -|a|^2 + e^2 = 0 & (10) \\ 1 + \overline{a}c + e\overline{a} - e\overline{k} = 0 & (11) \\ 1 - \overline{a}c - \underline{|e|^2} - \overline{a}k = 0 & (12) \end{cases} \qquad \begin{cases} 1 - \overline{a}c - \overline{ea} - \overline{e}\overline{k} = 0 & (13) \end{cases}$$

From equation (4) we see that $k = -\overline{c}$. Using $k = -\overline{c}$ in equation (8), we get $c^2 = 1$. It follows that $c = \pm 1$, which implies $k = \mp 1$. Put $c = \pm 1$ and $k = \mp 1$ in equation (11), we get $e = \mp 1$. We get the following two one-parameter families

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & a & -a & \pm 1 & \mp 1 \\ 1 & \overline{a} & 0 & \mp 1 & -\overline{a} & \pm 1 \\ 1 & -\overline{a} & \mp 1 & 0 & \overline{a} & \pm 1 \\ 1 & \pm 1 & -a & a & 0 & \mp 1 \\ 1 & \mp 1 & \pm 1 & \pm 1 & \mp 1 & 0 \end{pmatrix}$$

The third and last possible choice is:

Case 1.2.3

$$b = -a, d = -c$$

$$f=-\overline{a}, g=-e$$
$$j=\overline{a}, h=-\overline{e}$$

system (6.3) reduces to:

$$\begin{cases}\overline{c}-a-e+k=0 & (4)\\ -\overline{c}-\overline{e}+a+\overline{k}=0 & (5)\end{cases} \qquad \begin{cases}1-a\overline{e}-ca+c\overline{e}=0 & (6)\\ 1+ae-ce-ca=0 & (7)\\ a\overline{e}-c\overline{k}=0 & (8)\\ \underline{1}-ae-\underline{|a|^2}+ck=0 & (9)\end{cases}$$

(6.3)

$$\begin{cases}\overline{a}e-ea=0 & (10)\\ \underline{1}+\overline{a}c-\underline{|e|^2}-e\overline{k}=0 & (11)\\ 1-\overline{a}c+e\overline{a}-\overline{a}k=0 & (12)\end{cases} \qquad \begin{cases}1-\overline{a}c-\overline{ea}+\overline{a}\overline{k}=0 & (13)\end{cases}$$

Equation (10) implies $a\in\mathbb{R}$, so, $a=\pm1$. Summing up equations (4) and (5), we get $Re(k)=Re(e)\Rightarrow k=e$ or $k=\overline{e}$.

Case 1.2.3.1

$$k=e$$

Substituting $k=e$ in the above system of equations, we get

$$\begin{cases}\overline{c}-a-e+e=0 & (4)\\ -\overline{c}-\overline{e}+a+\overline{e}=0 & (5)\end{cases} \qquad \begin{cases}1-a\overline{e}-ca+c\overline{e}=0 & (6)\\ 1+ae-ce-ca=0 & (7)\\ a\overline{e}-c\overline{e}=0 & (8)\\ ae+ce=0 & (9)\end{cases}$$

$$\begin{cases}\overline{a}e-ea=0 & (10)\\ \overline{a}c-\underline{|e|^2}=0 & (11)\\ 1-\overline{a}c+e\overline{a}-\overline{a}e=0 & (12)\end{cases} \qquad \begin{cases}1-\overline{a}c-\overline{ea}+\overline{ae}=0 & (13)\end{cases}$$

From equation (8), we get $c=a=\pm1$. Hence, we get

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1\\ 1 & 0 & \pm1 & \mp1 & \pm1 & \mp1\\ 1 & \pm1 & 0 & e & \mp1 & -e\\ 1 & \mp1 & \overline{e} & 0 & -\overline{e} & \pm1\\ 1 & \pm1 & \mp1 & -e & 0 & e\\ 1 & \mp1 & -\overline{e} & \pm1 & \overline{e} & 0\end{pmatrix}$$

Case 1.2.3.2

$$k=\overline{e}$$

Substituting $k=\overline{e}$ into the system (6.4), we get

$$\begin{cases}\overline{c}-a-e+\overline{e}=0 & (4)\\ -\overline{c}-\overline{e}+a+e=0 & (5)\end{cases} \qquad \begin{cases}1-a\overline{e}-ca+c\overline{e}=0 & (6)\\ 1+ae-ce-ca=0 & (7)\\ a\overline{e}-ce=0 & (8)\\ -ae+c\overline{e}=0 & (9)\end{cases}$$

(6.4)

$$\begin{cases} \overline{a}e - ea = 0 & (10) \\ \overline{a}c - e^2 = 0 & (11) \\ 1 - \overline{a}c + e\overline{a} - \overline{ae} = 0 & (12) \end{cases} \qquad \begin{cases} 1 - \overline{a}c - \overline{ea} + \overline{a}e = 0 & (13) \end{cases}$$

Now put $a = 1$ in the above system of equations, we get

$$\begin{cases} \overline{c} - 1 - e + \overline{e} = 0 & (4) \\ -\overline{c} - \overline{e} + 1 + e = 0 & (5) \end{cases} \qquad \begin{cases} 1 - \overline{e} - c + c\overline{e} = 0 & (6) \\ 1 + e - ce - c = 0 & (7) \\ \overline{e} - ce = 0 & (8) \\ -e + c\overline{e} = 0 & (9) \end{cases}$$

$$\begin{cases} e - e = 0 & (10) \\ c - e^2 = 0 & (11) \\ 1 - c + e - \overline{e} = 0 & (12) \end{cases} \qquad \begin{cases} 1 - c - \overline{e} + e = 0 & (13) \end{cases}$$

Subtracting equation (7) from equation (6), we get $c = 1 \Rightarrow e \in \mathbb{R} \Rightarrow e = \pm 1$.
By putting $a = -1$ in (6.5), we get

$$\begin{cases} \overline{c} + 1 - e + \overline{e} = 0 & (4) \\ -\overline{c} - \overline{e} - 1 + e = 0 & (5) \end{cases} \qquad \begin{cases} 1 + \overline{e} + c + c\overline{e} = 0 & (6) \\ 1 - e - ce + c = 0 & (7) \\ -\overline{e} - ce = 0 & (8) \\ e + c\overline{e} = 0 & (9) \end{cases}$$

$$\begin{cases} -e + e = 0 & (10) \\ -c - e^2 = 0 & (11) \\ 1 + c - e\overline{a} + \overline{e} = 0 & (12) \end{cases} \qquad \begin{cases} 1 + c + \overline{e} - e = 0 & (13) \end{cases}$$

Subtracting equation (7) from equation (6), we get $c = -1 \Rightarrow e \in \mathbb{R} \Rightarrow e = \pm 1$. Summing up we get the two families:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \mp 1 & \pm 1 & \mp 1 \\ 1 & \pm 1 & 0 & \pm 1 & \mp 1 & \mp 1 \\ 1 & \mp 1 & \pm 1 & 0 & \mp 1 & \pm 1 \\ 1 & \pm 1 & \mp 1 & \mp 1 & 0 & \pm 1 \\ 1 & \mp 1 & \mp 1 & \pm 1 & \pm 1 & 0 \end{pmatrix}$$

Now we consider **<u>Case1.3</u>**

$$b = -a, d = -c$$

$$g = -\overline{a}, f = -e$$

The system (6.2) of 13 equations reduces to the system of 12 equations:

$$\begin{cases} -\overline{a} + \overline{e} + h + j = 0 & (3) \\ \overline{c} - \overline{e} + \overline{h} + k = 0 & (4) \\ -\overline{c} - a + \overline{j} + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 - a\overline{e} - c\overline{e} + ca = 0 & (6) \\ 1 + ae + c\overline{h} - c\overline{j} = 0 & (7) \\ 1 - ae - ah - c\overline{k} = 0 & (8) \\ \underline{1 - |a|^2} - aj + ck = 0 & (9) \end{cases}$$

(6.5)

$$\begin{cases} e\overline{h} + \overline{a}\overline{j} = 0 & (10) \\ 1 + \overline{a}c + eh - \overline{a}\overline{k} = 0 & (11) \\ 1 - \overline{a}c + ej - ek = 0 & (12) \end{cases} \qquad \begin{cases} 1 - \overline{a}c - |e|^2 + j\overline{k} = 0 & (13) \\ 1 + \overline{a}c - \overline{ea} + hk = 0 & (14) \end{cases}$$

The equatoin (13) $-\overline{a}c + j\overline{k} = 0$ can be deleted because it can be obtaind by multiplying equation (9) by nonzero factor $\overline{a}\overline{k}$. We see that equation (3) is satisfied if and if one of the following three equalities holds: $e = a, j = -h$ or $h = \overline{a}, j = -\overline{e}$ or $j = \overline{a}, h = -\overline{e}$. Looking at the first equality:

Case 1.3.1

$$b = -a, d = -c$$

$$g = -\overline{a}, f = -e$$

$$e = a, j = -h$$

leading to the system of 10 equations:

$$\begin{cases} \overline{c} - \overline{a} + \overline{h} + k = 0 & (4) \\ -\overline{c} - a - \overline{h} + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 - |a|^2 - c\overline{a} + ca = 0 & (6) \\ 1 + a^2 + c\overline{h} + c\overline{h} = 0 & (7) \\ 1 - a^2 - ah - c\overline{k} = 0 & (8) \\ ah + ck = 0 & (9) \end{cases}$$

$$\begin{cases} a\overline{h} - \overline{a}\overline{h} = 0 & (10) \\ 1 + \overline{a}c + ah - \overline{a}\overline{k} = 0 & (11) \\ 1 - \overline{a}c - eh - ek = 0 & (12) \end{cases} \qquad \begin{cases} 1 + \overline{a}c - |a|^2 + hk = 0 & (14) \end{cases}$$

