SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Étale homotopy and rational points

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2013 - No 16

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

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Handledare: Rikard Bøgvad

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Abstract

In this thesis, we try to understand some recent results of Harpaz-Schlank [10] where they apply étale homotopy theory to unify obstruction theories in algebraic geometry. We start by showing how one can put a model category on certain pro-categories after [2] and then we apply this to understand the constructions of Harpaz-Schlank in a model categorical way. We end by discussing some future research directions and how to calculate the obstructions in some concrete cases.

" And the end of all our exploring
will be to arrive where we started
and know the place for the first time "
-T.S Eliot , "Little Gidding".

Acknowledgements

I am very grateful to Andreas Holmström for introducing me to this wonderful thesis subject. Further, I'd like to thank Rikard Bögvad for being the inspiration that he is, and Alexander Berglund for reading this paper. I am also very much in debt to Per Salberger. Some parts of this thesis was written while I was visiting the Hausdorff Research Institute for Mathematics which provided ideal working conditions. Tomer Schlank has been more than kind in answering my many questions on the subject and sharing his knowledge.

Contents

1 Introduction

This thesis started in the summer of 2012 when Andreas Holmstrom told me about some very interesting new work that had been done on the étale homotopy type and suggested that it might make a good thesis subjectThe main purpose of this paper is to give the reader with little knowledge of recent developments in obstruction theory an overview of some recent developments. To do this, I have tried to include a lot of detail in the first part of the paper, where I cover the constructions of Barnea-Schlank in [2]. In this section almost all proofs are included. I give some basic examples that are not found in the literature to better aid the reader in understanding the constructions.

The second part of the paper is not as detailed and some proofs can not be found in my paper, but are instead passed to the references. I have, when I omitted to prove the theorem myself, tried to give some motivation as to why the statement should be true and I always give a precise reference if one wants a proof of the theorem in question. However, while the first part is mainly an exposition of [2], the second part tries to use their material to see the paper of Harpaz-Schlank [10] in new light. The idea to apply the material of [2] to Harpaz-Schlank is not my own, but will appear in a future paper of Harpaz-Schlank. Much of what I know about the subject comes much from discussions with Tomer Schlank. I claim no originality in the ideas on how to apply [2] to [10]. I have however included some examples that are not to be found in the literature and I have also added some discussion on the obstructions, how this could be calculated and future research directions.

2 The Jerusalem Machine

In 2010 Harpaz-Schlank wrote a paper that framed arithmetic obstructions in terms of homotopy theory, and with this, unified several classical obstruction principles under one framework. They did this by constructing a relative étale homotopy type for a variety X over a field K. Later, Barnea-Schlank put the results of Harpaz-Schlank in a suitable model category. They put a model categorical construction on Pro(C), where C is a "weak fibration category". Further, they showed that given a geometric morphism (f_*, f^*) between two topoi $S \rightleftarrows T$, we have a Quillen adjunction between $Pro(T^{\Delta^{op}})$ and $Pro(S^{\Delta^{op}})$. We then have a **relative topological realization of T over S** , defined as $|T|_S = \mathbb{L}L_{f^*}(*_T)$ where $*_T$ is a terminal object of $T^{\Delta^{op}}$. This gives us a completely new approach of defining a homotopy theory of schemes, which we get by using the so called **The Jerusalem machine**. A specialized version of this machine produces as output Artin-Mazur's étale homotopy type, only better in some aspects. The Jerusalem machine gives us a way of constructing a vast number of relatives to Artin-Mazur's étale homotopy type. To be more precise, we have analogolus constructions for every geometric morphism of topoi! Further, the construction of Artin-Mazur is an object of $Pro(Ho(S))$ instead of $Pro(S)$.

2.1 Weak fibration categories

We will here freely use language and definitions from model categories. For the definitions, the reader is referred to [3] section 2.2 . A weak fibration category should be seen as a category where we have a good notion of a weak equivalence and fibration, but for some reason, the model categorical axioms are not fulfilled. We could, for example, not have a functorial factorization of maps .

Definition 1. A weak fibration category is a category C with all finite limits and two subcategories $\mathcal{W}$ and $\mathcal{F}$ that are subject to the following axioms:
(i) $\mathcal{W}$ satisfies the 2 out of 3-property, that is, if $g, f \in mor(C)$ and $f \circ g$ is well-defined and 2 of f, g, $f \circ g$ is in $\mathcal{W}$, so is the third.
(ii) $\mathcal{F}$ and $\mathcal{F} \cap \mathcal{W}$ are closed under pullbacks.
(iii) We can factor every map $h : X \to Y$ as $h = f \circ g$ where $g \in \mathcal{W}$ and $f \in \mathcal{F}$.

Remark. We say that a category C with two subcategories $\mathcal{W}$ and $\mathcal{C}$ is a weak cofibration category if C^{op} with $\mathcal{W}^{op}$ and $\mathcal{C}^{op}$ is a weak fibration category.

Example 2.1. Any model category disregarding the cofibrations is a weak fibration category. It is easy to see that (i) and (iii) are immediately satisfied by the axioms of a model category. (ii) follows since we can categorize fibrations and acyclic fibrations as a class of morphisms having the right lifting property with respect to acyclic cofibrations resp. cofibrations, and the right lifting property is closed under pullback.

Example 2.2. Let G be any profinite group, and let us look at $(G - set)^{\Delta^{opp}}$. I claim that this is a weak fibration category. Let $f : X \to Y$ be a weak equivalence iff $for(f) : for(X) \to for(Y)$ is a weak equivalence of simplicial sets, where for is the forgetful functor, and similarily with fibrations. Let us briefly verify that this satisfy the above axioms. We se that (i) follows from the fact that we are looking at weak equivalences of simplicial sets, and the same with (ii). The only one that needs some verification is (iii), which can be proved using a small objects argument.

Now, let C be a weak fibration category. There is in general no sensible way of defining a model category structure on C, since we do not have a functorial factorization of maps. We could ask ourselves if it is possible to get functorial factorization in any way. There is a general method to achieve functorial factorizations for weak fibration categories , but we have to leave C and go to $Pro(C)$. In $Pro(C)$, we go through all possible factorisations of $A \to B$ such that $A \to C \to B$ is a cofibration in C followed by an acyclic fibration in C. This gives us an inverse system of factorizations, and results in a pro-object which modulo some technical details defines the factorization. We will make this more precise soon.

2.2 Some classes of morphisms

We now define some classes of morphisms between pro-objects that will be of crucial importance later on in the model category structure.

Definition 2. Let I be a poset. We say that I is cofinite if for every $x \in I$ the set $I_{\leq x} = \{y \in I | y \leq x\}$ is finite.

Definition 3. Let C be a category with finite limits, I a small category, and $\mathcal{W}$ some class of morphisms in C. Let $F : A \to B$, $A, B \in C^I$.
(i) If for every $i \in I$, $F_i : A(i) \to B(i)$ is in $\mathcal{W}$ we call F a levelwise $\mathcal{W}$-map.
(ii) If I is a cofinite poset and $X_t \to Y_t \times_{\lim_{s<t} Y_s} \lim_{s<t} X_s$ is in $\mathcal{W}$ for all t, then we call F a special $\mathcal{W}$-map.

Let us try to give an example of where the motivation for special $\mathcal{W}$-maps come from.

Definition 4. A Reedy category is a category R which has two subcategories R_+ and R_- containing all the objects, together with a function $d : Ob(R) \to \alpha$ where α is some ordinal number such that the function d satisfies the following properties:
1) If $f : X \to Y$ is a morphism in R_+, then $d(X) < d(Y)$
2) If $f : X \to Y$ is a morphism in R_- then $d(X) > d(Y)$
3) Every morphism f in R can be factored uniquely as $f = g \circ h$, where h is in R_- and g is in R_+.

We can view any totally ordered set as a Reedy category in the obvious way. The most important Reedy category is the category Δ, with objects natural numbers and morphism between them order-preserving maps. One of the main reasons for considering Reedy categories is that if we have a model category $\mathcal{M}$, then the category $\mathcal{M}^R$ has a particularly nice model structure. Let us sketch how the model structure on $\mathcal{M}^R$ is obtained, for more details the reader can consult ch. 15 of [11].

Definition 5. Let R be a Reedy category and $\mathcal{M}$ a model category. Given a functor $X : R \to \mathcal{M}$ and an object $r \in R$ we define its latching object to be $L_rX = colim_{s \xrightarrow{+} r} X_s$, where the colimit goes over the full subcategory of the comma category $R_+ \downarrow r$ that contains all objects except the identity arrow of r. Dually, we define its matching object to be $M_r = lim_{r \xrightarrow{-} s} X_s$ where the limit goes over the full subcategory of $r \downarrow R_-$ that contains all objects except the identity arrow of r.

Given this, the model structure on $\mathcal{M}^R$ is defined as follows. Let the weak equivalences be the levelwise weak equivalences. We say that a morphism $f : X \to Y$ is a cofibration or a trivial cofibration if for all r, the canonical map $L_rY \amalg_{L_rX} X_r \to Y_r$ is a cofibration or a trivial cofibration in $\mathcal{M}$ respectively, and a fibration or a trivial fibration if the canonical map $X_r \to M_rX \times_{M_rY} Y_r$ is a fibration or a trivial fibration in $\mathcal{M}$ respectively. Now, with this, let us note that considering a totally ordered poset I as a Reedy category, the fibrations are exactly special $\mathcal{F}$-maps. So we see that the notion of special $\mathcal{M}$ maps stems from this and tries to abstract this for our purposes.

Definition 6. Let C be a category with finite limits, M a class of morphisms of C. We denote by :
1. $Lw^{\cong}(M)$ morphisms in $Pro(C)$ isomorphic to morphisms induced by natural transformations that are levelwise M-maps.
2. $Sp^{\cong}(M)$ morphisms in $Pro(C)$ isomorphic to morphisms induced by natural transformations that are special M-maps.

$Lw^{\cong}(M)$ and $Sp^{\cong}(M)$ will play a central part in the model structure on pro-categories later and we will need to see that they satisfy some certain properties to be able to define the model structure. This lemma is important for showing that in the model structure on $Pro(C)$, cofibrations are closed under retracts.

Theorem 7. *Let C be a category with finite limits and M a class of morphisms closed under pullbacks and containing all isomorphisms. Then if $F : X \to Y$ and $F \in Sp^{\cong}(M)$, then we have that $F \in Lw^{\cong}(M)$.*

Proof. See [2] Proposition 2.12 □

We denote by $^{\perp}M$ the morphisms that have the left lifting property with respect to maps in M. The following lemma will be important later on for showing that the fibrations we'll define in $Pro(C)$ have the right lifting property with respect to acyclic cofibrations.

Lemma 8. $^{\perp}M = ^{\perp} Sp^{\cong}(M)$.

Proof. [2] Lemma 2.22. □

2.3 The factorization argument

In many categories we encounter it would be higly desireable to endow it with a model category structure. In many cases however, there might not exist a model category with the morphisms we demand. For example, let Γ be a pro-finite group, and $\Gamma - Set$ the category consisting of sets where Γ acts continuously, i.e each element has an open stabilizer. We have a model category on simplicial sets, let us say that a model category structure on $Sset^{\Gamma}$ is projective if all weak equivalences are levelwise weak equivalences and fibrations levelwise fibrations in $Sset$. If Γ is infinite, there is no projective model structure. It would be highly desireable to try to achieve one here, however. Let us first sketch why a projective model structure can't exist on $Sset^{\Gamma}$.

Theorem 9. *For Γ an infinite pro-finite group, there is no projective model structure on $Sset^{\Gamma}$.*

Proof. Let us suppose to the contrary that there is a projective model structure. We claim that in this case, every cofibrant object must have a free action of Γ. However, such an action can't exist in $Sset^{\Gamma}$ since we restricted ourselves to continuous actions. For $G = \Gamma/H$ a finite (continuous) quotient, we can consider EG (see 3.5 Def. 70) as an object of $Sset^{\Gamma}$, and it is easy to check that $EG \to *$ is a fibration and that EG is contractible. Since cofibrations have the left lifting property with respect to trivial fibrations, they must in particular satisfy them for $EG \to *$, implying that a cofibrant object X would admit Γ-equivariant maps to EG. The action of G on EG is free, so all stabilizers on X will be in the kernel of the projection $\Gamma \to G$, and since G was arbitrary the action of Γ on X must be free. □

So, in some sense the reason for why there is no model structure is simply the lack of cofibrant replacements. The solution is then to approximate a cofibrant replacement to an object X by an inverse family $\{X_i\}$, each approximating the cofibrant replacement better and better. The main purpose of this section is to show how we can put a projective model category on the pro-category of certain categories C. It turns out that C can be, for example, $Sset^{\Gamma}$, so the above example gives us some motivation. The main technical argument used by Barnea-Schlank in [2] is what we call the factorization argument, which is very akin to Quillen's cosmall object argument. The factorization argument helps us in finding functorial factorizations in our model structure on $Pro(C)$, which is the most involved of the axioms for a model category to prove that $Pro(C)$ satisfies.

We will thus seek refuge in $Pro(C)$ to create a model category out of a weak fibration category $(C, \mathcal{W}, \mathcal{F})$.

Definition 10. (Pro-factorizable category)
A pro-factorizable category consists of a category C and $M \subset C$ a subcategory closed under pullbacks and a class of morphisms N such that:
(i) C is small and has all finite limits.

(ii) Every map $f : X \to Y$ can be factored as

$$X \xrightarrow{g} Z \xrightarrow{h} Y,$$

where $g \in \mathcal{N}$ and $h \in \mathcal{M}$.

Lemma 11. *Any map $f : X \to Y$ in $Pro(C)$ can be represented up to isomorphism by a inverse system of maps $\{f_i : X_i \to Y_i\}_{i \in I}$ such that I is a directed and cofinite set. That is, f is isomorphic to a map induced by a morphism g in the functor category C^I.*

Proof. See [14] 6.13 and [8] 2.1.6 . □

Theorem 12. *Let (C, M, N) be a pro-factorizable category. Then every map $f : X \to Y$ in $Pro(C)$ has a functorial factorization as $X \xrightarrow{g} Z \xrightarrow{h} Y$ where $g \in Sp^{\cong}(\mathcal{M})$ and $h \in Lw^{\cong}(\mathcal{N}) \cap^{\perp} \mathcal{M}$.*

Convention for the proof We will view a poset as a category here, but where the morphisms are in the order that is reverse to the usual one, that is, we have a map $a \to b$ iff $a \geq b$. We will for a functor $p : C \to D$ and a functor $X : D \to E$ let $p^*X : C \to E$ denote the functor X pulled back by p.

Proof. We can by the previous lemma assume that the map $f : \{X_t\}_{t \in T} \to \{Y_t\}_{t \in T}$ is induced from a natural transformation C^T, T a directed category. The proof can now put into the following steps:
Step 1: Find a cofinite set A_f together with a functor $p : A_f \to T$.
Step 2: Find a factorization

$$p^*X \xrightarrow{g} H_f \xrightarrow{h}\to p^*Y$$

of p^*f in C^{A_f}.
Step 3: Show that A_f is a directed set.
Step 4: Show that $p : A_f \to T$ is cofinal.
Step 5: Show that in the factorization

$$p^*X \xrightarrow{g} H_f \xrightarrow{h}\to p^*Y$$

of p^*f in C^{A_f} from (2) we have that $h \in Sp^{\cong}(M)$ and $g \in Lw^{\cong}(N) \cap^{\perp} M$.

