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Creation of strange attractors in the quasi-periodically forced quadratic family

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ABSTRACT. In this paper we will study the creation of strange non-chaotic attractors, the invariant, attracting graph of a nowhere continuous measurable $\psi : \mathbb{T} \to [0,1]$, in certain families of quasi-periodically forced quadratic maps

$$\begin{aligned}\Phi_{\alpha,\beta} &: \mathbb{T} \times [0,1] \to \mathbb{T} \times [0,1] \\ &: (\theta, x) \mapsto (\theta + \omega, c_{\alpha,\beta}(\theta) \cdot x(1-x)),\end{aligned}$$

where ω is a Diophantine irrational, and $c_{\alpha,\beta}(\theta) : \mathbb{T} \to [\frac{3}{2}, 4]$ is a prescribed family of maps.

The same model was studied by Bjerklöv in [2] for $\beta = 1$, where it was shown to possess a strange non-chaotic attractor for a certain critical value of $\alpha = \alpha_c$. There it was also shown that $\inf_{\theta \in \mathbb{T}} \psi(\theta) = 0$.

In this paper, we will show that, whenever $0 \le \beta < 1$, the attractor for that same value of $\alpha = \alpha_c$ is the invariant, attracting graph of a continuous measurable $\psi : \mathbb{T} \to [0,1]$. Moreover, for the value $\alpha = \alpha_c$, we will establish asymptotic bounds on the minimum distance $\delta(\beta)$, as β goes to 1, from the attractor to the repelling set $\mathbb{T} \times \{0,1\}$; more precisely, we show that there are a $\delta > 0$, and constants $0 \le a_1 \le a_2$ such that

$$a_1(1-\beta) \le \delta(\beta) \le a_2(1-\beta)$$

whenever $1 - \delta \le \beta \le 1$.

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Contents

1. Introduction

A (smooth) continuous dynamical system is essentially a smooth flow[1] $f^t(x) : \mathbb{R} \times X \to X$ on a smooth manifold[2] X. The field of dynamical systems was born out of a desire to understand the long-term behaviour of a physical system governed by certain laws. It is often of interest to consider also discrete systems, where $f(x) : X \to X$, such as when looking at the state of a continuous system at discrete time-intervals. One can easily reduce a continuous system to a discrete one by setting, for some $\tau > 0$,

$$F(x) = \int_0^\tau f^t(x)dt.$$

An important part in studying dynamical systems is classifying the invariant sets. A set A is invariant if $f^t(A) \subseteq A$ for all t in $\mathbb{R}$ ($f(A) \subseteq A$, when the system is discrete). The orbit of a point x in X is the set

$$\{f^t(x) : t \in \mathbb{R}\} \quad (\{f^n(x) : n \geq 0\}).$$

An invariant set is simply a collection of orbits. Of particular interest among the invariant sets are the so-called attracting sets, and repelling sets. An *attractor* is an attracting set containing no smaller attracting set. In [10], Milnor discusses alternative definitions of attractors. We will consider the following one:

Definition 1.1 (Attracting set)**.** A closed invariant subset A is called an attracting set if it satisfies:

- the *realm of attraction* $\rho(A)$, the set of points x in X such that the orbit of x eventually stays in A, has positive measure (in the sense of Lebesgue).

An attracting set A is called an attractor if it satisfies:

- there is no strictly smaller closed invariant set $A' \subset A$ such that $\rho(A')$ coincides with $\rho(A)$ up to a set of measure zero.

An attracting set is one which attracts nearby orbits. A repelling set repells (sends away) almost every orbit coming close to it. A repellor is similarly an indecomposable repelling set. Knowing the attractors and the repellors in a system, and what they look like, gives a lot of information about the long-term behaviour of the system.

Another important concept is that of *chaos*, where the central idea is a sensitive dependence on initial conditions (orbits of nearby points will likely diverge). We will not be directly concerned with chaos, but will be interested in *nonchaotic* systems. Actually, the (strange) attractors we will be interested in will exhibit behaviour somewhere on the boundary between chaotic and non-chaotic systems, usually having some sensitivity to initial dependence (see [6]).

In recent decades, it has become apparent that an attractor can have a strange geometry. Such attractors are called *strange* attractors. One of the earliest uses of the term is in the article [13], by Ruelle and Takens in 1971, where a possible connection is made between the appearance of turbulence in fluids and the existence of strange attractors in such systems. There is no

[1] A global solution to a differential equation.

[2] A generalized smooth surface.

clear definition of what constitutes a strange attractor, but the term has been used to describe attractors with strange geometrical properties, such as a noninteger fractal dimension or nowhere differentiability. In [11, 12], the notion of strange attractors is discussed in more detail.

Following the article by Ruelle and Takens, the existence of strange attractors were discussed in the context of chaotic systems, such as the Hénon map, and the Lorenz system. Perhaps it was believed initially that strange attractors were connected to chaos; however soon enough, in the article [7] from 1984, the notion of *strange nonchaotic attractors* was introduced. The authors of that article presented numerical evidence of a strange attractor which was nonchaotic, that is, the dynamics considered on the attractor as an isolated system is not chaotic.

In the article [9], Keller proved rigorously the existence of strange nonchaotic attractors in a class of systems, called *pinched* (will be explained below). The model in [7] was a special case of that class.

We will be interested in a type of systems called *quasiperiodically forced*. The motivation for considering such systems arise from physics. A simple example consists of two forces acting linearly on an object with periodically varying amplitudes, where their respective periods are incommensurate[3]. In [5], this physical connection is discussed at more length.

Having seen that quasiperiodicity is of physical relevance, it becomes interesting to study such systems from a mathematical viewpoint. A way to model a one-dimensional quasiperiodically forced system (as a discrete system), is to let

$$\Phi : \mathbb{T} \times X \to \mathbb{T} \times X : (\theta, x) \mapsto (\theta + \omega, g(\theta) \cdot f(x)),$$

where $\mathbb{T}$ is the circle $(\mathbb{R} \backslash \mathbb{Z})$, ω is irrational, and g is called the *forcing map*. A system where $\mathbb{T} \times \{0\}$ is invariant is called *pinched* if the forcing map is 0 for at least one θ (and hence every orbit going through such a θ will get stuck at 0).

Quasiperiodically forced systems are very interesting, since they provide many examples of strange nonchaotic attractors. In fact, in accordance with [1, 2], we make the following definition:

Definition 1.2 (Strange nonchaotic attractor)**.** Let

$$\Phi : \mathbb{T} \times [0,1] \to \mathbb{T} \times [0,1] : (\theta, x) \mapsto (\theta + \omega, g(\theta) \cdot f(x)),$$

where ω is irrational. The graph of a measurable function $\psi : \mathbb{T} \to [0,1]$ is a called a strange attractor for the system if

- it is invariant, that is $\Phi(\theta, \psi(\theta)) = (\theta + \omega, \psi(\theta + \omega))$, for a.e. θ;
- it is discontinuous (almost) everywhere; and
- it attracts the orbits of a set of points of positive measure.

In [2] Bjerklöv proved the existence of a strange nonchaotic attractor in a *non-pinched* system. In a later article [3] by the same author, it was shown that the attractor is dense in a "regular" surface.

In the article [8] Haro and de la Llave made numerical studies of a family of quasiperiodic hyperbolic system[4], where the expanding and contracting directions where merged for certain

[3]The ratio of the respective frequencies, $\frac{\omega_1}{\omega_2}$, is irrational; that is, their periods will never align, but they will repeatedly get closer and closer to aligning; the resulting force is *quasiperiodic*.

[4]At each point there is an expanding direction, and a contracting one

critical parameter values. This caused the attracting set and the repelling set to merge at certain points. They found numerical evidence suggesting that the minimum distance between the attracting and repelling sets was asymptotically linear in the parameter, when the parameter was sufficiently close to the critical value. In [4] Bjerklöv and Saprykina analytically proved the claim for certain models.

In this paper, we will establish similar asymptotic behaviour for the system considered in [2].

2. Our Model

In the article [2], the existence of a strange nonchaotic attractor for the following quasi-periodically forced quadratic map, for a particular α, was established:

$$\Phi_\alpha : \mathbb{T} \times [0,1] \to \mathbb{T} \times [0,1] : (\theta, x) \mapsto (\theta + \omega, c_\alpha(\theta) \cdot p(x)),$$

where ω is a (Diophantine, see further down) irrational number,

$$p(x) = x(1-x)$$

is the quadratic (or logistic) map, and $c_\alpha(\theta)$ is a smooth forcing map. The map $c_\alpha(\theta)$ was fashioned to be $\approx \frac{3}{2}$, except at two peaks 0 and $\alpha \approx \omega$, where $c_\alpha(\theta)$ "suddenly" hits 4. The expression used for $c_\alpha(\theta)$ was

$$c_\alpha(\theta) = \frac{3}{2} + \frac{5}{2}\left(\frac{1}{1+\lambda(\cos 2\pi(\theta - \alpha/2) - \cos \pi\alpha)^2}\right),$$

where λ is assumed to be sufficiently large, in order for the peaks to be narrow. Below (fig. 1) is a figure showing what the graph of $c(\theta)$ might look like. In this paper, we will introduce another parameter β to the system, where the peaks are scaled down by that constant, and study what happens to the attractor as the parameter is perturbed.

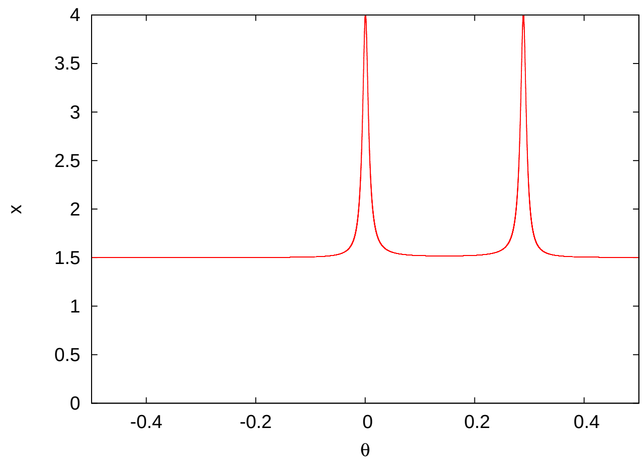

Figure 1. The graph of $c(\theta)$.

In order to understand this particular choice of $c_\alpha(\theta)$, we have to make a brief detour into the dynamics of the quadratic map. Consider the map $f_\omega : \mathbb{T} \times [0,1] \to [0,1] : (\theta, x) \mapsto (\theta + \omega, \frac{3}{2}x(1-x))$. By using the results in section 3, it is possible to show that every $(\theta, x) \in \mathbb{T} \times (0,1)$ will converge to $\mathbb{T} \times \{\frac{1}{3}\}$, and that this is an attractor if ω is irrational.

Actually, it is possible to show that, as long as the forcing map is within a small $\varepsilon > 0$ of $\frac{3}{2}$ (the actual ε may even vary for different θ), there will be an attractor $(\theta, \psi(\theta))$, where $\psi(\theta) \approx \frac{1}{3}$ is continuous.

We can see that, for every $\theta \in \mathbb{T}$,

$$f_\omega(\theta, 1) = f_\omega(\theta, 0) = (\theta + \omega, 0).$$

The set $\mathbb{T} \times \{0,1\}$ is not only invariant, but is actually a repelling set. It is true that the subset $\mathbb{T} \times \{0\}$ is a repellor (indecomposable repelling set), but since $x = 1$ is mapped directly to $x = 0$, we will be interested in the whole set $\mathbb{T} \times \{0,1\}$.

So, our c_α was made to be $c_\alpha(\theta) \approx \frac{3}{2}$, except when θ is *very* close to 0 and α. Note that $p(x) = x(1-x)$ is symmetric around the maximum $\frac{1}{4}$ at $x = \frac{1}{2}$, and so in order to "hit" the repellor, we want orbits to tend to $x = \frac{1}{2}$ at the second peak α.

In order to produce the strange attractor, the α was tweaked to a critical value α_c, whereby the limits of certain orbits would enter the chain $(\alpha_c - \omega, \approx \frac{1}{3}) \mapsto (\alpha_c, \frac{1}{2}) \mapsto (\alpha_c + \omega, 1) \mapsto (\alpha_c + 2\omega, 0)$, causing the attractor to "merge" with the repelling set $\mathbb{T} \times \{0,1\}$ (and hence get stuck at the repellor $\mathbb{T} \times \{0\}$).

The strange attractor that was found has been approximated in simulations, and is shown below (fig. 2). Note that, for ease of visualization, the value of α used in fig. 1 is different from the one producing the strange attractor in fig. 2.

Also note that the effect of the peaks are felt one step later in the attractor. This is simply because

$$\Phi(\theta, x) = (\theta + \omega, c(\theta) \cdot x(1-x)),$$

and so the effect of the peak is seen in the next iteration. The last thing to note is that, as we choose λ larger, the peaks of $c(\theta)$, and hence also the peaks of the attractor, will become more narrow, and the parts around $\approx \frac{1}{3}$ will become flatter.

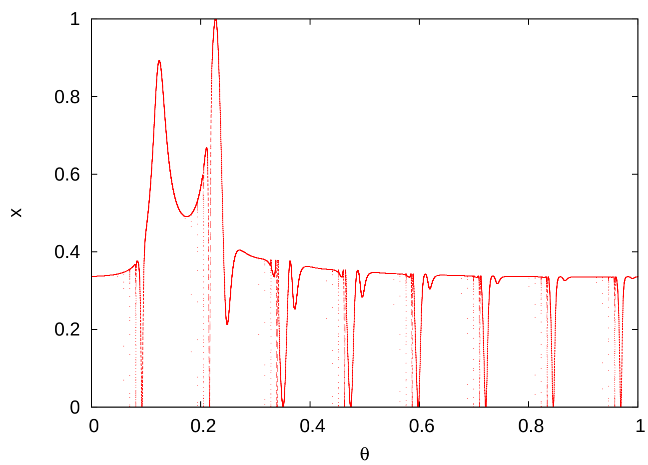

FIGURE 2. The strange attractor.

