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Economical Applications of Mathematical Control Theory

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Paweena Surapolbhichet

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MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

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Handledare: Yishao Zhou

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Abstract

The aim of this report is to make an easier access to mathematical control theory by working on certain types of problems which have economical relevance, illustrated by completely solving some problems after presentation of Pontryagin's Maximum Principle with various end point conditions and discussion on under what conditions this principle is also sufficient for optimal solution.

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1 Introduction

1.1 A historical view of Mathematical Control Theory

Control theory was famous among mathematicians during World War II for the benefit of fire-control systems and electronics. But already in the early Roman time, control mechanisms were used for engineering in the field of feedback control when they kept water by using several combinations of valves. Later in 1769, James Watt was known to be the one who came up with the famous steam engines. This theory applied in different areas. In 1868, James Clerk Maxwell, Scottish physicist and mathematician, performed the first mathematical analysis of the stability properties of the steam engine. After his work went on public, the increment of the interest in control theory has resulted in more and more research in control and its applications. The theory of feedback amplifiers was developed by the scientists at Bell Telephone Laboratories in 1930s. Nowadays there are mainly two approaches in optimal control theory. One is the Optimality Principle, also called dynamic programming introduced by Richard Bellman and the other is Pontryagn's Maximum (Minimum) Principle by the Russian mathematician L. Pontryagin. The so-called modern control theory can be dated back to the end of 1950s or beginning of 1960s by a Swiss mathematician Rudolf Kalman who invented the celebrated Kalman filter, based on linear-quadratic Gaussian optimal control. Kalman introduced the basic control theoretic concepts known as reachability, controllability, and their dual concepts, constructibility and observability which are central in all kinds of control problems. Kalman has also brought research of control theory into the study of algebraic analysis. Perhaps the main distinction between classical and modern control theory is the treatment of single- input/single-output and multi-input/multi-output. Mathematics behind such a treatment is linear algebra.

So what is mathematical control theory? We cite the answer from Sontag's book [4]

> Mathematical control theory is the area of application-oriented mathematics that deals with the basic principles underlying the analysis and design of control systems. To control an object means to influence, engineers build devices that incorporate various mathematical techniques.

Nowadays, mathematicians and scientists utilize control theory in broad fields such as biology, engineering, programming and economy.

1.2 Formulation of a simple control problem

In this report, we concentrate on the study of control theory in economical applications. The topics cover the calculus of variations and the theory of differential equations.

The simplest problem in the *calculus of variation*, where the function $x(t)$ is real-valued, continuous and differentiable for $t \in [t_0, t_1]$, is

$$\max \int_{t_0}^{t_1} f(t, x(t), x'(t)) dt,$$

subject to

$$x(t_0) = x_0$$

where $x(t)$ is the state variables, x_0 is fixed, and the prime $'$ denotes the derivative of a function of t, and $f : \mathbb{R} \to \mathbb{R}$ is continuous. This problem can be transformed to the following control problem by letting $u(t) = x'(t)$.

$$\max \int_{t_0}^{t_1} f(t, x(t), u(t)) dt,$$

subject to

$$x'(t) = u(t), \quad x(t_0) = x_0,$$

where $x(t)$ is a state variable and $u(t)$ is a control function defined for $t \in [t_0, t_1]$. In control applications the values of the state at the terminal time $x(t_1) = x_1$, can be free, or it can be fixed, or mixture of partial free components and partial fixed components at the terminal time. Later in this report we shall present various constraints on the state variables at the terminal time.

2 Basic Control Theory in Economical Terms

We begin by considering a system with a real value *state variable*, $x(t)$, where t represents time. The state variables of the system describe, for example, the stock of goods present in the economy. During the process, the value of function $x(t)$ may not work as you wanted to achieve the goal, thus we have to control the system by using a *control function,* $u(t)$. During the operation of the system, the rate of change over time (in the value of $x(t)$) can be controlled since it may depend on that variable t or some other variables. For example the flow of goods consumed at any instant.

The rate of change of the state variable is defined by the derivative of the function $x(t)$. Let t_0 be an initial time such that $x(t_0) = x_0$ is given. Thus the pattern of $x(t)$ can be represented by a differential equation

$$x'(t) = g(t, x(t), u(t)), \quad x(t_0) = x_0 \tag{1}$$

where the initial point is fixed with the given x_0. If the control function $u(t)$ is defined for $t \geq t_0$, then there is a unique solution for (1).

Assume that there is a real valued function f depending on variables t, $x(t)$, and $u(t)$, i.e. $f : [t_0, t_1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The value of this function $f(t, x(t), u(t))$ can be measured by the integral under the expected time-period, $[t_0, t_1]$, as follows

$$\int_{t_0}^{t_1} f(t, x(t), u(t))dt. \tag{2}$$

The integral in (2) is called the *objective* or the *criterion* and in economic analysis this introduces the benefits produced under each period of continuous time of f controlled by $u(t)$. Different control function depends on its own time-period which leads to particular value in the objective (2). Note that the terminal time t_1 is not necessarily fixed.

The basic problem in this section is that we want to study the maximization of the integral (2) which satisfies the differential equations (1) and the constraints imposed on $x(t_1)$. For instance, the capital stock aggregation over time, because the consumption path of the economy determines net investment. Hence the aim of the controlling system is usually to contribute to a given objective.

Example 2.1: *Optimal control problems*

- The values of all the relevant variables determine the electricity consumption of household at any time, and we want to minimize the total electricity consumption so that monthly earning covers within a given time period.

- Capital stock (the values of consumption) and time, may determine the welfare of a company at each instant. Assume that there is a given specific values of the stock at the beginning and the end, then the objective is to maximize total welfare over a fixed time horizon.

■

Example 2.2: Consider the optimal control problem in *Economic growth*

$$\max \int_0^T (1-s)f(k)dt, \quad K' = sf(k), \quad k(0) = 0, \quad k(T) \geq k_T, \quad 0 \geq s \geq 1$$

where we have the real capital stock of a country $k = k(t)$, its production function $f(k)$ and $s = s(t)$ is the control variable with $s \in [0, 1]$, i.e. s is the fraction of production set aside for investment. The quantity $(1-s)f(k)$ is the flow of consumption per unit of time. The constant k is the capital stock, hence the initial and terminal capital stock is k_0 respectively k_T. The condition $k(T) \geq k_T$ means that we wish to leave a capital stock of at least k_T to those who live after time T. So in this problem we wish to maximize the integral of this quantity over the planning horizon [0, T], i.e. to maximize total consumption over the period [0, T].

■

Example 2.3: The optimal control problem in *Oil extraction*

Let $x(t)$ be the amount of oil in a reservoir at time t. Assume that K is amount of oil at the beginning time $t = 0$, so $x(0) = K$. Let $u(t)$ be the rate of extraction, then for each time $t > 0$ gives a different of the amount of oil

$$x(t) - x(0) = -\int_0^t u(\tau)d\tau$$

or

$$x(t) = K - \int_0^t u(\tau)d\tau,$$

where

$$x'(t) = -u(t), \quad x(0) = K. \tag{3}$$

Hence $x(t)$, is the amount of oil which left at time t, equals to the different of the initial amount K distracts by the total amount extracting during the time $[0, t]$.

Moreover, assume that the cost per unit of time, denoted by C, depends on the variables t, x and u, so $C = C(t, x, u)$. Further $p(t)$ is the market price of oil at the time t. The instantaneous profit of time t is then

$$\phi(t, x(t), u(t)) = p(t)u(t) - C(t, x(t), u(t))$$

where pu is the sale revenue per unit of time at t. Now if the discount rate is denoted by r, the total discounted profit over time $t \in [0, T]$ can be calculated as follows

$$\int_0^T [p(t)u(t) - C(t, x(t), u(t))]e^{-rt} dt \tag{4}$$

where $x(T) \geq 0$ and $u(t) \geq 0$.

There may be the following types of control problems

- Fixed terminal time:
  Find the rate of extraction $u(t) \geq 0$ that maximizes (4) subject to (3) and $x(T) \geq 0$ over an extraction period $[0, T]$.
- Free terminal time:
  Find the rate of extraction $u(t) \geq 0$ and the optimal terminal time T that maximizes (4) subject to (3) and $x(T) \geq 0$.

■

3 Control Problems in Simple Cases

In this section, let us consider control problems where the control variable and the terminal state have no restrictions, i.e. the values of $u(t)$ are in $-\infty, \infty)$ and $x(t_1)$ is free, thus we have the following problem:

$$\max \int_{t_0}^{t_1} f(t, x(t), u(t))dt, \quad u \in (-\infty, \infty) \tag{5}$$

subject to

$$x'(t) = g(t, x(t), u(t)), \tag{6}$$

$$t_0, t_1, \quad x(t_0) = x_0, \quad x_0 \text{ fixed}, \quad x(t_1) \text{ free}, \tag{7}$$

where the function $g : [t_0, t_1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and the control function is defined on the interval $[t_0, t_1]$. A pair of a state variable and a control function $(x(t), u(t))$ is called an *admissible pair*, and we call such a pair that maximizes the integral in (5) for an *optimal pair* $(x^*(t), u^*(t))$. The solution of the differential equation defined by (6), together with any given control function $u \in (-\infty, \infty)$ will usually be uniquely established for the whole time interval $[t_0, t_1]$.

With the constraint from (6) for $t \in [t_0, t_1]$ there is a *co-state variable* $p(t) \in \mathbb{R}$, also called the *adjoint function*, $p = p(t)$ whose values are in $\mathbb{R}$. This can be compared with Lagrange multiplier in constrained optimization problems, but here it is a function of t and it is processed through the ***Hamiltonian*** function defined by

$$H(t, x, u, p) = f(t, x, u) + pg(t, x, u) \tag{8}$$

For more details it will be presented later.

Theorem 3.1: The Maximum Principle

Suppose that $(x^*(t), u^*(t))$ is an optimal pair for problem (5)-(7). Then there exists a non-zero continuous function $p(t)$ such that, for each t in $[t_0, t_1]$,

$$p(t_1) = 0 \quad \text{and} \quad p'(t) = -H_x^{'}(t, x^*(t), u^*(t), p(t)) \tag{9}$$

and

$$u = u^*(t) \text{ maximizes } H(t, x^*(t), u, p(t)), \text{ for } u \text{ in } (-\infty, \infty). \tag{10}$$

Note that (i) the optimality condition for (10) is

$$H_u^{'}(t, x^*(t), u^*(t), p(t)) = 0, \tag{11}$$

where $H_\cdot^{'}$ is the partial derivatives of H with respect to $\cdot$.

(ii) The condition $p(t_1) = 0$ is called a ***transversality*** condition. When the adjoint variable vanishes at the terminal time, it means that $x(t_1)$ is free.