From equation (10) it follows that $a \in \mathbb{R}$, therefore $a = \pm 1$. Using the value of a in equation (7), we get $h = -c$. Equation (5) implies $k = \pm 1$. We get the following two one-parameter families:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \mp 1 & c & -c \\ 1 & \pm 1 & 0 & \pm 1 & \mp 1 & \mp 1 \\ 1 & \mp 1 & \pm 1 & 0 & -c & c \\ 1 & \overline{c} & \mp 1 & -\overline{c} & 0 & \pm 1 \\ 1 & -\overline{c} & \mp 1 & \overline{c} & \pm 1 & 0 \end{pmatrix}$$

The second choice is:

Case1.3.2

$$b = -a, d = -c$$

$$g = -\overline{a}, f = -e$$

$$h = \overline{a}, j = -\overline{e}$$

system (6.6) leads to 10 equations:

$$\begin{cases} \overline{c} - \overline{e} + a + k = 0 & (4) \\ -\overline{c} - a - e + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 - a\overline{e} - c\overline{e} + ca = 0 & (6) \\ 1 + ae + ca + ce = 0 & (7) \\ 1 - ae - |a|^2 - c\overline{k} = 0 & (8) \\ a\overline{e} + ck = 0 & (9) \end{cases}$$

(6.6)

$$\begin{cases} ea - \overline{a}e = 0 & (10) \\ 1 + \overline{a}c + e\overline{a} - \overline{a}\overline{k} = 0 & (11) \\ \underline{1} - \overline{a}c - \underline{|e|^2} - ek = 0 & (12) \end{cases} \qquad \begin{cases} 1 + \overline{a}c - \overline{ea} + \overline{a}k = 0 & (14) \end{cases}$$

From equation (10) we see that $a \in \mathbb{R}$, therefore $a = \pm 1$. By adding equations (4) and (5), we get $Re(k) = Re(e) \Rightarrow k = e$ or $k = \overline{e}$.

<u>Case 1.3.2.1</u>

$$k = e$$

Subtitute $k = e$ in the above system of equations, we have

$$\begin{cases} \overline{c} - \overline{e} + a + e = 0 & (4) \\ -\overline{c} - a - e + \overline{e} = 0 & (5) \end{cases} \qquad \begin{cases} 1 - a\overline{e} - c\overline{e} + ca = 0 & (6) \\ 1 + ae + ca + ce = 0 & (7) \\ -ae - c\overline{e} = 0 & (8) \\ a\overline{e} + ce = 0 & (9) \end{cases}$$

(6.7)

$$\begin{cases} ea - \overline{a}e = 0 & (10) \\ 1 + \overline{a}c + e\overline{a} - \overline{ae} = 0 & (11) \\ -\overline{a}c - e^2 = 0 & (12) \end{cases} \qquad \begin{cases} 1 + \overline{a}c - \overline{ea} + \overline{a}e = 0 & (14) \end{cases}$$

Put $a = 1$ in the above system, we get

$$\begin{cases} \overline{c} - \overline{e} + 1 + e = 0 & (4) \\ -\overline{c} - 1 - e + \overline{e} = 0 & (5) \end{cases} \qquad \begin{cases} 1 - \overline{e} - c\overline{e} + c = 0 & (6) \\ 1 + e + c + ce = 0 & (7) \\ -e - c\overline{e} = 0 & (8) \\ \overline{e} + ce = 0 & (9) \end{cases}$$

$$\begin{cases} e - e = 0 & (10) \\ 1 + c + e - \overline{e} = 0 & (11) \\ -c - e^2 = 0 & (12) \end{cases} \qquad \begin{cases} 1 + c - \overline{e} + e = 0 & (14) \end{cases}$$

Subtracting equation (7) from equation (6), we get $c = -1$. Using the value of c in equation (4), we get $e = \pm 1$. Now substitute $a = -1$ in (6.8), we get the following system of equations

$$\begin{cases} \overline{c} - \overline{e} - 1 + e = 0 & (4) \\ -\overline{c} + 1 - e + \overline{e} = 0 & (5) \end{cases} \qquad \begin{cases} 1 + \overline{e} - c\overline{e} - c = 0 & (6) \\ 1 - e - c + ce = 0 & (7) \\ e - c\overline{e} = 0 & (8) \\ -\overline{e} + ce = 0 & (9) \end{cases}$$

$$\begin{cases} -e + e = 0 & (10) \\ 1 - c - e + \overline{e} = 0 & (11) \\ c - e^2 = 0 & (12) \end{cases} \qquad \begin{cases} 1 - c + \overline{e} - e = 0 & (14) \end{cases}$$

Subtracting equation (7) from equation (6), we get $c = 1$. Equation (4) implies $e = \pm 1$. Summing up, we have if $a = \pm 1 \Rightarrow c = \mp 1$. We get the following two one-parameter families

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm1 & \mp1 & \mp1 & \pm1 \\ 1 & \pm1 & 0 & \pm1 & \mp1 & \mp1 \\ 1 & \mp1 & \pm1 & 0 & \pm1 & \mp1 \\ 1 & \mp1 & \mp1 & \pm1 & 0 & \pm1 \\ 1 & \pm1 & \mp1 & \mp1 & \pm1 & 0 \end{pmatrix}$$

<u>Case 1.3.2.2</u>

$$k = \overline{e}$$

Subtitute $k = \overline{e}$ into the system (6.7), we have

$$\begin{cases} \overline{c} - \overline{e} + a + \overline{e} = 0 & (4) \\ -\overline{c} - a - e + e = 0 & (5) \end{cases} \qquad \begin{cases} 1 - a\overline{e} - c\overline{e} + ca = 0 & (6) \\ 1 + ae + ca + ce = 0 & (7) \\ \underline{1} - ae - \underline{|a|^2} - ce = 0 & (8) \\ a\overline{e} + c\overline{e} = 0 & (9) \end{cases}$$

$$\begin{cases} ea - \overline{a}e = 0 & (10) \\ 1 + \overline{a}c + e\overline{a} - \overline{a}e = 0 & (11) \\ \underline{1} - \overline{a}c - \underline{|e|^2} - \underline{|e|^2} = 0 & (12) \end{cases} \qquad \begin{cases} 1 + \overline{a}c - \overline{ea} + \overline{ae} = 0 & (14) \end{cases}$$

Put $a = 1$ in the above system, we get from equation (4) $c = -1$. Put $a = -1$ into the system, we get from equation (4) $c = 1$. Hence if $a = \pm1 \Rightarrow c = \mp1$. We get the following two one-parameter families

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm1 & \mp1 & \mp1 & \pm1 \\ 1 & \pm1 & 0 & e & -e & \mp1 \\ 1 & \mp1 & \overline{e} & 0 & \pm1 & -\overline{e} \\ 1 & \mp1 & -\overline{e} & \pm1 & 0 & \overline{e} \\ 1 & \pm1 & \mp1 & -e & e & 0 \end{pmatrix}$$

The third possible choice is

<u>Case 1.3.3</u>

$$b = -a, d = -c$$

$$g = -\overline{a}, f = -e$$

$$j = \overline{a}, h = -\overline{e}$$

We have the following system (6.5) of equations:

$$\begin{cases} \overline{c} - \overline{e} - e + k = 0 & (4) \\ -\overline{c} - a + a + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 - a\overline{e} - c\overline{e} + ca = 0 & (6) \\ 1 + ae - ce - ca = 0 & (7) \\ 1 - ae + a\overline{e} - c\overline{k} = 0 & (8) \\ -|a|^2 + ck = 0 & (9) \end{cases}$$

$$\begin{cases} -e^2 + |a|^2 = 0 & (10) \\ \underline{1} + \overline{a}c - \underline{|e|^2} - \overline{a}\overline{k} = 0 & (11) \\ 1 - \overline{a}c + e\overline{a} - ek = 0 & (12) \end{cases} \qquad \begin{cases} 1 + \overline{a}c - \overline{ea} - \overline{e}k = 0 & (14) \end{cases}$$

Equation (5) implies $c = k$. Using $k = c$ in equation (9), we get $c = \pm 1$. From equation (10) it follows that $e = \pm 1$ We have, the two one-parameter families

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & a & -a & \pm 1 & \mp 1 \\ 1 & \overline{a} & 0 & \pm 1 & \mp 1 & -\overline{a} \\ 1 & -\overline{a} & \pm 1 & 0 & \mp 1 & \overline{a} \\ 1 & \pm 1 & \mp 1 & \mp 1 & 0 & \pm 1 \\ 1 & \mp 1 & -a & a & \pm 1 & 0 \end{pmatrix}$$

Now we shall consider:

Case 2.0

$$c = -a, d = -b$$

In system (6.1) the 15 orthogonilaty conditions reduces to 14 equations:

$$\begin{cases} \overline{a} + e + f + g = 0 & (2) \\ \overline{b} + \overline{e} + h + j = 0 & (3) \\ -\overline{a} + \overline{f} + \overline{h} + k = 0 & (4) \\ -\overline{b} + \overline{g} + \overline{j} + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 + b\overline{e} - a\overline{f} - b\overline{g} = 0 & (6) \\ 1 + ae - a\overline{h} - b\overline{j} = 0 & (7) \\ 1 + af + bh - b\overline{k} = 0 & (8) \\ 1 + ag + bj - ak = 0 & (9) \end{cases}$$

(6.8)

$$\begin{cases} 1 + \overline{a}b + f\overline{h} + g\overline{j} = 0 & (10) \\ \underline{1 - |a|^2} + eh + g\overline{k} = 0 & (11) \\ 1 - \overline{a}b + ej + fk = 0 & (12) \end{cases} \qquad \begin{cases} 1 - \overline{b}a + \overline{e}f + j\overline{k} = 0 & (13) \\ \underline{1 - |b|^2} + \overline{e}g + hk = 0 & (14) \end{cases}$$

$$\begin{cases} 1 + \overline{a}b + \overline{f}g + \overline{h}j = 0 & (15) \end{cases}$$

Equation (14) can be deleted, because it can be obtained from equation (11) by multiplying non-zero factor $\overline{e}k$. 13 equations left. Let us study the first case from equation (2).