The conclusion we come to is that, what is producing the strange attractor is the "merging" of the attractor and the repelling set $\mathbb{T} \times \{0,1\}$ at a dense set of values of θ.

Now, let $\beta \in [0,1]$. The extended system we will be interested in is given by

$$\Phi_{\alpha,\beta} : \mathbb{T} \times [0,1] \to \mathbb{T} \times [0,1] : (\theta, x) \mapsto (\theta + \omega, c_{\alpha,\beta}(\theta) \cdot p(x)),$$

where

$$c_{\alpha,\beta}(\theta) = \frac{3}{2} + \beta \frac{5}{2} \left(\frac{1}{1 + \lambda (\cos 2\pi(\theta - \alpha/2) - \cos \pi\alpha)^2} \right).$$

In the original article [2] the model was studied for the value $\beta = 1$, where it was shown that there exists a critical $\alpha = \alpha_c$ (the index c is for critical) such that the system possesses a strange nonchaotic attractor. The purpose of this paper is to study what happens to the attractor when $\alpha = \alpha_c$ is fixed, but β is varied close to 1.

We think of the $\lambda > 0$ as some sufficiently large constant, depending only on ω.

An example of what the graph of $c(\theta)$ might look like for $\beta = 0.5$ can be seen in fig. 3. The corresponding attractor can be seen in fig. 4. The figure suggests that the attractor is continuous, and in fact, as we will show, the attractor is continuous when $0 \leq \beta < 1$.

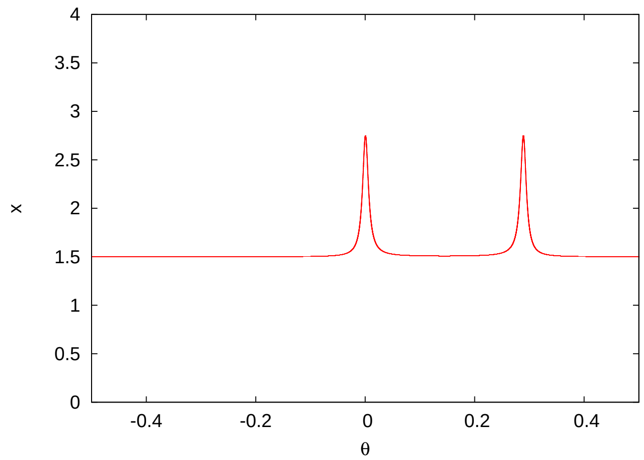

FIGURE 3. The graph of $c(\theta)$ when $\beta = 0.5$.

Since the strange attractor appears when the minimum distance between the attractor and the repelling set is 0, it would be interesting to see how this distance depends on the parameter β. In fig. 5, we have plotted this minimum distance as obtained in the simulations. The graph seems to suggest that the distance is asymptotically linear as β approaches 1.

For technical reasons (see lemma 3.1 for the consequences of this assumption), we will assume that $\omega \in \mathbb{T}$ is an irrational number satisfying the Diophantine condition

$$\inf_{p \in \mathbb{Z}} |q\omega - p| > \frac{\kappa}{|q|^{\tau}} \text{ for all } q \in \mathbb{Z} \backslash \{0\}, \qquad (DC)_{\kappa,\tau}$$

for some $\kappa > 0, \tau \geq 1$ (note that it is sufficient to consider only $q \in \mathbb{Z}_+ = \{1,2,3,\dots\}$ by symmetry in $p \in \mathbb{Z}$). This is no severe restriction, since (Lebesgue) almost every irrational ω satisfies the Diophantine condition, for at least some $\kappa > 0$, and $\tau \geq 1$. Indeed, let $\mathcal{D}$ be the set of all Diophantine irrationals in $[0,1)$. Then the complement $\mathcal{D}^c$ is included in the decomposition

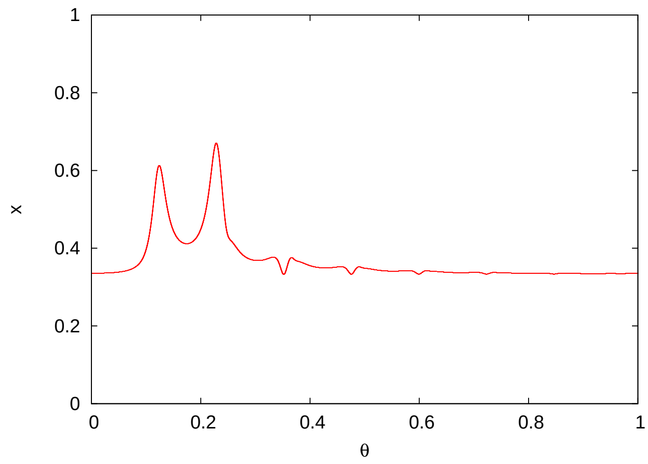

FIGURE 4. The continuous (!) attractor when $\beta = 0.5$.

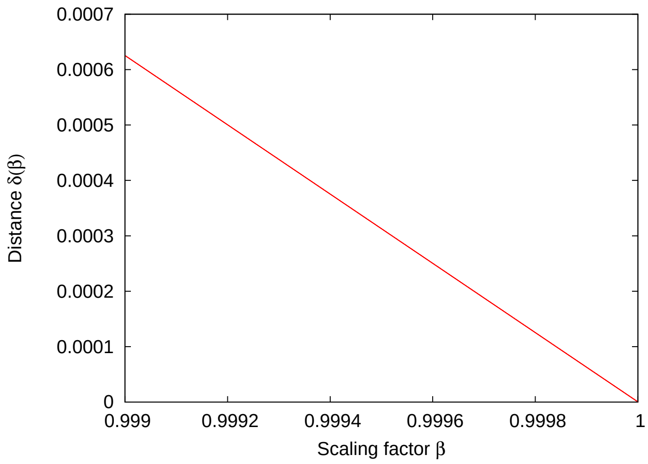

FIGURE 5. The minimum distance as a function of β, when β is close to 1.

$\mathcal{D}^c \subset \bigcup_{q=1}^{\infty} \mathcal{D}_q$, where $\mathcal{D}_q = \{\omega \in [0,1) : |q\omega - p| \leq \frac{\kappa}{q^\tau}$, for every $p \in \mathbb{Z}, \kappa > 0, \tau > 1\}$. By a quick rearrangement, we obtain

$$\mathcal{D}_q = \{\omega \in [0,1) : |\omega - \frac{p}{q}| \leq \frac{\kappa}{q^{\tau+1}}, \text{ for every } p \in \mathbb{Z}, \kappa > 0, \tau > 1\},$$

and so $\mathcal{D}_q$ is the set of irrationals ω such that the smallest distance from ω to a rational with denominator q is smaller than $\frac{\kappa}{q^{\tau+1}}$ for every $\kappa > 0, \tau > 1$. Now, since $\omega \in [0,1)$, it is sufficient to consider only $p = 0, 1, \ldots, q-1$, and so the set $\mathcal{D}_q$ can be covered by q intervals of length $2\frac{\kappa}{q^{\tau+1}}$ (notice that the length is maximized as τ decreases), so it has a cover of total length $2\frac{\kappa}{q^\tau}$. Hence there is a cover of $\mathcal{D}^c$ of total length

$$\sum_{q=1}^{\infty} 2\frac{\kappa}{q^{\tau}} = 2\kappa \sum_{q=1}^{\infty} \frac{1}{q^{\tau}} < 2\kappa \cdot S,$$

where $\kappa > 0$ is arbitrary, and $\sum\limits_{q=1}^{\infty} \frac{1}{q^{\tau}}$ converges to some S, since $\tau > 1$. Hence, the complement $\mathcal{D}^c$ has zero measure. Since $\inf\limits_{p\in\mathbb{Z}} |q\omega - p|$ is invariant under $\omega \mapsto \omega + n$ for $n \in \mathbb{Z}$, the complement of all Diophantine irrationals is $\bigcup\limits_{n\in\mathbb{Z}} \mathcal{D}^c + n$, which has zero measure (a countable union of null sets is a null set, just choose covers $\frac{\varepsilon}{2^n}$).

We mentioned that the system is non-chaotic. This can be established by looking at the Lyapunov exponents. Let the following two-dimensional dynamical system be given:

$$\Phi : \mathbb{T} \times [0,1] \to \mathbb{T} \times [0,1] : (\theta, x) \mapsto (\theta + \omega, c(\theta) \cdot p(x)), \tag{2.1}$$

where ω is fixed. The Lyapunov exponents at a point (θ_0, x_0) measure the long-term effect of slightly perturbing the initial points. There is a good treatment of this case in the book [5]. Let $(\theta_0, x_0) + (\varepsilon_\theta, \varepsilon_x)$ be a small perturbation of the initial point (θ_0, x_0). We will use the notation $(\delta\theta_n, \delta x_n)$ for the perturbation of the n-th iterate (in particular, $(\delta\theta_0, \delta x_0) = (\varepsilon_\theta, \varepsilon_x)$). Using the short-hand $(\theta_n, x_n) = \Phi^n(\theta_0, x_0)$, then the perturbation of the n-th iterates can be calculated by

$$\begin{pmatrix} \delta\theta_n \\ \delta x_n \end{pmatrix} = \begin{pmatrix} \frac{\partial x_n}{\partial x_0} & \frac{\partial x_n}{\partial \theta_0} \\ \frac{\partial \theta_0}{\partial x_0} & \frac{\partial \theta_n}{\partial \theta_0} \end{pmatrix} \begin{pmatrix} \delta\theta_0 \\ \delta x_0 \end{pmatrix} = \begin{pmatrix} \frac{\partial x_n}{\partial x_0} & \frac{\partial x_n}{\partial \theta_0} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \delta\theta_0 \\ \delta x_0 \end{pmatrix},$$

and since the Jacobian matrix is upper-triangular, the eigenvalues will be $\frac{\partial x_n}{\partial x_0}$, and $\frac{\partial \theta_n}{\partial \theta_0}$, in the x-direction and the θ-direction, respectively. The derivative $\frac{\partial x_n}{\partial x_0}$ simply measures the rate at which the n-th iteration of x_0 changes, when we perturb x_0 from a particular value $x_0 = x$. Above, we have used the fact that

$$\frac{\partial \theta_n}{\partial \theta_0} = \frac{\partial(\theta_0 + n\omega)}{\partial \theta_0} = 1.$$

Now the Lyapunov exponents γ_x, and γ_θ are defined as

$$\gamma_x(\theta_0, x_0) = \lim_{n\to\infty} \frac{1}{n} \log \left| \frac{\partial x_n}{\partial x_0} \right|$$
$$\gamma_\theta(\theta_0, x_0) = \lim_{n\to\infty} \frac{1}{n} \log \left| \frac{\partial \theta_n}{\partial \theta_0} \right|,$$

whenever the limits exist. In our case, at least γ_θ is well-defined and equal to 0. In case the first limit doesn't exist, we define the upper Lyapunov exponent

$$\overline{\gamma_x}(\theta_0, x_0) = \limsup_{n\to\infty} \frac{1}{n} \log \left| \frac{\partial x_n}{\partial x_0} \right|.$$

As we can see, the Lyapunov exponents measure the average separation/contraction of nearby orbits. The system is said to be nonchaotic if the (upper) Lyapunov exponents are non-positive (≤ 0) for almost every point (θ_0, x_0). Later, we will see that this is the case for us.

By $\delta_\alpha(\beta)$ we mean the minimum distance between the attractor ψ^β below, and the repelling set $\mathbb{T} \times \{0,1\}$, where α is fixed. We are now ready to state the main theorem of this paper.

Main Theorem. *Assume that ω satisfies the Diophantine condition $(DC)_{\kappa,\tau}$ for some $\kappa > 0$ and $\tau \geq 1$. Then for all sufficiently large $\lambda > 0$, there is a parameter value $\alpha = \alpha_c$ such that the following holds for the map $\Phi_{\alpha,\beta}$:*

- *i) When $\beta = 1$, there is a strange attractor, the graph of a nowhere continuous measurable function $\psi : \mathbb{T} \to [0,1]$, which attracts points (θ, x), for a.e. $\theta \in \mathbb{T}$, and every $x \in (0,1)$.*
- *ii) When $0 \leq \beta < 1$, there is a curve, the graph of a C^1 function $\psi : \mathbb{T} \to [0,1]$, which attracts every point $(\theta, x) \in \mathbb{T} \times (0,1)$.*
- *iii) The (minimum) distance $\delta_{\alpha_c}(\beta)$ between the attractor and the repelling set, considered as a function of β, is asymptotically bounded by linear functions as $\beta \to 1$, that is*
$$a_1(1-\beta) \leq \delta_{\alpha_c}(\beta) \leq a_2(1-\beta),$$
for some constant $0 \leq a_1 \leq a_2$ as $\beta \to 1$.
- *iv) The system (and hence the attractor) is nonchaotic for $0 \leq \beta \leq 1$, since $\overline{\gamma_x}(\theta_0, x_0) < \frac{1}{2}\log(3/5) < 0$ for (almost, when $\beta = 1$) every $\theta \in \mathbb{T}$ and for every $x \in (0,1)$.*

This theorem extends the results obtained in [2] by introducing the parameter β, showing that there is an attractor for $0 \leq \beta \leq 1$ ($\beta = 1$ is proved in that article), and that it is continuous (even C^1) whenever $0 \leq \beta < 1$. Moreover, the bounds on the asymptotics of the distance are new.

The proof of the theorem is arranged in three parts, and the proofs are quite technical. The first one (section 3) is a collection of numerical results we will use in the later parts, and are not important for the flow of ideas in the latter parts. However, throughout the latter sections there will be a large emphasis on products of derivatives, as in lemma 3.9.

In section 4, the induction, the idea is to look at successively smaller scales around the peaks. Orbits with the same θ-coordinate can be shown to contract with time by looking at the appropriate scale, as long as $0 \leq \beta < 1$. This is done by following the orbits until they enter the peaks, and then showing that if we are looking at the appropriate scale, it will just return to a "good" state after some relatively short time.