Theorem 3.2: Mangasarian

If the requirements in Theorem 3.1 are given together with the following requirement, a ***sufficient*** condition,

$$H(t, x, u, p(t)) \text{ is concave in } (x, u) \text{ for each } t \text{ in } [t_0, t_1], \tag{12}$$

then $(x^*(t), u^*(t))$ is optimal solution that satisfies (6),(10)and (9).

Remark: When the optimal problem is to minimize the objective in (5), we can solve the problem the same as it is to maximize the negative corresponding (original) objective function. The other alternative is that we can reformulate the maximum principle for the minimization problem: an optimal control $(x^*(t), u^*(t))$ minimizes the Hamiltonian (8). The sufficient condition for this is a convexity of $H(t, x, u, p)$ with respect to x, u.

From now on keep it in mind that the optimal problem can have a concavity and a convexity of of $H(t, x, u, p)$ with respect to x, u. But most of the problems in this paper refer to the maximum problem that extends from Theorem 3.1 and Theorem 3.2.

3.1 Necessary Condition

Let us study in more details Theorem 3.1. Why is the condition in this theorem a necessary condition for problem (5)-(7)? That is to ask: why is the condition satisfied provided $(x^*(t), u^*(t))$ is a maximizing solution for $t \in [t_0, t_1]$? The explanation is illustrated in this subsection.

Consider *Lagrange multiplier* and assume that the function $p(t)$ is a single Lagrange multiplier (since we now associate with a single constraint.). Let $p(t)$ be a continuously differentiable function of $t \in [t_0, t_1]$. For any functions $x(t), u(t)$ satisfying (6) and (7), we have $p(t)g(t, x(t), u(t)) = p(t)x'(t)$ and hence

$$\int_{t_0}^{t_1} f(t, x(t), u(t))dt = \int_{t_0}^{t_1} [f(t, x(t), u(t)) + p(t)g(t, x(t), u(t)) - p(t)x'(t)]dt \tag{13}$$

The integration of the last term in the right side of (13) by part gives

$$-\int_{t_0}^{t_1} p(t)x'(t)dt = -p(t_1)x(t_1) + p(t_0)x(t_0) + \int_{t_0}^{t_1} p'(t)x(t)dt. \tag{14}$$

Substituting (14) into (13), we have

$$\begin{aligned}\int_{t_0}^{t_1} f(t, x(t), u(t))dt = &\int_{t_0}^{t_1} [f(t, x(t), u(t)) + p(t)g(t, x(t), u(t)) + x(t)p'(t)]dt \\ &-p(t_1)x(t_1) + p(t_0)x(t_0).\end{aligned} \tag{15}$$

Since a control function $u(t), t \in [t_0, t_1]$, together with the condition (6) and (7), determines the path of the corresponding state variable $x^*(t), t \in [t_0, t_1]$, it also determines the value of (15).

Let $h(t)$ be a fixed modification in the control $u(t)$. Suppose that $u^*(t)$ is a optimal control, $h(t)$ is a fixed function and γ is a parameter. A one-parameter family of comparison controls is then $u^*(t)+\gamma h(t)$. Now let $y(t,\gamma), t \in [t_0, t_1]$, represent the state variable that satisfies (6) and (7) with control function $u^*(t) + \gamma h(t), t \in [t_0, t_1]$. Furthermore assume that $y(t, \gamma)$ is a smooth function for both arguments, t and γ. Thus the optimal path $x^*(t)$ occurs when $\gamma = 0$

$$y(t, 0) = x^*(t), \quad y(t_0, \gamma) = x_0.$$

Suppose that u^*, x^* and h are fixed. Rewrite the integral in (5) with the control function $u^*(t) + \gamma h(t)$ and state $y(t, \gamma)$. We have a function of a single parameter γ as

$$J(\gamma) = \int_{t_0}^{t_1} f(t, y(t, \gamma), u^*(t) + \gamma h(t))dt.$$

Reformulating and using (15) give

$$J(\gamma) = \int_{t_0}^{t_1} [f(t, y(t, \gamma), u^*(t)+\gamma h(t))+p(t)g(t, y(t, \gamma), u^*(t)+\gamma h(t))+y(t, \gamma)p'(t)]dt$$
$$-p(t_1)y(t_1, \gamma) + p(t_0)y(t_0, \gamma). \tag{16}$$

The function $J(\gamma)$ has its maximum value when $\gamma = 0$, because we have the optimal control as u^*. Differentiating (16) with respect to γ and inserting the value $\gamma = 0$,

$$J'(\gamma) = \int_{t_0}^{t_1} [(f'_x + pg'_x + p')y'_\gamma + (f'_u + pg'_u)h]dt - p(t_1)y'_\gamma(t_1, 0), \tag{17}$$

Since $\gamma = 0$, we have $y_\gamma(t_0, \gamma) = 0$ for all γ. Then the function has value along (t, x^*, u^*) and the last term in (16) is independent of γ. Until now function $p(t)$ is assumed to be differentiable. Let $p(t)$ satisfies the linear differential equation,

$$p'(t) = -[f'_x(t, x^*, u^*) + p(t)g'_x(t, x^*, u^*)], \quad \text{with} \quad p(t_1) = 0. \tag{18}$$

Combining (18) with (17), it is necessary that

$$J'(\gamma) = \int_{t_0}^{t_1} [f'_u(t, x^*, u^*) + pg'_u(t, x^*, u^*)]hdt = 0. \tag{19}$$

Since the function $h(t)$ is arbitrary we can choose $h(t) = f'_u(t, x^*, u^*)+pg'_u(t, x^*, u^*)$. Then we have

$$\int_{t_0}^{t_1} [f'_u(t, x^*, u^*) + pg'_u(t, x^*, u^*)]^2 dt = 0. \tag{20}$$

Eventually, this implies that

$$f'_u(t, x^*, u^*) + pg'_u(t, x^*, u^*) = 0, \quad t \in [t_0, t_1]. \tag{21}$$

Summary: If the function $(x^*(t), u^*(t))$ maximize (5) subject to (6) and (7) for $t \in [t_0, t_1]$, then there is a continuously differentiable function $p(t)$ such that (x^*, u^*) and p simultaneously satisfy the *state equation*

$$x'(t) = g(t, x(t), u(t)), \quad x(t_0) = x_0, \tag{22}$$

the *multiplier equation*

$$p'(t) = -[f'_x(t, x(t), u(t)) + p(t)g'_x(t, x(t), u(t))], \quad p(t_1) = 0, \tag{23}$$

and the *optimality condition*

$$f'_u(t, x(t), u(t)) + p(t)g'_u(t, x(t), u(t)) = 0. \tag{24}$$

Note that the multiplier equation in (23) is also known as the *auxiliary*, *adjoint*, *costate* or *influence* equation.

There is an easier way to memorize all this, by the Hamiltonian

$$H(t, x(t), u(t)) = f(t, x, u) + p(t)g(t, x, u), \tag{25}$$

as follows:
(24):

$$H'_u = f'_u + pg'_u = 0 \tag{26}$$

(23):

$$-H'_x = -(f'_x + pg'_x) = p'(t) \tag{27}$$

(22):

$$H'_p = x'(t) = g(t, x, u). \tag{28}$$

3.2 Sufficient Condition

In the calculus of variation, when the integrand $f(t, x, x')$ is concave in x and x' the necessary condition is also sufficient for optimality. What happens when functions of x, u in the optimal control problem are both concave? The results are similar.

Suppose functions $f(t, x(t), u(t))$ and $g(t, x(t), u(t))$ are both differentiable and concave functions of x, u, consider the problem

$$\max \int_{t_0}^{t_1} f(t, x(t), u(t)) dt, \quad u \in (-\infty, \infty) \tag{29}$$

subject to

$$x'(t) = g(t, x(t), u(t)), \quad x(t_0) = x_0. \tag{30}$$

Suppose also that the functions x^*, u^* and p satisfy the necessary conditions

$$f'_u(t, x, u) + p g'_u(t, x, u) = 0, \tag{31}$$

$$p'(t) = -(f'_x(t, x, u) + p g'_x(t, x, u)), \tag{32}$$

$$p(t_1) = 0, \tag{33}$$

and the constraint (30) for $t \in [t_0, t_1]$. Now let $x(t)$ and $p(t)$ be continuous functions with

$$p(t) \geq 0, \tag{34}$$

for all t if $g(t, x, u)$ is concave in x or u. Thus a solution of the problem (29) with constraint (30) is $(x^*(t), u^*(t))$. If the functions f and g are both concave in (x, u) then the necessary conditions (30)-(33) are also sufficient for optimality.

Proof: Suppose the solution $(x^*(t), u^*(t))$ and p satisfy (30)-(33). Let the function (x, u) satisfies (30). Let f^*, g^* be defined for (t, x^*, u^*) and f, g defined for the path (t, x, u). We have to show that

$$I := \int_{t_0}^{t_1} (f^* - f) dt \geq 0, \tag{35}$$

Because of the concavity of (x, u) in f we get

$$f^* - f \geq (x^* - x) f_x^{*\prime} + (u^* - u) f_u^{*\prime}, \tag{36}$$

and it follows as

$$I \geq \int_{t_0}^{t_1} [(x^* - x) f_x^{*\prime} + (u^* - u) f_u^{*\prime}] dt =$$

$$= \int_{t_0}^{t_1} [(x^* - x)(-p g_x^{*\prime} - p') + (u^* - u)(-p g_u^{*\prime})] dt =$$

$$= \int_{t_0}^{t_1} p[g^{*\prime} - g - g_x^{*\prime}(x^* - x) - g_u^{*\prime}(u^* - u)] dt =$$

$$\geq 0. \tag{37}$$

The explanation of (37) is that the second line in (37) is a substitution of $f_x^{*\prime}$ by (32) and substitution of $f_u^{*\prime}$ by (31). Then integrate by part the term involving p' to get the third line and finally the last line is due to the concavity of x and u in $g(t,x,u)$. From (37) we see that if p is positive then the value in the last square bracket in (37) must be positive. Since g is assumed to be concave (convex) in x, u, the last bracket will instead be equal to zero. Thus p satisfies (34). But if the function g is linear (not concave/convex) in x, u, the function p can have any sign.

Furthermore, if the function f is concave, g is convex and $p \leq 0$, then it follows that the necessary conditions will also be sufficient for optimality. To prove that, follow the same process as above: It will lead to a negative p and coefficients in the next last line, which gives a positive product.

□

Example 3.1: Solve the problem

$$\max \int_0^T [1 - tx(t) - u(t)^2]dt, \quad x'(t) = u(t), \quad x(0) = x_0, \quad x(T) \text{ free}, \quad u \in \mathbb{R}$$

where x_0 and T are given positive constants.