Case2.1

$$c = -a, d = -b$$

$$e = -\overline{a}, g = -f$$

The system reduces to 12 equations:

$$\begin{cases} \overline{b} - a + h + j = 0 & (3) \\ -\overline{a} + \overline{f} + \overline{h} + k = 0 & (4) \\ -\overline{b} - \overline{f} + \overline{j} + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 - ba - a\overline{f} + b\overline{f} = 0 & (6) \\ \underline{1 - |a|^2} - a\overline{h} - b\overline{j} = 0 & (7) \\ 1 + af + bh - b\overline{k} = 0 & (8) \\ 1 - af + bj - ak = 0 & (9) \end{cases}$$

(6.9)

$$\begin{cases} 1 + \overline{a}b + f\overline{h} - f\overline{j} = 0 & (10) \\ -\overline{a}h - f\overline{k} = 0 & (11) \\ 1 - \overline{a}b - \overline{a}j + fk = 0 & (12) \end{cases} \qquad \begin{cases} 1 - \overline{b}a - af + j\overline{k} = 0 & (13) \end{cases}$$

$$\begin{cases} \underline{1} + \overline{a}b - \underline{|f|^2} + \overline{h}j = 0 & (15) \end{cases}$$

Equation (15) can be deleted, since it can be obtained from equation (7) by multiplying nonzero factor by $\overline{b}h$. Let us consider Equation (3). It is satisfied if and only if one of the following three equalities holds: $a = \overline{b}, j = -h$ or $h = -\overline{b}, j = a$ or $j = -\overline{b}, h = a$. Consider the first equality:

Case2.1.1

$$c = -a, d = -b$$
$$e = -\overline{a}, g = -f$$
$$a = \overline{b}, j = -h$$

leading to the system of 10 equations:

$$\begin{cases} -b + \overline{f} + \overline{h} + k = 0 & (4) \\ -\overline{b} - \overline{f} - \overline{h} + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} \underline{1 - |b|^2} - \overline{bf} + b\overline{f} = 0 & (6) \\ -\overline{bh} + b\overline{h} = 0 & (7) \\ 1 + \overline{b}f + bh - b\overline{k} = 0 & (8) \\ 1 - \overline{b}f - bh - \overline{b}k = 0 & (9) \end{cases}$$

$$\begin{cases} 1 + |b|^2 + f\overline{h} + f\overline{h} = 0 & (10) \\ -bh - f\overline{k} = 0 & (11) \\ 1 - b^2 - bj + fk = 0 & (12) \end{cases} \qquad \begin{cases} \underline{1 - |b|^2} - \overline{b}f - h\overline{k} = 0 & (13) \end{cases}$$

From equation (6), we get $b = \pm 1$. From equation (10) it follows that $h = -f$. Put $b = \pm 1$ and $h = -f$ in equation (9), we get $k = \pm 1$. We get two one-parameter of families:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \pm 1 & \mp 1 & \mp 1 \\ 1 & \pm 1 & 0 & \mp 1 & f & -f \\ 1 & \pm 1 & \mp 1 & 0 & -f & f \\ 1 & \mp 1 & \overline{f} & -\overline{f} & 0 & \pm 1 \\ 1 & \mp 1 & -\overline{f} & \overline{f} & \pm 1 & 0 \end{pmatrix}$$

Now consider the second equality:

Case2.1.2

$$c = -a, d = -b$$
$$e = -\overline{a}, g = -f$$
$$h = -\overline{b}, j = a$$

(6.10) leading to the system of 10 equations:

$$\begin{cases} -\overline{a} + \overline{f} - b + k = 0 & (4) \\ -\overline{b} - \overline{f} + \overline{a} + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 - ba - a\overline{f} + b\overline{f} = 0 & (6) \\ ab - b\overline{a} = 0 & (7) \\ \underline{1} + af - \underline{|b|^2} - b\overline{k} = 0 & (8) \\ 1 - af + ba - ak = 0 & (9) \end{cases}$$

(6.10)

$$\begin{cases} 1 + \overline{a}b - fb - f\overline{a} = 0 & (10) \\ \overline{a}\overline{b} - f\overline{k} = 0 & (11) \\ \underline{1} - \overline{a}b - \underline{|a|^2} + fk = 0 & (12) \end{cases} \qquad \begin{cases} 1 - \overline{b}a - af + \overline{k}a = 0 & (13) \end{cases}$$

From equation (7) it follows that $a \in \mathbb{R}$, therefore $a = \pm 1$. By adding equations (4) and (5), we get $Re(b) = Re(k) \Rightarrow k = b$ or $k = \overline{b}$.

Case 2.1.2.1

$$k = b$$

Put $k = b$ in the above system, we have

$$\begin{cases} -\overline{a} + \overline{f} - b + b = 0 & (4) \\ -\overline{b} - \overline{f} + \overline{a} + \overline{b} = 0 & (5) \end{cases} \qquad \begin{cases} 1 - ba - a\overline{f} + b\overline{f} = 0 & (6) \\ ab - b\overline{a} = 0 & (7) \\ \underline{1} + af - \underline{|b|^2} - \underline{|b|^2} = 0 & (8) \\ 1 - af + ba - ab = 0 & (9) \end{cases}$$

$$\begin{cases} 1 + \overline{a}b - fb - f\overline{a} = 0 & (10) \\ \overline{a}\overline{b} - f\overline{b} = 0 & (11) \\ \underline{1} - \overline{a}b - \underline{|a|^2} + fb = 0 & (12) \end{cases} \qquad \begin{cases} 1 - \overline{b}a - af + \overline{b}a = 0 & (13) \end{cases}$$

Put $a = 1$ in equation (4), we get $f = 1$. Put $a = -1$ in equation (4), we get $f = -1$. Summin up, we get if $a = \pm 1 \Rightarrow f = \pm 1$. We get two-one parameter family

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & b & \mp 1 & -b \\ 1 & \pm 1 & 0 & \mp 1 & \pm 1 & \mp 1 \\ 1 & \overline{b} & \mp 1 & 0 & -\overline{b} & \pm 1 \\ 1 & \mp 1 & \pm 1 & -b & 0 & b \\ 1 & -\overline{b} & \mp 1 & \pm 1 & \overline{b} & 0 \end{pmatrix}$$

Case 2.1.2.2

$$k = \overline{b}$$

Substitute $k = \overline{b}$ in the system (6.11) of equations, we get

$$\begin{cases} -\overline{a} + \overline{f} - b + \overline{b} = 0 & (4) \\ -\overline{b} - \overline{f} + \overline{a} + b = 0 & (5) \end{cases} \qquad \begin{cases} 1 - ba - a\overline{f} + b\overline{f} = 0 & (6) \\ ab - b\overline{a} = 0 & (7) \\ \underline{1} + af - \underline{|b|^2} - b^2 = 0 & (8) \\ 1 - af + ba - a\overline{b} = 0 & (9) \end{cases}$$

(6.11)

$$\begin{cases} 1 + \overline{a}b - fb - f\overline{a} = 0 & (10) \\ \overline{a}\overline{b} - fb = 0 & (11) \\ \underline{1} - \overline{a}b - \underline{|a|^2} + f\overline{b} = 0 & (12) \end{cases} \qquad \begin{cases} 1 - \overline{b}a - af + ab = 0 & (13) \end{cases}$$

Put $a = 1$ in the above system, we get

$$\begin{cases} -1 + \overline{f} - b + \overline{b} = 0 & (4) \\ -\overline{b} - \overline{f} + 1 + b = 0 & (5) \end{cases} \qquad \begin{cases} 1 - b - \overline{f} + b\overline{f} = 0 & (6) \\ b - b = 0 & (7) \\ f - b^2 = 0 & (8) \\ 1 - f + b - \overline{b} = 0 & (9) \end{cases}$$

$$\begin{cases} 1 + b - fb - f = 0 & (10) \\ \overline{b} - fb = 0 & (11) \\ -b + f\overline{b} = 0 & (12) \end{cases} \qquad \begin{cases} 1 - \overline{b} - f + b = 0 & (13) \end{cases}$$

Equation (6) implies $f = 1$. Then from equation (11), we get $b \in \mathbb{R} \Rightarrow b = \pm 1$.