Then, since p is cofinal, we have $p^*X \cong X$, $p^*Y \cong Y$ in Pro(C) and we have our desired factorization with all our required properties.
Step 1 and 2:
We'll now start by defining A_f and the factorization inductively. A_f will be a cofinite poset. We define $A_f^0 = \emptyset$ and the factorisation of

$$p_0^*X \xrightarrow{g} H \xrightarrow{h} p_0^*Y$$

the only (trivial) way. Let us now suppose that we have defined a n-level cofinite poset A^n_f and a factorization

$$p_n^*X \xrightarrow{g} H \xrightarrow{h} p_n^*Y$$

with the desired properties. For a poset R, we let $R^{\lhd}$ denote the poset R with a greatest element adjoined to it. Let us now define B^{n+1} as the set of all tuples

$$(R, p : R^{\lhd} \to T, p^*X \xrightarrow{g} H \xrightarrow{h} Y)$$

where R is a finite downward closed set in A^n_f and $p : R^{\lhd} \to T$ is a functor whose restriction to R is just p^n_R. Now, we define a n+1-level cofinite poset $A^{n+1}_f = B^{n+1} \amalg A^n_f$ by declaring that for $c \in A^n_f$,

$$c < (R, p : R^{\lhd} \to T, p^*X \xrightarrow{g} H \xrightarrow{h} Y)$$

iff $c \in R$. We define $p^{n+1} : A^{n+1}_f \to T$ as p^n on A^n_f and

$$p^{n+1}(R, p : R^{\lhd} \to T, p^*X \xrightarrow{g} H \xrightarrow{h} Y) = p(\lhd).$$

It is then clear that from the factorization $p_n^*X \to gH \xrightarrow{h} p_n^*Y$ we have an induced factorization

$$p_{n+1}^* : X \xrightarrow{g} H \xrightarrow{h} p_{n+1}^*Y$$

with $h \in Sp^{\cong}(M)$ and $g \in Lw^{\cong}(N)$. This concludes step 1 and 2.

This is all very abstract, so let us try to elucidate the first step in the construction. We see first that $A^1_f = B^1_f$. B^f_1 has objects consisting of all factorizations $(t, X_t \xrightarrow{g} H \xrightarrow{h} Y_t)$ where $g \in N$ and $h \in M$. $p^1_f : A^1_f \to T$ is then just $p(t, X_t \xrightarrow{g} H \xrightarrow{h} Y_t) = t$. As a set, define $A_f = \cup_i A^i_f$. Take the limit of all p^n and we obtain a functor $p : A_f \to T$ and a factorization $p^*X \xrightarrow{g} H \xrightarrow{h} p^*Y$ so that $g \in Lw(N)$, $h \in Sp(M)$.

Step 3:

We will now prove that A_f is directed, first we need two lemmas.

Lemma 13. *A cofinite poset A is directed iff for every finite down-set $R \subset A$ there exists an element $c \in A$ such that $c \geq r$ for all $r \in R$.*

Proof. Assume that every finite down set has such an upper bound. Take $a, b \in A$. Consider the finite down-sets
$R_{\leq a} = \{r \in A | r \leq a\}$ and $R_{\leq b}$. Then, $R_{\leq a} \cup R_{\leq b}$ is a finite down-set and as such, by hypothesis, we have $c \in A$ such that c is greater than all elements in $R_{\leq a} \cup R_{\leq b}$ and in particular, c is greater than a and b. So A is directed. Conversly, if A is directed, each finite set of elements has an upper bound by definition, so a fortiori, all finite down sets has upper bounds. ☐

Lemma 14. *Let R be a finite poset and $f : X \to Y$ a map in $C^{R^{\triangleleft}}$ and suppose that we have a factorization of f_R, $X_R \xrightarrow{g} H \xrightarrow{h} Y_R$ such that $g \in Lw(N)$, $h \in Sp(M)$. Then all lifts of this factorization to $R^{\triangleleft}$ are in natural bijective correspondence with factorizations of*

$$X(\triangleleft) \to \lim_R H \times_{\lim_R Y} Y(\triangleleft)$$

into

$$X(\triangleleft) \xrightarrow{a} H'(\triangleleft) \xrightarrow{b} \lim_R H \times_{\lim RY} Y(\triangleleft)$$

such that $a \in N$ and $b \in M$.

Proof. To define a lift of our factorization we need to define $H(\triangleleft)$, compatible maps $H(\triangleleft) \to H(r)\ \forall r \in R$, and a factorization of $f(\triangleleft)$ such that the induced factorization $f = g'h'\ h' : H^{\triangleleft} \to Y$ and $g' : X \to H^{\triangleleft}$, $g'_R = g$, $h'_R = h$, come from natural transformations and $g' \in Lw(N)$, and $h' \in Sp(M)$. Note now that to say that maps $H(\triangleleft) \to H(r)\ \forall r$ are compatible is the same as defining a morphism $H(\triangleleft) \to \lim_R H(r)$. Further, that g' and h' are natural transformations is tantamount to asking that

$$\begin{array}{ccccc} X(\triangleleft) & \longrightarrow & H(\triangleleft) & \longrightarrow & Y(\triangleleft) \\ \downarrow & & \downarrow & & \downarrow \\ \lim\limits_R X & \longrightarrow & \lim\limits_R H & \longrightarrow & \lim\limits_R Y \end{array}$$

commutes (since we know that they are natural transformations when restricted to R, we only need to check that they're compatible with the restriction and eachother). To check that $g' : X^{\triangleleft} \to H^{\triangleleft}$ is levelwise N, it suffices to check that $g'(\triangleleft) \in N$, and similarly to check that $h' : H^{\triangleleft} \to Y^{\triangleleft}$ is a special M map.So, to conclude, we need :
1. To define an object $H(\triangleleft)$.
2. Make sure that

$$\begin{array}{ccccc} X(\triangleleft) & \longrightarrow & H(\triangleleft) & \longrightarrow & Y(\triangleleft) \\ \downarrow & & \downarrow & & \downarrow \\ \lim\limits_R X & \longrightarrow & \lim\limits_R H & \longrightarrow & \lim\limits_R Y \end{array}$$

commutes. We see that to say that this boils down to saying (since our category has finite limits) that we have a factorization

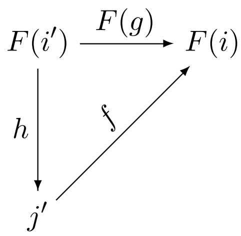
$$X(\triangleleft) \to H(\triangleleft) \to Y(\triangleleft) \times \lim_R H.$$

So our lemma follows. □

Now, let us prove that A_f is directed. For this it suffices, by the above lemma, to show that each finite down set has an upper bound.So let $S \subset A_f$ be a finite down set. For some n we have that $S \subset A_f^n$, since S is finite. So, an upper bound will then be achieved by taking $A = (S, p : S^{\triangleleft} \to T, p^*X \to H \to p * Y)$ in B_f^{n+1} which we claim exists. Indeed, T is directed so we can take an upper bound of the finite set S, call the upper bound t. Then, define $P(\triangleleft) = t$. Then, by the above lemma, gives us a factorization of our desired form. So A_f is directed and Step 3 is done.

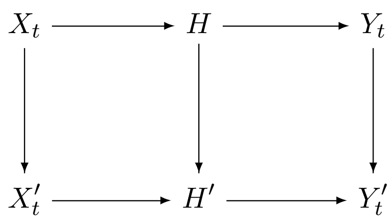
Step 4:
Now we need to prove that p is cofinal.

Definition 15. (Pre cofinal) A functor $F : I \to J$ is pre cofinal if for every morphism $f : j \to F(i)$ there exists a morphism $g : i' \to i$ such that there is a $h : F(i') \to j$ so that



commutes.

Definition 16. (The factorization category $\mathcal{F}_f$ of f) The factorization category $\mathcal{F}_f$ of $f : X \to Y$, $X, Y \in C^T$ has as objects $(t, X_t \xrightarrow{g} H \xrightarrow{h} Y_t)$, such that $h \circ g = f_t$, and $g \in \mathcal{N}$, $h \in \mathcal{M}$ and morphisms commutative diagrams



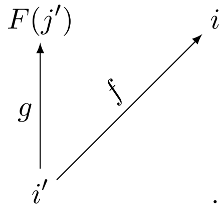
such that the outer morphisms are induced by a morphism $t \to t'$ and the induced map $H \to H' \times_{Y_{t'}} Y_t$ is in $\mathcal{M}$.

Lemma 17. *If we have a functor $p : A_f \to T$ and a factorization $p^*X \xrightarrow{g} H \xrightarrow{h} Y$ of p^*f in C^{A_f} such that $h \in Lw(N)$, $g \in Sp(M)$, we have an induced functor $q : A_f \to \mathcal{F}_f$ such that the composition of q with the natural projection $\mathcal{F}_f \to T$ is p.*

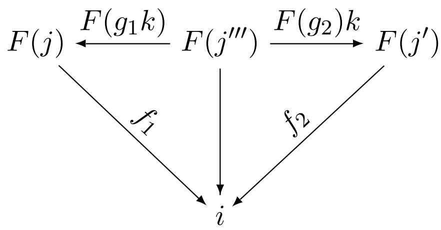
Proof. This follows immediately from the inductive definition of A_f. Indeed, A_f^1 has as objects $(t, X_t \xrightarrow{g} H \xrightarrow{h} Y_t)$, such that $h \circ g = f_t$, and $g \in \mathcal{N}$, $h \in \mathcal{M}$. □

Lemma 18. *A functor $F : I \to J$ between directed categories is pre cofinal iff it is cofinal.*

Proof. We will only prove one direction, the one we will use. The proof of the other implication can be found in Barnea-Schlank [3.11]. Remember that F is cofinal iff for every $i \in I$ the comma category (i/F) is non-empty and connected. To see that it is non-empty, pick $j' \in J$ and consider $F(j')$. We now want to find a morphism $F(j') \to i$. We have, since I is directed, a diagram



Since F is pre cofinal there exists a morphism $h : j \to j'$ such that $F(h) = g \circ l$. So $g \circ l$ is an object of (i/F) and thus, it is non-empty. Now we need to show that it is connected. So let $f_1 : F(j) \to i$ and $f_2 : F(j') \to i$ be given.J is directed so we have $g_1 : j'' \to j$, $g_2 : j'' \to j$, which gives rise to parallell morphisms $f_i g_i : F(j'') \to i$. I is directed, so these parallel morphisms have a equalizer $e : i' \to F(j'')$, that is, we have $f_1 g_1 e = f_2 g_2 e$. F is pre cofinal so we have $k : j''' \to j''$ such that $F(k) = el$ for some l. This gives rise to a commutative diagram



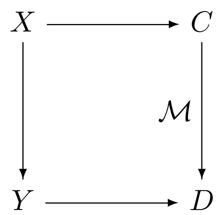
and thus, (i/F) is connected. ☐

Lemma 19. *If $F : J \to I$ and $G : I \to K$ is pre cofinal, then $GF : J \to K$ is pre cofinal.*

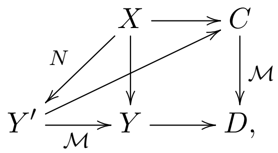
Proof. Routine. ☐

So, we now need to show that the induced morphism $A_f \to \mathcal{F}_f$ is pre-cofinal and that $\mathcal{F}_f \to T$ is pre-cofinal then by the two above lemmas, the natural functor $A_f \to T$ is cofinal.

Lemma 20. *A commutative diagram in C*

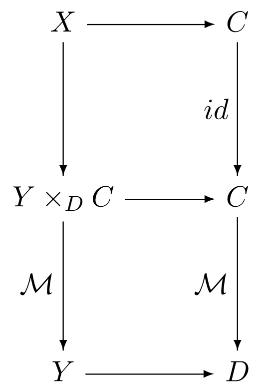


can be embedded into a larger commutative diagram



so that the induced morphism $Y' \to Y \times_D C$ is in $\mathcal{M}$.

Proof. We can in the following diagram, since $\mathcal{M} \circ N = Mor(C)$



factor $X \to Y \times_D C$ as $X \xrightarrow{g} Y' \xrightarrow{h} Y \times_D C$ such that $g \in N$ and $h \in \mathcal{M}$. This shows our claim. □

Lemma 21. *The projection $p : \mathcal{F}_f \to T$ is pre cofinal.*

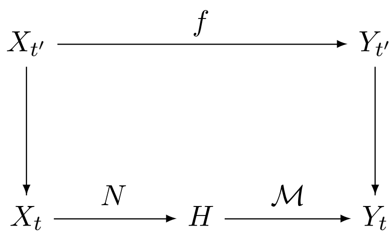
Proof. We need to show that if $(t, X_t \to H \to Y_t)$ is an object in $\mathcal{F}_f$ and $t' \to t$ is any morphism, there is a morphism

$$h : (t'', X_{t''} \to H' \to Y_{t''}) \to (t, X_t \to H \to Y_t)$$

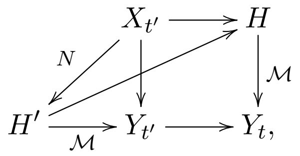
such that the induced morphism $t'' \to t$ factors through $t' \to t$. Note that it is enough to find a morphism

$$(t', X_{t'} \to H' \to Y_{t'}) \to (t, X_t \to H \to Y_t)$$

for this to hold. We clearly have a commutative diagram



such that the vertical morphisms are induced by $t' \to t$. Applying the above lemma, we get a commutative diagram



which clearly proves our claim. ☐

Lemma 22. *The induced morphism $q : A_f \to \mathcal{F}_f$ is pre cofinal.*

Proof. Say that we have an object $a = (R, p : R^{\triangleleft} \to T, p^*X \xrightarrow{g} H \xrightarrow{h} p^*Y)$ and a morphism

$$s : (t, X_t \xrightarrow{g} H \xrightarrow{h} Y_t) \to (p(\triangleleft), X_{p(\triangleleft)} \xrightarrow{g'} H(\triangleleft) \xrightarrow{h'} Y_{p(\triangleleft)}).$$

Note that it is enough to find a $b = (S, r : S^{\triangleleft} \to T, r^*X \to H' \to r^*Y) \in A_f$ such that $b > a$, and the induced morphism

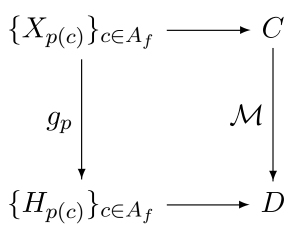
$$(p(\triangleleft), X_{p(\triangleleft)} \xrightarrow{g'} H(\triangleleft) \to h'Y_{p(\triangleleft)}) \to (S, r : S^{\triangleleft} \to T, r^*X \to H' \to r^*Y)$$

is just s. Let us set $M = R^{\triangleleft}$. We will use $M^{\triangleleft}$ to define our b. We will define $r : M^{\triangleleft} \to T$ by setting $r_{R^{\triangleleft}} = p$ and $r(\triangleleft) = t$. We define a morphism $t = r(\triangleleft) \to p(\triangleleft)$ and extend the factorization $p^*X \to H \to p^*Y$ to $r^*X \xrightarrow{g} H \to^h r^*Y$ by using our morphism s above.All that remains to show that $b \in A_f$ is that $H \to Y \in Sp(M)$. It will be enough to show that

$$H(r(\triangleleft)) \to Y(r(\triangleleft)) \times_{Y_{p(\triangleleft)}} H_{r(\triangleleft)}$$

is in $\mathcal{M}$, but this follows from our definition of s as a morphism in $\mathcal{F}_f$. ☐