Solution: Let $f(t,x,u) = 1 - tx(t) - u(t)^2$, hence the Hamiltonian is

$$H(t,x,u,p) = f + px' = 1 - tx(t) - u(t)^2 + pu.$$

Since $H'_u = -2u + p \geq 0$, by using Theorem 3.1 we have that the control $u = u^*(t)$ maximizes $H(t, x^*(t), u, p(t))$ for u if and only if it satisfies $H'_u = 0$ and this gives $u^*(t) = \frac{1}{2}p(t)$. From (10) and (9), $p'(t) = -H'_x = t$, $p(T) = 0$. In this problem $H''_{xx} = 0, H''_{uu} = -2 < 0$ and $H''_{xx}H''_{uu} - (H''_{xu})^2 = 0 \iff H$ is concave in x and u. Integrating $p'(t)$ yields $p(t) = \frac{1}{2}t^2 + C$. The terminal condition $p(T) = \frac{1}{2}T^2 + C = 0$ gives $C = -\frac{1}{2}T^2$. Combining these two functions $p(t)$ and $p(T)$, we have

$$p(t) = -\frac{1}{2}(T^2 - t^2) \text{ and then } u^*(t) = -\frac{1}{4}(T^2 - t^2).$$

Integrating $u^*(t)$,

$$x^*(t) = x_0 - \frac{1}{4}T^2 t + \frac{1}{12}t^3.$$

We have now found the solution pair $(x^*(t), u^*(t))$ that satisfies all given conditions.

■

Example 3.2: *A macroeconomic control problem*

Consider the simple macroeconomic problem. Consider the state function $y(t)$ of the economy over the course of a planning period $[0,T]$. The state is to be steered toward the desire level $\hat{y}$, independent of t, by mean of the control $u(t)$, where $y'(t) = u(t)$. Using control is costly, thus we have to minimize the integral $\int_0^T [(y(t)-\hat{y})^2 + c(u(t))^2]dt$ with $y(T) = \hat{y}$ and c being a positive constant. Denote now the difference between the original state variables and the target level by $y(t)-\hat{y} = x(t)$. Let, at the terminal time, the target value of x be free and $u(t) = x'(t)$. So we have the control problem as follows

$$\min \int_0^T (x^2 + cu^2(t))dt, \quad x'(t) = u(t), \quad x(0) = x_0, \quad x(T) \text{ free}$$

where $u(t) \in \mathbb{R}$ and $c > 0$ and x_0 and T are given.

Solution: Corresponding to the given problem, it is equivalent to maximize $-\int_0^T [x(t)^2 + c(u(t))^2]dt$. The Hamiltonian is

$$H(t,x,u,p) = -x^2 - cu^2 + pu,$$

so

$$H'_x = -2x \quad \text{and} \quad H'_u = -2cu + p.$$

The necessary condition $H'_u = 0$ gives that

$$-2cu^*(t) + p(t) = 0.$$

Thus $u^*(t) = p(t)/2c$. The differential equation for $p(t)$ is

$$p'(t) = -H'_x(t, x^*(t), u^*(t), p(t)) = 2x^*(t). \tag{38}$$

Since $x'^*(t) = u^*(t)$ so

$$x'^*(t) = p(t)/2c. \tag{39}$$

Insert (39) into the derivative of (38) with respective to t we have

$$p''(t) = 2x'^*(t) = p(t)/c.$$

The general solution for the homogeneous differential equation is

$$p(t) = Ae^{rt} + Be^{-rt} \quad \text{where } r = 1/\sqrt{c}.$$

The boundary conditions $p(T) = 0$ and $p'(0) = 2x^*(0) = 2x_0$ gives

$$p(T) = Ae^{rT} + Be^{-rT} = 0 \quad \text{and} \quad p'(0) = r(A-B) = 2x_0$$

which yields $A = 2x_0 e^{-rT}/r(e^{rT}+e^{-rT})$ and $B = -2x_0 e^{rT}/r(e^{rT}+e^{-rT})$, hence

$$p(t) = \frac{2x_0}{r}\frac{e^{-r(T-t)} - e^{r(T-t)}}{e^{rT} + e^{-rT}},$$

$$u^*(t) = \frac{p(t)}{2c} = \frac{x_0}{cr}\frac{e^{-r(T-t)} - e^{r(T-t)}}{e^{rT} + e^{-rT}},$$

and

$$x^*(t) = \frac{1}{2}p'(t) = x_0\frac{e^{r(T-t)} + e^{-r(T-t)}}{e^{rT} + e^{-rT}}.$$

Note that $H(t, x, u, p) = -x^2 - cu^2 + pu$ is concave in (x, u) since $H''_{xx} = -2 < 0, H''_{uu} = -2c < 0$ where c is given as a positive constant and $H''_{xx}H''_{uu} - (H''_{xu})^2 = 4c - 0 > 0$. This satisfies Mangasarian's theorem so the last expressions for $u^*(t)$ and $x^*(t)$ are the pair solution for this problem.

■

4 Regularity Conditions

Assume that the control function $u(t)$ has values in the fixed subset $U \subset \mathbb{R}$. We call U as the *control region.* Normally in applied economics the control functions can vary in different ways. In example 2.3 about oil extraction, we have seen that the value of the control function was restricted to $u(t) \geq 0$. Nothing is extraneous for explanation (the oil can not pump back into the reservoir.) and the control region in this case is then $U = [0, \infty)$. The important thing here is that the control region can be a closed set, for example $u(t)$ can have the value at the boundary of U.

Regularity conditions for the control function $u(t)$ include continuous in most of the economical literature. It is of no exception in this report the control function is given to be continuous for all problems that we deal with except the last one where the control is of the form

$$u(t) = \begin{cases} 1 & \text{for } t \text{ in} \quad [t_0, t_i] \\ 0 & \text{for } t \text{ in} \quad (t_i, t_1] \end{cases}$$

which at time $t = t_i$ exhibits a jump, thus $u(t)$ is discontinuous and in this case $u(t)$ is called *piecewise continuous.*

Assume that there is a function $u(t)$. If this function has one-sided limits from both above and below at one point of discontinuity, t_i, where the function is also defined of this point. Then this function has a *finite jump* at the point t_i. In each finite interval, if a function has at most a finite number of discontinuities then this function is *piecewise continuous* with a finite jump at each point of discontinuity. At a point of discontinuity t_i, if the value of $u(t_i)$ is a left-limit of $u(t)$ at t_i then $u(t)$ is called for *left-continuous.* Furthermore, if the control function is defined in the interval $[t_0, t_1]$ of time, then the assumption on $u(t)$ is that it is continuous at both ends.

So when $u = u(t)$ has discontinuities, what should the explanation for the solution of $x'(t) = g(t, x, u)$ be? Well, at the point where $u(t)$ is discontinuous, the continuous function $x(t)$ is not differentiable, but it does have a derivative in the other points that satisfies the equation.

Up to now we have not put any restrictions on the functions $f(t, x, u)$ and $g(t, x, u)$. *Let assume from now on that functions f, g and their first-order partial derivatives with respective to x and u are continuous in (t, x, u).*

4.1 Theorems about finding possible global solutions

We close this section with a short summary on how to find a possible global solutions. There are essentially three results that can be used:

4.1.1 Necessary conditions

This condition is given in the theorems of *Pontryagin's Maximum Principle* and it provides candidates to an optimal control. Rigorously speaking, this condition does not guarantee that there is a solution for the maximization problem.

4.1.2 Sufficient conditions

This condition in the theorem gives a *sufficiency result*, which was originally developed by Mangasarian. The requirements in this type of condition involve concavity/convexity of functions. Assume that we have a state variable $x^*(t)$ and an adjoint variable $p(t)$. If a control function $u^*(t)$ satisfies the sufficient conditions then the solution of the maximization problem is given by $(x^*(t), u^*(t))$. But this condition is not necessary for solving the problem. Even if the sufficient conditions are not satisfied, in many control problems, there are optimal solutions.

4.1.3 Existence

The use of the *existence theorem* goes as follows: At first find all the possible solutions by using the necessary conditions. Thereafter examine those possible solutions. The optimal solution is the one that gives the largest value of the objective function. This theorem ensures that the given conditions solve the maximization problem.

5 Interpretations in Economical Terms

5.1 A general interpretation in Economics

What is the meaning of the multiplier in economics? In control problems the multiplier $p(t)$ is the marginal valuation of the associated state variables at t, and it has an economically meaningful interpretation.

Consider

$$\max \int_{t_0}^{t_1} f(t, x(t), u(t))dt, \tag{40}$$

subject to

$$x'(t) = g(t, x(t), u(t)), \quad x(t_0) = x_0. \tag{41}$$

Let $V(x_0, t_0)$ denote the maximum of (40) where x_0 represents an initial state at initial time t_0. If $p(t)$ is the marginal valuation of the state variable at t then it also constitutes the definition of the derivative of V with respect to x. That is,

$$V_x(x(t), t) = p(t), \quad t_0 \leq t \leq t_1,$$

where $V_x = V'_x$.

Proof: Let x^* and u^* be the optimal state and the optimal control function for (40) and let the $p(t)$ be the corresponding multiplier. Consider an initial state $x_0 + h$ where h is a number close to zero. Suppose u^* is a continuous function of t.

For a continuously differentiable multiplier function $p(t)$ and the differential equation for x,

$$V(x_0, t_0) = \int_{t_0}^{t_1} f(t, x^*, u^*)dt =$$

$$= \int_{t_0}^{t_1} [f(t, x^*, u^*) + g(t, x^*, u^*)p - px']dt. \tag{42}$$

Integrating the last term along (t, x^*, u^*) by parts and using the assumption that x, u are optimal for this problem give

$$V(x_0, t_0) = \int_{t_0}^{t_1} (f^* + pg^* + p'x^*)dt - p(t_1)x^*(t_1) + p(t_0)x^*(t_0). \tag{43}$$

Similarly,

$$V(x_0 + h, t_0) = \int_{t_0}^{t_1} f dt =$$

$$= \int_{t_0}^{t_1} (f + pg + p'x)dt - p(t_1)x(t_1) + p(t_0)[x(t_0) + h]. \tag{44}$$

Subtracting,

$$V(x_0 + h, t_0) - V(x_0, t_0) = \int_{t_0}^{t_1} [f(t, x, u) - f(t, x^*, u^*)]dt =$$

$$= \int_{t_0}^{t_1} (f + pg + p'x - f^* - pg^* - p'x^*)dt$$
$$+p(t_0)h - p(t_1)x(t_1) - x^*(t_1)]. \tag{45}$$

Using Taylor series for the integrand around (t, x^*, u^*)

$$V(x_0 + h, t_0) - V(x_0, t_0) = \int_{t_0}^{t_1} [(f_x^* + pg_x^* + p')(x - x^*)+$$

$$+(f_u^* + pg_u^*))(u - u^*)]dt + p(t_0)h - p(t_1)[x(t_1) - x^*(t_1)] + R_n \tag{46}$$

where R_n is a reminder term. For optimal x^*, u^*, p the necessary conditions (22), (23) and (24) hold. Now let p be the multiplier that satisfies the necessary conditions for (40) i.e.

$$p' = -(f_x^* + pg_x^*), \qquad f_u^* + pg_u^* = 0 \qquad p(t_1) = 0$$

Hence (46) is reduced to

$$V(x_0 + h, t_0) - V(x_0, t_0) = p(t_0)^T h + R_n. \tag{47}$$

Now divide (47) by the parameter h and then let h approach zero, we have

$$\lim_{h\to 0}[V(x_0 + h, t_0) - V(x_0, t_0)]/h = V_x(x_0, t_0) = p(t_0). \tag{48}$$

It is now proved that the limit exists for the initial time t_0 but not for all t. Since $p(t)$ is the marginal valuation of the associated state variable at time t then the problem must be modified optimally thereafter.