In section 5 the results on the rate of contraction are used to prove the main theorem, which is done through various bounds on derivatives of the attracting curve. There is a correspondence between each proposition in this section and a statement in the main theorem.

3. SOME PREPARATIONS AND LEMMAS FOR LATER

The reason for choosing a Diophantine ω is that we then get a lower bound on the number of iterations required by the map $\theta \mapsto \theta + \omega$ to return to a small interval of $\mathbb{T}$ (lemma 3.1). This is a very important assumption used in our techniques.

Lemma 3.1. *If $\omega \in \mathbb{T}$ satisfies the Diophantine condition $(DC)_{\kappa,\tau}$, and $I \subset \mathbb{T}$ is an interval of length $\varepsilon > 0$, then*

$$I \cap \bigcup_{0<|m|\leq N} (I + m\omega) = \emptyset$$

with $N = [(\kappa/\varepsilon)^{1/\tau}]$[5].

Proof. Let $I = [a, a+\varepsilon]$ (the argument is similar when the interval is not closed). Suppose that $0 < |m| \leq N$, then

$$\inf_{p\in\mathbb{Z}} |m\omega - p| > \frac{\kappa}{|m|^\tau} \geq \frac{\kappa}{((\kappa/\varepsilon)^{1/\tau})^\tau} = \varepsilon.$$

However, if $I \cap (I + n\omega) \neq \emptyset$ for some n, then there are $0 \leq \delta_1, \delta_2 \leq \varepsilon$, such that

$$a + \delta_1 = a + n\omega + \delta_2 \mod 1$$
$$\Leftrightarrow$$
$$\delta_1 = n\omega + \delta_2 \mod 1$$
$$\Leftrightarrow$$
$$n\omega + (\delta_2 - \delta_1) \in \mathbb{Z}.$$

Since $(\delta_2 - \delta_1) \in [-\varepsilon, \varepsilon]$,

$$\inf_{p\in\mathbb{Z}} |n\omega - p| \leq \varepsilon,$$

and so $|n| > N$. □

We will fix, for the remainder of this paper, the following notation.

$$\Phi_{\alpha,\beta} : \mathbb{T} \times [0,1] \to \mathbb{T} \times [0,1] : (\theta, x) \mapsto (\theta + \omega, c_{\alpha,\beta}(\theta) \cdot p(x)),$$

where $\beta \in [0,1]$, ω is a Diophantine irrational number,

$$p(x) = x(1-x)$$

is the quadratic map, and

$$c_{\alpha,\beta}(\theta) = \frac{3}{2} + \beta\frac{5}{2}\left(\frac{1}{1 + \lambda g(\theta,\alpha)^2}\right),$$

where

$$g(\theta, \alpha) = \cos 2\pi(\theta - \alpha/2) - \cos \pi\alpha.$$

The constant λ will be assumed sufficiently large throughout this paper. We will often suppress the parameters α, β in our notation whenever they can be understood from context.

[5] $[x]$ denotes the integer part of x.

Given (θ_0, x_0), we will use the notation

$$(\theta_n, x_n) = \Phi^n(\theta_0, x_0), \quad n \geq 0.$$

We will introduce a few intervals and constants of importance later in the induction. We let

$$I_0 = [-\lambda^{-1/7}, \lambda^{-1/7}];$$
$$\mathcal{A}_0 = [\omega - \lambda^{-2/5}/2, \omega - 2\lambda^{-2/3}].$$

The interval I_0 contains most of the θ where c has its first peak, and is the first zooming interval in the induction. The interval $\mathcal{A}_0$ is where some of the interesting values of α lie. In particular $\alpha_c \in \mathcal{A}_0$. There is one more such interesting interval, situated slightly to the right of ω, but to keep derivatives positive, we have chosen to focus on the left side of the peak at 0. Needless to say, the same techniques apply to the other interval, except that some constants might have to be tweaked.

The constants are

$$M_0 = [\lambda^{1/(14\tau)}];$$
$$K_0 = [\lambda^{1/(28\tau)}],$$

where $[x]$ denotes the integer part of x. They have been chosen to be $M_0 \approx \sqrt{N}$, and $K_0 \approx N^{1/4}$, where N is the minimal return time to I_0 in lemma 3.1.

Also, given an interval I, and a $\theta_0 \in \mathbb{T}$, we denote by $N(\theta_0; I)$ the smallest non-negative integer N such that $\theta_N = \theta_0 + N\omega \in I$. Note that $N(\theta_0; I) = 0$ if $\theta_0 \in I$.

The "contracting" region C is given by

$$C = [1/3 - 1/100, 1/3 + 1/100],$$

and corresponds to the values of x where there is strong contraction, as long as $\theta \notin I_0 \cup (I_0 + \omega)$. This is the desirable place to be, and the whole induction step is devoted to showing that orbits spend almost all their time in this region.

Below is a list of a number of important numerical lemmas from [2]. We refer to that article, in case the proof has been omitted here, but rest assure they can all be verified by straight-forward computations.

Lemma 3.2 ([2, Lemma 2.1]). *Let $P(x) = (3/2 + \varepsilon)x(1-x)$. If $|\varepsilon| > 0$ is sufficiently small, then $P(C) \subset C$, where C is the interval $[1/3 - 1/100, 1/3 + 1/100]$. Moreover, $0 < P'(x) < 3/5$ for every $x \in C$.*

Lemma 3.3 ([2, Lemma 2.2]). *Let P be as in the previous lemma. If $1/100 \leq x \leq 99/100$, then $1/100 < P(x) < 2/5$, provided that $|\varepsilon| > 0$ is sufficiently small. Furthermore, under the same assumptions on ε, $P(x) < 2/5$, for every $x \in [0,1]$.*

Lemma 3.4 ([2, Lemma 2.3]). *Assume that $|\varepsilon_1|, |\varepsilon_2|, \ldots, |\varepsilon_{20}| < \varepsilon$. Let $P_i(x) = (3/2 + \varepsilon_i)x(1-x)$ $(i = 1, \ldots, 20)$. Then $P_{20} \circ P_{19} \circ \cdots \circ P_1(x) \in [1/3 - 1/100, 1/3 + 1/100] = C$, for every $x \in [1/100, 99/100]$, provided that $|\varepsilon| > 0$ is sufficiently small.*

Lemma 3.5 ([2, Lemma 2.4]). *If $P(x) = ax(1-x)$ $(a \geq 3/2)$, then $P(x) \geq \frac{5}{4}x$ for all $x \in [0, 1/10]$.*

The following lemmas will ascertain that the perturbations of the constant in the quadratic map $c(\theta)p(x)$ will be small when $\theta \notin I_0 \cup (I_0 + \omega)$.

If the proof of a statement in the following lemma is omitted, it may be proved in a similar way as the other statements, and those proofs can all be found in [2].

Lemma 3.6. *For all sufficiently large $\lambda > 0$ the following hold for $\alpha \in \mathcal{A}_0$:*

a) $c(\mathcal{A}_0 - \omega, \alpha) \supsetneq [2,3]$.

b) $|\partial_\theta c_\alpha(\theta)|, |\partial_\beta c_\alpha(\theta)| < 1/\sqrt{\lambda}$ *for every* $\theta \notin I_0 \cup (I_0 + \omega)$.

c) For any $0 \le \delta \le 1$, $\{\theta : c(\theta) \ge \left(\frac{3}{2} + \beta\frac{5}{2}\right)(1-\delta)\} \cap (I_0 + \omega) \subseteq [\alpha - \sqrt{\delta}\lambda^{-1/4}, \alpha + \sqrt{\delta}\lambda^{-1/4}]$.

Proof. The first statement can be found in the article.

For the function $g(\theta,\alpha) = \cos 2\pi(\theta - \alpha/2) - \cos \pi\alpha$ we have

$$g(\theta,\alpha) = (2\pi \sin \pi\alpha)\theta + O(\theta^2) \text{ as } \theta \to 0$$
$$g(\theta,\alpha) = (2\pi \sin \pi\alpha)(\theta - \alpha) + O((\theta-\alpha)^2) \text{ as } \theta \to \alpha$$

Since $|I_0| = 2\lambda^{-1/7}$

As $\theta \in I_0$, we have

$$\begin{aligned}
|g(\theta,\alpha)| &= |(2\pi \sin \pi\alpha)\theta + O(\theta^2)| \\
&\le |const| \cdot |\theta| + |O(\theta^2)| \\
&\le |const| \cdot \lambda^{-1/7} + |const(\lambda)| \cdot |\lambda^{-2/7}| \\
&\le |const(\lambda)| \cdot |\lambda^{-1/7}| \\
&\le const \cdot \lambda^{-1/7}
\end{aligned}$$

where the dependence of $const(\lambda)$ on λ is purely one of distance from $\theta \in I_0$ to 0, which decreases with λ, and so $const(\lambda)$ is uniformly bounded by some constant b, for sufficiently large λ. For $\theta \in I_0 + \omega$, a similar argument holds, and we may choose some constant $b > 0$ such that

$$g^{-1}([-b\lambda^{-1/7}, b\lambda^{-1/7}]) \subset I_0 \cup (I_0 + \omega).$$

Now, differentiating c with respect to β yields, for $\theta \notin I_0 \cup (I_0 + \omega)$,

$$\partial_\beta c(\theta,\alpha) = \frac{5}{2}\left(\frac{1}{1 + \lambda g(\theta,\alpha)^2}\right) < \frac{5}{2}\left(\frac{1}{1 + \lambda b^2 \lambda^{-2/7}}\right) < \frac{1}{\sqrt{\lambda}}.$$

For the last statement, we calculate the Taylor series at $\theta = \alpha$, to obtain

$$c(\theta) = \frac{3}{2} + \beta\frac{5}{2} - 10\beta\lambda\pi^2 \sin^2(\pi\alpha)(\theta-\alpha)^2 + \beta\lambda O((\theta-\alpha)^3)$$

So,

$$c(\theta) \ge \left(\frac{3}{2} + \beta\frac{5}{2}\right)(1-\delta)$$

implies that

$$\beta\lambda\left(10\pi^2\sin^2(\pi\alpha)(\theta-\alpha)^2+O((\theta-\alpha)^3)\right)\leq\left(\frac{3}{2}+\beta\frac{5}{2}\right)\delta$$

Now, $c(\alpha\pm\sqrt{\delta}\lambda^{-1/4})<\left(\frac{3}{2}+\beta\frac{5}{2}\right)(1-\delta)$, since

$$\begin{aligned}\beta\lambda\left(10\pi^2\sin^2(\pi\alpha)\delta\lambda^{-1/2}+O(\delta^{3/2}\lambda^{-3/4})\right)&=\left(10\pi^2\sin^2(\pi\alpha)\beta\lambda^{1/2}+\cdot\beta O(\delta^{1/2}\lambda^{1/4})\right)\delta\\&>\left(\frac{3}{2}+\beta\frac{5}{2}\right)\delta\end{aligned}$$

when $\lambda>0$ is large (independent of δ). Since c is smaller further away from the peak at α, we are done. □

In the proof of the following lemma, the idea is that we can use the above lemmas about $P(x)=(\frac{3}{2}+\varepsilon)x(1-x)$ as long as $\theta\notin I_0\cup(I_0+\omega)$, since $|c_\alpha(\theta)-\frac{3}{2}|<1/\sqrt{\lambda}<\varepsilon$ when λ is sufficiently large.

Lemma 3.7 ([2, Lemma 3.2])**.** *Provided that $\lambda>0$ is sufficiently large, the following statements hold:*

- *If $\theta_0\notin I_0\cup(I_0+\omega)$, and $x_0\in C$, then $x_1\in C$, and $|c(\theta_0)p'(x_0)|<3/5$.*
- *If $\theta_0,\dots,\theta_{19}\notin I_0\cup(I_0+\omega)$, and $x_0\in[1/100,99/100]$, then $x_{20}\in C$.*

Lemma 3.8 ([2, Lemma 3.3])**.** *If $\theta_0\in\mathbb{T}$, $x_0\geq 1/100$, and if $x_{-1}\in(0,1/100)\cup(99/100,1)$, then $x_2\in[1/100,99/100]$.*

We will now establish bounds on the partial derivatives $\partial_\theta x_n$, and $\partial_\beta x_n$. Applying the product rule and the chain rule, we obtain

$$\partial x_{n+1}=(\partial c(\theta_n))\cdot p(x_n)+c(\theta_n)\cdot p'(x_n)\cdot\partial x_n,$$

where ∂ denotes partial differentiation with respect to either θ or β. We find inductively that

$$\partial x_{n+1}=(\partial c(\theta_n))\cdot p(x_n)+\partial x_0\prod_{j=0}^{n}c(\theta_j)\cdot p'(x_j)+\sum_{k=1}^{n}\left(\partial\theta_{k-1}p(x_{k-1})\prod_{j=k}^{n}c(\theta_j)\cdot p'(x_j)\right).$$

It will be very important in section 4 to keep good control on products such as $\prod_{j=0}^{n}c(\theta_j)\cdot p'(x_j)$. They will also come into play when approximating derivatives in section 5.

The following lemma is an adaptation of [2, Lemma 3.5].

Lemma 3.9. *Assume that $x_0\in[0,1]$, $\partial_\theta x_0=\partial_\beta x_0=0$, and $\prod_{j=k}^{T}|c_jp'(x_j)|<(3/5)^{(T-k+1)/2}$ for every $k\in[0,T]$, where $T>10\log\lambda$ is an integer. Assume moreover that $|\partial_\theta c_k|,|\partial_\beta c_k|<1/\sqrt{\lambda}$ for $k\in[T-10\log\lambda,T]$. Then $|\partial_\theta x_{T+1}|,|\partial_\beta x_{T+1}|<\lambda^{-1/4}$ provided that λ is sufficiently large.*

Proof. Exactly as in the proof of [2, Lemma 3.5]. □

The following lemma is a restatement of [2, Lemma 3.4] to include the parameter β, and is used in the proof of the main theorem to give a lower bound on how long it takes x_0 to return to C after having come really close to the peaks in the θ-direction.