So, finally, we see that the functors $q : A_f \to \mathcal{F}_f$, $p : \mathcal{F}_f \to T$ are pre-cofinal, and the composition of pre cofinal functors are precofinal, and since A_f and T are directed, $p \circ q : \mathcal{A}_f \to T$ is cofinal. This concludes Step 4.
Step 5:
All that remains for the proof of theorem 1.13 is to show that $g_p \in^{\perp} \mathcal{M}$ as a map in $Pro(C)$, since we have in the previous steps shown that $h \in Sp^{\cong}(M)$ and $g \in Lw^{\cong}(N)$. So , say that we have a commutative diagram



in Pro(C), where $A \in \mathcal{C}$ and $D \in \mathcal{C}$ are considered as objects in Pro(C) by the natural embedding. We can without further loss of realization assume that it factors through a morphism in $\mathcal{C}$ of the form: $g_{p(a)} : X_(p(a)) \to H_a$, $a \in A_f$. We have a commutative diagram in $\mathcal{C}$

$$\begin{array}{ccccc} & & X_{p(a))} & \longrightarrow & C \\ & \overset{N}{\swarrow} & \downarrow & \nearrow & \downarrow \mathcal{M} \\ Z & \underset{\mathcal{M}}{\longrightarrow} & H_a & \longrightarrow & D. \end{array}$$

by lemma 3.19. This gives us a morphism

$$\begin{array}{ccccc} X_{p(a)} & \xrightarrow{N} & Z & \xrightarrow{\mathcal{M}} & Y_{p(a)} \\ id \downarrow & & \mathcal{M} \downarrow & & id \downarrow \\ X_{p(a)} & \xrightarrow{g_{p(a)}} & H_a & \xrightarrow{h_a} & Y_{p(a)} \end{array}$$

in $\mathcal{F}_f$ and by the pre cofinality of $p : A_f \to \mathcal{F}_f$ there exists $b \to a$ such that we have a big commutative square of the form

$$\begin{array}{ccccc} X_{p(b)} & \xrightarrow{g_b} & H_b & \xrightarrow{h_b} & Y_{p(b)} \\ \downarrow & & \downarrow & & \downarrow \\ X_{p(a)} & \xrightarrow{N} & Z & \xrightarrow{\mathcal{M}} & Y_{p(a)} \\ \downarrow id & & \downarrow \mathcal{M} & & \downarrow id \\ X_{p(a)} & \underset{g_{a)}}{\longrightarrow} & H_a & \underset{h_a}{\longrightarrow} & Y_{p(a)}. \end{array}$$

and composing $H_b \to Z$ with $Z \to C$ above gives us a lift. Our proof is finally done. □

2.4 The model category structure on Pro(C)

The main technical work for defining a model category structure on Pro(C) for a certain kind of weak fibration category has now been overcome by the factorization argument. We will be a bit brief in this section on some of the techincal conditions of proving that the structure we put on $Pro(C)$ is actually a model category and refer to [2] for further details. The factorization argument is a crucial technique that we feel the reader however should have been exposed to, and which is what everything else rests on. In this section, $Ar(C)$ denotes the category of morphisms of C . For definition of retracts and other terminology regarding model categories see [3] section 2.2.

Definition 23. (Admissible weak fibration category) A weak fibration category $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is admissible if $Lw^{\cong} \subset Pro(\mathcal{C})$ satisfies the 2-out-of-3 property.

Theorem 24. *Let $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ be an admissible weak fibration category. There exists a model structure on $Pro(C)$ such that:*
1. Weak equivalences are maps that are isomorphic in $Ar(Pro(C)$ to levelwise fibrations, that is $\mathbf{W} = Lw^{\cong}(\mathcal{W})$
2. Fibrations are retracts of maps isomorphic in $Ar(Pro(C))$ to special $\mathcal{F}$ maps, that is $\mathbf{F} = R(Sp^{\cong}(\mathcal{F}))$
3. Cofibrations are what they have to be, namely $\mathbf{C} =^{\perp} (\mathbf{F} \cap \mathbf{W})$.
Further, the cofibrations can be identified with ${}^{\perp}Sp^{\cong}(\mathcal{F} \cap \mathcal{W})$.

Let us note for further use that $Sp^{\cong}(\mathcal{F}) \subset \mathbf{F}$.

2.5 A Quillen Adjunction

To define the Jerusalem Machine, it will be crucial to understand that certain functors $F : C \to D$ between pro-admissible weak fibration categories induces a Quillen adjunction between $Pro(C)$ and $Pro(D)$. We first recall that in a Quillen adjunction (F, G) , $F : C \to D$, $G : D \to C$ we call F a left Quillen functor - if it preserves cofibrations and trivial cofibrations. Similiarily, G is a right Quillen functor preserving fibrations and trivial fibrations. The adjunction ensures that F commutes with colimits and that G commutes with limits. This inspires the following definition:

Definition 25. (Weak Quillen functors) Let $F : D \to C$ be a functor between two weak fibration categories. If F preserves fibrations and trivial fibrations and commutes with finite limits, we say that F is a weak right Quillen functor. In the same way, if $F : D \to C$ is a functor between two weak cofibration categories and commutes with finite colimits and preserves cofibrations and trivial cofibrations, we call F a weak left Quillen functor.

Example 2.3. Let C be a Grothendieck site with enough points (that is, isomorphy can be checked on stalks) and let us consider $T = Sh(C)^{\Delta^{op}}$, the site of simplicial sheaves on C. We define a weak equivalence to be one that induces weak equivalences of simplicial sets stalkwise, and fibrations the ones inducing Kan fibrations stalkwise. Then a weak right Quillen functor $x^* : T \to Sset$ is induced from any point $x^* : C \to Set$.

Theorem 26. *Let $F : C \to D$ be a weak right Quillen functor between two admissible weak fibration categories. Then the induced functor $Pro(F) : Pro(C) \to Pro(D)$ preserves fibrations and trivial fibrations as well. This implies that $Pro(F)$ has a left adjoint $L_F : Pro(D) \to Pro(C)$, and $(Pro(F), L_F)$ form a Quillen adjunction. If F has a left adjoint G, then $L_F \cong Pro(G)$.*

Proof. [2] Prop. 6.3. □

2.6 Simplicial presheaves

In this subsection we will use material from [3], mainly chapter 2.4 and chapter 3.1 . We will next to notation not defined in this paper provide a direct reference to [3].

If we consider the category of schemes, we might be inclined to consider the category of simplicial schemes, considering that simplicial objects tend to be well-behaved and maybe we can perform homotopy theory through this construction. This will not be a model category, it won't have all colimits and limits. So, we look for a larger category in which the category of simplicial schemes embeds, and where all limit exists. A natural solution is simplicial presheaves.

Definition 27. Let C be an arbitrary category. A simplicial presheaf on C is a functor $C^{opp} \to Sset$. The category of simplicial presheaves has as objects simplicial presheafs and morphisms natural transformations inbetween them. We denote the category by $sPre(C)$.

There is a way to put a model category structure on $sPre(C)$ (see [13]) if C is a small Grothendieck site . We will however mostly be concerned with showing that you can view $sPre(C)$ as a weak fibration category. What is a natural candidate for the set of weak equivalences, $\mathcal{W}$? Well, we would like to extend the usual notion of weak equivalences on simplicial sets. We know that if a map $f : X \to Y$ of simplicial sets induces isomorphisms on all homotopy groups of the realization of X and Y, it is a weak equivalence. It will be necessary for our purposes to make this "local' and combinatorially.

Definition 28. Let $f : X \to Y$ be a morphism between simplicial sets. f is a combinatorial weak equivalence if :
1. The function $f_* : \pi_0(X) \to \pi_0(Y)$ is a bijection.
2. The induced maps $\pi_m(X, x) \to \pi_m(Y, f(x))$ are all isomorphisms for all choice of basepoints and $m \geq 1$.

For a simplicial set X let $\pi_m(X) = \amalg_{x \in X_0} \pi_m(X, x)$. We have a natural map $\pi_m(X) \to X_0$. Then with this notation, the above definition is equivalent to
1. The function $f_* : \pi_0(X) \to \pi_0(Y)$ is a bijection.
2 . For all $m \geq 1$

$$\begin{array}{ccc} \pi_n(X) & \longrightarrow & \pi_n(Y) \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Y_0 \end{array}$$

is a pullback diagram.
The merit of this description is that it is a functorial construction, so that we can extend it to simplicial presheaves. That is, we have for a simplicial presheaf $X : C^{op} \to Sset$ for every $n \geq 0$ a presheaf $\pi_n(X)$, and morphisms $\pi_n(X) \to X_0$ for $n \geq 1$. As previously stated, we want to extend the above construction and we want to do it "locally" in some fitting sense. We know that sheaves tend to capture local information, so , let $\tilde{\pi_n}(X)$ be the sheaf associated to $\pi_n(X)$.

Definition 29. (Local weak equivalence) Let X and Y be simplicial presheaves. $f : X \to Y$ is a local weak equivalence iff:

1. $\tilde{\pi_0}(X) \to \tilde{\pi_0}(Y)$ is a sheaf isomorphism
2. For all $n \geq 1$, the following is a pullback diagram in the category of sheaves:

$$\begin{array}{ccc} \tilde{\pi_n}(X) & \longrightarrow & \tilde{\pi_n}(Y) \\ \downarrow & & \downarrow \\ \tilde{X_0} & \longrightarrow & \tilde{Y_0}. \end{array}$$

It is easily seen that this implies that a local weak equivalence induces an isomorphism of all sheaves of homotopy groups of X and Y, for choice of baisepoints. In fact, the two conditions can be shown to be equivalent if one defines what an isomorphism of all sheaves of homotopy groups means more properly. It is obvious that levelwise weak equivalences (see 2.2, def. 4) are local weak equivalences since they satisfy the above conditions already on the presheaf level.

So, we now want to find fibrations for simplicial presheaves. Remembering that a fibration should satisfy the right lifting property with respect to some certain class of morphisms, usually inclusions of horns (see [3] 2.4.1). Since we are, in some sense, working in a local structure where we have a notion of sheaves, we can generalize this a bit. Say that we have the inclusion $i : \wedge_n^k \to \Delta^n$ and say that we have a map $f : X \to Y$ of simplicial presheaves. We can ask ourselves, given any $U \in C$, and square

$$\begin{array}{ccc} \wedge_n^k & \longrightarrow & X(U) \\ {\scriptstyle i}\downarrow & & \downarrow \\ \Delta^n & \longrightarrow & Y(U) \end{array}$$

does there exist a lift of i? This turns out to be, a bit to restrictive. A possibility can be that we don't have a global lift, but maybe lifts on some covering of U. We will make this more precise. We say that f satisfies the local right lifting property (with respect to the inclusion of $\wedge_n^k \to \Delta^n$) if for every $U \in C$ and commutative square as above there is some covering sieve R of U such that for every $\phi : V \to U$ in R there is a lift

$$\begin{array}{ccccc} \wedge_n^k & \longrightarrow & X(U) & \longrightarrow & X(V) \\ {\scriptstyle i}\downarrow & & \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & Y(U) & \longrightarrow & Y(V). \end{array}$$

We define analogously that f satisfies the local right lifting property with respect to $\partial\Delta^n \to \Delta^n$.

Definition 30. Let $f : X \to Y$ be a map of simplicial presheaves. We say that f is a fibration if f satisfies the local right lifting property with respect to all inclusions $\wedge_n^k \to \Delta^n$ for all $n \geq 0$, $0 \leq k \leq n$. We say that f is a local acyclic fibration if f satisfies the local right lifting property with respect to all inclusions of the form $\partial\Delta^n \to \Delta^n$ for all $n \geq 0$.

Theorem 31. *Letting $\mathcal{W}$ be the local weak equivalences and $\mathcal{F}$ be the local fibrations, simplicial presheaves with $(\mathcal{W}, \mathcal{F})$ as weak equivalences and fibrations becomes a weak fibration category.*

Proof. [2] Prop. 9.6 □

Definition 32. Let C be a Grothendieck site. We define the category of simplicial sheaves on C, $Sh(C)^{\Delta^{op}}$, to be the category consisting of functors $\Delta^{op} \to Sh(C)$ with natural transformations between them.

For simplicial sheaves, we can see them as embedded in $SPS(C)$ and we say that a map is a local fibration or a local weak equivalence if it is so in simplicial presheaves. We can also view simplicial sheaves as a weak fibration category with these morphisms. We will mainly be applying this material in the next section.

2.7 First glimpse of the Jerusalem Machine

We are now finally ready to define the Jerusalem machine. As an application, we shall show that we can derive (an improved version) of the étale homotopy type of Artin-Mazur.

Theorem 33. *(The Jerusalem Machine) Let $f : C \to D$ be a map of sites. Then there is a Quillen adjunction*

$$L : Pro(Sh(C)^{\Delta op}) \to Pro(Sh(D))^{\Delta^{opp}} : Pro(f^*),$$

where f^* is the geometric morphism induced by f.

Proof. It is enough to show that f induces a weak right Quillen functor $f^* : Sh(C)^{\Delta^{op}} \to Sh(D)^{\Delta^{op}}$ relative to the previously defined weak fibration structure. It preserves finite limits by the definition of being an inverse image functor, and it preserves local epimorphisms (f^* preserves epimorphisms on merit of being a left adjoint) so that it preserves local acyclic fibrations and local fibrations. This gives us the theorem. □

Example 2.4. Let us put the Jerusalem machine to test for a certain easy case one can think of, namely where C and D are trivial sites. The category of simplicial sheaves over these sites is simply $Set^{\Delta^{opp}}$, that is, simplicial sets. One has the terminal morphism of sites $F : D \to C = *$, defined in the only possible way.This induces a geometric morphism $f^* : Set \to Set : f_*$, which has to be the identity functor (by terminality). This induces in its turn a functor $f^* : Sset \to Sset$, given by the identity on each object. Now, every object is cofibrant with the standard model structure on $Sset$ (cofibrations are levelwise monomorphisms) .

The Jerusalem machine gives us a Quillen adjunction $Pro(f_1^*) : Pro(Sset) \to Pro(Sset) : Pro(f_1^*)$, and say that we want to compute $Pro(f_2^*)(X)$ for $X \in Pro(SSet)$. I claim that it is simply X, since in our model structure each object is already cofibrant. To see this, note that the unique map $\emptyset \to X$ is a cofibration iff there is a lift for every $g : Y \to Z$, $g \in \mathcal{F} \cap \mathcal{W}$ as the following diagram indicates:

$$\begin{array}{ccc} \emptyset & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow g \\ X & \longrightarrow & Z. \end{array}$$

but this is trivial.