Note that any portion of an optimal is itself optimal on an optimal path using Bellman's optimality principle. Let $\hat{t}$ be any time such that $t_0 \leq \hat{t} \leq t_1$. Suppose we follow the solution x^*, u^* of (40) for the period $t_0 \leq t \leq \hat{t}$, then stop and reconsider the next time period from $\hat{t}$ forwards:

$$\begin{aligned} &\max \quad \int_{\hat{t}}^{t_1} f(t, x, u)dt \\ &\text{subject to} \quad x'(t) = g(t, x, u), \quad x(\hat{t}) = x^*(\hat{t}). \end{aligned} \tag{49}$$

The same solution x^*, u^* to (40) for time $\hat{t} \leq t \leq t_2$, must be a solution for (49). Suppose that this is not true. Then there is a larger value than x^*, u^* for (49) where $\hat{t} \leq t \leq t_2$. The value of (40) can then be improved by following x^*, u^* on the path from an initial time to $\hat{t}$. Thereafter continue to follow the value from $\hat{t}$ to t_1, in other words integrate in (49) (since this coincides on $\hat{t} \leq t \leq t_2$). But this is a contradiction, therefore x^* and u^*, where $\hat{t} \leq t \leq t_2$, must solve (49). Combining (48) to (49), yields that

$$V_x(x(\hat{t}), \hat{t}) = p(\hat{t}).$$

This shows that the derivative exists and that it is the marginal valuation of the state variable at $\hat{t}$. But $\hat{t}$ is arbitrary, so for any t, $t_0 \leq t \leq t_1$,

$$V_x(x(t), t) = p(t), \quad t_0 \leq t \leq t_1$$

is the marginal valuation of the state variable at t, whenever this derivative exists. The proof is now complete.

□

Now consider t_1 where $p(t_1) = 0$, when there is no salvage term, and $p(t_1) = \alpha'(x_1)$, when there is a salvage term. This will be discussed in the last section. Let x be the stock of an asset and $f(t, x, u)$ be the current profit. Hence

$$p(t_1)x(t_1) = p(t_0)x(t_0) + \int_{t_0}^{t_1} (x'p + xp')dt =$$
$$= p(t_0)x(t_0) + \int_{t_0}^{t_1} (d(xp)/dt)dt.$$

Since $p(t)$ represents the marginal valuation of the state variable at t. The equation above implies that the value of the terminal stock of assets equals the value of the original stock plus the change in the value of assets over the control period $[t_0, t_1]$. And the explanation of

$$d(xp)/dt = x'p + xp'$$

is that the total rate of change in the value of assets (on the left side) equals to the value of additions (reductions) in the stock of assets (first term on the right side) add the change in the value of existing assets (second term on the right side). This leads to the changes in amount of assets and even the change in the value of all assets. From (44) the rate at which the total value enhances is

$$f + pg + xp' = H + xp' \quad \text{where} \quad H = f + pg. \tag{50}$$

The explanations of the equality in (50);
$f(t, x, u)$ - the current cash flow, pg - the change in state variable (note that

$pg = px'$), and xp' - the change valuation in current assets (the capital gain). Thus (50) represents the contribution rate at t toward the total value. Choose $u(t)$ to maximize H and hence to satisfy

$$\partial H/\partial u = f'_u + pg'_u = 0, \quad t_0 \le t \le t_1,$$

$$\partial^2 H/\partial u^2 = f''_{uu} + pg_{uu} \le 0.$$

We choose x to maximize (50)

$$f_x + pg_x + p' = 0.$$

Finally this implies that the problem

$$\max_{x,u} \quad [H(t,x,u,p(t)) + p'(t)^T x]$$

has $x = x^*(t), u = u^*(t)$ as a solution for all $t_0 \le t \le t_1$.

5.2 Adjoint variables (Shadow prices)

For many years economists have realized that the adjoint can be interpreted as the shadow price, which we have seen in the proof in 5.1. Let us summarize about this adjoint variable again. Suppose that $(x^*(t), u^*(t))$ is a solution for the problem in (5)-(6) with a unique adjoint function $p(t)$. Let V be the corresponding value of the objective function

$$V(x_0, x_1, t_0, t_1) = \int_{t_0}^{t_1} f(t, x^*(t), u^*(t))dt \tag{51}$$

where V depends on x_0, x_1, t_0 and t_1. So the function V is called the *optimal value function*.

At the time t_0, suppose x_0 is differentiable then the interpretation of $p(t)$ at the time $t = t_0$ is

$$\frac{\partial V(x_0, x_1, t_0, t_1)}{\partial x_0} = p(t_0) \tag{52}$$

which represents the marginal change in the optimal value function as x_0 increases. Note that the value of p in (52) is defined only at the initial time t_0. For an arbitrary $t \in [t_0, t_1]$ the value of p is determined by using the jump function $v = x(t^+) - x(t^-)$ for $t \in [t_0, t_1]$ and $x(t)$ is assumed to be differentiable everywhere. The function V will then depend on v. Suppose now that $(x^*(t), u^*(t))$ is the optimal solution for this problem when $v = 0$. Hence V is differentiable with respective to v at $v = 0$. This implies that the first-order approximate change in the value function in (51) with respective to an unit jump increase in $x(t)$, is the adjoint variable $p(t)$ and we have

$$\frac{\partial V(x_0, x_1, t_0, t_1)}{\partial v}|_{v=0} = p(t) \tag{53}$$

as a shadow price. If we consider small time interval $[t, t + \Delta t]$ thus $\Delta x \approx g(t, x, u)\Delta t$ and according to the Hamiltonian $H = f(t, x, u) + pg(t, x, u)$ we have

$$H\Delta t = f(t, x, u)\Delta t + p^T g(t, x, u)\Delta t \approx f(t, x, u)\Delta t + p^T \Delta x$$

where the maximum principle gives u to maximize H at each given time. Note that $f\Delta t$ is the instantaneous profit earned during time $[t, t+\Delta t]$ and $p\Delta t$ is the contribution to the total profit produced by an extra profit Δx at the terminal time of this period.

Consider the optimal value function (51) of problem (5)-(7) again. Let $H^*(t) = H(t, x^*(t), u^*(t), p(t))$. Since the function V is differentiable with respect to x_0, x_1, t_0 or t_1, we have

$$\frac{\partial V}{\partial x_0} = p(t_0), \quad \frac{\partial V}{\partial x_1} = -p(t_1), \quad \frac{\partial V}{\partial t_0} = -H^*(t_0), \quad \frac{\partial V}{\partial t_1} = H^*(t_1). \tag{54}$$

The economical explanations of the equations in (54):
(for this capital accumulation interpretation in subsection)

$\partial V/\partial x_0$: The initial capital stock x_0 increase by one unit the total profit will increase by approximately $p(t_0)$.
$\partial V/\partial x_1$: It is similar to the first one, but the effect of the state at time t_1 will have opposite sign compared with the effect of state at the time t_0. It means that increasing the capital decreases the total profit by approximately $p(t_1)$ since the capital will be left at the end.
$\partial V/\partial t_0$: The planning period t_0 extends, leads to shorter period and it decreases the total profit.
$\partial V/\partial t_1$: The planning period t_1 extends, time period will be longer which yields the increasing in the total profit.

Example 5.1: Use the problem in Example 3.1 to show that the equality in (52) is true.

Solution: The object function was $\int_0^T [1 - tx(t) - (u(t))^2]dt$, and the solution for this problem was $x^*(t) = x_0 - \frac{1}{4}T^2 t + \frac{1}{12}t^3$, $\quad u^*(t) = -\frac{1}{4}(T^2 - t^2)$, with $p(t) = -\frac{1}{2}(T^2 - t^2)$. By using (51) we get

$$V(x_0, x_1, 0, T) = \int_0^T [1 - tx^*(t) - (u^*(t))^2]dt =$$

$$= \int_0^T [1 - t(x_0 - \frac{1}{4}T^2 t + \frac{1}{12}t^3) - (-\frac{1}{4}(T^2 - t^2))^2] dt.$$

By using Leibniz's formula

$$F(x) = \int_{u(x)}^{v(x)} f(x,t)dt$$

$$\Longrightarrow F'(x) = f(x, v(x))v'(x) - f(x, u(x))u'(x) + \int_{u(t)}^{v(t)} \frac{\partial f(x,t)}{\partial x},$$

we get

$$\frac{\partial V(x_0, T)}{\partial x_0} = \int_0^T -tdt = -\frac{1}{2}T^2 = p(t_0).$$

Solving for the function $p(t)$ in Example 3.1, when the initial time $t = 0$, satisfies that $p(0) = -\frac{1}{2}(T^2 - 0)$ which gives the same value as $p(t_0)$ above. So we have shown that the equality (52) is true.

■

6 The Standard Type of Problems

In this section we consider a more realistic state variable at the terminal time in the following *standard end-constrained problem.*

6.1 The Pontryagin maximum principle

The problem is

$$\max \int_{t_0}^{t_1} f(t, x(t), u(t))dt, \qquad u \in U \subseteq \mathbb{R}^m \tag{55}$$

$$x'(t) = g(t, x(t), u(t)), \quad x(t_0) = x_0 \tag{56}$$

with one of the following terminal conditions

$$\text{(i) } x(t_1) = x_1, \quad \text{(ii) } x(t_1) \geq x_1, \quad \text{(iii) } x(t_1) \text{ free} \tag{57}$$

where the numbers t_0, t_1, and x_0, $x_1 \in \mathbb{R}^n$ are fixed and U is a fixed control region. For the control function $u \in U$, a pair $(x(t), u(t))$ is called an *admissible pair* if it satisfies (56) and (57). A pair that maximizes the integral in (55) is called an optimal pair.