Lemma 3.10. *Let $\alpha \in \mathcal{A}_0$, and $\beta \in [0,1]$ be fixed. Set*

$$J_M = \{\theta : c(\theta,\alpha) \geq \left(\frac{3}{2}+\beta\frac{5}{2}\right)(1-(4/5)^M)\} \cap (I_0+\omega).$$

Then, For all sufficiently large $\lambda > 0$, the following hold for $M \geq 10$:
Given $\theta_0 \in (I_0-\omega)\backslash(J_M-2\omega)$, and $x_0 \in [\frac{1}{100},\frac{99}{100}]$, there is a $3 \leq k \leq M-7$ such that $x_k \in [\frac{1}{100},\frac{99}{100}]$.
Given $\theta_0 \in I_0\backslash(J_M-\omega)$, and $x_0 \in C$, there is a $2 \leq k \leq M-7$ such that $x_k \in [\frac{1}{100},\frac{99}{100}]$.
Given $\theta_0 \in (I_0+\omega)\backslash J_M$, and $x_0 \in [\frac{1}{100},\frac{99}{100}]$, there is a $1 \leq k \leq M-7$ such that $x_k \in [\frac{1}{100},\frac{99}{100}]$.

The return time to the "good" region $[1/100, 99/100]$ is bounded by $M-7$ regardless of the value of β.

Proof. Suppose that $\theta_0 \in (I_0-\omega)\backslash(J_M-2\omega)$, and $x_0 \in [1/100,99/100]$. Then by lemma 3.3, $1/100 < x_1 < 2/5$ (note that $x_1 \in C$), and so $1/100 < x_2 < 4p(2/5) = 24/25 < 99/100$. In particular, this means that the first case subsumes the last two cases, and so the lemma follows if we can prove the first statement. For the next iterate, we obtain

$$1/100 \leq c(\theta)p(1/100) = c(\theta)p(99/100) < x_3 < \left(\frac{3}{2}+\beta\frac{5}{2}\right)\left(1-(4/5)^M\right)p(1/2) < 1-(4/5)^M.$$

If $x_3 \leq 99/100$, we just choose $k=3$ and are done. Suppose instead that $99/100 < x_3$, and set $y_3 = 1-x_3$, having the bounds $(4/5)^M < y_3 < 1/100$. We will make use of the simple relation $p(x) = p(1-x)$, to conclude that $x_4 = c(\theta_3)p(x_3) = c(\theta_3)p(y_3)$. Thus, we obtain

$$(5/4)(4/5)^M < x_4 < 99/100,$$

since $c(\theta)p(y_3) < 4p(1/100) < 99/100$, and more generally, for $4 \leq k \leq M$ (since $\theta_k \notin I_0 \cap (I_0+\omega)$),

$$(5/4)^k(4/5)^M < x_k,$$

unless for some k, $x_k \geq 1/100$ (implying that $x_k \in [1/100,99/100]$. Choosing $k = M-7$, we get

$$1/100 < (4/5)^7 < x_k < 2/5,$$

whence the statement follows. □

Keep in mind that $c(\theta) \leq (\frac{3}{2}+\beta\frac{5}{2})$ for every θ, and hence $c(\theta) < 4$ when $\beta < 1$.

Lemma 3.11. *For all sufficiently large $\lambda > 0$, we have the following lemma. Let $\alpha \in \mathcal{A}_0$, and $\beta \in [0,1)$ be fixed. Set*

$$J_M = \{\theta : c(\theta,\alpha) \geq \left(\frac{3}{2}+\beta\frac{5}{2}\right)(1-(4/5)^M)\} \cap (I_0+\omega).$$

Then, assuming that $M \geq 10$, there is a constant (integer) $M_C = M_C(\beta)$, depending only on β, such that:
Given $\theta_0 \in (J_M - 2\omega)$, and $x_0 \in [\frac{1}{100}, \frac{99}{100}]$, there is a $3 \leq k \leq M_C$ such that $x_k \in [\frac{1}{100}, \frac{99}{100}]$.
Given $\theta_0 \in (J_M - \omega)$, and $x_0 \in C$, there is a $2 \leq k \leq M_C$ such that $x_k \in [\frac{1}{100}, \frac{99}{100}]$.
Given $\theta_0 \in J_M$, and $x_0 \in [\frac{1}{100}, \frac{99}{100}]$, there is a $1 \leq k \leq M_C$ such that $x_k \in [\frac{1}{100}, \frac{99}{100}]$.

Proof. One satisfying, but not necessarily the smallest possible, value of M_C is the following:

$$M_C = \frac{\log \frac{1}{150 V_\beta (1-V_\beta)}}{\log \frac{5}{4}} + 4,$$

where $V_\beta = \frac{3}{8} + \beta \frac{5}{8}$. At the end of the proof, we will show that this constant is sufficient.

As in the previous proof, note that

$$1/100 \leq x_1 \leq 2/5, \quad 1/100 \leq x_2 \leq 99/100.$$

The difference is now that we get

$$1/100 \leq x_3 \leq (\frac{3}{8} + \beta \frac{5}{8}) = V_\beta,$$

and if $x_3 > 99/100$, setting $y_3 = 1 - x_3$ gives $1 - V_\beta \leq y_3 < 1/100$. Since $\theta_3 \notin I_0 \cup (I_0 + \omega)$, we get

$$\frac{3}{2} V_\beta (1 - V_\beta) \leq x_4 \leq 2/5.$$

As before, if $x_k < 1/100$, for $k \geq 3$ then by induction we get

$$x_{k+1} \geq (5/4) x_k \geq (5/4)^{k-4} \frac{3}{2} V_\beta (1 - V_\beta).$$

Thus, to get a lower bound on the constant needed, we solve

$$\frac{1}{100} \leq \left(\frac{5}{4}\right)^{k-4} \frac{3}{2} V_\beta (1 - V_\beta),$$

whose solution is

$$k \geq \frac{\log \frac{1}{150 V_\beta (1-V_\beta)}}{\log \frac{5}{4}} + 4.$$

That is, it is sufficient to set $M_C \geq \frac{\log \frac{1}{150 V_\beta (1-V_\beta)}}{\log \frac{5}{4}} + 4$, for the proof to hold. □

Remark. The preceding lemma is a complement to the one above it to include the effect in the x-direction of reaching the peaks in the θ-direction. Now the behaviour is crucially dependent on the value of β, and the return time is not necessarily uniformly bounded for $\beta < 1$ (not even necessarily defined for $\beta = 1$). In [2] it was shown that there is a value for $\alpha = \alpha_c \in \mathcal{A}_0$ such that there is no such $M_C(\beta)$ when $\beta = 1$, and this is what causes the attractor to be strange (some orbits get stuck at $\mathbb{T} \times \{0\}$).

4. The Induction

4.1. **Base case.** Recall the set I_0 we considered in the previous section. Here we will show that we have control on orbits as long as $\theta_k \notin I_0 \cup (I_0 + \omega)$. The inductive step then shows what happens inside $I_0 \cup (I_0 + \omega)$.

Proposition 4.1. *Let $\alpha \in \mathcal{A}_0$ be fixed. There is a $\lambda_1 > 0$ such that if $\lambda \geq \lambda_1$, then the following hold:*

(i)$_0$ If $\beta \in [0,1]$, $x_0, y_0 \in C$, and $\theta_0 \notin I_0 \cup (I_0 + \omega)$, then, letting $N = N(\theta_0; I_0)$, and $\xi_i \in \{tx_i + (1-t)y_i : t \in [0,1]\}$ be an arbitrary point between x_i and y_i, for every $i \in [0, N-1]$, the following hold:

$$\prod_{i=k}^{N-1} |c(\theta_i)p'(\xi_i)| < (3/5)^{N-k} \quad \text{for all } k \in [0, N-1]; \tag{4.1}$$

$$\prod_{i=0}^{k-1} |c(\theta_i)p'(\xi_i)| < (3/5)^{k} \quad \text{for all } k \in [1, N]; \tag{4.2}$$

$$x_k \in C \quad \text{for all } k \in [0, N]; \quad \text{and} \tag{4.3}$$

$$|x_k - y_k| < (3/5)^k |x_0 - y_0|, \quad \text{for all } k \in [1, N]. \tag{4.4}$$

(ii)$_0$ If $\beta \in [0,1]$, and $x_0 \in [1/100, 99/100]$, and $\theta_0 \notin I_0 \cup (I_0 + \omega)$, then

$$x_k \in [1/100, 99/100] \quad \text{for all } k \in [0, N].$$

Proof. By assumption, $\theta_i \notin I_0 \cup (I_0 + \omega)$ for every $i \in [0, N-1]$; by lemma 3.7 it is immediate that $x_i \in C$ for $i \in [0, N]$, which in particular implies $(ii)_0$.

It follows that $x_i, y_i \in C$, and hence $\xi_i \in C$ for every $i \in [0, N-1]$, and so by lemma 3.7, $|c(\theta_i)p'(\xi_i)| < 3/5$ for every $i \in [0, N-1]$, giving us that both

$$\prod_{i=k}^{N-1} |c(\theta_i)p'(x_i)| < (3/5)^{N-k},$$

for every $k \in [0, N-1]$, and

$$\prod_{i=0}^{k-1} |c(\theta_i)p'(x_i)| < (3/5)^{k},$$

for every $k \in [1, N]$.

By the mean value theorem there are points $\xi_i \in \{tx_i + (1-t)y_i : t \in [0,1]\}$, for each $i \in [0, N-1]$, such that

$$|x_k - y_k| = \prod_{i=0}^{k-1} |c(\theta_i)p'(\xi_i)| \cdot |x_0 - y_0|$$

for every $k \in [1,N]$. Together with (4.2), this implies that

$$|x_k - y_k| = \prod_{i=0}^{k-1} |c(\theta_i)p'(\xi_i)| \cdot |x_0 - y_0| < (3/5)^k |x_0 - y_0|.$$

□

4.2. **Inductive step.** The inductive step works by zooming in on intervals $I_n \subset I_0$, and showing that we have a good control on orbits as long as $\theta_k \notin I_n \cup (I_n + \omega)$. At some point we must ask ourselves what happens to orbits when they enter I_n. This is highly dependent on α and β, but the point is that, for suitable I_n, we will retain control throughout the interval I_n as long as $\beta < 1$.

We will begin by introducing some notation. Suppose that we are given intervals $I_0, \ldots, I_n$, and constants $K_0, \ldots, K_n, M_0, \ldots, M_n$. We then define the sets

$$\begin{aligned} \Theta_n &= \bigcup_{i=0}^{n} \bigcup_{m=-M_i}^{M_i} (I_i + m\omega), \Theta_{-1} = \mathbb{T} \setminus (I_0 \cup (I_0 + \omega)), \\ G_n &= \bigcup_{i=0}^{n} \bigcup_{m=0}^{3K_i} (I_i + m\omega), G_{-1} = \emptyset, \\ B_n &= \{\beta : M_C(\beta) \leq 2K_n - 7\}, \end{aligned}$$

where $M_C(\beta)$ is the constant in lemma 3.11.

The motivation for introducing this notation will be apparent in the induction. We see that, for every $n \geq 0$, the following hold

$$\begin{aligned} &\Theta_n \subseteq \Theta_{n-1} \\ &G_{n-1} \subseteq G_n \\ &B_n \subseteq B_{n+1}, \quad \text{and} \bigcup_{n=0}^{\infty} B_n = [0,1) \end{aligned}$$

The ideas behind the respective sets are:

- The set Θ_n consists of the points $\theta \in \mathbb{T}$ that are far away from each of the intervals $I_0, \ldots, I_n$. Starting with a $\theta_0 \in \Theta_n$ gives us some "breathing room" before we get close to the peaks.
- The set G_n consists of the points θ which have recently visited one of the intervals. The idea is that, if we hit the peak at I_0, but stay away from I_{n+1}, then expansion in the x-direction will stop shortly after we leave G_n (at most 20 iterations after), giving us a comparatively long (very much so) time for contraction before we hit the peaks again.
- The set B_n is the set of β for which it is necessary only to zoom as far as to the n-th scale (the interval I_n). That is, we get sufficiently good estimates on the contraction even if we only consider the intervals up to I_n. The conditions imposed on β are connected to the upper bound on the return time estimates in lemma 3.11.