Example 2.5. Let $\varphi : G \to H$ be a group homomorphism of discrete groups (i.e G and H are group objects in Set). This gives us a geometric morphism

$$\varphi_* : G - set \to H - set : \varphi^*.$$

Indeed, for a H-set A, we define the G-set A_G by the G-action induced from the homomorphism φ. We define $\varphi_*(A) = Hom_G(H, A)$,where Hom_G denotes morphisms of G-sets. There is an additional functor $\varphi_!(A) = X \times_G H$, where $X \times_G H = X \times G/ \sim$ where $\sim$ is the equivalence relation generated by $(gx, h) \sim (x, \varphi(g)h)$. It is easy to check that $\varphi_!$ is left adjoint to φ^* and thus φ^* is left exact. This implies that φ_* and φ^* is a geometric morphism. We have two natural weak fibration structures on simplicial G-sets, coming from the injective model structure for the first, and the latter from a certain weak fibration category . Note that we can see a simplicial G-set as a functor $G \to Sset$. The injective model structure then has weak equivalences and cofibrations are the morphism that levelwise has this property. Let us for simplicitly only consider the injective model structure on simplicial sets with an action of a group. I claim that in $Pro(H - Set^{\Delta^{op}})$ each object is cofibrant. This follows since the model category here is fibrantly generated, with generating acyclic fibrations $\mathcal{F} \cap \mathcal{W}$ i.e the acyclic fibrations in $H - Set$ and generating fibrations $\mathcal{F}$. So for $X \in Pro(H - Set^{\Delta^{op}})$ we need to check that the unique morphism $X \to *$ satisfies the left lifting property with respect to these morphisms. Let us spell it ut more clearly. Say that we have a commutative diagram in $Pro(H - Set^{\Delta^{op}})$

$$\begin{array}{ccc} \{X_i\}_{i \in I} & \longrightarrow & C \\ \downarrow & & \mathcal{F} \downarrow \\ * & \longrightarrow & D \end{array}$$

there is a lift making the diagram commutative. By the definition of a morphism in a pro-category, for some $i_0 \in I$, the above square factorizes as

$$\begin{array}{ccccc} \{X_i\}_{i\in I} & \longrightarrow & X_{i_0} & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \mathcal{F} \\ * & \longrightarrow & * & \longrightarrow & D \end{array} .$$

So it suffices to find a lift for X_{i_0}. But this lift exists since each object in H-set is cofibrant. It follows that every object is cofibrant. So, for the Quillen adjunction

$$\varphi_! : Pro(G - set^{\Delta^{op}}) \to Pro(H - Set^{\Delta^{op}}) : \varphi^*$$

it is easy to compute what $\mathbb{L}Pro(\varphi_!)$ is on an object, simply

$$\mathbb{L}Pro(\varphi_!)(X) = Pro(\varphi_!)(X).$$

Example 2.6. By [2] Remark 4.16, if the weak category stems from a model category, then the induced cofibrations in Pro(C) is just $Lw^{\cong}(Cof)$. This helps us here, when C is a small category with a discrete topology, and $D = *$ the trivial category. We have the unique geometric morphism

$$\Gamma_* : Set \to Set^{C^{op}} : \Gamma^*$$

here Γ^* is the constant sheaf functor, that in this case is just $\Gamma^* = const$, that assigns to a set the constant presheaf. It has a left adjoint, the colimit. Let us compute $\mathbb{L}Pro(colim)(*)$. We have that we can just take a cofibrant replacement to $*$ of simplicial presheaves. By [5] Lemma 2.7 a cofibrant replacement is given by , in degree k,

$$Q(*)_k = \amalg_{C_k \to C_{k-1} \to \cdots \to C_0 \to *} C_k$$

where C_k for $C_k \in C$ is seen as a simplicial presheaf by the Yoneda embedding, and the face and degeneracy maps are the obvious ones. $colimQ \cong N(C)$, the nerve of C. So the nerve of a category is an invariant computed by the Jerusalem machine. In general, for any simplicial presheaf A, , $hocolimA \cong colimQ(A)$, where $Q(A)$ is the diagonal of the bisimplicial presheaf we obtain if we apply Q to the presheaf A_n. So:

$$QA = (\cdots \amalg_{C_1 \to C_0 \to A_1} C_1 \rightrightarrows \amalg_{C_0 \to A_0} C_0).$$

So we have an easy way of computing what invariant the Jerusalem machine gives us for simplciial presheaves.

So, after these examples we are finally ready to prove our main theorem, connecting the Jerusalem machine and the classical étale homotopy type of Artin-Mazur. For material on hypercoverings and the étale homotopy type of Artin-Mazur see [3] Section 3.4 and 3.5 . Note first of all that if we have that X is a locally noetherian scheme and $X_{\text{ét}}$ is its étale site, then we have as usual a geometric morphism $\Gamma_* : X_{\text{ét}} \to Set : \Gamma^*$. Since X is locally noetherian, $X_{\text{ét}}$ is locally connected and thus, the constant sheaf functor Γ^* has a left adjoint $\Gamma_!$ that sends $U \to X$ to the scheme-theoretic connected components of U. The Jerusalem machine gives us a Quillen adjunction

$$Pro(\Gamma_!) : Pro(X_{\text{ét}}^{\Delta^{opp}}) \rightleftarrows Pro(Set^{\Delta^{op}}) : Pro(\Gamma^*),$$

and we define étale topological realization ,

$$|X_{\text{ét}}| = \mathbb{L}(Pro(\Gamma_!)(*_{\text{ét}}))$$

where $*_{\text{ét}}$ is the terminal simplicial sheaf on $Pro(X_{\text{ét}}^{\Delta^{op}})$. Since the étale homotopy type of Artin-Mazur is an object in the pro-homotopy category, while $|X_{\text{ét}}|$ is in the pro category, we will apply the natural functor

$$Ho : Pro({X_{\text{ét}}}^{\Delta^{opp}}) \to Pro(Ho(X_{\text{ét}}^{\Delta^{opp}}).$$

Theorem 34. *Let X be a locally Noetherian scheme, and $X_{ét}$ its étale topos. The image of $|X_{ét}|$ under the natural functor*

$$Ho : Pro(Sh(X_{ét})^{\Delta^{op}}) \to Pro(Ho(Sh(X_{ét})^{\Delta^{op}}))$$

is isomorphic in $Pro(Ho(Sh(X_{ét})^{\Delta^{op}}))$ to the étale homotopy type of Artin-Mazur.

Lemma 35. *In a topos with enough points, the hypercoverings can be identified with the contractible Kan objects.*

Proof. It clearly suffices to prove this in Set, where coverings are surjective families of maps, since in a topos with enough points isomorphisms can be checked on stalks. An hypercovering is non-empty, since the map $X_0 \to pt$ is a covering, and that $X_{n+1} \to (Cosk_n X.)_{n+1}$ is a covering (i.e surjective) can be interpreted as follows. First of all, note that a map is a acyclic fibrations if it has the right lifting property with respect to the inclusions $\partial\Delta^n \to \Delta^n$.Now, the condition that $X_{n+1} \to (Cosk_n X.)_{n+1}$ is surjective , gives, from the adjunction properties that the map

$$Hom(\Delta[n+1], X) \cong X_{n+1} \to Hom(sk\Delta[n+1], X) \cong (cosk_n X)_{n+1}$$

is surjective. This translates to the fact that every map

$$sk\Delta[n+1] = \partial\Delta[n+1] \to X$$

there is at least one map $\Delta[n+1] \to X$ inducing it. So, this gives that each map $\partial\Delta[n+1] \to X$ can be extended to a map $\Delta[n+1] \to X$, thus giving that X is contractible Kan. □

Proof. We will actually show something stronger, namely that this construction is isomorphic to the etale homotopy type of Artin-Mazur for any locally connected site. Let us factor the unique map $f : \emptyset \to *$ in $Pro(X_{ét}^{\Delta^{op}})$ as $\emptyset \xrightarrow{C} H^A \xrightarrow{F \cap W} *$.Then H^A is a cofibrant replacement of $*$, where $H^A : A \to X_{ét}^{\Delta^{op}}$. Such a factorization gives us an associated factorization category $\mathcal{F}_f$, and we can associate this with the full subcategory of $X_{ét}^{\Delta^{op}}$ consisting of locally fibrant and contractible objects, and acyclic fibrations between the objects. We call it T_{fw}. By the above, each hypercovering is in this factorization category. We can see H^A as a functor $A \to T_{fw}$. To proceed, we will need a lemma.

Lemma 36. *Let C be a weak fibration category . Then, with notation as above, $\pi_{\mathcal{T}_{fw}}$is a directed category, and the image of the middle object H_f^A in the factorization argument under the natural map $\pi : Pro(C) \to Pro(\pi(C))$ is isomorphic to πH (H restricted to the homotopy category).*

Proof. We will not prove in full that $\pi\mathcal{F}_f$ is directed, but sketch some parts. Let us take two objects

$$A_1 = (t, X_t \to H_t \to Y_t)$$

and

$$A_2 = (s, X_s \to H_s \to Y_s),$$

and show that there is an object $B = (u, X_u \to H_u \to Y_u)$ with morphisms $B \to A_1$, $B \to A_2$. Let us apply lemma (2.2) on the discrete poset R of two elements, the two objcets identified with t and s. Since T is directed, choose some u dominating both s and t. We can then identify $\triangleleft$ with u. We then have a natural morphism in $C^{R^{\triangleleft}}$, $f : X \to Y$, and a factorization of f_R as $h' \circ g'$ where g' is levelwise N and h is special M. By lemma 3.13 this can be extended to a factorization of f into the desired form, and we get an object B of the desired form. We should show that the morphism $B \to A_i$ is a morphism in $\pi\mathcal{F}_f$, i.e that the map $H_u \to H_{t_i} \times_{Y_{t_i}} Y_u$ is in $\mathcal{M}$. But this follows from the following lemma:

Lemma 37. *Let $X : T \to C$ be a special M-diagram (i.e the morphism $X_t \to \lim_{s<t} X_s$ is in $\mathcal{M}$ for all t). Then for any finite down-sets $A \subseteq B \subseteq T$, the map $\lim_{s\in B} X_s \to \lim_{s\in A} X_s$ is in $\mathcal{M}$.*

Proof. An easy induction proof. □

The case of finding a equalizer for two maps follows from a similar application of lemma 3.13, with some addition homotopy theoretical machinery. We refer the reader to [Barnea-Schlank 8.3] for a full proof. □

With this, we have an associated homotopy category πT_{fw}, which is directed. πT_{fw} has as objects the locally fibrant and contractible objects, and homotopy classes of morphisms between the objects. We have that $\pi_0 = \Gamma_!$ is a homotopy functor,, i.e factors through πT_{fw}. By this we have a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{H^A} & T_{fw} & \xrightarrow{\Gamma_! = \pi_0} & Sset \\ & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle Ho} \\ & & \pi T_{fw} & \xrightarrow{r} & Ho(Sset). \end{array}$$

$\gamma : T_{fw} \to \pi T_{fw}$ is the natural functor. .This gives that $Ho(\pi_0 H^A) : A \to Ho(Sset)$ factors through πT_{fw}.Since the composition $A \to \pi T_{fw}$ is cofinal (see the discussion in Section 2.3, after lemma 23), we have that $Ho(\pi_0 H^A)$ is isomorphic in $Pro(Ho(Sset))$ to a pro-object in $\pi T_{fw} \to Ho(Pro(Sset))$ which for a locally fibrant and contractible objects applies π_0 . This construction is naturally isomorphic to Artin-Mazurs construction, the objects in πT_{fw} consists of fibrant and contractible objects $U \in X_{\text{ét}}^{\Delta^{op}}$ and homotopy classes of maps between them. We have that $X_{\text{ét}}$ has enough points, so πT_{fw} consists of hypercoverings and we are done. For the general case of a locally connected site, the theorem follows since the hypercoverings are cofinal in the category of locally fibrant contractible sheaves ([12] 2.2). □

For a geometric morphism $f^* : Sh(C) \rightleftarrows Sh(D) : f_*$ we define the relative topological realization of $Sh(C)$ over $Sh(D)$ to be $|T|_S = \mathbb{L}L_{f^*}(*) \in Pro(Sh(D)^{\Delta^{op}})$.

2.8 A comparision of Artin-Mazur's construction and the derived construction.

Since we just showed that the construction Artin-Mazur gave of the étale homotopy type and the derived functor approach of Barnea-Schlank are isomorphic. So, a natural question might be - Why bother with this abstract approach? I shall here briefly discuss the merits of the derived functor approach. The construction of Artin-Mazur is very ad-hoc and it is very hard to get sense of what is really going on. Further, since the étale homotopy type is an object in the pro-homotopy category of simplicial sets, we could gain a lot more knowledge by seeing that it descends from some functor in $Pro(Sset)$ under the natural homotopy functor. For some applications the étale homotopy type isn't enough and we need the extra structure that Friedlander introduced in [9]. One spectacular theorem that seems to need the rigid étale topological type is Quillen's proof of Adam's Conjecture in Algebraic Topology. It should be true that one could use the étale topological realization instead of the rigid version of Friedlander [9]. Since we don't need to pass to the homotopy category in the derived functor approach, we will also have more structure that maybe can tell us more about our topos.

Another strength is that this construction lends itself to generalizations and extensions. Harpaz-Schlank first constructed the relative étale homotopy type in a way very akin to Artin-Mazur's étale homotopy type. Many of the arguments in that paper can be substantially shortened by viewing it through the lens of model categories, and the construction is then very easy. Harpaz-Schlank used the relative étale homotopy type for unifying the known obstructions for finding rational points on a variety over a field. This construction gives us a way of doing the same for varieties over any base scheme. Work relating the étale homotopy type to the section conjecture have been done in [1]. This leds us to question as to whether any new progress on Grothendieck's section conjecture could be done using say the flat homotopy type.

We will try to use this machinery to put the construction of Harpaz-Schlank under a more conceptual framework, under the language of model categories.

3 Applications of étale homotopy to obstruction theory

As previously mentioned in this thesis, the work of Schlank-Barnea on the Jerusalem machine began with trying to put the work of Harpaz-Schlank in a suitable model categorical structure. We will define a variety over a field k to be a separated scheme of finite type over $Speck$. A curve is a variety of dimension 1, a surface a variety of dimension 2. A beautiful and fundamental problem in diophantine geometry is the following:
"Given a diophantine equation $p(x_1, \dots x_n) = 0$, does there exist a solutions in the rational numbers?"

Let $X = \text{Spec } \mathbb{Z}[x_1, \dots, x_n]/(p(x_1, \dots, x_n)$. In modern scheme theory this translates to the question as to whether $Hom_{Sch}(\text{Spec } \mathbb{Q}, X) = X(\mathbb{Q})$ is empty or not. Surprisingly little is known about the set $X(\mathbb{Q})$ and some experts in the field such as Poonen has speculated that the problem of deciding whether $X(\mathbb{Q}) = \emptyset$ is undecideable. The problem of finding integer points, i.e , determing whether $X(\mathbb{Z}) = \emptyset$ is known to be undecideable, it is Hilbert's tenth problem. The problem of rational solutions to a diophantine equation generalizes in the obvious way, given a variety X over the field k, does X have a k-rational point? That is, is there a k-morphism Spec $k \to X$?

3.1 The Hasse Principle

Definition 38. (Global fields) Let k be a field. We say that k is a global field if k is either a number field (that is, a finite extension of $\mathbb{Q}$) or a global function field (a finite extension of $\mathbb{F}_p(t)$).