As in the basic control problem, to deduce the Maximum principle we go through the same procedure as in the problem with end state free. Thus we need to form the modified objective functional in (17), where we see that if the final state is fixed, i.e., $x(t)$ has a specified value at the terminal time t_1 as stated in (i) there is no variation for $x_i(t_1)$, that is, $y'_\gamma(t_1, 0) = 0$, thus the terminal condition for $p(t)$ is unconstrained. It seems that all we need to do for the maximum principle is to change the boundary condition of p at t_1 to the state variable x at t_1. However, there are cases where the problem is ill-conditioned. Let us study the following problem

$$x'(t) = u^2(t),$$

$$x(0) = 0, \quad x(1) = 0.$$

We want to maximize the functional

$$J = \int_0^1 u(t)dt.$$

Clearly we can solve the equation for x by integrating both sides of the equation $x'(t) = u^2(t)$, that gives

$$x(t) = \int_0^t u^2(s)ds.$$

It is evident that $x(0) = 0$, and

$$x(1) = \int_0^1 u^2(t)dt.$$

But $x(1) = 0$ so $u(t) = 0$. Thus the solution is $u = 0$. However this solution does not satisfy the necessary conditions of the maximum principle if we follow the above argument. This can be seen as follows:

$$H(t, x, u, p) = u + pu^2.$$

The optimality condition for the Hamiltonian is

$$H_u = 1 + 2pu = 0 \text{ i.e. } u = -1/2p$$

and the costate equation is

$$p' = -H_x = 0$$

giving that p is a constant without any boundary constraint. Now we see that this constant u is either non-zero or infinity (if $p = 0$). So this u cannot be the solution of the given problem because then it does not satisfy the boundary conditions for x. However, if we modify the Hamiltonian to

$$H(t, x, u, p) = p_0 f(t, x, u) + pg(t, x, u)$$

with $p_0 \geq 0$ we see that the solution satisfy the necessary conditions with $p_0 = 0$ after the similar computation.

This example gives us some hints how to reformulate the Pontryagin's Maximum Principle. In the light of previous argument we make a "small" correction for the necessary conditions by modifying the Hamiltonian function to

$$H(t, x, u, p) = p_0 f(t, x, u) + p^T g(t, x, u) \tag{58}$$

where $p_0 \in \mathbb{R}$. So if $p_0 \neq 0$ in (58) we can divide the equation in (58) by p_0 to get the Hamiltonian with $p_0 = 1$.

Theorem 6.1: The maximum principle: Standard end constraints

Let $x^*(t), u^*(t) \in U$ be an optimal solution to the problem terminal connstraints in (55)-(57). Then there is an adjoint trajectory $p(t)$ and a constant $p_0 \geq 0$ with $(p_0, p) \not\equiv 0$ such that

1. The control $u^*(t)$ maximizes $H(t, x^*, u, p(t))$ where $u \in U$, i.e.

$$H(t, x^*(t), u, p(t)) \leq H(t, x^*, u^*(t), p(t)) \quad \text{for all } u \in U, \forall t \in [t_0, t_1] \tag{59}$$

2. $$p'(t) = -H_x^{'}(t, x^*(t), u^*(t), p(t)) \tag{60}$$

3. For each terminal conditions in (57), there is a corresponding transversality condition on $p(t_1)$ as follows:

$$\begin{gathered}\text{(i')} \quad p(t_1) \quad \text{no condition} \\ \text{(ii')} \quad p(t_1) \geq 0, \quad \text{with} \quad p(t_1) = 0 \quad \text{if} \quad x^*(t_1) > x_1 \\ \text{(iii')} \quad p(t_1) = 0.\end{gathered} \tag{61}$$

Note:

- The conditions of the optimal problem do not change when $p_0 = 0$ thus when $p_0 = 0$ the inequality in (59) can be formulated as

  $$p(t)^T g(t, x^*(t), u, p(t)) \leq p(t)^T g(t, x^*, u^*(t), p(t)), \quad \text{for all} \quad u \in U.$$

  Under the condition $x(t_1)$ is free, by (61)(iii') we have $p(t_1) = 0$, but $p_0 \neq 0$ and $p_1 \neq 0$ then $p(0) = 1$ for this condition.

- The inequality in (61)(ii') is reversed when the condition in (57) is (ii).

Next we consider the control problem defined by (55)-(57) with the scalar state x and the scalar control u.

Theorem 6.2: Mangasarian

Suppose that $(x^*(t), u^*(t))$ is an admissible pair with a corresponding adjoint function $p(t)$ such that the conditions (i)-(iii) in Theorem 6.1 are satisfied with $p_0 = 1$. Suppose also that $H(t, x, u, p(t))$ is concave in (x, u) for every $t \in [t_0, t_1]$ and the control region is convex. Then $(x^*(t), u^*(t))$ is an optimal pair.

Theorem 6.3: The maximum principle with a variable final time

Suppose that $(x^*(t), u^*(t))$ is defined on the time interval $[t_0, t_1^*]$ and that it is an admissible pair that solve the problem (55)-(57) with free $t_1 \in (t_0, \infty)$. Then all the conditions in the maximum principle in Theorem 6.1 are satisfied on $[t_0, t_1^*]$ and in addition

$$H(t_1^*, x^*(t_1^*), u^*(t_1^*), p^*(t_1^*)) = 0. \tag{62}$$

Example 6.1: a) Solve the control problem

$$\max \int_0^T (x - \frac{1}{2}u^2) dt, \quad x' = u, \quad x(0) = x_0, \quad x(T) \text{ free} \quad u \in \mathbb{R}.$$

b) Compute the optimal value function $V(x_0, T)$ and verify the equality in (54) for this problem.

Solution: a) The Hamiltonian is

$$H(t, x, u) = x - \frac{1}{2}u^2 + pu.$$

Applying the maximal principle yields the following

$$H'_u = -u + p = 0 \tag{63}$$

$$H''_{uu} = -1 < 0 \tag{64}$$

$$p' = -H'_x = -1. \tag{65}$$

Note that the inequality in (64) satisfies the maximum principle.
From (63) we have

$$u^*(t) = p(t) \tag{66}$$

and according to (65) $p' = -1$ and $p(T) = 0$, hence the integration yields

$$p(t) = -t + c_1, \quad c_1 \quad \text{is constant,} \quad \Longrightarrow \quad c_1 = T$$

$$\Longrightarrow \quad p(t) = -t + T. \tag{67}$$

Since $x'^* = u^*$, substituting (67) in (66) gives

$$x'^*(t) = u^*(t) = -t + T,$$

by integrating, this implies that

$$x^*(t) = -\frac{t^2}{2} + Tt + c_2, \quad c_2 \quad \text{is a constant.}$$

Together with condition $x(0) = x_0 = 0$ we have $c_2 = 0$. Hence

$$x^*(t) = Tt - \frac{t^2}{2},$$

$$u^*(t) = T - t. \tag{68}$$

■

b) Compute the optimal solution (68) and inserting into $V(x_0, T)$ we get

$$V(x_0, T) = \int_0^T (x_0 - \frac{1}{2}(T - t)^2)dt = \left[x_0 t - \frac{t^3}{6}\right]_0^T = x_0 T - \frac{T^3}{6}.$$

The equality in (54) verifies as follow

$$\frac{\partial V}{x_0} = T \quad \Longleftrightarrow \quad p(t_0) = T - 0 = T$$

$$\frac{\partial V}{x_1} = 0 \quad \Longleftrightarrow \quad p(t_1) = -T + T = 0$$

$$\frac{\partial V}{t_0} \quad \text{not function of } t_0 \quad \Longleftrightarrow \quad p(t_0) \quad \text{not function of } t_0$$

$$\frac{\partial V}{T} = x_0 + \frac{T^2}{2} = \frac{T^2}{2} \quad \Longleftrightarrow \quad p(t_0) = \frac{T^2}{2}. \qquad ■$$

Example 6.2: Solve the control problem

$$\max \int_0^T (x - t^3 - \frac{1}{2}u^2)dt, \quad x' = u, \quad x(0) = x_0, \quad x(T) \text{ free} \quad u(t) \in \mathbb{R}.$$

and determine the value of T.

Solution: a) The Hamiltonian equation is

$$H(t, x, u) = x - \frac{1}{2}u^2 + pu.$$

Applying the maximal principle yields the following

$$H'_u = -u + p = 0 \tag{69}$$

$$H''_{uu} = -1 < 0 \tag{70}$$

$$p' = -H'_x = -1 \tag{71}$$

$$H^*(T) = x^* - T^3 - \frac{1}{2}u^{*2} + pu^* = 0. \tag{72}$$

Note that the inequality in (70) satisfies the maximum principle.
Equation (69) gives

$$u^*(t) = p(t) \tag{73}$$

and according to (71) $p' = -1$, $p(T) = 0$ and hence by the same process as in Example 6.1 we have

$$x^*(t) = Tt - \frac{t^2}{2},$$

$$u^*(t) = T - t. \tag{74}$$

Furthermore, apply $p(T) = u^*(T) = 0$ in (72) then

$$H^*(T) = x^*(T) - T^3 - \frac{1}{2}u^{*2}(T) + p(T)u^*(T) =$$

$$= T^2 - \frac{T^2}{2} - T^3 = 0,$$

which gives $T = T^* = \frac{1}{2}$, hence the solutions are

$$x^*(t) = \frac{1}{2}(t - t^2) \quad \text{and} \quad u^*(t) = \frac{1}{2} - t.$$

■

6.2 Control problems with fixed initial and final states

In this section we sketch a proof of the Maximum principle for the optimal problem with a state variable specified at both the initial and the terminal time. The aim is to show how to deal with vector-valued functions involved in control problems and how to derive the optimality condition (59).

We consider the optimization problem

$$\max \int_{t_0}^{t_1} f(t, x(t), u(t))dt, \tag{75}$$

subject to

$$x'(t) = g(t, x(t), u(t)), \tag{76}$$

$$x(t_0) = x_0, \quad x(t_1) = x_1, \quad t_0, t_1 \quad \text{fixed}, \tag{77}$$

$$u \in U \subset \mathbb{R}^m,$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are continuous and continuously differeiable with respect to x.