Now put $a=-1$ in (6.12), we get

$$\begin{cases}1+\overline{f}-b+\overline{b}=0 & (4)\\ -\overline{b}-\overline{f}-1+b=0 & (5)\end{cases} \qquad \begin{cases}1+b+\overline{f}+b\overline{f}=0 & (6)\\ -b+b=0 & (7)\\ \underline{1}-f-\underline{|b|}^2-b^2=0 & (8)\\ 1+f+b+\overline{b}=0 & (9)\end{cases}$$

$$\begin{cases}1-b-fb+f=0 & (10)\\ -\overline{b}-fb=0 & (11)\\ b+f\overline{b}=0 & (12)\end{cases} \qquad \begin{cases}1+\overline{b}+f-b=0 & (13)\end{cases}$$

From equation (6), we get $f=-1$. Then from equation (11), we get $b\in\mathbb{R}\Rightarrow b=\pm1$. We get the two families:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1\\ 1 & 0 & \pm1 & \pm1 & \mp1 & \mp1\\ 1 & \pm1 & 0 & \mp1 & \pm1 & \mp1\\ 1 & \pm1 & \mp1 & 0 & \mp1 & \pm1\\ 1 & \mp1 & \pm1 & \mp1 & 0 & \pm1\\ 1 & \mp1 & \mp1 & \pm1 & \pm1 & 0 \end{pmatrix}$$

The third possible choice is:

<u>Case2.1.3</u>

$$c=-a, d=-b$$
$$e=-\overline{a}, g=-f$$
$$j=-\overline{b}, h=a$$

(6.10) leading to the system of 10 equations:

$$\begin{cases}-\overline{a}+\overline{f}+\overline{a}+k=0 & (4)\\ -\overline{b}-\overline{f}-b+\overline{k}=0 & (5)\end{cases} \qquad \begin{cases}1-ba-a\overline{f}+b\overline{f}=0 & (6)\\ -|a|^2+b^2=0 & (7)\\ 1+af+ba-b\overline{k}=0 & (8)\\ \underline{1}-af-\underline{|b|}^2-ak=0 & (9)\end{cases}$$

$$\begin{cases}1+\overline{a}b+f\overline{a}+fb=0 & (10)\\ -|a|^2-f\overline{k}=0 & (11)\\ 1-\overline{a}b+\overline{a}\overline{b}+fk=0 & (12)\end{cases} \qquad \begin{cases}1-\overline{b}a-af-\overline{bk}=0 & (13)\end{cases}$$

From equation (7) it follows that $b=\pm1$. Equation (4) implies that $\overline{f}=-k$. From equation (11), we get $f=-k$. Using in equation (4) implies that $k=\pm1\Rightarrow f=\mp1$. So, we get two one-parameter of families

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1\\ 1 & 0 & a & \pm1 & -a & \mp1\\ 1 & \overline{a} & 0 & -\overline{a} & \mp1 & \pm1\\ 1 & \pm1 & -a & 0 & a & \mp1\\ 1 & -\overline{a} & \mp1 & \overline{a} & 0 & \pm1\\ 1 & \mp1 & \pm1 & \mp1 & \pm1 & 0 \end{pmatrix}$$

Now we shall consider the second option

<u>Case2.2</u>

$$c=-a, d=-b$$
$$f=-\overline{a}, g=-e$$

So, the system (6.9) of 13 equations reduces to the system of 12 equations:

$$\begin{cases} \overline{b}+\overline{e}+h+j=0 & (3)\\ -\overline{a}-a+\overline{h}+k=0 & (4)\\ -\overline{b}-\overline{e}+\overline{j}+\overline{k}=0 & (5)\end{cases} \qquad \begin{cases} 1+b\overline{e}+a^2+b\overline{e}=0 & (6)\\ 1+ae-a\overline{h}-b\overline{j}=0 & (7)\\ \underline{1-|a|^2}+bh-b\overline{k}=0 & (8)\\ 1-ae+bj-ak=0 & (9)\end{cases}$$

$$\begin{cases} 1+\overline{a}b-\overline{a}\overline{h}-e\overline{j}=0 & (10)\\ eh-e\overline{k}=0 & (11)\\ 1-\overline{a}b+ej-\overline{a}k=0 & (12)\end{cases} \qquad \begin{cases} 1-\overline{b}a-\overline{a}\overline{e}+\overline{k}j=0 & (13)\end{cases}$$

$$\begin{cases} 1+\overline{a}b+ae+\overline{h}j=0 & (15)\end{cases}$$

From equation (11) it follows that $h=\overline{k}$. Using $h=\overline{k}$ in the above system, we get

$$\begin{cases} \overline{b}+\overline{e}+\overline{k}+j=0 & (3)\\ -\overline{a}-a+2k=0 & (4)\\ -\overline{b}-\overline{e}+\overline{j}+\overline{k}=0 & (5)\end{cases} \qquad \begin{cases} 1+b\overline{e}+a^2+b\overline{e}=0 & (6)\\ 1+ae-ak-b\overline{j}=0 & (7)\\ b\overline{k}-b\overline{k}=0 & (8)\\ 1-ae+bj-ak=0 & (9)\end{cases}$$

$$\begin{cases} 1+\overline{a}b-\overline{a}k-e\overline{j}=0 & (10)\\ e\overline{k}-e\overline{k}=0 & (11)\\ 1-\overline{a}b+ej-\overline{a}k=0 & (12)\end{cases} \qquad \begin{cases} 1-\overline{b}a-\overline{a}\overline{e}+\overline{k}j=0 & (13)\end{cases}$$

$$\begin{cases} 1+\overline{a}b+ae+kj=0 & (15)\end{cases}$$

From equation (4), we get $k=Re(a)\Rightarrow a=\pm1\Rightarrow k=\pm1\Rightarrow h=\pm1$. Equation (6) implies $e=-b$. Put in equation (3), we get $\overline{k}=-j\Rightarrow j=\mp1$. We have the following two one-parameter of families

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1\\ 1 & 0 & \pm1 & b & \mp1 & -b\\ 1 & \pm1 & 0 & -b & \mp1 & b\\ 1 & \overline{b} & -\overline{b} & 0 & \pm1 & \mp1\\ 1 & \mp1 & \mp1 & \pm1 & 0 & \pm1\\ 1 & -\overline{b} & \overline{b} & \mp1 & \pm1 & 0 \end{pmatrix}$$

Consider

<u>Case2.3</u>

$$c=-a, d=-b$$

$$g=-\overline{a}, f=-e$$

The system (6.9) of 13 equations reduces to the system of 12 equations:

$$\begin{cases} \overline{b}+\overline{e}+h+j=0 & (3)\\ -\overline{a}-\overline{e}+\overline{h}+k=0 & (4)\\ -\overline{b}-a+\overline{j}+\overline{k}=0 & (5)\end{cases} \qquad \begin{cases} 1+b\overline{e}+a\overline{e}+ba=0 & (6)\\ 1+ae-a\overline{h}-b\overline{j}=0 & (7)\\ 1-ae+bh-b\overline{k}=0 & (8)\\ \underline{1-|a|^2}+bj-ak=0 & (9)\end{cases}$$

(6.12)

$$\begin{cases} 1+\overline{a}b-e\overline{h}-\overline{a}\overline{j}=0 & (10)\\ eh-\overline{a}\overline{k}=0 & (11)\\ 1-\overline{a}b+ej-ek=0 & (12)\end{cases} \qquad \begin{cases}\underline{1}-\overline{b}a-\underline{|e|^2}+j\overline{k}=0 & (13)\end{cases}$$

$$\begin{cases}1+\overline{a}b+\overline{ea}+\overline{h}j=0 & (15)\end{cases}$$

The equation (13) $-\overline{b}a+j\overline{k}=0$ can be deleted, because it can be obtained from equation (9) $bj-ak=0$ by multiplying by the non-zero factor $\overline{bk}$. 11 equations remain. Now equation (3) is satisfied if and only if one of the following three equalities holds: $e=-b, j=-h$ or $h=-\overline{b}, j=-\overline{e}$ or $j=-\overline{b}, h=-\overline{e}$. From the first equality, we consider

Case2.3.1

$$c=-a, d=-b$$

$$g=-\overline{a}, f=-e$$

$$e=-b, j=-h$$

leading to the system of 10 equations:

$$\begin{cases}-\overline{a}+\overline{b}+\overline{h}+k=0 & (4)\\ -\overline{b}-a-\overline{h}+\overline{k}=0 & (5)\end{cases} \qquad \begin{cases}\underline{1-|b|^2}-a\overline{b}+ba=0 & (6)\\ 1-ab-a\overline{h}+b\overline{h}=0 & (7)\\ 1+ab+bh-b\overline{k}=0 & (8)\\ -bh-ak=0 & (9)\end{cases}$$

(6.13)

$$\begin{cases}1+\overline{a}b+b\overline{h}+\overline{a}\overline{h}=0 & (10)\\ -bh-\overline{a}\overline{k}=0 & (11)\\ 1-\overline{a}b+bh+bk=0 & (12)\end{cases} \qquad \begin{cases}\underline{1}+\overline{a}b-\overline{b}\overline{a}-\underline{|h|^2}=0 & (15)\end{cases}$$

From equation (15) it follows that $b\in\mathbb{R}$ therefore $b=\pm1$. By adding equations (4) and (5), $k=a$ or $k=\overline{a}$.