When faced with trying to show that $X(k) = \emptyset$, one method is trying to inject $X(k)$ in some larger set S and show that $S = \emptyset$. We will construct a ring of adèles, but for this, we will first need a definition.

Definition 39. (Restricted product) Let I be an index set , and S a finite subset of I. If we for each $i \in I$ have a locally compact group G_i and for each $i \in I \setminus S$ a compact open subgroup K_i of G_i, we define the restricted product (with respect to S) $\Pi'_S G_i$ to be the subset of the product $\Pi_i G_i$ that consists of elements $(g_i)_{i \in I}$ such that $g_i \in K_i$ for cofinitely many $i \in I \setminus S$. In many situations, S and K_i are obvious and we will in this case not mention them and simply write $\Pi' G_i$ for the restricted product.

Definition 40. (The ring of adèles) Let k be a global field. Let $p \in k$ be a place and let k_p denote the completion of k in p. An adéle of k is a family $\alpha = (\alpha_p)$ where $\alpha_p \in k_p$ and p varies over the places of k, and further , α_p is integral in k_p for cofinitely many p. They form a ring, $\mathbb{A}_k = \Pi'_p k_p$ where addition and multiplication are defined componentwise. Here Π' denotes the restricted product.

Remark. There is also a natural topology on the ring of adèles, which we won't use here, but which is useful when considering some topics in obstruction theory such as weak approximation.

Now note that for each place p, there is a field homomorphism $i : k \to k_p$ which gives us a morphism i : Spec $k_p \to$ Spec k. The associated map $X(\text{Spec } k) \to X(\text{Spec } k_p)$ is an injection, since Spec $k_p \to$ Spec k is an epimorphism in the category of affine schemes. A way to show that $X(\text{Spec } k) = \emptyset$ is then to consider the injection $X(\text{Spec } k) \to \Pi_p X(\text{Spec } k_p)$.

Definition 41. (The Hasse Principle) Let X be a k-variety for a global field k. X satisfies the Hasse principle (or the local-global principle) if $X(k_p) \neq \emptyset$ for all places p implies that $X(k) \neq \emptyset$.

Thus, for varieties satisfying the Hasse Principle we can look at the completions and see if it has solutions there. The Hasse Principle was first used by Helmut Hasse to show

that for a quadratic form $q(x_1, \dots, x_d)$, the equation $q(x_1, \dots, x_d) = 0$ has a solution over any number field k iff it does so if we go the completion k_p where p ranges over all places. If we restrict our number field to $\mathbb{Q}$, then it translates to that $q(x_1, \dots x_d) = 0$ has a rational solution iff it does so in real numbers and in the p-adics. John H. Conway describes quadratic forms beautifully in his The Sensual Quadratic Form:
"A quadratic form over $\mathbb{Q}$ is rather like a bouqet of flowers, each flower being the corresponding form over one of the fields $\mathbb{Q}_p$. From the fragrances of these flowers we can recover the structure of the rational form." We shall not prove this theorem here. However, there are many counterexamples to the Hasse Principle.

Example 3.1. (A simple counterexample to the Hasse Principle) Let $f(x) = (x^2 - 3)(x^2 - 13)(x^2 - 39)$. $f(x) = 0$ has the roots $x \pm \sqrt{3}, \pm\sqrt{13}, \pm\sqrt{39}$. It is clear that it has no solutions in $\mathbb{Q}$. Now, we have that $x^2 - 13 \equiv 0(3)$ has the solutions $x \equiv 1, 2$ and 1 can be extended to solutions in the 3-adics (by an easy corollary of Hensel's lemma, namely that $b \in \mathbb{Z}$ is a square in the ring of integers of the p-adics for p an odd prime iff $b = p^{2r}c$, $r \in \mathbb{Z}$ and $x^2 \equiv c$ has a solution, that is, c is a quadratic residue mod p). Now, we have that $x^2 - 3 \equiv 0(13)$ has the solutions $x = 4, 9$. Both 4 and 9 can be extended to solutions in the 13-adics. Now, for any other prime p, I claim that $f(x)$ has a solution in $\mathbb{Q}_p$. It is enough to show that at least one of 3, 13 or 39 is a square in the p-adics. Let us now suppose that $(3/p) = -1$, $(13/p) = -1$, this yields $(39/p) = (3/p)(13/p) = 1$.Thus $f(x) = 0$ yields a counterexample to the Hasse principle.

Now, there are various obstruction theories trying to explain the failure of the Hasse Principle. As this example has shown, that $X(\mathbb{A}_k) \neq \emptyset$ does not imply $X(k) \neq \emptyset$. This is seen since for the natural map

$$X(k) \to \prod_p X(k_p),$$

a point $f \in X(k)$ is not integral only for cofinitely many places, so that this map factors through the adeles. Now a natural strategy would be to try to find some intermediate set $X(k) \subseteq X(k_{\text{obstr.}}) \subseteq X(\mathbb{A}_k)$, and show that $X(k_{\text{obstr.}}) = \emptyset$, and in this case, we say that the failure of the Hasse Principle is explained by the obstruction. We will now briefly describe the classical obstruction theories.

3.2 The Brauer-Manin obstruction

We here assume that the reader has basic familiarity with étale cohomology theory and knows that étale cohomology over a field coincides with Galois Cohomology. Let $\mathbb{G}_m = \text{Spec } (\mathbb{Z}[t, t^{-1}])$, for a scheme X and $U \to X$, we define $\mathbb{G}_m(U) = \Gamma(U, \mathcal{O}_U)^*$, and it is then easy to see that this forms an étale sheaf (briefly, this follows since Spec $\mathbb{Z}[t, t^{-1}]$ is a group scheme).

Definition 42. (The Brauer Group of a field) Let k be a field. The Brauer group has as elements equivalence classes of central simple algebras over k which are Brauer equivalent. Two central simple algebras B,C are Brauer equivalent , $B \cong C$ if $B \cong$

$M_n(D)$ and $C \cong M_m(D)$ for some division algebra D, where M_n is the matrix ring.The group operation is induced by the tensor product, and the inverse of an element is the opposite algebra.

Definition 43. (The cohomological Brauer Group of a scheme) Let X be a scheme. The cohomological Brauer group $Br(X)$ is $H^2_{\text{ét}}(X, \mathbb{G}_m)$. For the case $X = \text{Spec } k$ a field, it can be shown to be equivalent to the definition given above.

We will use the cohomological Brauer group when we define the Brauer-Manin obstruction, but since $Br(X)$ can be quite hard to grasp intuitively, I will introduce a certain subgroup of $Br(X)$ which has a concrete description and will be highly useful for computations.

Definition 44. An Azumaya algebra $\mathcal{A}$ on a scheme X is a $\mathcal{O}_\mathcal{X}$-algebra that is locally free and coherent as a $\mathcal{O}_\mathcal{X}$-module with $\mathcal{A}_x \neq 0 \ \forall x$ that satisfies one of the following two equivalent conditions:
(i) The fiber $\mathcal{A}(x)$ is a central simple algebra over $k(x)$ for every $x \in X$.
(ii) There is an open covering $\{U_i \to X\}_i$ in the étale topology (or the fppf topology, short for flat, finitely presented, quasifinite), such that for each i, $U_i \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{A} \cong M_{n_i}(\mathcal{O}_{U_i})$ for some $n_i \geq 1$. Here $M_{n_i}(\mathcal{O}_{U_i}) = End(\mathcal{O}_{U_i}$.

So, an Azumaya algebra on a scheme X can be seen as a family of central simple algebras that are parametrized by the points of X. It is easy to see that the set of isomorphism classes of Azumaya algebras on X have a natural group structure with the tensor product as the group operation, and we will denote this group by $Br'X$, the Azumaya Brauer Group. A natural question now is: What is the relation between $Br'X = BrX$? The answer is not as easy as one might hope, but there are some theorems that help us out. First, we will need a cohomological interpretation of $Br'X$.

Let us start with the observation that an Azumaya algebra is really an $\mathcal{O}_\mathcal{X}$ -algebra that locally looks like the $\mathcal{O}_\mathcal{X}$-algebra M_{n_i} (where n_i may vary!) . So, it is then easy to see that the isomorphism $U_i \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{A} \cong M_{n_i}(\mathcal{O}_{U_i})$ gives rise to an automorphism $\rho \in Aut(M_{n_i}(\mathcal{O}_{U_i}))$. Now, assuming that $\mathcal{A}$ is of constant rank n^2, we see that the collection of automorphisms gives rise to an element of $H^1(X_{\text{ét}}, Aut(M_n))$ and conversely, that every element in $H^1(X_{\text{ét}}, Aut(M_n))$ gives rise to an Azumaya algebra of rank n^2. So, we can interpret each element of $Br'(X)$ cohomologically on each connected component in a nice manner. I claim that there is a canonical map:

$$Br'(X) \to Br(X).$$

For this, we will need the following lemma:

Lemma 45. *(Noether-Skolem for local rings)*
Let A be a central simple algebra over the local ring R. Then every automorphism of A is of the form

$$a \to uau^{-1},$$

that is, an inner automorphism.

Proof. [15] Chapter 4 Prop.1.4. ☐

corollary 46. $Aut(M_n(R)) = PGL_n(R) = GL_n(R)/R^*$.

Proof. Indeed, we have a morphism $GL_n(R) \to Aut(M_n(R))$ with kernel R^*. ☐

This implies that there is a short exact sequence of sheaves

$$0 \longrightarrow \mathbb{G}_m \longrightarrow GL_n \longrightarrow PGL_n \longrightarrow 0$$

in the étale topology (actually, it is exact in the Zariski and fppf topology too). This induces a map $H^1(X, PGL_n) \to H^2(X, \mathbb{G}_m) = BrX$ so that each Azumaya algebra of rank n^2 gives an element of BrX. It can be shown that when X is not connected, one can define the map over each connected component. One can show that this map in general will be injective. For details, we refer the reader to [15], chapter 4.

The following is often useful for calculating the Brauer-Group.

Theorem 47. *If X is a regular quasiprojective variety over a field, then* $Br'X \cong BrX$.

Proof. Follows from [4] ☐

Finally, to define the Brauer-Manin obstruction we will need to define a certain map, the map of local invariants. Let K be a non-archimedean local field (the reader uneasy with local fields may take K to be the field of p-adic numbers) . We assume that the valuation v associated to K is normalized, i.e that K is complete with respect to the valuation, the valuation group satisfies $v(K^*) = \mathbb{Z}$ (where K^* is the non-zero elements) and that the residue field k of K with respect to v is a finite field. What the latter property mean can be seen easily, by considering that associated to K, we have a ring of integers $\mathcal{O}$ consisting of the elements $x \in K$ such that $v(x) \geq 0$, and with a maximal ideal m consisting of 0 and the elements x such that $v(x) > 0$. The residue field k is simply $\mathcal{O}/m$. We have that this will be a finite field (since it is a compact and discrete as a topological space), and thus $card(k)$=q is a finite number. We can with this define an absolute value on K, by for $x \in K$, $|x| = q^{-v(x)}$.

Let us now consider a division algebra with center K, of finite dimension as a vector space over K. We can extend v to a valuation v_D on D such that D is complete (i.e all Cauchy sequences converges). Note that every element $\alpha \in D$ lives in a subfield of D, for example, $K[\alpha]$ where $K[\alpha]$ will be finite by the merit of D finite. So, we extend $||$ to $K[\alpha]$ by $|\alpha|_D = |N_{K[\alpha]/K}(\alpha)|^{1/m}$ where m is the dimension of $K[\alpha]$ as a vector space over K. Note that the norm is in K .We can then extend the absolute value $||$to D, and we shall denote it as $||_D$. . We define the order of an element $\alpha \in D$ as the number given by

$$|x|_D = q^{(-v(N_{K[\alpha]/K}(\alpha))/m} = q^{-ord(\alpha)}.$$

Note that this gives that if we denote by D^* the nonzero elements, $ord(D^*) \subset \mathbb{Z}/n$. We need some further theorems on the Brauer group to define the local invariant map.

Theorem 48. *For every field k, $Br(k) = \cup_L Br(L/k)$ where L runs over the finite Galois extensions of k in some separable closure of k.*

Proof. [16] Chapter 3, prop. 3.10 □

Let $A \in Br(K)$, and represent A by some division algebra D with center K. It can be shown that D contains a maximal subfield L that is unramified over K of the same dimension as K (see [16] Chapter 4, Section 4). Letting $\mathcal{O}_L$ be the ring of integers of L and $\mathcal{O}_K$ that of K, we have a maximal ideal p of $\mathcal{O}_K$, and residue field k. Choose some $q \in \mathcal{O}_L$ lying over p. Then $l = \mathcal{O}_L/q$ is a finite field of order $card(k)^f$ where $f = [L : K]$. We then define the Frobenius automorphism of L by being the unique automorphism satisfying $Frob(x) \equiv x^q mod p$ for $x \in \mathcal{O}_K$. We will now give the following theorem:

Theorem 49. *Noether-Skolem for fields Let A be a simple k-algebra and B a central-simple k-algebra. If $f, g : A \to B$ are k-algebra morphisms then there is $b \in B$ that is a unit such that $f(a) = bg(a)b^{-1}$*

Proof. [16], chapter 4, prop. 2.10] □

This gives that the Frobenius automorphism $Frob : L \to L$ gives rise to a map $\varphi = i \circ Frob : L \to D$, where i is the inclusion, such that $\varphi(x) = \alpha x \alpha^{-1}$ for all $x \in L$. The order $ord(\alpha)$ of α is uniquely determined mod $\mathbb{Z}$, since if β has the same property, then $\beta = \alpha \cdot u$ for $u \in L$, and $ord(u) \equiv 0 \text{mod } \mathbb{Z}$ (since L has the same degree as D).

Definition 50. Let K be a local field and $A \in Br(K)$. Then A is represented by some finite dimensional division algebra D with center K. We can associate to D the Frobenius automorphism $Frob_D$, and as above, we see that it gives rise to a unique element $ord(\alpha) \in \mathbb{Z}/n$. We define the local invariant map $inv_K : Br(K) \to \mathbb{Q}/\mathbb{Z}$ as $inv_K(A) = ord(\alpha)$.

Theorem 51. *(The Brauer-Hasse-Noether Theorem) There is an exact sequence*

$$0 \longrightarrow H^2_{\acute{e}t}(Spec\ k, \mathbb{G}_m) \longrightarrow \oplus_p H^2_{\acute{e}t}(Spec\ k_p, \mathbb{G}_m) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

where the sum ranges over all places of k, and the map $inv : H^2_{\acute{e}t}(Spec\ k_p, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z}$ is the sum of the invariant maps.

Proof. [16] Chapter 7 , Theorem 4.2 □

This is a very deep theorem and a generalization of classical reciprocity theorems in number theory. We are now finally in position to define the Brauer-Manin obstruction. Note that we for a scheme X and an element $C \in Br(X)$ and a scheme-morphism $f : Y \to X$ we can define the pullback $f^*C \in Br(Y)$. We now define the Brauer-Manin pairing:

$$Bp : Br(X) \times X(\mathbb{A}_k) \to \mathbb{Q}/\mathbb{Z}$$

by

$$Bp(C, \{\alpha_p\}) = \sum_p inv_p(x_v^* C).$$

Intuitively, in the case of an Azumaya-algebra, we evaluate the Azumaya algebra at the adelic point and compute the local invariants of the central simple algebras it gives rise to.