Sketch of the proof: Without loss of generality we assume that the functions f and g do not explicitly dependent on t. The reason is as follows. We can introduce a new variable $y = t$ and stack it with x as $\tilde{x} = (y, x^T)^T$. Then the problem can be reformulated as

$$\max \int_{t_0}^{t_1} f(\tilde{x}(t), u(t))dt,$$

subject to

$$\tilde{x}'(t) = \tilde{g}(\tilde{x}(t), u(t)),$$

$$\tilde{x}(t_0) = \tilde{x}_0, \quad \tilde{x}(t_1) = \tilde{x}_1, \quad t_0, t_1 \quad \text{fixed},$$

$$u \in U \subset \mathbb{R}^m,$$

where $\tilde{g} = (1, g^T)^T$, $\tilde{x}_0 = (t_0, x_0^T)^T$ and $\tilde{x}_1 = (t_1, x_1^T)^T$. Accordingly we defined $H(x, u, p) = f(x, u) + p^T g(x, u)$.

Suppose that (x^*, u^*) is an optimal solution for the problem (75)-(77) with the objective value G^*, and (x, u) is any feasible solution of the same problem with the objective value G. Now we shall compute $\Delta G = G - G^*$ to derive the necessary condition in the Maximum Principle.

By integration by part

$$\begin{aligned}
\Delta G = & \int_{t_0}^{t_1} [f(x, u) + p^T g(x, u) + x^T p' - f(x^*, u^*) - p^T g(x^*, u^*) - (x^*)^T p']dt. \\
& + p(0)^T (x(t_0) - x^*(t_0)) - p(t_1)^T (x(t_1) - x^*(t_1)) \\
= & \int_{t_0}^{t_1} [H(x, u, p) - H(x^*, u^*, p) + x^T p' - (x^*)^T p']dt \\
& + p(0)^T (x(t_0) - x^*(t_0)) - p(t_1)^T (x(t_1) - x^*(t_1)).
\end{aligned} \tag{78}$$

Set $\delta x = x - x^*$, $\delta u = u - u^*$. Then

$$\Delta G = \int_{t_0}^{t_1} [H(x^*+\delta x, u^*+\delta u, p) - H(x^*, u^*+\delta u, p) + H(x^*, u^*+\delta u, p) - H(x^*, u^*, p) - (x^*)^T p']dt$$

$$+p(0)^T \delta x(t_0) - p(t_1)^T \delta x(t_1).$$

Suppose that "(x, u) is sufficiently close to (x, u)" (roughly speaking). Then Taylor series around $(x^*, u^* + \delta u)$ yields

$$\begin{aligned}
\Delta G = & \int_{t_0}^{t_1} [(H_x(x^*, u^* + \delta u, p) + (p')^T)\delta x + H(x^*, u^* + \delta u, p) - H(x^*, u^*, p)]dt \\
& + p(0)^T \delta x(t_0) - p(t_1)^T \delta x(t_1) + O(\epsilon) \\
= & \int_{t_0}^{t_1} [(H_x(x^*, u^*, p) + (p')^T)\delta x + (H_x(x^*, u^* + \delta u, p) - H_x(x^*, u^*, p))\delta x \\
& + (H(x^*, u^* + \delta u, p) - H(x^*, u^*, p))]dt \\
& + p(0)^T \delta x(t_0) - p(t_1)^T \delta x(t_1) + O(\epsilon) \\
= & \int_{t_0}^{t_1} [(H_x(x^*, u^*, p) + (p')^T)\delta x + (H(x^*, u^* + \delta u, p) - H(x^*, u^*, p))]dt \\
& + p(0)^T \delta x(t_0) - p(t_1)^T \delta x(t_1) + O(\epsilon).
\end{aligned}$$

where $O(\epsilon)$ is the higher order of ϵ with sufficiently small ϵ. The last line follows by noting that both δx and the integral of $H_x(x^*, u^* + \delta u, p) - H_x(x^*, u^*, p)$ are of order ϵ. Now since the initial and the final states are fixed we have $\delta x(t_0) = \delta x(t_1) = 0$. Thus

$$\Delta G = \int_{t_0}^{t_1} [(H_x(x^*, u^*, p) + (p')^T)\delta x + (H(x^*, u^* + \delta u, p) - H(x^*, u^*, p))]dt + O(\epsilon)$$

Now choose $p(t)$ to make the first term on the right hand side of the previous equation vanish. As before we require $p(t)$ to satisfy the adjoint equation

$$p'(t) = -(H_x)(x^*, u^*)^T = -[f_x(t, x^*, u^*) + (g_x(t, x^*, u^*))^T p(t)].$$

However, there is no boundary conditions for p. Hence

$$\Delta G = \int_{t_0}^{t_1} [(H(x^*, u^* + \delta u, p) - H(x^*, u^*, p))]dt + O(\epsilon).$$

If u^* is optimal it follows that for all t

$$H(x^*, u^* + \delta u, p) \le H(x^*, u^*, p)$$

for all $v = u^* + \delta u \in U$. To verify this, suppose that for some t there were a $v \in U$ with

$$H(x^*, u^* + \delta u, p) > H(x^*, u^*, p).$$

Then we could change the function u^* so as to make the integrand in the last integral positive over a small interval (say of width ϵ) containing this t. The integral itself would be positive (and of order ϵ). Hence ΔG would be positive, contradicting the fact that the function u produces the maximal objective value.

Note that if the problem has unconstrained control and/or the Hamiltonian is not linear in u, we can reduce the optimality condition for the Hamiltonian to

$$H_u'^* = f_u(t, x^*, u^*) + (g_u(t, x^*, u^*))^T p = 0.$$

To see this we can Taylor expand $H(x, u, p)$ at (x^*, u^*) in the first step of computation of ΔG. Then

$$\Delta G = \int_{t_0}^{t_1} [(H_x(x^*, u^*, p))^T \delta x + (H_u(x^*, u^*, p))^T \delta u] dt + O(\epsilon).$$

To account for the degenracy situation discussed in the beginning of this section we introduce a constant p_0 into the Hamiltonian

$$H(x, u, p) = p_0 f(x, u) + p^T g(x, u).$$

These degeneracy situations (where the terminal constraint is overwhelmingly imposeing) correspond to $p_0 = 0$. In these cases the objective does not enter the conditions. Nevertheless, in well-formulated problems, $p_0 > 0$ and without loss of generality we may set it to 1. Therefore, in practice, we always try to apply the maximum principle with $p_0 = 1$. We shall give an example to show why.

A more natural way to see how p_0 enters the procedure is demonstrated as follows. Introduce a new variable $y(t) = \int_{t_0}^t f(x(s), u(s)) ds$. Then $y(t_0) = 0$ and $y(t_1)$ is free. Let

$$\tilde{x}(t) = \begin{pmatrix} y \\ x \end{pmatrix}, \tilde{g} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

This leads to the following equivalent problem

$$\max y(t_1),$$

subject to

$$\tilde{x}'(t) = \tilde{g}(\tilde{x}(t), u(t)), \quad u \in U$$

with

$$\tilde{x}(t_0) = \begin{pmatrix} 0 \\ x_0 \end{pmatrix} \quad \text{and} \quad \tilde{x}(t_1) \in \mathbb{R} \times \{x_1\}.$$

As before we introduce an extended muliplier $\tilde{p} = (p_0, p^T)^T$. Do the same calculation we shall see that $H(x, u, \tilde{p}) = p_0 f + p^T g$. The rest calculation is formally the same except that the final state constraints are mixed. The Maximum Principle for this problem is discussed in the last section.

Note finally that all arguments above are not mathematically precis and have to be justified. So it is not a rigorous proof. Such a proof, however, is much involved and we are going to omit it.

Now we give some more examples.

Example 6.3: *Optimal Consumption*

Find a function $c(t)$ that maximize

$$J = \int_0^1 \ln[c(t)4s(t)]dt \tag{79}$$

subject to

$$s' = 4s(t)(1 - c(t)), \quad s(0) = 1, \quad s(1) = e^2. \tag{80}$$

Solution: The Hamiltonian of the problem is

$$H = \ln 4 + \ln c + \ln s + p(4s(1 - c)).$$

Applying the maximal principle yields the following equations

$$\frac{\partial H}{\partial c} = \frac{1}{c(t)} - 4p(t)s(t) = 0, \tag{81}$$

$$p'(t) = -\frac{\partial H}{\partial s} = -\frac{1}{s(t)} - 4p(t)(1 - c(t)), \tag{82}$$

$$s'(t) = \frac{\partial H}{\partial p} = 4s(t)(1 - c(t)). \tag{83}$$

From (81) we have $c = 1/4ps$. Substituting this c in (82) and (83) we obtain a differential equations as follows

$$p' = -\frac{1}{s} - 4p\left(1 - \frac{1}{4ps}\right) = -4p$$

and

$$s' = 4s\left(1 - \frac{1}{4ps}\right) = 4s - (1/p).$$

The first differential equation yields

$$p(t) = p(0)e^{-4t}, \tag{84}$$

then substituting (84) into the second equation gives

$$s' = 4s - e^{4t}/p(0).$$

Move all s terms to the left-hand side, thereafter multiply the equation by integrating factor e^{-4t}. Then

$$s'e^{-4t} - ase^{-4t} = -1/p(0),$$

where the left-hand side of the equation is the derivative of se^{-4t}. Integrating this we get the general solution

$$se^{-4t} = -t/p(0) + C, \quad C \quad \text{is a constant.}$$

According to (80) $C = 1$ and $p(0) = -1/(e^{-2} - 1) \backsimeq 1.156$. Hence the solution is

$$s(t) = e^{4t} - 0.865te^{4t}. \tag{85}$$

Substituting (84) and (85) in (81) we obtain

$$c(t) = \frac{1}{4.624 - 4t} \tag{86}$$

which describes the increasing of the consumption ration during the given time (to reach the maximum).

■

Example 6.4: Solve the control problem

$$\max \int_0^T (-9 - \frac{1}{4}u^2)dt, \quad x' = u, \quad x(0) = x_0, \quad x(T) = 16 \quad u(t) \in \mathbb{R}.$$

and determine the value of T.