Proposition 4.2. *Let $\alpha \in \mathcal{A}_0$ be fixed. There is a $\lambda_1 > 0$ such that if $\lambda \geq \lambda_1$, then the following hold:*

Suppose that for some $n \geq 0$, we have constructed closed intervals $I_0 \supset I_1 \supset \cdots \supset I_n$, and chosen integers $M_0 < M_1 < \cdots < M_n$ and $K_0 < K_1 < \cdots < K_n$, satisfying

$$|I_k| = (4/5)^{K_{k-1}}, \quad K_k \in [(5/4)^{K_{k-1}/(4\tau)}, 2(5/4)^{K_{k-1}/(4\tau)}], \quad \text{for } k = 1,2,\ldots,n; \tag{4.5}$$

$$M_k \in [(5/4)^{K_{k-1}/(2\tau)}, 2(5/4)^{K_{k-1}/(2\tau)}], \quad \text{for } k = 1,2,\ldots,n; \quad \text{and} \tag{4.6}$$

$$I_n \supseteq [\alpha - (4/5)^{K_n}, \alpha + (4/5)^{K_n}]. \tag{4.7}$$

Assume further that the following holds:

$(i)_n$ *If $\beta \in [0,1]$, $x_0, y_0 \in C$, and $\theta_0 \in \Theta_{n-1}$, then, letting $N = N(\theta_0; I_n)$, and $\xi_i \in \{tx_i + (1-t)y_i : t \in [0,1]\}$ be an arbitrary point between x_i and y_i, for every $i \in [0, N-1]$, the following hold:*

$$\prod_{i=k}^{N-1} |c(\theta_i)p'(\xi_i)| < (3/5)^{(1/2+1/2^{n+1})(N-k)} \quad \text{for all } k \in [0, N-1]; \tag{4.8}$$

$$\prod_{i=0}^{k-1} |c(\theta_i)p'(\xi_i)| < (3/5)^{(1/2+1/2^{n+1})k} \quad \text{for all } k \in [1, N]; \tag{4.9}$$

$$x_k \notin C \quad \text{for some } k \in [0,N] \Rightarrow \theta_k \in G_{n-1}; \quad \text{and} \tag{4.10}$$

$$|x_k - y_k| < (3/5)^{(1/2+1/2^{n+1})k}|x_0 - y_0|, \quad \text{for all } k \in [1, N], \tag{4.11}$$

$$\bigcup_{k=0}^{20} (I_n + (2K_n + k)\omega) \subseteq \Theta_{n-1}, I_n - M_n\omega \in \Theta_{n-1}. \tag{4.12}$$

$(ii)_n$ *If $\beta \in [0,1]$, $x_0 \in [1/100, 99/100]$, and $\theta_0 \notin I_0 \cup (I_0 + \omega)$, then*

$$x_k \notin [1/100, 99/100] \text{ and } k \in [0, N(\theta_0; I_n)] \Rightarrow \theta_k \in G_{n-1}.$$

$(iii)_n$ *If $\beta \in [0,1]$, $x_0 \in C$, and $\theta_0 \notin I_n$, then, letting $N = N(\theta_0; I_n)$*

$$x_N \in C. \tag{4.13}$$

Then there is a closed interval $I_{n+1} \subset I_n$, and integers M_{n+1}, K_{n+1} satisfying (4.5 - 4.7)$_{n+1}$ such that $(i - iv)_{n+1}$ hold.

Moreover, under the same assumptions, the following holds:

$(iv)_n$ *If $\beta \in B_n$, $x_0, y_0 \in C$, and $\theta_0 \in I_n \cup (I_n + \omega)$, then, letting $N = N(\theta_0; I_n)$,*

$$\theta_{2K_n+k} \in \Theta_{n-1}, \quad \text{for every } k \in [0, 20]; \quad \text{and} \tag{4.14}$$

$$x_{2K_n+20} \in C. \tag{4.15}$$

Proof. Lemma 3.1 gives minimal return times

$$\begin{cases} [(\kappa(4/5)^{K_{k-1}})^{1/\tau}] := N_k & k \geq 1 \\ [(2\kappa\lambda^{1/7})^{1/\tau}] := N_0 & k = 0 \end{cases}$$

N_k to the respective intervals I_k. The constants M_k, K_k have been chosen to be $M_k \approx \sqrt{N_k}, K_k \approx \sqrt{M_k}$. By choosing λ sufficiently large, we see that $N_k \gg M_k \gg K_k$.

In particular, lemma 3.1 implies that

$$I_k \cap \bigcup_{0<|m|\leq 10M_k} (I_k + m\omega) = \emptyset, \tag{4.16}$$

for every $k = 0, 1, \ldots, n$. Also, since $3K_i < M_i$,

$$\bigcup_{m=0}^{3K_i} (I_i + m\omega) \subset \bigcup_{m=-M_i}^{M_i} (I_i + m\omega)$$

for every $k = 0, 1, \ldots, n$, implying that

$$\Theta_n \cap G_n = \emptyset, \tag{4.17}$$

for $n \geq -1$. Moreover, since $I_n \subset I_k$ $(k = 0, 1, \ldots, n-1)$, and $(I_k - \omega) \cap \left(\bigcup_{m=0}^{3K_k} (I_k + m\omega) \right)$ for $k = 0, 1, \ldots, n-1$, we get that

$$(I_n - \omega) \cap G_n = \emptyset. \tag{4.18}$$

Constructing the interval I_{n+1}:
Let

$$I_{n+1} = [\alpha - (4/5)^{K_n}/2, \alpha + (4/5)^{K_n}/2].$$

We have the inclusion

$$J_{2K_n} = \{\theta : c(\theta) \geq (\frac{3}{2} + \beta\frac{5}{2})(1 - (4/5)^{2K_n})\} \subseteq [\alpha - \frac{(4/5)^{K_n}}{\lambda^{1/4}}, \alpha + \frac{(4/5)^{K_n}}{\lambda^{1/4}}] \subseteq I_{n+1}.$$

This means, in particular, that by lemma 3.10, as long as $\theta_k \notin \bigcup_{m=-1}^{1} (I_{n+1} + m\omega)$, we have good control on the contraction.

Choosing the constants K_{n+1}, and M_{n+1}:
See [2, Proposition 4.2], where it is also shown that they satisfy $(4.12)_{n+1}$.

Verifying $(i)_{n+1}$

We want to prove that, for $N = N(\theta_0; I_{n+1})$,

$$\prod_{i=k}^{N-1} |c(\theta_i)p'(x_i)| < (3/5)^{(1/2+1/2^{n+2})(N-k)} \quad \text{for all } k \in [0, N-1]; \tag{4.19}$$

$$\prod_{i=0}^{k-1} |c(\theta_i)p'(x_i)| < (3/5)^{(1/2+1/2^{n+2})k} \quad \text{for all } k \in [1, N]; \tag{4.20}$$

$$x_k \notin C \quad \text{for some } k \in [0, N] \Rightarrow k \in G_{n-1}; \quad \text{and} \tag{4.21}$$

$$|x_k - y_k| \leq (3/5)^{(1/2+1/2^{n+1})k} |x_0 - y_0|, \quad \text{for all } k \in [1, N]. \tag{4.22}$$

We will designate, by (4.19)[T]-(4.22)[T], the corresponding statements with N replaced by an integer $T > 0$.

Begin by dividing the interval $[0, N]$ into parts

$$0 < s_1 < s_2 < \cdots < s_r = N,$$

where the s_l are the times when $\theta_{s_l} \in I_n$ (and $\theta_k \notin I_n$ for $k \neq s_i$ for any i, and $0 \leq k \leq N$). It might very well be the case that $r = 1$ (i.e. we never visit any of the bigger intervals $I_i \supset I_n$ before we visit I_n).

By the induction hypothesis, (4.20)[s_1] holds. Hence, if $r = 1$, we are done. Suppose instead that $r > 1$, and that (4.20)[s_l] holds for $k \in [1, s_l]$, where $1 \leq l < r$.

Proceeding as in the verification of $(iii)_{n+1}$ below, we obtain that $\theta_{s_l+2K_n+20} \in \Theta_{n-1}$, and $x_{s_l+2K_n+20} \in C$. Hence

$$\prod_{i=s_l+2K_n+20}^{k-1} |c(\theta_i)p'(\xi_i)| < (3/5)^{(1/2+1/2^{n+1})(k-s_l+2K_n+20)} < (3/5)^{(1/2+1/2^{n+2})(k-s_l+2K_n+20)} \tag{4.23}$$

for $k \in [s_l + 2K_n + 20 + 1, s_{l+1}]$. since $|c(\theta)p'(x)| \leq 4 < (5/3)^3$ for every pair (θ, x), we obtain the following bounds, for $k \in [s_l + 1, s_l + 2K_n + 20]$

$$\prod_{i=s_l}^{k-1} |c(\theta_i)p'(\xi_i)| < (5/3)^{3k}.$$

Hence, for $k \in [s_l + 1, s_l + 2K_n + 20]$, we have

$$\prod_{i=0}^{k-1} |c(\theta_i)p'(\xi_i)| < (3/5)^{(1/2+1/2^{n+1})s_l} \cdot (5/3)^{3(k-s_l)} \leq (3/5)^{(1/2+1/2^{n+1})s_l - 3k}.$$

If we can show that $(1/2 + 1/2^{n+1})s_l - 3k > (1/2 + 1/2^{n+2})(s_l + k)$, we obtain the inequality, for $k \in [s_l + 1, s_l + 2K_n + 20]$,

$$\prod_{i=0}^{k-1} |c(\theta_i)p'(\xi_i)| < (3/5)^{(1/2+1/2^{n+2})(s_l+k)}. \tag{4.24}$$

This follows, since $s_l \geq N_n > K_n^2$, and $K_n \gg 8 \cdot 2^{n+2}$, for λ large enough, by the inequality

$$(1/2+1/2^{n+1})s_l - 3k - (1/2+1/2^{n+2})(s_l+k) > 1/2^{n+2}N_n - 4k \tag{4.25}$$
$$> 1/2^{n+2}K_n^2 - 8K_n - 160 = K_n(1/2^{n+2}K_n - 8) - 160 > 0. \tag{4.26}$$

Combining (4.23) and (4.24), we obtain, for $k \in [1, s_{l+1}]$, that

$$\prod_{i=0}^{k-1} |c(\theta_i)p'(\xi_i)| < (3/5)^{(1/2+1/2^{n+2})k}.$$

By induction, (4.20)[N] holds, as was to be shown.

The statement (4.19)[N] is proved in a similar fashion as above, but instead one assumes that

$$\prod_{i=k}^{s_l-1} |c(\theta_i)p'(x_i)| < (3/5)^{(1/2+1/2^{n+2})(s_l-k)}$$

holds for $k \in [0, s_l - 1]$, where $1 \leq l < r$. One then uses (4.26) to show that

$$\prod_{i=k}^{s_{l+1}-1} |c(\theta_i)p'(x_i)| < (3/5)^{(1/2+1/2^{n+2})(s_{l+1}-k)}$$

holds for $k \in [s_l, s_{l+1} - 1]$.

In order to verify (4.22)[N], let $k \in [1, N]$ be given. By the mean value theorem, there is for every $i = 0, 1, \ldots, k-1$, a ξ_i between x_i and y_i, such that

$$|x_{i+1} - y_{i+1}| \leq |c(\theta_i)p'(\xi_i)| \cdot |x_i - y_i|.$$

By induction, we see that

$$|x_k - y_k| \leq \prod_{i=0}^{k-1} |c(\theta_i)p'(\xi_i)| \cdot |x_0 - y_0| < (3/5)^{(1/2+1/2^{n+2})k}.$$

To verify $(4.10)_{n+1}$, we divide the interval $[0, N]$ as above. Now, (4.21)[s_1] holds, and we suppose that (4.21)[s_l] holds for some $1 \leq l < r$. We know that $\theta_{s_l+k} \in G_n$ for $k \in [0, 2K_n + 19]$. Argue as in the proof of $(iii)_{n+1}$ below, that $\theta_{s_l+2K_n+20} \in \Theta_{n-1}$, and $x_{s_l+2K_n+20} \in C$. Apply (4.10) again, to obtain (4.21)[s_{l+1}]. By induction, $(4.10)_{n+1}$ holds.

Verifying $(ii)_{n+1}$

As above, we begin by dividing the interval $[0, N]$ into parts

$$0 < s_1 < s_2 < \cdots < s_r = N,$$

where the s_l are the times when $\theta_{s_l} \in I_n$.

By the induction hypothesis, the following holds:

$$x_k \notin [1/100, 99/100] \text{ and } k \in [0, s_1] \Rightarrow \theta_k \in G_{n-1} \subset G_n.$$

Suppose that for some $1 \le l < r$, we have for every $k \in [1, s_l]$ that

$$x_k \notin [1/100, 99/100] \Rightarrow \theta_k \in G_n.$$

Since $(I_n - \omega) \cap G_n = \emptyset$, we see that $x_{s_l - 1} \in [1/100, 99/100]$, and so there is a $3 \le k \le 2K_n - 7$ such that $x_{s_l + k} \in [1/100, 99/100]$ by lemma 3.10. Arguing as in the proof of $(iii)_{n+1}$ below, we see that $\theta_{2K_n} \in \Theta_{n-1}$, and $x_{2K_n} \in [1/100, 99/100]$. Hence, by $(ii)_n$, we have

$$x_k \notin [1/100, 99/100] \text{ and } k \in [2K_n, s_1] \Rightarrow \theta_k \in G_{n-1} \subset G_n.$$

Of course, since $\theta_k \in G_n$ for $0 \le k \le 3K_n$, we see that

$$x_k \notin [1/100, 99/100] \text{ and } k \in [0, s_1] \Rightarrow \theta_k \in G_{n-1} \subset G_n.$$

By induction, $(ii)_{n+1}$ holds.

Verifying $(iii)_{n+1}$:
Let $0 \le s_1 < s_2 < \cdots < s_r = N$ be the return times to I_n. If $s_1 = 0$, then by assumption, $x_{s_1} = x_0 \in C$. If $s_1 > 0$, then the induction hypothesis implies that $x_{s_1} \in C$. If $r = 1$, then we are done.

Suppose instead that we have proved that, for some $1 \le l < r$ we have $x_{s_l} \in C$. Since $\theta_{s_l} \in I_n \backslash I_{n+1}$, applying lemma 3.10, we get a $3 \le t \le 2K_n - 7$ such that $x_{s_l + t} \in [1/100, 99/100]$.

If $\theta_{s_l + t} \notin I_0 \cup (I_0 + \omega)$, then by $(ii)_n$, $x_{s_l + k} \notin [1/100, 99/100]$ implies that $\theta_{s_l + k} \in G_{n-1}$. Since, by (4.12), $\theta_{s_l + 2K_n + i} \in \Theta_{n-1}$ $(i = 0, 1, \ldots, 20)$, (4.17) implies that $x_{s_l + 2K_n} \in [1/100, 99/100]$, and so $x_{s_l + 2K_n + 20} \in C$ by lemma 3.7.

If, however, $\theta_{s_l + t} \in I_0 \cup (I_0 + \omega)$, then assume that this t is the smallest such time. Now, $x_{s_l + t - 1} \notin [1/100, 99/100]$ by our assumption on t, and by lemma 3.8, $x_{s_l + t + 2} \in [1/100, 99/100]$. Since $\theta_{s_l + t + 2} \notin I_0 \cup (I_0 + \omega)$, we may proceed as in the above paragraph to obtain $x_{s_l + 2K_n + 20} \in C$.