Theorem 52. *For any choice* $(C, \{\alpha_p\}_p) \in Br(X) \times X(\mathbb{A}_k)$*, the sum* $Bp(C, \{\alpha_p\}) = \sum_p inv_p(x_v^* C)$ *is finite.*

Proof. See [19] 5.2 □

We now say that $C \in Br(X)$ is in the left kernel of the Brauer-Maning pairing if for all $\alpha_p \in X(\mathbb{A}_k)$ the Brauer-Manin pairing is zero. We denote this left kernel by $kerB_m$.

Theorem 53. *Let* $x \in X(k)$ *Then we have that* $(\{x\}_p) \in kerBm$*.*

Proof. We have a commutative diagram :

$$\begin{array}{ccccccccc} & & X(k) & \longrightarrow & X(\mathbb{A}_k) & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & Brk & \longrightarrow & \sum_p Brk_p & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \end{array}$$

and this clearly gives that an element $x \in X(k)$ will get mapped to 0. □

Definition 54. (The Brauer-Manin obstruction) The Brauer-Manin obstruction $X(\mathbb{A}_k)^{Br}$ of a smooth and geometrically integral variety over k is defined to be ker Bm as above.

We say that the failure of the Hasse Principle is accounted for by the Brauer-Manin obstruction if $X(\mathbb{A}\neg) \neq \emptyset$ but $X(\mathbb{A}\neg^{Br})) = \emptyset$. It is obvious by the above theorem that $X(k) \subset X(\mathbb{A}_k)^{Br}$, so that if $X(\mathbb{A}_k)^{Br} = \emptyset$, then $X(k)$ must be empty too.

Example 3.2. Swinnerton-Dyer's Del Pezzo surface
Let us consider the Del Pezzo surface X in $\mathbb{P}_{\mathbb{Q}}{}^4$ given by the equations

$$uv = x^2 - 5y^2$$

and

$$(u+v)(u+2v) = x^2 - 5z^2.$$

This variety gives a failure of the Hasse Principle that is accounted for by the Brauer-Manin obstruction. Let us first begin by showing that this surface has points in every completion of $\mathbb{Q}$. Note that if $p \neq 2$, one of -1, 5 or -5 will be a p-adic square. We can check that this is true as follows. Let us first suppose that $p \neq 2$. Then we have the polynomial $f(x) = x^2 - b \in \mathbb{Q}_p[x]$ where b is either -1,5 or -5. We note that at least one of our elements are squares mod p , and by Hensel's lemma it is a p-adic square.So, it is then easy to see that one of the three points $(1, 1, 1, 0, \sqrt{-1})$, $(10, -10, 5, 5, \sqrt{5})$ and $(5, 0, 0, 0, \sqrt{-5})$ is a rational point in the completion. For the completion at 2, $(-25, 5, 0, 5, 2\sqrt{-15})$ is a point lying on our surface.

Let us suppose that it has a rational point, then we can easily see that not both u and v can be zero, and we can assume them to be coprime integers. With this condition, it is not true that x y or z has to be integers however. For the 5-adics, it is easy to see x and y are integers there, and that if $5|uv$ then 5 will also divide x, and 5 then divides $x^2 - 5z^2$. But this is absurd, u and v are coprime and therefore 5 can not divide $(u+v)(u+2v)$. So, 5 can't divide uv and we can do the same argument and show that 5 can't divide $(u+2v)(u+v)$.If we consider a prime $p \equiv 2, 3 \text{mod } 5$ then, since mod 5, we have $uv = x^2$ and u and v are coprime no odd power of p can divide u or v. This gives us that $u, v \equiv \pm 1 \text{mod } 5$. We can now see that mod 5, since both $u+v$ and $u+2v$ are coprime, no odd power of a prime p equivalent to 2 or 3 mod 5 can divide either of them. This gives that $u + v \equiv \pm 1$ which is a contradiction, so there are no rational points.

We will now construct an Azumaya algebra and show that the Brauer-Manin obstruction accounts for this failure of the local-global principle. We will give some basic properties of quaternion algebras over general rings for this.

Definition 55. Let K be a field. A Quaternion algebra C is then a central simple algebra that has dimension 4 over K. When the characteristic of K is not 2, one can describe a quaternion as a vector space over K with basis $\{1, i, j, k\}$ that satisfies the following multiplication rules:

$$i^2 = a$$
$$j^2 = b$$
$$ij = k$$
$$ji = -k.$$

for some non-zero $a, b \in K$. One sees that $k^2 = -ab$, and we let (a, b) (where $a, b \in K$ and non-zero) be the quaternion algebra defined as above.

We say that a quaternion algebra C is trivial if it is isomorphic to the matrix ring $M_2(K)$.

Definition 56. Hilbert Symbol Let us define a function $K^* \times K^* \to \{-1, 1\}$ for a field K by $(a, b)_K = 1$ if $1 = ax^2 + by^2$ has a solution in $K^* \times K^*$ and -1 otherwise.

The Hilbert Symbol is a central tool in local class field theory. For our purposes, we will reinterpret it in terms of quaternion algebras. Let K be a local field with $charK \neq 2$. Then $(a, b)_K = 1$ iff the quaternion algebra (a, b) over K is trivial, which is easy to prove given the above descriptions.

In can be shown that $(5, u/(u + v))$ is an Azumaya algebra on X [20]. Now, for a local field K $(5, u/(u + v))$,where u and $u + v$ are not zero, can be interpreted as a Hilbert Symbol, giving 1 if it is isomorphic to the matrix algebra of 2 by 2 matrices. We have that, by the usual properties of the Hilbert symbols that since 5 is a square in $\mathbb{R}$, that for an infinite place P, $(5, u(P)/(u(P) + v(P)))$ is isomorphic to $M_2(\mathbb{R})$) (one sees that the equation $1 = 5x^2 + (u(P))/(u(P) + v(P))y^2$ has a solution in $\mathbb{R}$. This gives that the invariant map is zero in this case, since it will represent the trivial element (this can easily be seen by noting that the invariant map is an isomorphism and the only division algebras over the real numbers are the reals itself and the quaternions).

Now, the map taking a point to the invariant can be shown to be continuous, implying that the invariant map is zero for all points in $X(\mathbb{R})$. Now, since for an odd prime p where 5 is a square, the same argument as we did for the reals can be done mutatis mutandis for $X(\mathbb{Q}_p)$ to show that the invariant map then is 0. For the case when 5 is not a square, we must be more delicate. Say that p is an odd number not equal to 5 and that 5 is not a square mod p. A slightly tricky computation shows that the hilbert symbol is 1, giving that the invariant map is zero. When p is 5 , it can be shown that the Hilbert symbol is -1 and that thus, the invariant map takes our quaternion algebra to 1/2. When p is 2, it can be shown that the invariant map is zero, once again by a computation by the hilbert symbol and the defining equations for X. We see that for any adelic point P, there is an Azumaya algebra not being zero when pairing P with it, so that $X(\mathbb{A}_{\mathbb{Q}})^{Br} = \emptyset$.

3.3 Descent obstruction

Definition 57. (Algebraic group) An algebraic group G over a field k is a group scheme which is smooth as a k-variety . An algebraic group is affine if the following extra condition is satisfied:
$G = \text{Spec } A$ for some ring A.

So, let X be a k-variety. To be able to define the descent obstruction we need some machinery from algebraic geometry.

Definition 58. Let G be a group scheme over X. An action of G on a X-scheme Y is a morphism

$$G \times_X Y \to Y$$

that gives us a group action of $G(T)$ on $Y(T)$ for every X-scheme T.

Definition 59. (Torsors)

Let G be a group scheme over X. A G-torsor over X is an X-scheme Y on which G acts, together with a faithfully flat and finitely presented morphism $Y \to X$, and further, the map $Y \times_X G \to Y \times_X Y$ is an isomorphism.

This is also equivalent (see [16] Ch. III sect. 4) to: There is a covering $U_i \to X$ for the flat topology on X such that for each i, S_{U_i} with its induced G_{U_i}-action is isomorphic to G_{U_i}. By the Yoneda lemma, that the morphism $Y \times_X G \to Y \times_X Y$ is an isomorphism translates to the fact that for all X-schemes Z, $G(Z)$ acts transitively on $Y(Z)$. We say that a G-torsor Y over X is trivial if it is isomorphic to the G-torsor $G \to X$ with the right action of G. This gives us that a G-torsor Y is trivial iff $Y(X) \neq \emptyset$, where $Y(X) = Hom_{\text{X-sch}}(X, Y)$. Indeed, an element $f \in Y(X)$ gives rise to asection of $Y \to X$. This gives us an isomorphism with G, the isomorphism $G \to Y$ given by $g \to f(g)$. It would now be nice if we could classify all G-torsors over X up to isomorphism in terms of some "nice" pointed set. This will be essential later in defining the descent obstruction and to find this pointed set, we need to use some notions from non-abelian cohomology.

Let $\mathcal{F}$ be a group sheaf on X , and let us consider some covering $U = \{U_i \to X\}_{i \in I}$ with respect to some topology.Let us write U_J for the fiber products of U_i with $i \in J \subset I$. A 1-cocycle with respect to $\mathcal{F}$ is a family of elements $\{g_{ij}\}_{I \times I}$ where $g_{ij} \in \mathcal{F}(U_i \times_X U_j)$ such that the family satisfy the cocycle conditions, that is on triple intersections $U_{ijk} = U_i \times_X U_j \times_X U_k$ we have that $(g_{ij}) \cdot (g_{jk}) = g_{ik}$. We form the set $\check{H}^1(U, \mathcal{F})$ by taking all the 1-cocycles and identifying cocycles that are cohomologous. Two cocycles g and g' are said to be cohomologous if there is a (h_i) where $h_i \in \mathcal{F}(U_i)$ such that on the intersection $U_{ij} = U_i \times_X U_j$ we have that $g'_{ij} = h_i g_{ij}$.

We define the first non-abelian sheaf cohomology group $H^1(X, \mathcal{F})$ of $\mathcal{F}$ as $\lim_{\rightarrow} \check{H}^1(U, \mathcal{F})$ where U ranges over the category of coverings on X. The distinguished element is the element which is the identity everywhere, so that this is a pointed set.This allows us to classify torsors of G over X as follows. First, we make the assumption that G is smooth over X, so that every G-torsor is locally trivial for some étale covering .

Theorem 60. *($H^1(X, G)$ corresponds to G-torsors over X) Let G be an affine algebraic group scheme. There is a bijection:*

$$\{\textit{elements } g \in H^1(X, G)\} \rightleftarrows \{ \textit{ isomorphism classes of G-torsors over } X \}$$

where the trivial G-torsor (i.e a torsor that is isomorphic to G acting on itself) corresponds to the distinguished element of $H^1(X, G)$.

Proof. Let Y be a G-torsor over X, and $U_i \to X$ be an étale cover on which it is locally trivial. Then, we have that for each U_i, the set $Y(U_i)$ is non-empty, so take an element

$x_i \in Y(U_i)$ for all i. By the condition that Y is a G-torsor, there is an unique $g_{ij} \in G(U_{ij})$ such that $g_{ij}x_i = x_j$. It is easy to check that picking such a g_{ij} for all $(i,j) \in I \times I$ we get a G-cocycle . If we change the choice of x_i we will get a cohomologous cocycle, and it is invariant under isomorphism classes, thus well-defined. We define an inverse mapping to this. Let us take $U = (U_i \to X)$ to be an étale covering of X and form the sheaves $\mathcal{C}_0, \mathcal{C}_1$ by $\mathcal{C}_0(W) \to \Pi_i G(U_i \times W)$ and $\mathcal{C}_1 W \to \Pi_i G(U_i \times_X U_j \times_X W)$. There is an obvious map $d : \mathcal{C}_0 \to \mathcal{C}_1$ which takes $g_i \in \mathcal{C}_0(W)$ to $g_i^{-1}g_j$. Then, if we take a 1-cocycle g on U, we define a subsheaf $S \subset \mathcal{C}_0$ by $S(V_i) =$ sections of $\mathcal{C}_0$ that arises as inverse images under d of the natural element g gives on $\mathcal{C}_1(V_i)$. We have an action of G on this sheaf, $(x_i, g) \to (g^{-1}x_i)$. By this we can easily see that the trivial cocycle gets mapped to the sheaf S that on each U_i is G_{U_i} . We now have one operation where we take a G-cocycle and get a certain sheaf of sets S (a G-fppf torsor over X) that is, a sheaf of sets together with an action of G on S satisfying analogous conditions that a G-torsors where Y is a scheme had to satisfy, that is , $G \times_X S \to S \times_X S$ is an isomorphism .Now, by the analogous process before the statement of this theorem, we get a G 1-cocycle for each G fppf-torsor over X and these are mutually inverse, as one easily checks. Further, since G is affine, every G-fppf torsor over X is represantable by a scheme. This gives the theorem. □

It turns out that for a k-variety X, the torsors partition the k-valued points. Let us take $f : Y \to X$ to be a G-torsor for an affine algebraic group G, by the above, it represents an element $x \in H^1(X, G)$.For each k-valued point $y \in X(k)$ we have the fiber of Y over y, $Y_y \to y$ which is a k-torsor as can easily be checked. So to summarize, for each torsor $f : Y \to X$ we get a mapping $\lambda_f : X(k) \to H^1(k, G)$. We have that this partitions $X(k)$ as follows. Let us take a G-torsor f and look at the set $\{x \in X(k)$ such that $\lambda_f(x) = y\}$ for $y \in H^1(k, G)$, that is, $\lambda_f^{-1}(y)$.

Theorem 61. *Let X be a k-variety. For each $x \in \check{H}^1(k, G)$ for an affine algebraic group G, x represented by $f : Y \to X$, we can define for each $y \in H^1(k, G)$ a twisting operation on torsors, producing a "twisted torsor" $f^y : Z^y \to X$ such that $\{z \in X(k) : \lambda_f(z) = y\} = f^y(Z^y(k))$, this gives us that the partition of $X(k)$ given above can be written as $X(k) = \amalg_{x \in H^1(k,G)} f^x(Z^x(k))$.*

Proof. [17] Theorem 8.3.1. □

We are now ready to start to define the descent obstruction.

Definition 62. (Descent set determined by a G-torsor) Let X be a smooth geometrically integral variety over a global field k. For a smooth affine algebraic group G over X, the torsor $f : Y \to X$ gives us a set $X(\mathbb{A}_k)^f = \cup_{\rho \in H^1(k,G)} f^\rho(Y^\rho(\mathbb{A}_k))$, that is, elements $(x_i) \in X(\mathbb{A}_k)$ such that the image of (x_i) under the natural evaluation map $X(\mathbb{A}_k) \to \Pi_{\text{p places of k}} H^1(k_p, G)$ comes from an element of $H^1(k, G)$.