Solution: a) The Hamiltonian equation is

$$H(t, x, u) = -9 - \frac{1}{4}u^2 + pu.$$

Applying the maximal principle yields the following

$$H'_u = -\frac{1}{2}u + p = 0, \tag{87}$$

$$H''_{uu} = -\frac{1}{2} < 0 \quad \text{satisfies the maximum principle,} \tag{88}$$

$$p' = -H'_x = 0, \quad p(T) \text{ no condition,} \tag{89}$$

$$H^*(T) = -9 - \frac{1}{4}u^{*2}(T) + p^*(T)u^*(T) = 0. \tag{90}$$

From equation (87) we have $u^* = 2p$. According to (89) $p = c_1$, where c_1 is a constant. Since $x'^* = u^*$ then $x'^* = 2c_1$, the integration yields

$$x^*(t) = 2c_1t + c_2,$$

and the fixed values

$$x^*(0) = c_2 = 0, \quad x^*(T) = 2c_1T + c_2 = 16,$$

so $c_1 = 16/2T$, $c_2 = 0$. Hence $x^*(t) = 16t/T$ and we obtain that $u^* = 16/T$. According to (90) we have

$$H^*(T) = -9 - \frac{1}{4}(\frac{16}{T})^2(T) + \frac{16}{2T}(\frac{16}{T})(T) = 0 \quad \Longrightarrow \quad T = \frac{8}{3}.$$

Thus the solution is

$$x^*(t) = \frac{16t}{T} = \frac{16t}{8/3} = 6t,$$

$$u^*(t) \equiv 6.$$

■

As the previous example shows it is sometimes necessary to have p_0 in the Hamiltonian. This is crucial in non-linear control systems. However if the system is linear then we can almost always assume that $p_0 = 1$. This can be illustrated by the following LQ control with fixed end condition:

Minimize

$$J = \int_0^{t_1} u(t)^2 dt \quad \text{subject to} \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ x(0) = x_0, x(t_1) = 0. \end{cases}$$

Assume that the system is completely controllable. We start by maximizing the Hamiltonian

$$H(\tilde{p}, x, u) = -p_0 u^2 + p'(Ax + Bu).$$

There are two cases: $p_0 = 0$ an $p_0 \neq 0$.
Case 1: $p_0 = 0$.
If this is the case, we have $\arg\min_u H(\tilde{p}, x, u) = \arg\min_u [p'(Ax + Bu)] = \pm\infty$ unless $p'B = 0$. It is however, impossible to have $u = \pm\infty$ on a nonzero time interval since then the cost would be infinite, which clearly cannot be the minimum since we know that the system can be driven to origin with finite energy expenditure. The other alternative $p(t)'B = 0$ for $t \in [0, t_1]$ is also impossible. To see this we note that the adjoint equation

$$\dot{p}(t) = -A'p(t)$$

has the solution $p(t) = e^{-A't}p(0)$. Hence, in order for $p(t)'B = 0$ for $t \in [0, t_1]$ we need

$$\begin{cases} p(0)'B = 0 \\ \dot{p}(0)'B = 0 \\ \vdots \\ p^{(n-1)}(0)'B = 0 \end{cases} \Leftrightarrow \begin{cases} p(0)'B = 0 \\ p(0)'AB = 0 \\ \vdots \\ p(0)'A^{n-1}B = 0 \end{cases} \Leftrightarrow p(0)'[B, AB, ..., A^{n-1}B] = 0.$$

If the system is assumed to be completely controllable or (A, B) is controllable, then the matrix $[B, AB, A^{n-1}B]$ has full rank, which implies that $p(0) = 0$.

(See e.g. [4].) However, then $p(t) = 0$ and $\tilde{p}(t) = 0$ which contradicts the theorem. This leads to the conclusion that $p_0 = 0$ is impossible for a controllable system. Since the controllability is generic for linear system ([4]) we can say that this case is almost neglectible.

Case 2: $p_0 = 1$. Now we assume that the system is completely controllable. We have that $u(t) = \frac{1}{2}B'p$ maximizes the Hamiltonian. The adjoint equation is

$$\dot{p}(t) = -A'p(t)$$

which has the solution $p(t) = e^{-A't}p(0), x(0) = x_0$. By the variation of constants formula we obtain

$$\begin{aligned} x(t_1) &= e^{At_1}x_0 - \frac{1}{2}\int_0^{t_1} e^{A(t_1-s)}BB'e^{-A's}ds p(0) \\ &= e^{At_1}x_0 - \frac{1}{2}W(t_1,0)e^{-A't_1}p(0) \end{aligned}$$

where the reachability Grammian is

$$W(t_1, 0) = \int_0^{t_1} e^{A(t_1-s)}BB'e^{A'(t_1-s)}ds.$$

In our case the system is controllable and therefore $W(t_1, 0)$ is positive definite and thus invertible. We can solve for $p(0)$, which gives

$$p(0) = -2e^{A't_1}W(t_1, 0)^{-1}e^{At_1}x_0.$$

This gives the optimal control

$$u(t) = \frac{1}{2}B'e^{-A't}p(0) = -B'e^{A'(t_1-t)}W(t_1, 0)^{-1}e^{At_1}x_0$$

and the optimal cost becomes (after some calculations)

$$J^* = x_0'e^{A't_1}W(t_1, 0)^{-1}e^{At_1}x_0.$$

6.3 Control problems with inequality at the endpoint

Sometimes the terminal value of the state variable is generalized to be not less/more than some fixed constant. This section is about the optimal problems with bounded constraints on the endpoint.

In general, the economical problems in the control applications often have lower bound constraint in the inequality final state value. For instance, the cosmetic firm is required not to leave the firm until the stock of cosmetic $x(t_1)$

is higher than a lower bound x_L. Let us consider a problem with a lower bound constraint:

$$\max \int_{t_0}^{t_1} f(t, x(t), u(t))dt, \qquad u \in U \subseteq \mathbb{R} \tag{91}$$

$$\text{subject to} \quad x'(t) = g(t, x(t), u(t)), \tag{92}$$

$$x(t_0) = x_0, \quad x(t_1) \geq x_L, \quad t_0, t_1 \quad \text{fixed.} \tag{93}$$

Theorem 6.3.1: For the problem (91) with (92)-(93) the following conditions are necessary,

$$p(t_1) \geq 0, \quad x(t_1) - x_L \geq 0, \quad p(t_1)^T[x(t_1) - x_L] = 0. \tag{94}$$

Proof: Assume that there is a solution for the problem (91). Let V be equal to the integral of $f(t, x, u)$ in (91) with $x(t_1) = x_1$ and let x^* be a solution to $V^*(x_L)$ with the constraint,

$$x(t_1) - x_L \geq 0.$$

The necessary condition that characterizes x^* is

$$\frac{\partial V^*}{\partial x_1} \leq 0, \quad x_1^* - x_L \geq 0, \quad (\frac{\partial V^*}{\partial x_1})[x_1^* - x_L] = 0. \tag{95}$$

The condition (95) together with the partial differential equation of the optimal value function V w.r.t. x_1 in (54) imply the condition (94) in Theorem 6.3.1. $\square$

7 The Maximum Principle and The Calculus of Variations

In this section let us study the relation between optimal control theory and the calculus of variations. Using the maximum principle it is possible to extend the calculus of variations. A problem in calculus of variations is

$$\max \int_{t_0}^{t_1} F(t, x(t), x'(t))dt, \quad x(t_0) = x_0, \quad \begin{cases} (a) \quad x(t_1) = x_1 \\ (b) \quad x(t_1) \geq x_1 \\ (c) \quad x(t_1) \quad \text{free} \end{cases}, \tag{96}$$

where the condition $x(t_1)$ can be one of $(a) - (c)$. The problem has no other constraints. Let $u(t) = x'(t)$. Then it is unconstrained. Together with the Hamiltonian $H(t, x, u, p) = p_0F(t, x, u) + pu$ we can transform to this problem to a control problem. Since $u \in \mathbb{R}$, a necessary condition for the maximum is

$$H'_u(t, x^*(t), u^*(t), p(t)) = p_0F'_u(t, x^*(t), u^*(t)) + p(t) = 0, \tag{97}$$

for $(p_0, p(t)) \neq (0, 0)$. We write F'_u for $\partial F/\partial u$. In (97) we have $p_0 \neq 0$ thus we can take $p_0 = 1$. As we have seen before the Hamiltonian gives the differential equation for $p(t)$ as

$$p'(t) = -H'_x(t, x^*(t), u^*(t), p(t)) = -F'_x(t, x^*(t), u^*(t)). \tag{98}$$

Now differentiating $p(t)$ in (97) with respect to t gives

$$\frac{d}{dt}(F'_u(t, x^*(t), u^*(t))) + p'(t) = 0. \tag{99}$$

Combining (98), (99) and $u^* = x'^*$ we have

$$F'_x(t, x^*, x'^*) - \frac{d}{dt}(F'_{x'}(t, x^*, x'^*)) = 0. \tag{100}$$

as an Euler equation. We write $F'_{x'}$ for $\partial F/\partial \dot{x}$. Furthermore, (97) yields

$$p(t) = -F'_{x'}(t, x^*, x'^*). \tag{101}$$

With respect to (x, u), the concavity of $F(t, x, x')$ is equivalent to the concavity of the Hamiltonian. In the maximum principle the Hamiltonian has its maximum at $u^*(t)$ for $t \in [t_0, t_1]$. If F is in C^2 then $H'_u = 0$ implies $F'_{x'} = 0$. And $H''_{uu} \leq 0$ implies $F''_{x'x'} \leq 0$ (In the calculus of the variation, this is called the *Legendre condition*).

Example 7.1: Solve the problem by illustrating the use of the Euler's equation.

$$\max \int_0^T e^{-2t} \ln c(t)dt \quad \dot{x}(t) = 3x - c, \quad x(0) = x_0, \quad x(T) = x_T$$

Solution: Formulate this as a problem in the calculus of variation, by substituting $3x - \dot{x}$ into $c(t)$ in the integral, then we have to maximize

$$\int_0^T e^{-2t} \ln(3x - \dot{x}) dt \tag{102}$$

subject to $x(0) = x_0$ and $x(T) = x_T$. We apply Euler's equation

$$\frac{\partial F}{\partial x} - \frac{d}{dt}(\frac{\partial F}{\partial \dot{x}}) = 0 \quad \text{where} \quad F = e^{-2t} \ln(3x - \dot{x})$$

to (102). It becomes

$$\frac{e^{-2t}3}{3x - \dot{x}} = \frac{d}{dt}(-\frac{e^{-2t}}{3x - \dot{x}}) =$$

$$= \frac{2e^{-2t}}{3x - \dot{x}} + \frac{e^{-2t}(2\dot{x} - \ddot{x})}{(3x - \dot{x})^2},$$

which yields a second-order linear differential equation as following,

$$\ddot{x} - 4\dot{x} + 3s = 0. \tag{103}$$

Transform (103) into a system of two first-order linear differential equations we have

$$\begin{pmatrix} \dot{y} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix},$$

where $y = \dot{x}$ and $\dot{y} = \ddot{x}$. The eigenvalues of the matrix are 1 and 3 so the solution is,

$$x(t) = Ae^{3t} + Be^t.$$

According to $x(0) = x_0$ and $x(T) = x_T$, the constants A and B can be determined:

$$x_0 = A + B, \quad x_T = Ae^{3T} + Be^T \Longrightarrow$$

$$\Longrightarrow A = (x_T e^{-3T} - x_0 e^{-2T})/(1 - e^{-2T}),$$

$$B = (x_0 - x_T e^{-3T})/(1 - e^{-2T}).$$

Compute A and B in (103) we can get the final solution. Note that by solving this problem with control theory it will give the same solution.