Case 2.3.1.1

$$k=a$$

Substitute $k=a$ and $b=1$ in the above system, we get

$$\begin{cases}-\overline{a}+1+\overline{h}+a=0 & (4)\\ -1-a-\overline{h}+\overline{a}=0 & (5)\end{cases} \qquad \begin{cases}-a+a=0 & (6)\\ 1-a-a\overline{h}+\overline{h}=0 & (7)\\ 1+a+h-\overline{a}=0 & (8)\\ -h-a^2=0 & (9)\end{cases}$$

(6.14)

$$\begin{cases}1+\overline{a}+\overline{h}+\overline{a}\overline{h}=0 & (10)\\ -h-\overline{a}\overline{a}=0 & (11)\\ 1-\overline{a}+h+a=0 & (12)\end{cases} \qquad \begin{cases}\overline{a}-\overline{a}=0 & (15)\end{cases}$$

Equation (7) implies $h=-1$. Equation (4) implies $a=\pm 1$. Now substitute $k=a$ and $b=-1$, we get

$$\begin{cases} -\overline{a}-1+\overline{h}+a=0 & (4) \\ 1-a-\overline{h}+\overline{a}=0 & (5) \end{cases} \qquad \begin{cases} +a-a=0 & (6) \\ 1+a-a\overline{h}-\overline{h}=0 & (7) \\ 1-a-h+\overline{a}=0 & (8) \\ h-a^2=0 & (9) \end{cases}$$

$$\begin{cases} 1+\overline{a}-\overline{h}-\overline{a}\overline{h}=0 & (10) \\ h-\overline{a}\overline{a}=0 & (11) \\ 1\overline{a}-h-a=0 & (12) \end{cases} \qquad \begin{cases} -\overline{a}+\overline{a}=0 & (15) \end{cases}$$

Equation (7) implies $h=1$. Summing up, we get $b=\pm 1 \Rightarrow h=\mp 1$. We can notice $a=\pm 1$. We get two families

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \pm 1 & \mp 1 & \mp 1 \\ 1 & \pm 1 & 0 & \mp 1 & \pm 1 & \mp 1 \\ 1 & \pm 1 & \mp 1 & 0 & \pm 1 & \mp 1 \\ 1 & \mp 1 & \pm 1 & \pm 1 & 0 & \pm 1 \\ 1 & \mp 1 & \mp 1 & \mp 1 & \pm 1 & 0 \end{pmatrix}$$

Case 2.3.1.2

$$k=\overline{a}$$

Substitute $k=\overline{a}$ and $b=1$ in (6.14), we get

$$\begin{cases} -\overline{a}+1+\overline{h}+\overline{a}=0 & (4) \\ -1-a-\overline{h}+a=0 & (5) \end{cases} \qquad \begin{cases} -a+a=0 & (6) \\ 1-a-a\overline{h}+\overline{h}=0 & (7) \\ 1+a+h-a=0 & (8) \\ -h-|a|^2=0 & (9) \end{cases}$$

$$\begin{cases} 1+\overline{a}+\overline{h}+\overline{a}\overline{h}=0 & (10) \\ -h-|a|^2=0 & (11) \\ 1-\overline{a}+h+\overline{a}=0 & (12) \end{cases} \qquad \begin{cases} \overline{a}-\overline{a}=0 & (15) \end{cases}$$

From equation (4), we see that $h=-1$. Similarly for $b=-1$, we get $h=1$. Summing up, we have if $b=\pm 1 \Rightarrow h=\mp 1$. We get two-one paramete families

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & a & \pm 1 & -a & \mp 1 \\ 1 & \overline{a} & 0 & \mp 1 & \pm 1 & -\overline{a} \\ 1 & \pm 1 & \mp 1 & 0 & \mp 1 & \pm 1 \\ 1 & -\overline{a} & \pm 1 & \mp 1 & 0 & \overline{a} \\ 1 & \mp 1 & -a & \pm 1 & a & 0 \end{pmatrix}$$

Consider the second option

Case2.3.2

$$c=-a, d=-b$$

$$g=-\overline{a}, f=-e$$

$$h=-\overline{b}, j=-\overline{e}$$

From (6.13), we get the following system of 10 equations

$$\begin{cases} -\overline{a}-\overline{e}-b+k=0 & (4)\\ -\overline{b}-a-e+\overline{k}=0 & (5)\end{cases}\qquad \begin{cases} 1+b\overline{e}+a\overline{e}+ba=0 & (6)\\ 1+ae+ab+be=0 & (7)\\ \underline{1}-ae-\underline{|b|^2}-b\overline{k}=0 & (8)\\ -b\overline{e}-ak=0 & (9)\end{cases}$$

(6.15)

$$\begin{cases} 1+\overline{a}b+eb+\overline{a}e=0 & (10)\\ -e\overline{b}-\overline{a}\overline{k}=0 & (11)\\ \underline{1}-\overline{a}b-\underline{|e|^2}-ek=0 & (12)\end{cases}\qquad \begin{cases}1+\overline{a}b+\overline{ea}+b\overline{e}=0 & (15)\end{cases}$$

We can delete equation (5) because it is same as equation (4). Similarly we can delete equation (9) because it is same as equaion (11). 8 equations remain:

$$\begin{cases} -\overline{a}-\overline{e}-b+k=0 & (4)\end{cases}\qquad \begin{cases} 1+b\overline{e}+a\overline{e}+ba=0 & (6)\\ 1+ae+ab+be=0 & (7)\\ \underline{1}-ae-\underline{|b|^2}-b\overline{k}=0 & (8)\end{cases}$$

$$\begin{cases} 1+\overline{a}b+eb+\overline{a}e=0 & (10)\\ -e\overline{b}-\overline{a}\overline{k}=0 & (11)\\ \underline{1}-\overline{a}b-\underline{|e|^2}-ek=0 & (12)\end{cases}\qquad \begin{cases}1+\overline{a}b+\overline{ea}+b\overline{e}=0 & (15)\end{cases}$$

From equation (4), we shall consider the following three more cases:

Case2.3.2.1

$$e=-a, k=b$$

Case2.3.2.2

$$b=-\overline{a}, k=\overline{e}$$

Case2.3.2.3

$$e=-\overline{b}, k=\overline{a}$$

First consider the case:

Case 2.3.2.1

$$e=-a, k=b$$

We get the following system of 7 equations

$$\begin{cases} \underline{1}-b\overline{a}-\underline{|a|^2}+ba=0 & (6)\\ 1-a^2+ab-ba=0 & (7)\\ a^2-|b|^2=0 & (8)\end{cases}\qquad \begin{cases} \underline{1}+\overline{a}b-ab-\underline{|a|^2}=0 & (10)\\ a\overline{b}-\overline{a}\overline{b}=0 & (11)\\ -\overline{a}b+ab=0 & (12)\end{cases}$$

$$\begin{cases}1+\overline{a}b-\overline{aa}-b\overline{a}=0 & (15)\end{cases}$$

From equation (8), we get $a=\pm1$. We get two-one parameter families:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1\\ 1 & 0 & \pm1 & b & \mp1 & -b\\ 1 & \pm1 & 0 & \mp1 & \pm1 & \mp1\\ 1 & \overline{b} & \mp1 & 0 & -\overline{b} & \pm1\\ 1 & \mp1 & \pm1 & -b & 0 & b\\ 1 & -\overline{b} & \mp1 & \pm1 & \overline{b} & 0 \end{pmatrix}$$

Consider the case:

Case 2.3.2.2

$$b = -\overline{a}, k = \overline{e}$$

From (6.16), we have

$$\begin{cases} 1 - \overline{ae} - |a|^2 + a\overline{e} = 0 & (6) \\ 1 + ae - \overline{|a|^2} - \overline{a}e = 0 & (7) \\ -ae + \overline{a}e = 0 & (8) \end{cases} \qquad \begin{cases} 1 - \overline{aa} - \overline{a}e + \overline{a}e = 0 & (10) \\ ae - \overline{a}e = 0 & (11) \\ \overline{aa} - |e|^2 = 0 & (12) \end{cases}$$

$$\begin{cases} 1 - \overline{aa} + \overline{ae} - \overline{ae} = 0 & (15) \end{cases}$$

From equation (11), it follows that $a = \pm 1$. We get the following two-one parameter famililies

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \mp 1 & \mp 1 & \pm 1 \\ 1 & \pm 1 & 0 & e & -e & \mp 1 \\ 1 & \mp & \overline{e} & 0 & \pm 1 & -\overline{e} \\ 1 & \mp 1 & -\overline{e} & \pm 1 & 0 & \overline{e} \\ 1 & \pm 1 & \mp 1 & -e & e & 0 \end{pmatrix}$$

Consider the last case **Case2.3.2.3**

$$e = -\overline{b}, k = \overline{a}$$

(6.16) implying

$$\begin{cases} 1 - b^2 - ab + ab = 0 & (6) \\ 1 - a\overline{b} + ab - |b|^2 = 0 & (7) \\ a\overline{b} - ba = 0 & (8) \end{cases} \qquad \begin{cases} 1 + \overline{a}b - |b|^2 - \overline{a}b = 0 & (10) \\ -\overline{b}\overline{B} - |a|^2 = 0 & (11) \\ -\overline{a}b - \overline{a}e = 0 & (12) \end{cases}$$

$$\begin{cases} 1 + \overline{a}b - b\overline{a} - |b|^2 = 0 & (15) \end{cases}$$

From equation (6), it follows that $b = \pm 1$. We have two-one parameter faimilies

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & a & \pm 1 & -a & \mp 1 \\ 1 & \overline{a} & 0 & \mp 1 & \pm 1 & -\overline{a} \\ 1 & \pm 1 & \mp 1 & 0 & \mp 1 & \pm 1 \\ 1 & -\overline{a} & \pm 1 & \mp 1 & 0 & \overline{a} \\ 1 & \mp 1 & -a & \pm 1 & a & 0 \end{pmatrix}$$

The last and third possible choice is:

Case2.3.3

$$c = -a, d = -b$$

$$g = -\overline{a}, f = -e$$

$$j = -\overline{b}, h = -\overline{e}$$

From (6.13) we getting the following system of 10 equations:

$$\begin{cases} -\overline{a} - \overline{e} - e + k = 0 & (4) \\ -\overline{b} - a - b + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 + b\overline{e} + a\overline{e} + ba = 0 & (6) \\ 1 + ae + ae + b^2 = 0 & (7) \\ 1 - ae - b\overline{e} - b\overline{k} = 0 & (8) \\ -|b|^2 - ak = 0 & (9) \end{cases}$$

$$\begin{cases} 1+\overline{a}b+e^2+\overline{a}b=0 & (10) \\ -|e|^2-\overline{a}\overline{k}=0 & (11) \\ 1-\overline{a}b-e\overline{b}-ek=0 & (12) \end{cases} \qquad \begin{cases} 1+\overline{a}b+\overline{ea}+e\overline{b}=0 & (15) \end{cases}$$