Since mapping an element $f \in X(k)$ first to $X(\mathbb{A}_k)$ under the natural map $X(k) \to X(\mathbb{A}_k)$ and then applying the natural evaluation $X(\mathbb{A}_k) \to \Pi_{\text{p places of k}} H^1(k_p, G)$ clearly lies in $X(\mathbb{A}_k)^f$, we have that $X(k) \subset X(\mathbb{A}_k)^f$.If we cut this set down with all torsors, that is, $X(\mathbb{A}_k)^{\text{desc}} = \cap_{G,\text{ G-torsors } f:Y\to X} X(\mathbb{A}_k)$ where the intersection runs over all possible affine algebraic group schemes G and all G-torsors over X.We see that $X(k) \subset X(\mathbb{A}_k)^{\text{desc}} \subset X(\mathbb{A}_k)$.It can be shown that the Brauer-Manin obstruction above is equivalent to an analogous descent obstruction where we let G run over all groups PGL_n. So, descent obstruction is strictly stronger than Brauer-Manin obstruction.

3.4 The best of two worlds - The étale Brauer set

We will now come to the last of the classical obstructions to work with before we can show how these diverse constructions can be unified in terms of étale homotopy. As above, let $x \in H^1(k, G)$ and X be a k-variety and G an affine algebraic k-group. For each G-torsor over X, each twist of $f : Y \to X$, we have that $Y^x(k) \subset Y^x(\mathbb{A}_k)$. This implies that we must also have $Y^x(k) \subset Y^{x,Br}(\mathbb{A}_k)$. This leads us to defining for a G-torsor f:

$$X(k) \subset X^f(\mathbb{A}_k)^{Br} = \cup_{x\in H^1(k,G)} f^x(Y^x(\mathbb{A}_k)^{Br}).$$

Let us now take the intersection of all these sets for all G-torsors f over X ,

$$X^G = \cap_{f \text{ G-torsor over X}} X^f(\mathbb{A}_k)^{Br}.$$

Doing this for each finite affine algebraic group G, we get

$$X(\mathbb{A})^{\text{ét,Br}} = \cap_{G \text{finite affine alg. group}} X^G$$

which we call the **étale Brauer-Manin set.**

3.5 The homotopy fixed point set

We shall now try to unify some of these obstructions. Let G be a finite group. For a G-set X we are used to thinking of the fixed points X^G of the G-action on X as the set $\{x \in X | gx = x \forall g \in G\}$. For our purposes, we want to study the homotopy groups of X^G but this is not the correct notion if we just use the regular definition, we will now briefly sketch why. The reason for X^G being the "wrong" space to study stems from the fact that X^G is not homotopy invariant. Indeed, let us consider $\mathbb{R}^2$ with the $\mathbb{Z}$-action given by $n(x, y) \to (x+n, y+n)$. Then, $(\mathbb{R}^2)^{\mathbb{Z}} = \emptyset$ but if we consider a point with trivial $\mathbb{Z}$-action, the fixed point set is not empty, but a point. $\mathbb{R}^2$ is homotopy equivalent to a point and the actions are compatible, so that the fixed point set is indeed not homotopy invariant . An idea would be to try to find a version of fixed points that is invariant under homotopy. For this, we might conjecture that we take a category with a nice model structure and construct some kind of homotopy fixed point set, by derived functors of some sort. We will see that indeed, in some favourable cases, this can be done. We start by noting that a fixed point of a G-set X is really a G-morphism $pt \to X$, which is equivariant and the point is given the trivial action. For this to be a homotopy invariant idea we must consider these notions to be the same for contractible spaces (which are homotopy equivalent to a point). We have an universal contractible G-space, EG.

Definition 63. (The universal contractible space) Let C be some category of spaces on which G acts and where we have some definition of weak equivalences of spaces. EG, the universal contractible space is defined, if it exists, by the following universal property: For any contractible space X with a G-action, there is a G-morphism $EG \to X$ unique up to homotopy .

Milnor gave a construction for EG for topological groups .An example in the category of simplicial sets with the standard model structure is that for a finite group G, we define $EG = cosk(G)$, where G is viewed as a functor $pt = \Delta^{op}_{\leq 0} \to Set_G$. This is a simplicial set which in degree n simply is G^{n+1}. It is easy to see that the action of G on EG is free and that EG is contractible, by our section on simplicial sets. That it is universal with this property follows from the universal property of the coskeleton functor.

So, with all this, what is a categorical description of fixed points? Well, given a G-object X the fixed point object should be an object X^G with a G-map $X^G \to X$ that is unchanged when we act on it by G. This clearly is analogous to our definition of a limit (!). So, given a G-set X,it is clear that we can consider it as a functor $X : G \to Sset$. The limit of this functor have all the properties we want of a fixed point object.We want to make this homotopical, and general philosophy suggests that we should try derived functors. We have the following theorem:

Theorem 64. *Let us endow $Sset^G$ with the injective model structure (see 2.7 Ex. 37) . We then have a Quillen adjunction $const : Sset \leftrightarrows Sset^G : lim$.*

Proof. It is trivial to note that the constant functor preserves cofibrations and trivial weak cofibrations, and as such the adjunction is a Quillen adjunction. ☐

With this, we make the following definition:

Definition 65. (The homotopy fixed point set) Let X be a simplicial G-set and assume $Sset^G$ has the injective model structure (see 2.7 Ex. 37). Then, the homotopy fixed point $X^{hG} \in Sset$ is given by $holimX$, where X is considered as a functor from G in the obvious way.

So to compute the homotopy fixed point set we only need to replace X with a fibrant replacement X^{fib} and then apply lim.We have that lim is simply given by $Map_G(*, -)$, where Map_G is the simplicial mapping space of G-equivariant maps. So, in this case, $X^{hG} = Map_G(*, X^{fib})$. But by the following section on derived mapping spaces,

$$Map_G(*, X^{fib}) \cong Hom_{Der}(*, X).$$

Now, the derived mapping space only depends on the weak equivalences, and as such, we can compute it in the projective model structure. We have that in the projective model structure, we have very handy fibrant replacements. In the projective model structure, $Hom_{Der}(*, X) \cong Map_G(EG, Ex^{\infty}X)$. So we have a well-defined notion of a homotopy fixed point set in either model structure on simplicial G-sets. For G infinite, there is no projective model structure on $Sset^G$. To remedy this, we go to the procategory. We see that $Sset^G$ can be seen as a weak fibration category, where weak equivalences are the ones inducing weak equivalences on the underlying simplicial sets and fibrations the ones inducing fibrations on the underlying simplicial set. Moreover, it is a admissible weak fibration category.

Let us briefly try to give the reader some vistas on the ideas of the obstructions Harpaz-Schlank gave, and where it all stems from. We have a well-developed obstruction theory for G-equivariant complexes, which is analogous to ordinary obstruction theory. Remember that for a CW-complex X , say that we have a map f on the (n-1)-skeleton, there are certain obstruction theories that tells us when this map can be extended to the n-skeleton. Let $p : E \to X$ be a fiber bundle with fiber F that we assume to be connected, and where all spaces are CW-complexes. Say that we want to find a section of p. We can assume each simplex of X is contained in some trivializing neighborhood for p. So, we try to inductively build this section. On the 0-skeleton of X^0 we just take any point over the fiber over each 0-cell. We extend it over the 1-skeelton, and continuing in this way, we can find an obstruction to the existence of a section of p over the 2-skeleton , an element of $H^2(X, \pi_1(F))$. Then the section can be extended if the element is 0 in the cohomology group. If we can extend the section over the 2-skeleton , we can find an obstruction to extending it over the 3-skeleton in $H^3(X, \pi_1(F))$ etc. Barnea-Schlank had an idea for extending this to schemes, by their Jerusalem machine. They give a functor F that takes each k-variety $p : X \to k$ to an object such that each section gives a map $F(speck) \to F(X)$ which has a well-defined obstruction theory. The first obstruction lives in $\pi_0(F(X)^{h\Gamma_k})$ where Γ_k is the absolute galois group, and we have obstructions $H^i(\Gamma_k, \pi_1(F(X))$ for each $i \geq 2$ which is defined if the previous one vanishes. They explore the first of these obstructions in their paper, and we will stop there too.

3.6 Hammock localization

The idea of Hammock localiation is more intricate than the idea of simply localizing a category. If we consider the category of topological spaces and we just localize with respect to weak equivalences, we'll identify homotopic maps. However, this might lose more information than we want, what if we actually want to know in what ways f and g are homotopic? This is done by the Hammock localization, which was defined by Dwyer-Kan in [6].

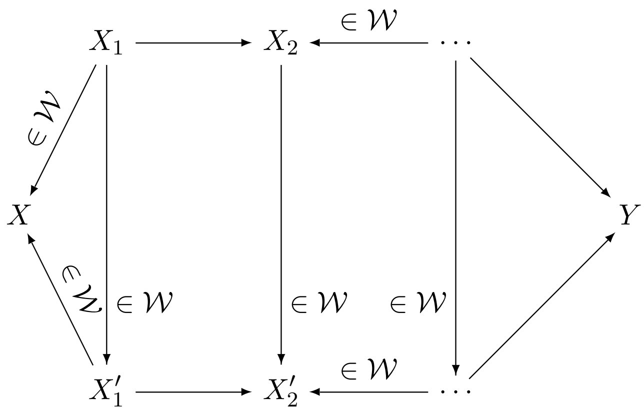
Definition 66. A relative category C is a category C together with a subcategory $W \subset C$ such that W contains all objects of C.

Definition 67. Let C be a relative category. Define a new category, $L^H C$ where $Ob(L^H C) = Ob(C)$, and morphisms between $X, Y \in Ob(L^H C)$ are given by a simplicial set, called a derived mapping space, $Hom_{Der}(X, Y)$, that is given as the nerve

of the category defined in the following manner:
1.The objects consists of zig-zags of morphisms in C,

$$X \xleftarrow{\in \mathcal{W}} X_1 \to X_2 \xleftarrow{\in \mathcal{W}} \to \cdots \to Y.$$

2. Morphisms are given as equivalence classes of diagrams:



where the endpoints are fixed, and two such hammock diagrams are equivalent if we can obtain one from another by composing two consecutive maps (and in the other way, taking a composed map into two), adding identities, or reversing isomorphisms. Composition of morphisms are given by concenating two hammock diagrams next to eachother.

This is all very abstract, so it might be soothing with some concreteness as we try to come to grip with this. We call the 0-simplicies in $Hom^{Der}(X,Y)$ vertices and 1-simplicies edges.

Theorem 68. *If C is a relative category, and we consider $L^H C$, there is an equivalence of categories $\pi_0(L^H C) \cong Ho(C)$, where π_0 takes the connected components of the derived mapping space between two objects.*

Proof. For an object

$$X \xleftarrow{\in \mathcal{W}} X_1 \to X_2 \xleftarrow{\in \mathcal{W}} \to \cdots \to Y$$

the object

$$X \xleftarrow{\in \mathcal{W}} X'_1 \to X'_2 \xleftarrow{\in \mathcal{W}} \to \cdots \to Y$$

lies in the same connected component iff there is a simplicial homotopy between them, by the exponential law. However, note now that every such 0-simplex can be seen as a map in $Hom_{Ho(C)}(X,Y)$, just invert the left arrows (which we can, since we're in the homotopy category). Further, if we do this, we see that if two 0-simplicies of

$Hom_{Der}(X,Y)$ are in the same component they must clearly also be homotopic as maps in $Hom_{Ho(C)}(X,Y)$. Every homotopy class of maps $f : X \to Y$ can also be seen as a 0-simplex in $\pi_0(Hom_{Der}(X,Y))$. So, this map which is given by inverting the left arrows is surjective and further, injective since we identify two objects iff they are homotopic as maps. This gives us in an obvious way a functor which is essentially surjective and fully faithful and thus our theorem follows. □

We also have a morphism of sets $Hom_C(X,Y) \to \pi_0(Map_{Der}(X,Y))$. That this is useful for our considerations can be seen through the following theorem.

Theorem 69. *If C is a simplicial model category, and $X, Y \in Ob(C)$, then $Hom_{Der}(X,Y)$ is weakly equivalent to $Map(X_*, Y^*)$ where X_* is the cofibrant replacement and Y^* the fibrant replacement.*

Proof. [6] □

We now define the notion of a homotopy fixed point in more generality.

Definition 70. Let $X \in C$, where C is a relative category. with a terminal object. Define the space of derived sections as $Hom_{Der}(*,X)$. We call the elements in π_0 of this space derived sections. When C is a category where each object has an action of the group G on it, we call the space of derived sections the space of homotopy fixed points, and the elements in π_0 homotopy fixed points and denote it by X^{hG}.

3.7 The homotopy obstruction

So, let us briefly consider our plan. We will want to do what we did for simplicial sets with a G-action and mimic this for schemes. The above hints that we should try to define homotopy fixed points as being π_0 of some derived mapping space. So we need to construct a derived mapping space and some analogous category to that of simplicial G-sets.

With this, let X and S be schemes, and $f : X \to S$ be a morphism, where we consider X as an S-scheme. Recall that we have the notion of the relative étale shape of X over S to be an object

$$\acute{\mathrm{E}}\mathrm{t}_{/S}(X) \in Pro(Sh(S_{\text{ét}}^{\Delta^{opp}}))$$

given by

$$\mathbb{L}L_{f^*}(*) \in Pro(Sh(S_{\text{ét}}))^{\Delta^{op}}$$

where $*$ is the terminal sheaf, using the Jerusalem machine. Here we will show how the existence of a section $g : S \to X$, gives an element of

$$\pi_0(Hom_{der}(*_{S_{\text{ét}}}, Et_{/S}(X)))$$

where $*_{S_{\text{ét}}}$ denotes the terminal pro-simplicial sheaf on the étale site of S. Note that since $\mathbb{L}L_{f^*}$ and $Pro(f^*)$ are adjoint , we have for each object $A \in Pro(Sh(S_{\text{ét}}^{\Delta^{opp}}))$ a map $A \to Pro(f^*)\mathbb{L}L_{f^*}A$. Noting that for a section we have $g^*f^* = (fg)^* = 1$, so if we apply g^* to the unit morphism

$$*_S^* \to Pro(f^*)\mathbb{L}L_{f^*}(*_S^*)$$

where $*_S^*$ is the cofibrant replacement, we get $g^*(*_S^*) \to \mathbb{L}L_{f^*}(*_S^*) = \text{Ét}_{/S}(X)$.$g^*$ is a weak Quillen functor, and as thus preserves trivial fibrations, and in particular

$$g^*(*_X^*) \to g^*(*_X) = *_S.$$

Noting that

$$\pi_0(Hom_{der}(*_{S_{\text{ét}}}, Et_{/S}(X)))$$

is in bijection with the homotopy classes of maps between $*_{S_{\text{ét}}}$ and $Et_{/S}(X)$, it follows that we get a homtopy class of a map

$$*_{S_{\text{ét}}} \to Et_{/S}(X)$$

for each section. So, we have a map

$$X(S) \to \pi_0(Hom_{der}(*_{S_{\text{ét}}}, Et_{/S}(X))).$$

If the latter is empty, we can thus have no sections, in other words, no S-points . The latter object is called the set of derived sections. This method can be tweaked to produce similar obstructions for sections of a map of topoi.To show that $\pi_0(Hom_{der}(*_{S_{\text{ét}}}, Et_{/S}(X)))$ is empty, we can find just one object of the inverse system $Et_{/S}(X)$ for which there is no derived section.