In any case, we have $\theta_{s_l + 2K_n + 20} \notin I_n$, and $x_{s_l + 2K_n + 20} \in C$, and so $(iii)_n$ applies again, to conclude that $x_{s_{l+1}} \in C$. Using induction, we obtain our conclusion.

Verifying $(iv)_n$:
Since $K \in B_n$, lemma 3.10 and lemma 3.11 together imply that there is a $1 \le t \le 2K_n - 7$ such that

$$x_t \in [1/100, 99/100].$$

Suppose that this t is the smallest such number. If $\theta_t \notin I_0 \cup (I_0 + \omega)$, invoking $(ii)_n$, and noting that $\theta_{2K_n + k} \in \Theta_{n-1}$ for $k \in [0, 20]$, and $\Theta_{n-1} \cap G_{n-1} = \emptyset$, we obtain that

$$x_{2K_n} \in [1/100, 99/100];$$

using (lemma about return to C) we see that $x_{2K_n + 20} \in C$, and $\theta_{2K_n + 20} \in \Theta_{n-1}$.

If $\theta_t \in I_0 \cup (I_0 + \omega)$, then as in the proof of $(iii)_{n+1}$ above, by lemma 3.8 implies that $x_{t+2} \in [1/100, 99/100]$. Since $\theta_{t+2} \notin I_0 \cup (I_0 + \omega)$, we just refer to the argument in the above paragraph, and conclude that the statement $(iv)_n$ holds true. □

Corollary 4.3. *By proposition 4.1, $(i - iii)_0$ hold, where $(iii)_0$ just corresponds to (4.3), and so by proposition 4.2 $(i - iv)_n$ hold for every $n \ge 0$.*

5. Proof of Main Theorem

In this section we will use the same notation as in section 4. Throughout this section we will assume that λ is sufficiently large for every result in the previous sections to hold.

5.1. **Constructing the attractor.** Here we show that, for every $0 \leq \beta < 1$, there is an attractor which is the graph of an invariant continuous (actually C^1) function $\psi^\beta : \mathbb{T} \to (0,1)$.

Lemma 5.1. *Suppose that $\beta \in B_n$ for some $n \geq 0$ is fixed (that is, $\beta \in [0,1)$). If $\theta_0 \in \mathbb{T}$, and $x_0 \in (0,1)$, then there is a $t \geq 0$, such that $\theta_t \in \Theta_{n-1}$, and $x_t \in C$.*

Proof. First of all, the assumption on β means that $\beta \in [0,1)$. Since $0 < c(\theta) < 4$ for every $\theta \in \mathbb{T}$ when $0 \leq \beta < 1$, it follows that $x_k \in (0,1)$ for every $k \geq 0$ ($0 < x_i < 4p(\frac{1}{2}) = 1$).

Suppose that $x_0 \notin [1/100, 99/100]$. Then, by lemma 3.5, there is an $s > 0$ such that $x_s \in [1/100, 99/100]$. Let s be the smallest such integer.

If $\theta_s \notin I_0 \cup (I_0 + \omega)$, then by $(ii)_n$,

$$x_k \notin [1/100, 99/100] \text{ and } k \in [s, N(\theta_s; I_n)] \Rightarrow \theta_{s+k} \in G_{n-1}.$$

Let $N = N(\theta_s; I_n)$, then $\theta_{s+N} \in I_n$. By (4.12), $\theta_{s+N+2K_n+i} \in \Theta_{n-1}$ for $i = 0, 1, \ldots, 19$, and since $\Theta_{n-1} \cap G_{n-1}$ by (4.17), $x_{s+N+2K_n+20} \in C$ by lemma 3.7.

If $\theta_s \in I_0 \cup (I_0 + \omega)$, then since s was the smallest such integer, $x_{s-1} \notin [1/100, 99/100]$, and so by lemma 3.8, $x_{s+2} \in [1/100, 99/100]$. Now the same argument as above applies.

If $x_0 \in [1/100, 99/100]$, and $\theta_0 \notin I_0 \cup (I_0 + \omega)$, just apply the above arguments. If instead $\theta_0 \in I_0 \cup (I_0 + \omega)$, then if $x_2 \in [1/100, 99/100]$, it follows by the previous sentence; if however $x_2 \notin [1/100, 99/100]$, just invoke the lemma for $y_0 = x_2 \notin [1/100, 99/100]$. $\square$

Lemma 5.2. *Let $n \geq 0$ be arbitrary. If $\beta \in B_n$, $\theta_0 \in \Theta_{n-1}$, and $x_0, y_0 \in C$, then for each $k > 1$*

$$|x_k - y_k| < (3/5)^{k/2} |x_0 - y_0|.$$

Moreover, if $0 \leq \beta < 1$ is fixed, $\theta_0 \in \mathbb{T}$, and $x_0, y_0 \in (0,1)$, then

$$|x_k - y_k| < const(x_0, y_0) \cdot (3/5)^{k/2} |x_0 - y_0|, \tag{5.1}$$

where the constant depends only on x_0, y_0 (hence the constant is independent of k, and is uniform with respect to θ). Actually, as long as y_0, x_0 are in a (any) fixed closed subinterval of $(0,1)$, the constant is uniformly bounded .

Proof. For the first statement, let $0 < s_1 < s_2 < \cdots$ be the times when $\theta_{s_l} \in I_n$. By (4.11)

$$|x_k - y_k| < (3/5)^{(1/2 + 1/2^{n+1})k} |x_0 - y_0| < (3/5)^{k/2} |x_0 - y_0|,$$

for $k \in [1, s_1]$. Suppose that $|x_k - y_k| < (3/5)^{(1/2+1/2^{n+1})k} |x_0 - y_0|$ holds for $k \in [1, s_l]$. Since $\beta \in B_n$, $(iv)_n$ implies that $\theta_{s_l + 2K_n + 20} \in \Theta_{n-1}$, and $x_{s_l + 2K_n + 20} \in C$. Since $|c(\theta) p'(x)| < 4$ for every $\theta \in \mathbb{T}$ and $x \in [0,1]$, it follows that

$$|x_k - y_k| < 4^k \cdot (3/5)^{(1/2+1/2^{n+1}) s_l} |x_0 - y_0| < (3/5)^{(1/2+1/2^{n+1}) s_l - 3k} |x_0 - y_0|,$$

for $k \in [s_1 + 1, s_l + 2K_n + 20]$. Proceeding exactly as in the proof of $(i)_{n+1}$, proposition 4.2, we obtain the first statement.

For the second one, assume that $n \geq 0$ is such that $\beta \in B_n$. By lemma 5.1 there are times $s,t \geq 0$ such that $\theta_s, \theta_t \in \Theta_{n-1}$ and $x_s, y_t \in C$. Let $r = \max\{s,t\}$, then $\theta_r \in \Theta_{n-1}$. By (4.10) and (4.17) $x_r, y_r \in C$.

Now, for every $k \geq r$, we have

$$|x_k - y_k| < 4^r \cdot (3/5)^{(k-r)/2} |x_0 - y_0| = (4^r \cdot (5/3)^{r/2}) \cdot (3/5)^{k/2} |x_0 - y_0|,$$

where the maximal constant is independent of θ_0 and k. □

By (5.1), orbits (outside $\mathbb{T} \times \{0,1\}$) with the same θ-component converge to each other, and we obtain the following corollary.

Corollary 5.3. *If there is an invariant curve in $\mathbb{T} \times (0,1)$, it is an attractor, it is unique and attracts every point $(\theta, x) \in \mathbb{T} \times (0,1)$.*

Corollary 5.4. *For every $x_0 \in (0,1)$, and every $\theta_0 = \theta \in \mathbb{T}$, the Lyapunov exponent in the x-direction is strictly negative since*

$$\left| \frac{\partial x_k}{\partial x_0} \right| < const(x_0) \cdot (3/5)^{k/2}$$

for every $k > 0$, where the constant is uniform in k.

Proof. We have for small enough $h > 0$ that $x_0 + h, x_0 \in (0,1)$. Considering $x_n(x_0)$ as a function of x_0, we have

$$\begin{aligned} \left| \frac{\partial x_k}{\partial x_0} \right| &= \left| \lim_{h \to 0} \frac{x_k(x_0+h) - x_k(x_0)}{h} \right| \\ &< \left| \lim_{h \to 0} \frac{const(x_0, x_0+h) \cdot (3/5)^{k/2} |x_0 + h - x_k(x_0)|}{h} \right| \\ &= const(x_0) \cdot (3/5)^{k/2}. \end{aligned}$$

□

Proposition 5.5. *Let $0 \leq \beta < 1$; then there is an invariant curve attracting every point $(\theta, x) \in \mathbb{T} \times (0,1)$. This curve is given as the graph of a continuous invariant function*

$$\psi^\beta : \mathbb{T} \to (0,1). \tag{5.2}$$

Proof. Suppose that $\beta \in B_n$. Let $\theta_0 = \theta \in \mathbb{T}$, and let $0 \leq t_1 < t_2 < \dots$ be the times when $\theta_{-t_k} \in I_n$. Set for every $k \geq 1$

$$\psi_k(\theta) = \pi_2 \circ \Phi^{t_k}(\theta_0 - t_k \omega, \frac{1}{3}).$$

By $(iv)_n$, $x_{-t_k} \in C$ implies that $x_{-t+2K_n+20} \in C$; in addition, we have $\theta_{-t+2K_n+20} \in \Theta_{n-1}$. Hence, we will have

$$\Phi^{t_k}(\{\theta_{-t_k}\} \times C) \subseteq \{\theta_0\} \times J_{t_k}$$

where $|J_{t_k}| \leq (3/5)^{(t-2K_n-20)/2}$.

Let $m > t_k$, and $x_{-m} \in C$. Either $\theta_{-m} \notin I_n$, or $\theta_{-m} \in I_n$. In the first case, the next time $s \geq t_k$ that $\theta_{-s} \in I_n$, $x_{-s} \in C$ by $(iii)_n$. Therefore $x_{-t_k} \in C$ by $(iii - iv)_n$. Hence $J_s \subseteq J_{t_k}$, and so

$$\Phi^m(\{\theta_{-m}\} \times C) \subseteq \{\theta_0\} \times J_s \subseteq \{\theta_0\} \times J_{t_k}.$$

In the second case, by the above argument we will also have $x_{-t} \in C$, and so

$$\Phi^m(\{\theta_{-m}\} \times C) \subseteq \{\theta_0\} \times J_m \subseteq \{\theta_0\} \times J_t.$$

Hence, for every $m_1, m_2 > t_k$

$$|\psi_{m_1}(\theta) - \psi_{m_2}(\theta)| \leq |J_{t_k}| \xrightarrow[k\to\infty]{} 0$$

uniformly in θ; thus ψ_n converges to a continuous function ψ^β for $\beta \in [0,1)$. $\square$

For any fixed $\beta \in [0,1)$ we will write

$$\psi_n^\beta, \text{ and } \psi^\beta \tag{5.3}$$

for the corresponding functions defined above, to show their dependence on the parameter β.

5.2. **The minimum distance between the attractor and the repelling set.**

Lemma 5.6. *If $\theta_0 \in \Theta_n$ for some $n \geq 0$, and $x_0 \in C$, then $|\partial_\beta x_N|, |\partial_\theta x_N| < \frac{1}{\lambda^{1/4}}$, where $N = N(\theta_0; I_0)$.*

Proof. Recall that $K_0 = \lambda^{1/28}$, and so $K_0 \gg 10 \log \lambda$ if λ is large. Since $\theta_0 \in \Theta_n$, it follows that $N = N(\theta_0; I_0) \geq M_0 \gg K_0$. Thus $\theta_k \notin I_0 \cup (I_0 + \omega)$ for $k = 0, 1, \dots, K_0, \dots, N$.

Using (4.8) to control the products, and lemma 3.6 to control the derivatives of c, lemma 3.9 implies that $|\partial_\beta x_N|, |\partial_\beta y_N| \leq \lambda^{-1/4}$. $\square$

In [2, Proposition 4.2, $(ii)_n$], the following lemma is implicit:

Lemma 5.7. *There is an $\alpha = \alpha_c \in \mathcal{A}_0$ such that, if $x_0 = x \in C$, and letting $\theta_0 = \alpha_c - M_n\omega$, we have the following:*

$$x_{M_n} \xrightarrow[n\to\infty]{} \frac{1}{2}.$$

Now, the measurable $\psi^1(\theta) : \mathbb{T} \to [0,1]$ constructed in that article (for $\beta = 1$) was only defined for almost every $\theta \in \mathbb{T}$, and in particular was never explicitly defined for $\theta = \alpha_c$. However, the above lemma will serve to show that $\psi^\beta(\alpha_c) \to \frac{1}{2}$ as $\beta \to 1$. This will be needed when we establish the asymptotics of the distance between the attractor and the repelling set as $\beta \to 1$.

Below we will show that $\beta \mapsto \psi^\beta$ is continuous with respect to β, but since ψ^β is not well-defined for every θ when $\beta = 1$, we may only conclude that it is continuous for $\beta \in [0,1)$. We will however need to show that $\lim\limits_{\beta\to 1} \psi^\beta(\alpha_c) = \frac{1}{2}$, and we will accomplish this by using the above lemma.