From equation (11) it follows that $k=-\overline{a}$. Substitute $k=-\overline{a}$ in the above system of equations, we get

$$\begin{cases} -\overline{a}-\overline{e}-e-\overline{a}=0 & (4) \\ -\overline{b}-a-b-a=0 & (5) \end{cases} \qquad \begin{cases} 1+b\overline{e}+a\overline{e}+ba=0 & (6) \\ 1+ae+ae+b^2=0 & (7) \\ 1-ae-b\overline{e}+ba=0 & (8) \\ -|b|^2+|a|^2=0 & (9) \end{cases}$$

$$\begin{cases} 1+\overline{a}b+e^2+\overline{a}b=0 & (10) \\ -|e|^2+|a|^2=0 & (11) \\ 1-\overline{a}b-e\overline{b}-e\overline{a}=0 & (12) \end{cases} \qquad \begin{cases} 1+\overline{a}b+\overline{ea}+e\overline{b}=0 & (15) \end{cases}$$

From equation (5), we see that $Re(b)=-a \Rightarrow a=\pm 1 \Rightarrow b=\mp 1 \Rightarrow k=\mp 1$. Put $a=\pm 1$ and the value of b in equation (7), we get $e=\mp 1$. We get the following two families

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \mp 1 & \mp 1 & \pm 1 \\ 1 & \pm 1 & 0 & \mp 1 & \pm 1 & \mp 1 \\ 1 & \mp 1 & \mp 1 & 0 & \pm 1 & \pm 1 \\ 1 & \mp 1 & \pm 1 & \pm 1 & 0 & \mp 1 \\ 1 & \pm 1 & \mp 1 & \pm 1 & \mp 1 & 0 \end{pmatrix}$$

Now we shall take into accout

<u>Case3.0</u>

$$d=-a, c=-b$$

In (6.1) the 15 orthogonality conditions reduces to 14 equations:

$$\begin{cases} \overline{a}+e+f+g=0 & (2) \\ \overline{b}+\overline{e}+h+j=0 & (3) \\ -\overline{b}+\overline{f}+\overline{h}+k=0 & (4) \\ -\overline{a}+\overline{g}+\overline{j}+\overline{k}=0 & (5) \end{cases} \qquad \begin{cases} 1+b\overline{e}-b\overline{f}-a\overline{g}=0 & (6) \\ 1+ae-b\overline{h}-a\overline{j}=0 & (7) \\ 1+af+bh-a\overline{k}=0 & (8) \\ 1+ag+bj-bk=0 & (9) \end{cases}$$

(6.16)

$$\begin{cases} 1+\overline{a}b+f\overline{h}+g\overline{j}=0 & (10) \\ 1-\overline{a}b+eh+g\overline{k}=0 & (11) \\ \underline{1-|a|^2}+ej+fk=0 & (12) \end{cases} \qquad \begin{cases} \underline{1-|b|^2}+\overline{e}f+j\overline{k}=0 & (13) \\ 1-\overline{b}a+\overline{e}g+hk=0 & (14) \end{cases}$$

$$\begin{cases} 1+\overline{b}a+\overline{f}g+\overline{h}j=0 & (15) \end{cases}$$

Equation (13) $\overline{e}f+j\overline{k}=0$ can be deleted, since it can be obtained from equation (12) $ej+fk=0$, by multiplying nonzero factor $\overline{e}\overline{k}$. From the second equation, we can choose

<u>Case3.1</u>

$$d=-a, c=-b$$

$$e=-\overline{a}, g=-f$$

leading to the system of 12 equations:

$$\begin{cases} \overline{b} - a + h + j = 0 & (3) \\ -\overline{b} + \overline{f} + \overline{h} + k = 0 & (4) \\ -\overline{a} - \overline{f} + \overline{j} + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 - ba - b\overline{f} + a\overline{f} = 0 & (6) \\ \underline{1 - |a|^2} - b\overline{h} - a\overline{j} = 0 & (7) \\ 1 + af + bh - a\overline{k} = 0 & (8) \\ 1 - af + bj - bk = 0 & (9) \end{cases}$$

(6.17)

$$\begin{cases} 1 + \overline{a}b + f\overline{h} - f\overline{j} = 0 & (10) \\ 1 - \overline{a}b - \overline{a}h - f\overline{k} = 0 & (11) \\ -\overline{a}j + fk = 0 & (12) \end{cases} \qquad \begin{cases} 1 - \overline{b}a + af + hk = 0 & (14) \end{cases}$$

$$\begin{cases} \underline{1} + \overline{b}a - \underline{|f|^2} + \overline{h}j = 0 & (15) \end{cases}$$

Equation (15) can be deleted because it can be obtained from equation (7) by multiplying nonzero factor $\overline{b}j$. Equation (3) will satisfy if and only if one of the following three equalities holds: $a = -\overline{b}, j = -h$ or $h = -\overline{b}, j = a$ or $j = -\overline{b}, h = a$.

Consider the first equality:

Case3.1.1

$$d = -a, c = -b$$
$$e = -\overline{a}, g = -f$$
$$a = \overline{b}, j = -h$$

The system of equations reduces to 10 equations:

$$\begin{cases} -\overline{b} + \overline{f} + \overline{h} + k = 0 & (4) \\ -b - \overline{f} - \overline{h} + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 + |b|^2 - b\overline{f} + \overline{bf} = 0 & (6) \\ -b\overline{h} + \overline{bh} = 0 & (7) \\ 1 + \overline{b}f + bh + \overline{bk} = 0 & (8) \\ 1 - \overline{b}f - bh - bk = 0 & (9) \end{cases}$$

$$\begin{cases} 1 + b^2 + f\overline{h} + f\overline{h} = 0 & (10) \\ 1 - b^2 - bh - f\overline{k} = 0 & (11) \\ bh + fk = 0 & (12) \end{cases} \qquad \begin{cases} 1 - \overline{bb} + \overline{b}f + hk = 0 & (14) \end{cases}$$

From equation (7), we get $b = \in \mathbb{R} \Rightarrow b = \pm 1$. Using the value of b in equation (10), we get $h = -f$. Now put $b = \pm 1$ and $h = -f$ in equation (8), we get $k = \pm 1$. So, we get two one-parameter families

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \pm 1 & \mp 1 & \mp 1 \\ 1 & \pm 1 & 0 & \mp 1 & f & -f \\ 1 & \pm 1 & \mp 1 & 0 & -f & f \\ 1 & \mp 1 & \overline{f} & -\overline{f} & 0 & \pm 1 \\ 1 & \mp 1 & -\overline{f} & \overline{f} & \pm 1 & 0 \end{pmatrix}$$

Consider the second option

Case3.1.2

$$d = -a, c = -b$$
$$e = -\overline{a}, g = -f$$
$$h = -\overline{b}, j = a$$

The system (6.18) reduces to 10 equations:

$$\begin{cases} -\overline{b} + \overline{f} - b + k = 0 & (4) \\ -\overline{a} - \overline{f} + \overline{a} + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 - ba - b\overline{f} + a\overline{f} = 0 & (6) \\ b^2 - 1 = 0 & (7) \\ \underline{1} + af - \underline{|b|^2} - a\overline{k} = 0 & (8) \\ 1 - af + ba - bk = 0 & (9) \end{cases}$$

$$\begin{cases} 1 + \overline{a}b - fb - f\overline{a} = 0 & (10) \\ 1 - \overline{a}b + \overline{a}\overline{b} - f\overline{k} = 0 & (11) \\ -\overline{a}a + fk = 0 & (12) \end{cases} \qquad \begin{cases} 1 - \overline{b}a + af - \overline{b}k = 0 & (14) \end{cases}$$

From equation (7), we get $b = \pm 1$. From equation (5), we get $k = f$. Using $k = f$ in equation (12) implies $f = \pm 1$. We get two parameter families:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & a & \pm 1 & \mp 1 & -a \\ 1 & \overline{a} & 0 & -\overline{a} & \pm 1 & \mp 1 \\ 1 & \pm 1 & -a & 0 & \mp 1 & a \\ 1 & \mp 1 & \pm 1 & \mp 1 & 0 & \pm 1 \\ 1 & -\overline{a} & \mp 1 & \overline{a} & \pm 1 & 0 \end{pmatrix}$$

The last possible choice is

<u>Case3.1.3</u>

$$d = -a, c = -b$$
$$e = -\overline{a}, g = -f$$
$$j = -\overline{b}, h = a$$

(6.18) leading to the system 10 equations:

$$\begin{cases} -\overline{b} + \overline{f} + \overline{a} + k = 0 & (4) \\ -\overline{a} - \overline{f} - b + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 - ba - b\overline{f} + \overline{f} = 0 & (6) \\ -b\overline{a} + ab = 0 & (7) \\ 1 + af + ba - a\overline{k} = 0 & (8) \\ \underline{1} - af - \underline{|b|^2} - bk = 0 & (9) \end{cases}$$

(6.18)

$$\begin{cases} 1 + \overline{a}b + f\overline{a} + fb = 0 & (10) \\ \underline{1} - \overline{a}b - \underline{|a|^2} - f\overline{k} = 0 & (11) \\ \overline{a}\overline{b} + fk = 0 & (12) \end{cases} \qquad \begin{cases} 1 - \overline{b}a + af + ak = 0 & (14) \end{cases}$$

From equation (7) it follows that $a \in \mathbb{R}$, therefore $a = \pm 1$. Adding equations (4) and (5), we get $Re(b) = Re(k) \Rightarrow k = b$ or $k = \overline{b}$.