3.8 More on the relative etale homotopy type

Clouded by all abstraction we ought to see how to apply this. There is a rather concrete way, but we will need to make sure that $\pi_0(Hom_{der}(*_{S_{\text{ét}}}, Et_{/S}(X)))$ is well-behaved. Let us work with the case that originally interested us, $S = \text{Spec } k$ for k a field. In this case, a good first step is to understand $Et_{/\text{Spec } k}(X)$. We have that the structure map $f : X \to speck$ gives us a geometric morphism of topoi as usual, that is

$$f_* : Sh(X) \to Sh(k) : f^*.$$

We will now find a left adjoint to f^*. Let us, as usual, consider sheaves on $Sh(k)$ as sets with a continuous action of the absolute galois group. Under this identification, the inverse image functor will be given by the constant sheaf functor. A left adjoint $f_!$ is then induced by the functor $\pi_0 : X_{et} \to \Gamma - set = Speck_{et}$ that takes an etale scheme U over X_{et} to the connected components of $U \otimes Spec\overline{k}$. Note that this has a natural action of Γ and we can view it as an element of $\Gamma - set$.

So, we have a description of the left adjoint $f_!$, induced by π_0 that is rather concrete. Going to the pro-category of simplicial sheaves on X_{et}, we want to consider what is done to the terminal sheaf, that is $* \in Pro(Sh(X_{et})^{\Delta^{Opp}})$.By an argument completly analogous to that of showing that the classical etale homotopy type of Artin-Mazur was isomorphic, in the homotopy category of prosimplicial sheaves, to the image of the derived functor approach, we see that $Et_{/\text{Spec }k}(X)$ is weakly equivalent to $\grave{\text{E}}\text{t}(X \times \overline{k})$. So, for studying derived sections, it is no loss to consider $\acute{\text{E}}\text{t}_{/\text{Spec }k}(X)$ as $Et(X \otimes \overline{K})$. Now, in the homotopy category of Pro-simplicial Γ-sets, the image of the cofibrant replacement of $*$ can be seen to be given by the inverse system of Kan contractible simplicial Γ-sets (that these objects really are an inverse system, if we go to the homotopy category, is shown in [10] Prop 8.3). Let us call the subcategory of $Ho(Set_\Gamma^\Delta)$ consisting of Kan contractible simplicial Γ-sets for D.

Theorem 71. *The full subcategory $D_\Gamma^{fin} \subset D$ consisting of Kan contractible objects that are levelwise finite is cofinal, if we go to the homotopy category.*

Proof. [10] Lemma 4.2 □

So, we can see a derived section as a homotopy class of a map in $Pro(\Gamma - Set^{\Delta^{opp}})$ from D^{fin} to a fibrant replacement $\{X'_\alpha\}$ of $Et(X \times \overline{K}) = \{X_\beta\}$. So, it is $[D_i, X_\alpha] = lim_\alpha colim_i [D_i, X_\alpha] = \lim_i \pi_0(X^{h\Gamma})$.The latter representation of the derived sections is less conceptual, but good for computational purposes. Let us set $X(hK) = \pi_0(Hom_{der}(*_{S_{\acute{e}t}}, Et_{/S}(X))$ for the set of homotopy fixed points.

3.9 The étale homotopy obstruction

Definition 72. (The # -functor.) Let

$$X = \{X_\alpha\}_{\alpha \in I}$$

be a Kan pro-simplicial set. Define $X^\# = \{P_n(X_\alpha)\}_{n \in \mathbb{N}, \alpha \in I} = \{cosk_{n+1} \circ tr_n X_\alpha\}_{n \in \mathbb{N}, \alpha}$. This consists of all Postnikov pieces of the objects in the inverse system defining X.

Let Let k be any global field, X a variety over k, and note that we have, for each place p a map $f_p : \text{Spec } k_p \to \text{Spec } k$ inducing a geometric morphism, and we can pull-back $Et_{/k}(X)^\#$ to $f^*(\acute{\text{E}}\text{t}_{/k}(X)^\#)$, and setting $X(hk_p) = \pi_0(Hom_{der}(*, f^*(Et_{/k}(X)^\#))$ we have that the following is a commutative diagram of sets, which can easily be checked:

$$\begin{array}{ccc} X(k) & \longrightarrow & \pi_0(Hom_{Der}(*, Et_{/k}(X))) = X(hk) \\ \downarrow & & \downarrow \\ X(k_p) & \longrightarrow & X(hk_p). \end{array}$$

Taking this over all places, we get a diagram

$$\begin{array}{ccc} X(k) & \longrightarrow & X(hk) \\ \Big\downarrow & & \Big\downarrow \eta \\ \Pi_p X(k_p) & \xrightarrow{j} & \Pi_p X(hk_p). \end{array}$$

Definition 73. Let us consider the diagram above. Say that $(x_p) \in \Pi_p X(k_p)$ is homotopy rational if it is in $X(\Pi_p K_p)^h = j^{-1}\eta(X(hk))$.

This is the first of the obstructions defined by Harpaz-Schlank, only this time dressed up in a different language. If we have $X(\Pi_p K_p)^h = \emptyset$ but $X(\mathbb{A}_k) \neq \emptyset$, we say that there is an homotopy rational obstruction to the local-global principle.

One wants to construct something similar for adelic points, but this is not so easy. Harpaz-Schlank did it originally with an entirely different (but in the end, yielding equivalent obstructions) constructions than the one using derived mapping spaces. I will mention how the adelic homotopy fixed points can be constructed. All ideas are from [18]. What we want is to find some notion of adelic homotopy fixed points. We want to see it as some π_0 of a space of derived sections. It turns out that there is an adelic topos, $\mathbb{A}_{\mathbb{T}}$ and we can construct the homotopy fixed point space as the derived maps from the terminal sheaf to the relative shape of X over this adelic topos.

Definition 74. Let K be a number field and let ${\mathbb{A}_{\mathbb{K}}}^{pre}$ be the following category. An object X of C is a collection of sets $\{x_p\}$ one for each place in p, with an action of Γ_p on it so that for almost all p, the action of Γ_p factors through the unramified quotient Γ_p/I_p, where I_p is the inertia group, and such that there is some natural number N such that the size of each $\{x_p\}$ is less than N.The morphisms between two objects X and Y of this category consists of a collection of morphisms $\{f_p\}$, one for each place in p, such that $f_p : x_p \to y_p$ is a morphism of Γ_p-sets .

Definition 75. With notation as above, we define the adelic topos of K ,${\mathbb{A}_{\mathbb{K}}}^{top}$ to be $Ind({\mathbb{A}_{\mathbb{K}}}^{pre})$.

It can be shown using Giraud's axioms, that this is a topos. There is a morphism f^* from the etale topos of K to ${\mathbb{A}_{\mathbb{K}}}^{top}$ as follows. Every Γ-set is a filtered colimit of finite Γ-sets, and so we only have to define the natural map on finite sets, and this is done by inclusion. The map is well-defined since an action of Γ on the finite set X is only ramified for finitely many places, which can be shown using some arguments from elementary algebraic number theory. This morphism will preserve limits and all small colimits, as can easily be checked, and thus it comes from a geometric morphism. Further, there is a natural map from ${\mathbb{A}_{\mathbb{K}}}^{et}$ to ${\mathbb{A}_{\mathbb{K}}}^{top}$, where ${\mathbb{A}_{\mathbb{K}}}^{et}$ is the etale topos of $\mathbb{A}_{\mathbb{K}}$, the ring of adeles

, that comes from a geometric morphism defined in a similar way. This gives us that we can define the adelic homotopy fixed points as $X(h\mathbb{A}_{\mathbb{K}}) = \pi_0(Map_{Der}(*, f^*\acute{\mathrm{E}}\mathrm{t}_{/K}(X))$, where $*$ is the terminal pro-simplicial sheaf on ${\mathbb{A}_{\mathbb{K}}}^{top}$. We have the following commutative diagram :

$$\begin{array}{ccc} X(K) & \longrightarrow & X(\mathbb{A}_{\mathbb{K}}) \\ \downarrow & & \downarrow \\ X(\mathbb{A}_{\mathbb{K}}) & \longrightarrow & X(h\mathbb{A}_{\mathbb{K}}). \end{array}$$

Theorem 76. *Let k be a number field. Let X be a geometrically connected smooth k-variety. Then:* $X(\mathbb{A})^{\acute{e}t,Br} = X(\mathbb{A})^h$.

Proof. [10] 11.1 □

This shows that we can unify some known obstructions using this method, but it doesn't seem to give any new obstructions. For number fields, this is certainly true, but one expects that over other fields one might be able to find stronger obstructions. Further, there is an analogous theory for flat morphisms, and it might be the case that it gives stronger obstructions.

3.10 The étale homology obstruction

We will now define the last obstruction for finding rational points given by Harpaz-Schlank. Let us suppose that we have applied the Postnikov functor to the relative etale homotopy type of X in what appears. If we have a simplicial Γ-set X, we can obtain a simplicial Γ-module by applying the functor $\mathbb{Z}$ levelwise. We have a terminal map $X \to *$ and this gives a map $\mathbb{Z}X \to \mathbb{Z}$. For a variety over the field k, we can apply this functor $\mathbb{Z}$ in the diagram defining the relative etale homotopy type, which we will call $\mathbb{Z}Et_{/k}(X)$. We have a natural transformation $Et_{/k}(X) \to \mathbb{Z}Et_{/k}(X)$, and since the latter is also a simplicial Γ-set, we can define $X^{\mathbb{Z}}(hK)$ as well as $X^{\mathbb{Z}}(h\mathbb{A}_k)$ as the homotopy fixed points and adelic homotopy fixed points of $\mathbb{Z}Et_{/k}(X)$ respectively.The main merit in this approach is that we, through the Dold-Kan correspondence can study the latter using methods of homological algebra.

We have a commutative diagram:

$$\begin{array}{ccccc} X(k) & \xrightarrow{i} & X(hk) & \xrightarrow{f} & X^{\mathbb{Z}}(hk) \\ \downarrow{\scriptstyle loc} & & \downarrow{\scriptstyle hloc} & & \downarrow{\scriptstyle hloc_{\mathbb{Z}}} \\ X(\mathbb{A}_k) & \xrightarrow{j} & X(h\mathbb{A}_k) & \xrightarrow{g} & X^{\mathbb{Z}}(h\mathbb{A}_k) \end{array} .$$

Definition 77. The étale homology obstruction is defined, by the diagram above, to be $X(\mathbb{A}_k)^{\mathbb{Z}h} = j^{-1}(g^{-1}(hloc_{\mathbb{Z}}(X^{\mathbb{Z}}(hk))))$.

Theorem 78. *If X is a smooth and connected variety over k,* $X(\mathbb{A}_k)^{\mathbb{Z}h} = X(\mathbb{A}_k)^{Br}$.

Proof. [10] 10.1 □

4 Final remarks

Let us in this section take a step back and try to put what we have done in a larger perspective and discuss the pros and cons of the homotopy obstructions compared to the classical obstructions and also sketch further directions for research. The main merits of the homotopy obstructions lie in their unifying power - seemingly disparate theories come together as instances of the same principle. One major implication of this unification is that we are able to construct new obstruction theories, not only for varieties but for topoi. The problem is not whether one can construct new obstructions, but more how effective the obstructions are for doing actual calculations. We will mainly here concentrate on the étale homology obstruction, since it being an "abelianized" version of the étale homotopy obstruction, seem to be the easiest obstruction to calculate. Let us start by noting that for a variety X over a field K with absolute galois group Γ, a first step to understanding $X^{\mathbb{Z}}(hK) = \pi_0(Hom_{der}(*, \mathbb{Z}Et_{/k}(X)^{\#}))$ is to understand $\mathcal{U}^{\mathbb{Z}}_{der} = Hom_{der}(*, \mathbb{Z}\pi_0(\mathcal{U})^{\#})$ for $\mathcal{U}$ a hypercovering of X. $\mathcal{U}^{\mathbb{Z}}_{der}$ will be a simplicial Γ-set so by the Dold-Kan correspondence one can study the associated normalized chain complex $N\mathcal{U}^{\mathbb{Z}}_{der}$ with tools from homological algebra. Further, from the Dold-Kan correspondence we know that $\pi_0(N\mathcal{U}^{\mathbb{Z}}_{der}) \cong H^0(\Gamma, N\mathcal{U}^{\mathbb{Z}}_{der})$, where the latter is the hypercohomology of the complex. To calculate hypercohomology an useful tool is often to use an appropriate spectral sequence. To try to calculate the étale homology obstruction one should then try to understand $X^{\mathbb{Z}}(h\mathbb{A}_K)$ where a first good step is once again to understand π_0 of $\mathcal{U}^{\mathbb{Z}}_{\mathbb{A}_K,der} = Hom_{der}(*_{\mathbb{A}_{\mathbb{K}}{}^{top}}, \mathbb{Z}\pi_0(\mathcal{U})^{\#})$. For this, one proceedes analogously by using the Dold-Kan correspondence to show that studying π_0 of $\mathcal{U}^{\mathbb{Z}}_{\mathbb{A}_K,der} = Hom_{der}(*_{\mathbb{A}_{\mathbb{K}}{}^{top}}, \mathbb{Z}\pi_0(\mathcal{U})^{\#})$ is the same as studying hypercohomology of a certain complex. One will then obtain a commutative diagram :

$$\begin{array}{ccc} X(k) & \xrightarrow{i} & \pi_0(\mathcal{U}^{\mathbb{Z}}_{\mathbb{A}_K,der}) \\ \downarrow{\scriptstyle loc} & & \downarrow{\scriptstyle hloc_{\mathbb{Z}}} \\ X(\mathbb{A}_K) & \xrightarrow{j} & \pi_0(\mathcal{U}^{\mathbb{Z}}_{der} \end{array}$$

where i and j are defined using the methods 10.3.10 and 10.3.11 mutatis mutandis. Now, one shows that there are no rational points if one shows that $hloc_{\mathbb{Z}}(\pi_0(\mathcal{U}^{\mathbb{Z}}_{\mathbb{A}_K,der})) \cap j(X(\mathbb{A}_{\mathbb{K}})) = \emptyset$. This is not an easy task by any measure but gives an idea how to calculate it concretely.

In the above method for trying to show that there are no rational points, one might ask how what hypercoverings to consider. I will give some heuristics on how to translate a known case of the Brauer-Manin obstruction into the étale homology obstruction. So let X be a variety and $\mathcal{A}$ an Azumaya algebra on X. It is shown in [9] that for X quasiprojective, to calculate the étale topological type it is enough to consider Cech coverings of the variety X. With our Azumaya algebra $\mathcal{A}$ one should then try to find an étale cover $\mathcal{U}$ of X such that $\mathcal{A}$ is trivialized by the cover. I then propose to do the above suggested calculations with the Cech covering that $\mathcal{U}$ defines.

Future directions of research fall in my mind into two different families, the arithmetical and topological. I believe that one should for arithmetical problems try to study the flat homotopy type and the associated obstruction theory and actually show that the calculations can be made effective. The most obvious topological aspect falls into seeing whether one can translate the proofs using Friedlander's Étale topological type (see [9]) into the étale topological realization. This would be a very interesting result, since the construction by Friedlander is quite unnatural in some sense. The theory has just been uncovered and we don't yet know what it will yield. I believe that in the near future we will learn how to apply all this abstraction to help us understand solution sets to equations better. What fascinates me is how these abstract tools help us understand something so concrete, and I hope the reader shares my excitement regarding what future research in this field might bring.

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