Lemma 5.8. *For every $\varepsilon > 0$, there is a $\delta > 0$ such that $|\psi^\beta(\alpha_c) - \frac{1}{2}| < \varepsilon$ whenever $\beta \in [1-\delta, 1)$.*

Proof. Note that, for $\beta < 1$,

$$\psi^{\beta}(\alpha_c) = \lim_{n\to\infty} x_{M_n},$$

where $\theta_0 = \alpha_c - M_n\omega$, and $x_0 = x \in C$. Since $x_{M_n}(\beta) = \pi_2 \circ \Phi_{\beta}^{M_n}(\theta_0, x_0)$ is continuous with respect to β, for β in $[0,1]$ we obtain, after invoking lemma 5.7, that for every $\varepsilon > 0$

$$|\psi^{\beta}(\alpha_c) - \frac{1}{2}| = \lim_{n\to\infty} |x_{M_n}(\beta) - x_{M_n}(1) + x_{M_n}(\beta) - \frac{1}{2}| < \varepsilon$$

if $\beta \in [1-\delta, 1]$ for $\delta > 0$ small enough. □

We first start off by mentioning that the minimum distance is not obviously well-defined for $\beta = 1$ if $\alpha = \alpha_c$, if we take $\psi^1 : \mathbb{T} \to (0,1)$ to be as in [2]. It is however shown there that for almost every starting value θ_0, and every $x_0 \in (0,1)$, $\inf_{k\geq 0} x_k = 0$, even though $x_k > 0$ for every $k \geq 0$. Thus, we define the minimum distance to be 0 when $\beta = 1$. Later, we will see that this is the value obtained by continuous extension of the distance function $\delta_{\alpha_c}(\beta)$ to $\beta = 1$.

In order to prove the asymptotics of the distance, we must first show that $\psi^{\beta}(\theta)$ is C^1 (continuously differentiable) with respect to both θ and β. Since the functions ψ^{β} are defined by limits of smooth functions, we must show that the respective derivatives of those functions converge uniformly.

Proposition 5.9. *For every $\beta \in [0,1)$, and $\alpha \in \mathcal{A}_0$, the functions $\beta \mapsto \psi^{\beta}(\theta)$ and $\theta \mapsto \psi^{\beta}(\theta)$ are both continuously differentiable considered as one-variable functions in each of the separate variables β and θ, respectively.*

Moreover, for every fixed $0 \leq \beta < 1$

$$|\partial_{\beta}\psi^{\beta}(\theta) - \partial_{\beta}\psi^{\beta}(\phi)| < \frac{2}{\lambda^{1/4}} + O(|\theta - \phi|),$$

and for every fixed $\theta \in \mathbb{T}$

$$|\partial_{\theta}\psi^{\beta}(\theta) - \partial_{\theta}\psi^{\widetilde{\beta}}(\theta)| < \frac{2}{\lambda^{1/4}} + O(|\beta - \widetilde{\beta}|).$$

Proof. We will be needing a few inequalities, valid for every $\theta \in \mathbb{T}$, and $x, y \in [0,1]$. The first one is:

$$\begin{aligned} |p(x) - p(y)| &= |x(1-x) - y(1-y)| = |(x-y)(1-x) - y(x-y)| \\ &= |x-y| \cdot |1-x-y| \leq |x-y|. \end{aligned}$$

The second one is:

$$\begin{aligned} |p'(x) - p'(y)| &= |(1-2x) - (1-2y)| \\ &= 2|y-x| = 2|x-y|. \end{aligned}$$

The third one is, for arbitrary $a, b \in \mathbb{R}$:

$$|p'(x)a - p'(y)b| = |(p'(x) - p'(y))a - p'(y)(a-b)| \leq 2|x-y| \cdot |a| + 2|a-b|.$$

Since $\partial_\beta c(\theta)$, and $\partial_\theta c(\theta)$ are both bounded (they are continuous on a compact set), the maximum norms $\|\partial_\beta c\|, \|\partial_\theta c\|$ and $\|c\|$ are all bounded (henceforth, $\|\cdot\|$ shall denote the maximum norm).

Let $\theta = \theta_0 \in \mathbb{T}$. Using the above inequalities, and letting $\psi_n^\beta(\theta_0) = x_0$, $\psi_m^\beta(\theta_0) = y_0$, we obtain

$$\begin{aligned}|\partial_\beta \psi_n^\beta(\theta_0) - \partial_\beta \psi_m^\beta(\theta_0)| &= |\partial_\beta c(\theta_0)\cdot p(x_0) + c(\theta_0)\cdot p'(x_0)\partial_\beta x_0 - \partial_\beta c(\theta_0)\cdot p(y_0) + c(\theta_0)\cdot p'(y_0)\partial_\beta y_0| \\ &\le |\partial_\beta c(\theta_0)|\cdot|p(x_0)-p(y_0)| + |c(\theta_0)|\cdot|p'(x_0)\partial_\beta x_0 - p'(y_0)\partial_\beta y_0| \\ &\le \|\partial_\beta c\|\cdot|x_0-y_0| + \|c\|\cdot\|\partial_\beta c\|\cdot|x_0-y_0| + 2|\partial_\beta x_0 - \partial_\beta y_0| \\ &\le \|\partial_\beta c\|\cdot|\psi_n^\beta(\theta_0) - \psi_m^\beta(\theta_0)| + |x_0-y_0| + \|c\|\cdot\|\partial_\beta c\|\cdot|\psi_n^\beta(\theta_0) - \psi_m^\beta(\theta_0)| \\ &+ 2|\partial_\beta x_0 - \partial_\beta y_0|.\end{aligned}$$

The same inequality also holds when ∂_β is replaced with ∂_θ. Now, the first two terms vanish uniformly in the limit, since ψ_n^β converges uniformly for every β. The last term also vanishes uniformly, but some care is needed to show this.

Let $t \ge 0$ be such that, $\theta_{-t} \in I_0$. Let $n_1, n_2 > t + K_0$, so that $\theta_{-n_1}, \theta_{-n_2} \in \Theta_0$. Let $x_{-n_1}, y_{-n_2} = \frac{1}{3} \in C$, then by lemma 5.6, $|\partial_\beta x_{-t}|, |\partial_\beta y_{-t}| < \frac{1}{\lambda^{1/4}}$.

$$\begin{aligned}|\partial_\beta x_0 - \partial_\beta y_0| = &|\partial_\beta c(\theta_{-1})(p(x_{-1}) - p(y_{-1})) + \partial_\beta x_{-t}\prod_{j=-t}^{-1} c(\theta_j)\cdot p'(x_j) - \partial_\beta y_{-t}\prod_{j=-t}^{-1} c(\theta_j)\cdot p'(y_j) \\ &+ \sum_{k=-t+1}^{-1}\partial_\beta c(\theta_{k-1})\left(p(x_{k-1})\prod_{j=k}^{-1} c(\theta_j)\cdot p'(x_j) - p(y_{k-1})\prod_{j=k}^{-1} c(\theta_j)\cdot p'(y_j)\right)| \\ &\le \|\partial_\beta c\|\cdot|x_{-1}-y_{-1}| + |\partial_\beta x_{-t}|\prod_{j=-t}^{-1} 2c(\theta_j)\cdot|x_j - y_j| \\ &+ |\partial_\beta y_{-t} - \partial_\beta x_{-t}|\prod_{j=-t}^{-1}|c(\theta_j)\cdot p'(y_j)| \\ &+ \|\partial_\beta c\|\cdot\sum_{k=-t+1}^{-1}\left(\prod_{j=k}^{-1} 2c(\theta_j)\cdot|x_j-y_j| + \|\partial_\beta c\|\cdot\sum_{k=-t+1}^{-1}|x_{k-1}-y_{k-1}|\prod_{j=k}^{-1}|c(\theta_j)p'(y_j)|\right)\end{aligned}$$

We may t as large as we want, since every orbit of the circle rotation is dense when ω is irrational. Everything term converges uniformly as $n_1, n_2 \to \infty$. Letting $n = \min\{n_1, n_2\}$, the first term is

$$\|\partial_\beta c\|\cdot|x_{-1}-y_{-1}| \le \|\partial_\beta c\|\cdot const\cdot(3/5)^{(n-1)/2},$$

where the constant is uniform. The second term is

$$|\partial_\beta x_{-t}|\prod_{j=-t}^{-1} 2c(\theta_j)\cdot|x_j-y_j| \le \frac{8}{\lambda^{1/4}}\cdot const\cdot\prod_{j=-t}^{-1}(3/5)^{(n+j)/2}|x_{-n}-y_{-n}| \le (3/5)^{(n-1)/2},$$

where the constant is again uniform. The third term is

$$\frac{2}{\lambda^{-1/4}} \prod_{j=-t}^{-1} |c(\theta_j) \cdot p'(y_j)| \leq (3/5)^{(t-1)/2}.$$

The last terms are similarly shown to be uniformly convergent. Also, note that the exact same argument works for the derivative with respect to θ, since we only used the bounds on the partial derivatives for $\theta \in I_m$ in lemma 3.9. □

We have the following bounds on the partial derivatives ∂_β considered as functions in the variable θ:

$$\begin{aligned}
|\partial_\beta \psi^\beta(\theta+\omega) - \partial_\beta \psi^\beta(\phi+\omega)| &= |\partial_\beta c(\theta) \cdot p(\psi^\beta(\theta)) + c(\theta) p'(\psi^\beta(\theta)) \partial_\beta \psi^\beta(\theta) \\
&\quad - \partial_\beta c(\phi) \cdot p(\psi^\beta(\phi)) + c(\phi) p'(\psi^\beta(\phi)) \partial_\beta \psi^\beta(\phi)| \\
&= |\partial_\beta c(\theta) \cdot \Big(p(\psi^\beta(\theta)) - p(\psi^\beta(\phi)) \Big) + \big(\partial_\beta c(\theta) - \partial_\beta c(\phi) \big) p(\psi^\beta(\phi)) \\
&\quad + c(\theta) p'(\psi^\beta(\theta)) \Big(\partial_\beta \psi^\beta(\theta) - \partial_\beta \psi^\beta(\phi) \Big) \\
&\quad + \Big(c(\theta) p'(\psi^\beta(\theta)) - c(\phi) p'(\psi^\beta(\phi)) \Big) \partial_\beta \psi^\beta(\phi)| \\
&\leq \|\partial_\beta c\| \cdot |\psi^\beta(\theta) - \psi^\beta(\phi)| + \|\partial_\theta \partial_\beta c\| \cdot |\theta - \phi| \cdot |\psi^\beta(\phi)| \\
&\quad + \|c\| \cdot |\psi^\beta(\theta)| \cdot \frac{2}{\lambda^{1/4}} \\
&\quad + \Big(\|c\| \cdot 2|\psi^\beta(\theta) - \psi^\beta(\phi)| + |c(\theta) - c(\phi)| \cdot |\psi^\beta(\phi)| \Big) \cdot \frac{1}{\lambda^{1/4}}.
\end{aligned}$$

Every term except for the second last one, $\|c\| \cdot |\psi^\beta(\theta)| \cdot \frac{2}{\lambda^{1/4}}$, is seen to be uniformly convergent to 0 as $|\theta - \phi| \to 0$.

Remark. Why do we have to include so many steps when analysing the term $|\partial_\beta x_0 - \partial_\beta y_0|$, instead of proceeding like:

$$\begin{aligned}
|\partial_\beta x_0 - \partial_\beta y_0| &= |\partial_\beta c(\theta_{-1})(p(x_{-1}) - p(y_{-1})) + c(\theta_{-1}) p'(x_{-1}) \partial_\beta x_{-1} - c(\theta_{-1}) p'(y_{-1}) \partial_\beta y_{-1}| \\
&= |\partial_\beta c(\theta_{-1})||(1 + x_{-1} - y_{-1})(y_{-1} - x_{-1}) + c(\theta_{-1})(p'(x_{-1})(\partial_\beta x_{-1} - \partial_\beta y_{-1}) \\
&\quad + c(\theta_{-1})(p'(x_{-1}) - p'(y_{-1})) \partial_\beta y_{-1}| \\
&\leq 2\partial_\beta c(\theta_{-1})|y_{-1} - x_{-1}| + c(\theta_{-1}) \left(|\partial_\beta x_{-1} - \partial_\beta y_{-1}| + 2|y_{-1} - x_{-1}| \cdot |\partial_\beta y_{-1}| \right) \\
&\leq const \cdot |y_{-1} - x_{-1}| + c(\theta_{-1}) |\partial_\beta x_{-1} - \partial_\beta y_{-1}| ?
\end{aligned}$$

This is because $|\partial_\beta x_{-1} - \partial_\beta y_{-1}| < \frac{2}{\lambda^{-1/4}}$ is the best possible direct approximation. We need to study the iterations right before the last step in more detail, in order to establish convergence. The "correct" version shows that the early effects of not having the same starting value is negligible in the long run, and that once the contraction is strong enough, the derivatives will be almost equal for the last few iterations.

We also have a practical way of expressing the derivatives as the limit

$$\partial_\beta \psi^\beta(\theta) = \lim_n \partial_\beta \psi_n^\beta(\theta) = \lim_n \partial_\beta c(\theta_{-1}) \cdot p(x_{-1}) + c(\theta_{-1}) p'(x_{-1}) \partial_\beta x_{-1}.$$

The last term is of order $\frac{1}{\lambda^{-1/4}}$, and in general negligible in comparison to the first one.

Lemma 5.10. *There is a $\delta > 0$, such that $\lambda^{1/7} \le \partial_\theta \psi^\beta(\theta + \omega) \le \lambda$ for every $\theta \in \mathcal{A}_0$, for every $1 - \delta \le \beta < 1$.*

Proof. By [2, Lemma 3.1, c)], $\lambda^{1/6} < \partial_\theta c_{\alpha,\beta=1}(\theta) < \lambda$ for every $\alpha \in \mathcal{A}_0$, and $\theta \in \mathcal{A}_0$. Since c is smooth for all parameter values, $\partial_\theta c$ is continuous in β, and so by continuity, there is a $\delta > 0$ such that

$$\lambda^{1/6} \le \partial_\theta c_{\alpha,\beta}(\theta) \le \lambda$$

for every $\beta \in [1 - \delta, 1)$, and $\alpha, \theta \in \mathcal{A}_0$.