<u>Case 3.1.3.1</u>

$$k = b$$

First consider $a = 1$ and $k = b$, the above system of equations becomes:

$$\begin{cases} -\overline{b} + \overline{f} + 1 + b = 0 & (4) \\ -1 - \overline{f} - b + \overline{b} = 0 & (5) \end{cases} \qquad \begin{cases} 1 - b - b\overline{f} + \overline{f} = 0 & (6) \\ -b + b = 0 & (7) \\ 1 + f + b - \overline{b} = 0 & (8) \\ f - b^2 = 0 & (9) \end{cases}$$

$$\begin{cases} 1b + f + fb = 0 & (10) \\ -b - f\overline{b} = 0 & (11) \\ \overline{b} + fb = 0 & (12) \end{cases} \qquad \begin{cases} 1 - \overline{b} + f + b = 0 & (14) \end{cases}$$

Equation (10) implies $f = -1$. Similarly if we put $a = -1$ and $k = b$ in the system of equations, then again from equation (10), we get $f = 1$. Summing up if $a = \pm 1 \Rightarrow f = \mp 1$. We get the two families

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \pm 1 & \mp 1 & \mp 1 \\ 1 & \pm 1 & 0 & \mp 1 & \mp 1 & \pm 1 \\ 1 & \pm 1 & \mp 1 & 0 & \pm 1 & \mp 1 \\ 1 & \mp 1 & \mp 1 & \pm 1 & 0 & \pm 1 \\ 1 & \mp 1 & \pm 1 & \mp 1 & \pm 1 & 0 \end{pmatrix}$$

Case 3.1.3.2

$$k = \overline{b}$$

Now consider the case when $a = \pm 1$ and $k = \overline{b}$. Substitute $k = \overline{b}$ into the system (6.19), we get

$$\begin{cases} -\overline{b} + \overline{f} + \overline{a} + \overline{b} = 0 & (4) \\ -\overline{a} - \overline{f} - b + b = 0 & (5) \end{cases} \qquad \begin{cases} 1 - ba - b\overline{f} + \overline{f} = 0 & (6) \\ -b\overline{a} + ab = 0 & (7) \\ 1 + af + ba - ab = 0 & (8) \\ -af - |b|^2 = 0 & (9) \end{cases}$$

$$\begin{cases} 1 + \overline{a}b + f\overline{a} + fb = 0 & (10) \\ -\overline{a}b - fb = 0 & (11) \\ \overline{a}\overline{b} + f\overline{b} = 0 & (12) \end{cases} \qquad \begin{cases} 1 - \overline{b}a + af + a\overline{b} = 0 & (14) \end{cases}$$

From equation (4), we get $f = -a \Rightarrow f = \mp 1$. So, we get, two one-parameter of families

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & b & -b & \mp 1 \\ 1 & \pm 1 & 0 & \mp 1 & \mp 1 & \pm 1 \\ 1 & \overline{b} & \mp 1 & 0 & \pm 1 & -\overline{b} \\ 1 & -\overline{b} & \mp 1 & \pm 1 & 0 & \overline{b} \\ 1 & \mp 1 & \pm 1 & -b & b & 0 \end{pmatrix}$$

Consider

Case3.2

$$d = -a, c = -b$$

$$f = -\overline{a}, g = -e$$

The system (6.17) reduces 13 equaions reduces to the system of 12 equations:

$$\begin{cases} \overline{b} + \overline{e} + h + j = 0 & (3) \\ -\overline{b} - a + \overline{h} + k = 0 & (4) \\ -\overline{a} - \overline{e} + \overline{j} + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 + b\overline{e} + ba + a\overline{e} = 0 & (6) \\ 1 + ae - b\overline{h} - a\overline{j} = 0 & (7) \\ \underline{1 - |a|^2} + bh - a\overline{k} = 0 & (8) \\ 1 - ae + bj - bk = 0 & (9) \end{cases}$$

(6.19)

$$\begin{cases} 1+\overline{a}b-\overline{a}\overline{h}-e\overline{j}=0 & (10)\\ 1-\overline{a}b+eh-e\overline{k}=0 & (11)\\ ej-\overline{a}k=0 & (12)\end{cases} \qquad \begin{cases}\underline{1}-\overline{b}a-\underline{|e|^2}+hk=0 & (14)\end{cases}$$

$$\begin{cases}1+\overline{b}a+ae+\overline{h}j=0 & (15)\end{cases}$$

Equation (14) can be deleted, because it can be obtaind from equation (8) by multiplying non-zero factor $\overline{b}k$. From the third equation consider

Case3.2.1

$$\begin{aligned} d&=-a, c=-b\\ f&=-\overline{a}, g=-e\\ e&=-b, j=-h\end{aligned}$$

leading to the system of 10 equations:

$$\begin{cases}-\overline{b}-a+\overline{h}+k=0 & (4)\\ -\overline{a}+\overline{b}-\overline{h}+\overline{k}=0 & (5)\end{cases} \qquad \begin{cases}\underline{1-|a|^2}+ba+a\overline{e}=0 & (6)\\ 1-ab-b\overline{h}+a\overline{h}=0 & (7)\\ bh-a\overline{k}=0 & (8)\\ 1+ab-bh-bk=0 & (9)\end{cases}$$

(6.20)

$$\begin{cases} 1+\overline{a}b-\overline{a}\overline{h}-b\overline{h}=0 & (10)\\ 1-\overline{a}b-bh+b\overline{k}=0 & (11)\\ -bj-\overline{a}k=0 & (12)\end{cases} \qquad \begin{cases}\underline{1}+\overline{b}a-ab-\underline{|h|^2}=0 & (15)\end{cases}$$

From equation (15), we get $b\in\mathbb{R}\Rightarrow b=\pm1$. Summing equations (4) and (5), we get $Re(a)=Re(k)\Rightarrow k=a$ or $k=\overline{a}$.

Case 3.2.1.1

$$k=a$$

Consider $k=a$, and substitute in the above system of equaions, we get

$$\begin{cases}-\overline{b}-a+\overline{h}+a=0 & (4)\\ -\overline{a}+\overline{b}-\overline{h}+\overline{a}=0 & (5)\end{cases} \qquad \begin{cases}\underline{1-|a|^2}+ba+a\overline{e}=0 & (6)\\ 1-ab-b\overline{h}+a\overline{h}=0 & (7)\\ bh-a\overline{a}=0 & (8)\\ 1+ab-bh-ba=0 & (9)\end{cases}$$

$$\begin{cases} 1+\overline{a}b-\overline{a}\overline{h}-b\overline{h}=0 & (10)\\ 1-\overline{a}b-bh+b\overline{a}=0 & (11)\\ -bj-\overline{a}a=0 & (12)\end{cases} \qquad \begin{cases}\underline{1}+\overline{b}a-ab-\underline{|h|^2}=0 & (15)\end{cases}$$

From equation (4), we get $h=b\Rightarrow h=\pm1$.

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1\\ 1 & 0 & a & \pm1 & \mp1 & -a\\ 1 & \overline{a} & 0 & \pm1 & -\overline{a} & \mp1\\ 1 & \pm1 & \pm1 & 0 & \pm1 & \mp1\\ 1 & \mp1 & -a & \pm1 & 0 & a\\ 1 & -\overline{a} & \mp1 & \mp1 & \overline{a} & 0\end{pmatrix}$$

Case 3.2.1.2

$$k=\overline{a}$$

Substitute $k = \overline{a}$ into the system (6.21), we get

$$\begin{cases} -\overline{b} - a + \overline{h} + \overline{a} = 0 & (4) \\ -\overline{a} + \overline{b} - \overline{h} + a = 0 & (5) \end{cases} \qquad \begin{cases} ba + a\overline{e} = 0 & (6) \\ 1 - ab - b\overline{h} + a\overline{h} = 0 & (7) \\ bh - a^2 = 0 & (8) \\ 1 + ab - bh - b\overline{a} = 0 & (9) \end{cases}$$

(6.21)

$$\begin{cases} 1 + \overline{a}b - \overline{a}\overline{h} - b\overline{h} = 0 & (10) \\ 1 - \overline{a}b - bh + ba = 0 & (11) \\ -bj - \overline{a}\overline{a} = 0 & (12) \end{cases} \qquad \begin{cases} \underline{1} + \overline{b}a - ab - \underline{|h|}^2 = 0 & (15) \end{cases}$$

Put $b = 1$ in the above system of equations, we get

$$\begin{cases} -1 - a + \overline{h} + \overline{a} = 0 & (4) \\ -\overline{a} + 1 - \overline{h} + a = 0 & (5) \end{cases} \qquad \begin{cases} a + a\overline{e} = 0 & (6) \\ 1 - a - \overline{h} + a\overline{h} = 0 & (7) \\ h - a^2 = 0 & (8) \\ 1 + a - h - b\overline{a} = 0 & (9) \end{cases}$$

$$\begin{cases} 1 + \overline{a} - \overline{a}\overline{h} - \overline{h} = 0 & (10) \\ 1 - \overline{a} - h + a = 0 & (11) \\ -j - \overline{a}\overline{a} = 0 & (12) \end{cases} \qquad \{$$

From equation (6), we get $e = -1$. From equation (7), we get $h = 1$. Using the value of h in equation (4), we get $a = \pm 1$. Similarly put $b = -1$ in (6.22) we get $e = 1$, $h = -1$ and $a = \pm 1$. We get the following two families

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \pm 1 & \mp 1 & \mp 1 \\ 1 & \pm 1 & 0 & \mp 1 & \mp 1 & \pm 1 \\ 1 & \pm 1 & \mp 1 & 0 & \pm 1 & \mp 1 \\ 1 & \mp 1 & \mp 1 & \pm 1 & 0 & \pm 1 \\ 1 & \mp 1 & \pm 1 & \mp 1 & \pm 1 & 0 \end{pmatrix}$$

Consider

<u>Case3.2.2</u>

$$d = -a, c = -b$$
$$f = -\overline{a}, g = -e$$
$$h = -\overline{b}, j = -\overline{e}$$