■

8 Multiple Endpoint Conditions

The control problems in this section cover the optimal problems with several state and control variables with fixed/free value of the state variable at the end point, a free value of the upper limit of integration. It can contain an arbitrary number of functions and even a salvage term from a single problem.

Consider the standard end-constrained problems that seek for the vector functions $\mathbf{x}(t) = (x_1(t), x_2(t), ..., x_n(t))$ with n state variables and control function $\mathbf{u}(t) = (u_1(t), u_2(t), ..., u_m(t))$ with m controls, defined on a time interval $[t_0, t_1]$ to

$$\max \int_{t_0}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t))dt + \psi(t_1, \mathbf{x}(t_1)) \tag{104}$$

subject to the constraints

$$\begin{aligned} \frac{dx_1(t)}{dt} &= g_1(t, \mathbf{x}(t), \mathbf{u}(t)) \\ &\vdots \\ \frac{dx_n(t)}{dt} &= g_n(t, \mathbf{x}(t), \mathbf{u}(t)) \end{aligned} \tag{105}$$

we can rewrite theses condition as $\mathbf{x}' = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t))$, and

$$x_i(t_0) = x_i^0, \quad i = 1, 2, ..., n \quad (\mathbf{x}^0 = (x_1^0, ..., x_n^0) \quad \text{is a given point in } \mathbb{R}^n) \tag{106}$$

$$x_i(t_1) = x_i^1, \quad i = 1, 2, ..., l \tag{107}$$

$$x_i(t_1) \quad \text{free}, \quad i = l + 1, ..., r \tag{108}$$

$$x_i(t_1) \geq 0, \quad i = r + 1, ..., s \tag{109}$$

$$K(x_{n+1}, .., x_n, t_1) \geq 0 \quad \text{at} \quad t_1 \tag{110}$$

where $1 \leq l \leq r \leq s \leq n$, and K is assumed to be a continuous and differentiable function. The problem (104)-(110) can be solved if it satisfies the conditions below.

Necessary Conditions:

a). State equations:

$$\mathbf{x}' = x_i' = g_i(t, \mathbf{x}(t), \mathbf{u}(t)) = \mathbf{g}, \quad i = 1, ..., n$$

b). Adjoint (auxiliary, costate or multiplier) equations:

$$\mathbf{p} = p_i' = -\left(\frac{\partial f}{\partial \mathbf{x}} + \sum_{j=1}^{n} p_j \frac{\partial g_j}{\partial \mathbf{x}}\right), \quad i = 1, ..., n.$$

c). Optimality conditions:

I). $\frac{\partial f}{\partial u_j} + \sum_{k=1}^{n} \frac{p_k \partial g_k}{\partial u_j} = 0$, $j = 1, ..., m$.

II). $H(t, \mathbf{x}^*, \mathbf{u}, \mathbf{p})$ is maximized by $\mathbf{u} = \mathbf{u}^*$.

d). Transversality conditions:

I). $p_i(t_1) = \frac{\partial \psi}{\partial x_i}$ if $x_i(t_1)$ is free;

II). $x_i(t_1) \geq 0$, $p_i(t_1) \geq \frac{\partial \psi}{\partial x_i}$,
$x_i(t_1)[p_i(t_1) - \frac{\partial \psi}{\partial x_i}] = 0$:

III). $p_i(t_1) = \frac{\partial \psi}{\partial x_i} + p\frac{\partial K}{\partial x_i}$,
$i = l, ..., n$, $p \geq 0$, $pK = 0$,
if $K(x_l(t_1), .., x_n(t_1)) \geq 0$ is required;

IV). $p_i(t_1) = \frac{\partial \psi}{\partial x_i} + p\frac{\partial K}{\partial x_i}$, $i = l, ..., n$,
if $K(x_l(t_1), .., x_n(t_1)) = 0$ is required;

V). $f + \sum_{i=1}^{n} p_i g_i + \psi_t = 0$ at t_1 if t_1 is free;

VI). $f + \sum_{i=1}^{n} p_i g_i + \psi_t \geq 0$ at t_1, with strict equality in case $t_1 < T$,
if $T - t_1 \geq 0$ is required;

VII). $p_i(t_1) = \frac{\partial^2 \psi}{\partial x_i} + p\frac{\partial K}{\partial x_i}$, $i = l, ..., n$,
$f + \sum_{i=1}^{n} p_i g_i + \psi_t + p\frac{\partial K}{\partial x_i} = 0$, $p \geq 0, K \geq 0, pK = 0$ at t_1 if
$K(x_l(t_1), .., x_n(t_1)) \geq 0$ is required.

We close this report by an example which includes a salvage term and its solution contains a piecewise continuous control. The discontinuity occurs typically when the Hamiltonian is linear in u.

Example 8.1: *The optimal investment plan of a production facility.*
Let the facility operate from initial time $t = 0$ until terminal time T. Let P be the production rate and I be the investment rate for such $0 \leq I \leq \overline{I}$, in other words, suppose I to be positive and bounded above. Assume that the facility is salvaged at a price proportional to its production rate at that time. The corresponding objective is

$$J = \beta P(T) + \int_0^T [P(t) - I(t)]dt,$$

where $\beta > 0$. Suppose the production rate of the facility decreases at a rate proportional to the production rate at the time at which no investment occurs, but that investment tends to increase the production rate. The production rate and the investment rate give the following equation

$$\dot{P} = -\alpha P + \gamma I, \quad P(0) = P_0,$$

where $\alpha > 0, \gamma > 0$.

Solution: The optimal problem above is to

$$\max \quad \beta P(T) + \int_0^T [P(t) - I(t)]dt, \quad 0 \leq I \leq \overline{I}$$

subject to

$$\dot{P} = -\alpha P + \gamma I, \quad P(0) = P_0,$$

where $\alpha > 0, \gamma > 0$ and $\beta > 0$. With function $p(t)$ as an adjoint variable for the time $[0, T]$, the Hamiltonian is

$$\begin{aligned} H(t, P(t), I(t), p(t)) &= P(t) - I(t) + p(-\alpha P + \gamma I) \\ &= P(1 - \alpha p) + I(p\gamma - 1), \end{aligned}$$

which has the necessary condition

$$\dot{p} = -H'_P = \alpha p - 1, \quad p(T) = \beta,$$

the first-order differential equation above with $p(T) = \beta$ yields

$$p(t) = e^{\alpha t} p(0) + \frac{1}{\alpha}(1 - e^{\alpha t}). \tag{111}$$

Computing $t = T$ in (111) we get

$$\beta = e^{\alpha T} p(0) + \frac{1}{\alpha}(1 - e^{\alpha T}),$$

thus

$$p(0) = e^{-\alpha T}(\beta - \frac{1}{\alpha}(1 - e^{\alpha T})).$$

It follows that the adjoint function is

$$p(t) = (\beta - \frac{1}{\alpha})e^{\alpha t - T} + \frac{1}{\alpha}. \tag{112}$$

According to the maximum condition for each $t \in [0, T]$, $I^*(t)$ must maximize the Hamiltonian H subject to $I \in [0, \overline{I}]$. Only the term $I(p\gamma - 1)$ depending on $I(t)$ so,

$$I^*(t) = \begin{cases} 0 & \text{if} \quad p\gamma - 1 < 0, \\ \overline{I} & \text{if} \quad p\gamma - 1 > 0. \end{cases}$$

In the case where $p\gamma - 1 < 0$, we have $H = P(1-\alpha p) + 0 = P(1-\alpha p)$ and the necessary condition reads as follows:

$$H'_I = 0, \quad p' = -H'_P = \alpha p - 1, \quad p(T) = 0.$$

For the case where $p\gamma - 1 > 0$, we have $H = P(1-\alpha p) + \overline{I}(p\gamma - 1)$ and the necessary condition gives:

$$H'_{\overline{I}} = p\gamma - 1, \quad p' = -H'_P = \alpha p - 1, \quad p(T) = 0.$$

Assume that we want to determine the range of γ, for $0 \le t \le T$, related to α and β for such that the optimal facility is constant. Let the optimal facility be constant. By using the condition $0 \le I \le \overline{I}$, we have case 1:

$$I = 0 \quad \Longrightarrow \quad \gamma p < 1 \quad \Longrightarrow \quad \gamma(\beta - \frac{1}{\alpha})e^{\alpha t - T} + \frac{1}{\alpha} < 1 \quad \Longrightarrow$$

$$\Longrightarrow \quad (\beta - \frac{1}{\alpha} + \frac{1}{\alpha}) < 1 \quad \Longrightarrow \quad \gamma < \frac{1}{\beta}.$$

case 2:

$$I = \overline{I} \quad \Longrightarrow \quad \gamma p > 1 \quad \Longrightarrow \quad \gamma > ((\beta - \frac{1}{\alpha})e^{\alpha t - T} + \frac{1}{\alpha})^{-1} > 1/\frac{1}{\alpha} = \alpha.$$

From case 1 and case 2, we conclude that when $\alpha < \gamma < 1/\beta$ the facility is constant, where $0 < t < T$.

Now let us consider the case when the optimal policy contains a switch, which means that $\gamma p - 1 = 0$. Computing the function $p(t)$ from (112), we have

$$\gamma(\beta - \frac{1}{\alpha})e^{\alpha t - T} + \frac{1}{\alpha} = 1$$

thus the switch time can be determined by solving t. We have

$$t_s = T - \frac{1}{\alpha} \ln \frac{1/\alpha - \beta}{1/\alpha - 1/\gamma},$$

as the switch time.

Finally let consider the problem when the facility were to operate from time T on without any further investment, the corresponding facility is

$$\int_T^\infty P(t)dt, \tag{113}$$

where $I = 0$, and it yields that $\dot{P} = -\alpha P$. By using a first-order differential equation together with the condition that the initial production rate $P(0) = P_0$, we get

$$P(t) = e^{-\alpha t} P_0. \tag{114}$$

Inserting (114) in (113), we get the facility as following

$$\int_T^{\infty} e^{-\alpha t} P_0 dt = -\frac{1}{\alpha} e^{-\alpha t} \int_T^{\infty} P_0 dt = \frac{1}{\alpha} e^{-\alpha T} P_0 = \frac{P(T)}{\alpha},$$

which produces a terminal revenue.

$\blacksquare$

From the final example, we can see that this problem illustrates many economical perspectives. This is only one of the simple problems that our knowledge from this paper will be sufficient enough to solve. In the real economical world there are much more advanced and complicated problems where mathematical control theory play an important role.

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