(6.20) leading to the system of 10 equations:

$$\begin{cases} -\overline{b} - a - b + k = 0 & (4) \\ -\overline{a} - \overline{e} - e + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 + b\overline{e} + ba + a\overline{e} = 0 & (6) \\ 1 + ae + b^2 + ae = 0 & (7) \\ -b\overline{b} - a\overline{k} = 0 & (8) \\ 1 - ae - b\overline{e} - bk = 0 & (9) \end{cases}$$

$$\begin{cases} 1 + \overline{a}b + \overline{a}b + e^2 = 0 & (10) \\ 1 - \overline{a}b - e\overline{b} - e\overline{k} = 0 & (11) \\ -e\overline{e} - \overline{a}k = 0 & (12) \end{cases} \qquad \begin{cases} 1 + \overline{b}a + ae + b\overline{e} = 0 & (15) \end{cases}$$

From equation (8), we get $a = -k$. Using $a = -k$ in equation (4), we get $k = Re(b) \Rightarrow b = \pm 1, k = \pm 1 \Rightarrow a \mp 1$. Now put $a = -1, b = 1$ and $k = 1$ in equation (11), we get $e = 1$. Put $a = 1, b = -1$ and $k = -1$ in equation (11), we get $e = -1$. Summing up, we have $a = \mp 1, b = \pm 1, k = \pm 1$ and $e = \pm 1$. So, we get the following two families

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \mp 1 & \pm 1 & \mp 1 & \pm 1 \\ 1 & \mp 1 & 0 & \pm 1 & \pm 1 & \mp 1 \\ 1 & \pm 1 & \pm 1 & 0 & \mp 1 & \mp 1 \\ 1 & \mp 1 & \pm 1 & \mp 1 & 0 & \pm 1 \\ 1 & \pm 1 & \mp 1 & \mp 1 & \pm 1 & 0 \end{pmatrix}$$

Consider the last possible option

Case3.2.3

$$d = -a, c = -b$$
$$f = -\overline{a}, g = -e$$
$$j = -\overline{b}, h = -\overline{e}$$

From (6.20), we have, the system of 10 equations:

$$\begin{cases} -\overline{b} - a - e + k = 0 & (4) \\ -\overline{a} - \overline{e} - b + \overline{k} = 0 & (5) \end{cases}$$

$$\begin{cases} 1 + b\overline{e} + ba + a\overline{e} = 0 & (6) \\ 1 + ae + be + ab = 0 & (7) \\ -b\overline{e} - a\overline{k} = 0 & (8) \\ \underline{1} - ae - \underline{|b|^2} - bk = 0 & (9) \end{cases}$$

(6.22)

$$\begin{cases} 1 + \overline{a}b + \overline{a}e + eb = 0 & (10) \\ \underline{1} - \overline{a}b - \underline{|e|^2} - e\overline{k} = 0 & (11) \\ -e\overline{b} - \overline{a}k = 0 & (12) \end{cases}$$

$$\begin{cases} 1 + \overline{b}a + ae + e\overline{b} = 0 & (15) \end{cases}$$

We shall consider the following three cases:

Case3.2.3.1

$$a = -\overline{b}, k = e$$

Case3.2.3.2

$$e = -\overline{b}, k = a$$

Case3.2.3.3

$$k = \overline{b}, e = -a$$

Case 3.2.3.1

Substitute $a = -\overline{b}, k = e$ in the above system of equations, we get

$$\begin{cases} -\overline{b} + \overline{b} - e + e = 0 & (4) \\ b - \overline{e} - b + \overline{e} = 0 & (5) \end{cases}$$

$$\begin{cases} 1 + b\overline{e} - 1 + a\overline{e} = 0 & (6) \\ 1 - \overline{b}e + be - 1 = 0 & (7) \\ -b\overline{e} - a\overline{e} = 0 & (8) \\ -ae - be = 0 & (9) \end{cases}$$

$$\begin{cases} 1 - b^2 - be + eb = 0 & (10) \\ b^2 - 1 = 0 & (11) \\ -e\overline{b} + be = 0 & (12) \end{cases}$$

$$\begin{cases} 1 + \overline{b}a + ae + e\overline{b} = 0 & (15) \end{cases}$$

From equation (6), we get $b = \pm 1$. We have two-one parameter families of matrices

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \mp 1 & \pm 1 & \mp 1 & \pm 1 \\ 1 & \mp 1 & 0 & e & \pm 1 & -e \\ 1 & \pm 1 & \overline{e} & 0 & -\overline{e} & \mp 1 \\ 1 & \mp 1 & \pm 1 & -e & 0 & e \\ 1 & \pm 1 & -\overline{e} & \mp 1 & \overline{e} & 0 \end{pmatrix}$$

<u>Case 3.2.3.2</u>

$$e = -\overline{b}, k = a.$$

Substitute this into the system (6.23) of equations, we get

$$\begin{cases} -\overline{b} - a + \overline{b} + a = 0 & (4) \\ -\overline{a} + b - b + \overline{a} = 0 & (5) \end{cases} \qquad \begin{cases} 1 - b^2 + ba - ab = 0 & (6) \\ 1 + a\overline{b} - 1 + ab = 0 & (7) \\ b^2 - 1 = 0 & (8) \\ a\overline{b} - ab = 0 & (9) \end{cases}$$

$$\begin{cases} 1 + \overline{a}b - \overline{a}\overline{b} - 1 = 0 & (10) \\ -\overline{a}b + \overline{b}\overline{a} = 0 & (11) \\ \overline{b}\overline{b} - 1 = 0 & (12) \end{cases} \qquad \begin{cases} 1 + \overline{b}a + a\overline{b} + \overline{b}\overline{b} = 0 & (15) \end{cases}$$

From equation (9) it follows that $b \in \mathbb{R} \Rightarrow b = \pm 1$. We have two families of matrices

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & a & \pm 1 & \mp 1 & -a \\ 1 & \overline{a} & 0 & \mp 1 & -\overline{a} & \pm 1 \\ 1 & \pm 1 & \mp 1 & 0 & \pm 1 & \mp 1 \\ 1 & \mp 1 & -a & \pm 1 & 0 & a \\ 1 & -\overline{a} & \pm 1 & \mp 1 & \overline{a} & 0 \end{pmatrix}$$

<u>Case 3.2.3.3</u>

$$k = \overline{b}, e = -a.$$

Substitute $k = \overline{b}, e = -a$ in system of equations (6.23), we get

$$\begin{cases} -\overline{b} - a + a + \overline{b} = 0 & (4) \\ -\overline{a} + \overline{a} - b + b = 0 & (5) \end{cases} \qquad \begin{cases} \underline{1} - b\overline{a} + ba - |a|^2 = 0 & (6) \\ 1 - a^2 - ba + ab = 0 & (7) \\ b\overline{a} - ab = 0 & (8) \\ a^2 - 1 = 0 & (9) \end{cases}$$

$$\begin{cases} 1 + \overline{a}b - 1 - ab = 0 & (10) \\ -\overline{a}b + ab = 0 & (11) \\ a\overline{b} - \overline{a}\overline{b} = 0 & (12) \end{cases} \qquad \begin{cases} 1 + \overline{b}a - a^2 - a\overline{b} = 0 & (15) \end{cases}$$

From equation (12) if follows that $a \in \mathbb{R} \Rightarrow b = \pm 1$. We have two families of matrices

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & b & -b & \mp 1 \\ 1 & \pm 1 & 0 & \mp 1 & \mp 1 & \pm 1 \\ 1 & \overline{b} & \mp 1 & 0 & \pm 1 & -\overline{b} \\ 1 & -\overline{b} & \mp 1 & \pm 1 & 0 & \overline{b} \\ 1 & \mp 1 & \pm 1 & -b & b & 0 \end{pmatrix}$$

Finally, we shall discuss the last choice:

Case3.3

$$d = -a, c = -b$$
$$g = -\overline{a}, f = -e$$

(6.17) leading to the system of 12 equations:

$$\begin{cases} \overline{b} + \overline{e} + h + j = 0 & (3) \\ -\overline{b} - \overline{e} + \overline{h} + k = 0 & (4) \\ -\overline{a} - a + \overline{j} + \overline{k} = 0 & (5) \end{cases} \qquad \begin{cases} 1 + b\overline{e} + b\overline{e} + a^2 & (6) \\ 1 + ae - b\overline{h} - a\overline{j} = 0 & (7) \\ 1 - ae + bh - a\overline{k} = 0 & (8) \\ \underline{1 - |a|^2} + bj - bk = 0 & (9) \end{cases}$$

$$\begin{cases} 1 + \overline{a}b - e\overline{h} - \overline{a}\overline{j} = 0 & (10) \\ 1 - \overline{a}b + eh - \overline{a}\overline{k} = 0 & (11) \\ ej - ek = 0 & (12) \end{cases} \qquad \begin{cases} 1 - \overline{b}a - \overline{ea} + hk = 0 (14) \end{cases}$$

$$\begin{cases} 1 + \overline{b}a + \overline{ea} + \overline{h}j = 0 & (15) \end{cases}$$

From equation (12) it follows that $j = k$. Put $j = k$ in equation (3) and then adding equations (3) and (4), we get $h = \mp 1$ and $k = \pm 1$. Using $j = k$ in equation (5), we get $a = \pm 1$ and $j = \pm 1$. Using the value of a in equation (6) implies that $e = -b$. Combining all, we get the two one-parameter families:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & b & -b & \mp 1 \\ 1 & \pm 1 & 0 & -b & b & \mp 1 \\ 1 & \overline{b} & -\overline{b} & 0 & \mp 1 & \pm 1 \\ 1 & -\overline{b} & \overline{b} & \mp 1 & 0 & \pm 1 \\ 1 & \mp 1 & \mp 1 & \pm 1 & \pm 1 & 0 \end{pmatrix}$$

7

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