Now, let $x_{-1} = x \in C$, and $\theta_{-1} = \theta - (M_n + 1)\omega \in \mathcal{A}_0 - (M_n + 1)\omega \in \Theta_n$. Then $\theta_{M_n} \in \mathcal{A}_0$, and $x_{M_n - 1} \in C$, and so

$$\frac{3}{10} < \frac{3}{2} \cdot p(1/3 + 1/100) \le x_{M_n} \le 4p(1/3 + 1/100) < 95/100.$$

By lemma 5.6, $|\partial_\theta x_{M_n}| < \lambda^{-1/4}$, and since

$$\partial_\theta x_{M_n+1} = (\partial_\theta c(\theta)) \cdot p(x_{M_n}) + c(\theta) \cdot p'(x_{M_n}) \cdot \partial_\theta x_{M_n},$$

assuming that λ is very large, we obtain after some straight-forward computations that

$$\lambda^{1/7} < \partial_\theta x_{M_n} < \lambda,$$

if $\beta \in [1 - \delta, 1)$. □

Lemma 5.11. *Let $\alpha_c \in \mathcal{A}_0$ be the α_c in [2], that is, $\lim_{\beta \to 1} \psi^\beta(\alpha_c) = \frac{1}{2}$ for the parameter value $\alpha = \alpha_c$, and hence $\lim_{\beta \to 1} \psi^\beta(\alpha_c + \omega) = 1$. Then there are*

- *a monotonically decreasing sequence of positive $\{\varepsilon_n\}$, converging to 0,*
- *and a corresponding monotonically decreasing sequence of sets $\mathcal{A}_n$ contained in $\mathcal{A}_0$, with limit $\lim_{n \to \infty} \mathcal{A}_n = \{\alpha_c\}$,*

such that, if $\beta \in \mathcal{E}_n = \{1 \ge \beta \ge 1 - \varepsilon_n\}$, then the point on the attractor closest to the repelling set $\mathbb{T} \times \{0, 1\}$ lies in $\mathcal{A}_n$. Furthermore, this point corresponds to a maximum of ψ_β, and is closer to 1 than 0.

Proof. By proposition 4.2, we see that $\psi^\beta(I_0) \subseteq C$ for $0 \le \beta < 1$. Recall that

$$\psi^\beta(\theta) = c_{\alpha,\beta}(\theta - \omega) p(\psi^\beta(\theta - \omega)).$$

This gives us the following bounds when $\theta \in I_0 + \omega$, valid for every $0 \le \beta < 1$,

$$\frac{3}{10} < \frac{3}{2} \cdot p(1/3 + 1/100) \le \psi^\beta(\theta) \le 4p(1/3 + 1/100) < 99/100. \tag{5.4}$$

Recall that $\mathcal{A}_0 \subset I_0+\omega$. Since, by lemma 3.6, $c_{\alpha,\beta}(\mathcal{A}_0-\omega) \supsetneq [2,3]$ for $\alpha_c \in \mathcal{A}_0$, and every $1-\delta \leq \beta \leq 1$ for some small $\delta > 0$ by continuity of c with respect to β, there are $\theta \in \mathcal{A}_0$ such that

$$\psi^\beta(\theta) \leq 2p(1/3+1/100) < 1/2-1/100,$$

and

$$\psi^\beta(\theta) \geq 3p(1/3-1/100) > 1/2+1/100.$$

Since ψ is continuous, there is a $\theta_h(\beta) \in \mathcal{A}_0$ (let this be the one closest to α_c if there are more than one) such that $\psi^\beta(\theta_h(\beta)) = 1/2$, for every $1-\delta \leq \beta < 1$, if $\delta > 0$ is sufficiently small.

It is necessarily true that $\theta_c(\beta) \xrightarrow[\beta\to 1]{} \alpha_c$. Suppose, to derive a contradiction, that it were not true. Then there would be a $\delta > 0$, and an open interval $\mathcal{A} = [\alpha-\eta, \alpha+\eta]$ such that $\frac{1}{2} \notin \mathcal{A}$ for every $1-\delta \leq \beta < 1$. Let δ be at least as small as the one in lemma 5.10. Then for $\theta \in \mathcal{A}$, we see that, by the mean value theorem

$$|\psi^\beta(\theta) - \psi^\beta(\alpha_c)| \geq \lambda^{1/7}|\theta-\alpha| > 0$$

for every $1-\delta \leq \beta < 1$. This means that there is an $\varepsilon > 0$ such that $\psi^\beta(\mathcal{A}) \subset (0, \frac{1}{2}-\varepsilon] \cup [\frac{1}{2}+\varepsilon, 1)$ for every $1-\delta \leq \beta < 1$. This would of course imply that

$$\lim_{\beta\to 1} \psi^\beta(\alpha_c) \neq \frac{1}{2}.$$

Hence $\theta_c(\beta)$ converges to α_c as $\beta \to 1$.

We will show that the maximum of $\psi^\beta(\theta)$, when $\theta \in I_0+2\omega$, is assumed for some critical $\theta_c(\beta)$, converging to α_c as $\beta \to 1$. Let $\delta(\beta)$ be such that

$$c_{\alpha_c,\beta}(\theta_h(\beta)) = \left(\frac{3}{2}+\beta\frac{5}{2}\right)(1-\delta(\beta)).$$

Since $\theta_h(\beta) \to \alpha_c$ as $\beta \to 1$, and c is continuous, it follows that $\delta(\beta) \to 0$. Now, we invoke lemma 3.6, to obtain the set inclusion

$$\{\theta \in I_0+\omega : c_{\alpha,\beta}(\theta) \geq \left(\frac{3}{2}+\beta\frac{5}{2}\right)(1-\delta(\beta))\} \subseteq [\alpha_c - \sqrt{\delta(\beta)}\lambda^{-1/4}, \alpha_c + \sqrt{\delta(\beta)}\lambda^{-1/4}] = \mathcal{A}_c(\beta).$$

We will argue that the maximum will be attained for some $\theta_c(\beta) \in \mathcal{A}_c(\beta)$. Suppose that $\theta \in (I_0+\omega)\backslash A_c(\beta)$, then

$$\psi^\beta(\theta_h(\beta)) = \left(\frac{3}{2}+\beta\frac{5}{2}\right)(1-\delta(\beta))\cdot p(\frac{1}{2}) > c(\theta)\cdot p(\frac{1}{2}) \geq \psi^\beta(\theta).$$

Hence the conclusion follows. Also note that the $\mathcal{A}_c(\beta)$ are monotone decreasing in β, and converge to $\{\alpha_c\}$. From this, we may extract a sequence as in the statement. The only part left is showing that the minimal distance is attained in this set. This however can be established rather quickly.

We see that for $\theta \in I_0 + 2\omega$, using the bounds on $\psi^\beta(\theta - \omega)$, we have

$$1/100 \le 95/2000 = \frac{3}{2} p(95/100) \le \psi^\beta(\theta - \omega).$$

Since we may choose the maximum value of $\psi^\beta(\theta_c(\beta))$ as close to 1 as we want, by choosing β closer to 1, the minimal distance in $I_0 + 2\omega$ is attained at a point in $\mathcal{A}_c(\beta)$. Now, by lemma 3.3, the maximum value for $\theta \in I_0 + 3\omega$ is

$$\psi^\beta(\theta) < 2/5.$$

Since $\psi^\beta(\theta_c(\beta)) \ge 9/10$ is the maximum in $I_0 + 2\omega$, the minimum value in $I_0 + 3\omega$ will be attained at the point $\theta_c(\beta) + \omega$, then by lemma 3.5

$$\psi^\beta(\theta + \omega) = c(\theta + \omega) p(\psi^\beta(\theta_c(\beta))) = c(\theta + \omega) p(1 - \psi^\beta(\theta_c(\beta))) \ge \frac{5}{4} \psi^\beta(\theta_c(\beta)).$$

Since, by the same argument, the x-coordinate is increasing if $0 < x < 1/10$ as long as $\theta \notin I_0 + (I_0 + \omega)$, the minimal distance must be attained for $\theta_c(\beta) \in \mathcal{A}_c(\beta)$. $\square$

Definition 5.12. The minimum distance between the attracting curve ψ^β and the repelling set $\mathbb{T} \times \{0, 1\}$, for a fixed α, as a function in β is denoted by

$$\delta_\alpha(\beta),$$

and is defined for at least $0 \le \beta < 1$.

Proposition 5.13. *The distance $\delta_{\alpha_c}(\beta)$ is asymptotically bounded, as $\beta \to 1$, from both sides by linear functions. Specifically,*

$$a_1(1 - \beta) \le \delta(\beta) \le a_2(1 - \beta), \tag{5.5}$$

where $0 < a_1 \le a_2$ are constants, when β is sufficiently close to 1.

Proof. Suppose that $\alpha = \alpha_c$, and β is in $\mathcal{E}_0$ from lemma 5.11. Then the minimum distance between the attractor and the repelling set is attained for some $\theta(\beta) \in \mathcal{A}_0$, and this minimum distance is furthermore to the point 1; hence the distance, for a fixed $\beta \in \mathcal{E}_0$, will be given by the expression

$$\delta_{\alpha_c}(\beta) = 1 - \psi^\beta(\theta(\beta)).$$

Also, $\theta(\beta)$ converges to $\alpha_c + \omega$ as $\beta \to 1$, and we will set $\mathcal{A}(\beta)$ to be the smallest interval containing both $\alpha_c + \omega$ and $\theta(\beta)$. It follows that the length $|\mathcal{A}(\beta)| \to 0$ as $\beta \to 1$.

Since $\partial_\beta \psi^\beta(\theta)$ is continuous with respect to β for every $\theta \in \mathbb{T}$, it is integrable. Also, since $|\partial_\beta \psi^\beta(\alpha_c + \omega) - \partial_\beta \psi^\beta(\theta)| < \frac{2}{\lambda^{1/4}} + O(\alpha_c + \omega - \theta)$ by proposition 5.9, we obtain the following inequalities, valid for every $\theta \in \mathcal{A}(\beta)$

$$\partial_\beta \psi^\beta(\alpha_c + \omega) - \frac{2}{\lambda^{1/4}} + C(\alpha_c + \omega - \theta) < \partial_\beta \psi^\beta(\theta) < \partial_\beta \psi^\beta(\alpha_c + \omega) + \frac{2}{\lambda^{1/4}} + C(\alpha_c + \omega - \theta),$$

where $C \geq 0$ is some constant. Now, $|\psi^{\beta}(\alpha_c) - \frac{1}{2}| < \varepsilon$, for some $\varepsilon = \varepsilon(\beta) > 0$ converging to 0 as β goes to 1, and so

$$\begin{aligned}\partial_\beta \psi^{\beta}(\alpha_c + \omega) &= \partial_\beta c(\alpha_c) \cdot p(\psi^{\beta}(\alpha_c)) + c(\alpha_c) p'(\psi^{\beta}(\alpha_c)) \partial_\beta \psi^{\beta}(\alpha_c) \\ &= \frac{5}{2} \cdot p(\psi^{\beta}(\alpha_c)) + 4|\varepsilon| \cdot \left(\partial_\beta c(\alpha_c - \omega) \cdot p(\psi^{\beta}(\alpha_c - \omega)) + c(\alpha_c) p'(\psi^{\beta}(\alpha_c)) \frac{1}{\lambda^{1/4}} \right) \\ &\xrightarrow[\beta \to 1]{} \frac{5}{2} \cdot \frac{1}{4} = \frac{5}{8}.\end{aligned}$$

Hence, for β sufficiently close to 1, $\alpha_c + \omega - \theta$ is to the order of some ε converging to 0 as β approaches 1, and so

$$0 < \psi^{\beta}(\alpha_c + \omega) - \frac{2}{\lambda^{1/4}} + C(\alpha_c + \omega - \theta).$$

Thus we obtain two constants $0 < b_1 \leq b_2$, both approximately equal to $\frac{5}{8}$, up to $\frac{2}{\lambda^{1/4}} + \varepsilon$, where ε converges to 0 as $\beta \to 1$, such that $b_1 \leq \partial_\beta \psi^{\beta}(\theta) \leq b_2$, for every $\theta \in \mathcal{A}(\beta)$ and β close to 1. Integrating, we obtain for every $\beta \leq \tilde{\beta} < 1$ that

$$b_1(\tilde{\beta} - \beta) \leq \int_{\beta}^{\tilde{\beta}} \partial_\beta \psi^{K}(\theta) dK \leq b_2(\tilde{\beta} - \beta),$$

where

$$\int_{\beta}^{\tilde{\beta}} \partial_\beta \psi^{K}(\theta) dK = \psi^{\tilde{\beta}}(\theta) - \psi^{\beta}(\theta).$$

Using the mean value theorem we obtain, for some $\tilde{\theta}$ between θ and $\alpha_c + \omega$, that

$$\psi^{\tilde{\beta}}(\theta) = \partial_\theta \psi^{\tilde{\beta}}(\tilde{\theta})(\theta - (\alpha_c + \omega)) + \psi^{\tilde{\beta}}(\alpha_c + \omega) = \psi^{\tilde{\beta}}(\alpha_c + \omega) + \partial_\theta \psi^{\tilde{\beta}}(\tilde{\theta})(\theta - (\alpha_c + \omega)).$$

Thus, in particular, recalling that $\lambda^{1/7} \leq \partial_\theta \psi^{\tilde{\beta}}(\theta) \leq \lambda$ for θ in our interval,

$$\psi^{\tilde{\beta}}(\alpha_c + \omega) + \lambda^{1/7}(\theta - (\alpha_c + \omega)) \leq \psi^{\tilde{\beta}}(\theta) \leq \psi^{\tilde{\beta}}(\alpha_c + \omega) + \lambda(\theta - (\alpha_c + \omega)).$$

We must now show how the distance $\theta(\beta) - (\alpha_c + \omega)$ is related to $1 - \beta$. By the mean value theorem there is a $\tilde{\theta}$ between $\theta(\beta)$ and α_c such that

$$\psi^{\beta}(\theta(\beta)) - \psi^{\beta}(\alpha_c) = \partial_\theta \psi^{\beta}(\tilde{\theta})(\theta(\beta) - \alpha_c)$$

and so

$$\theta(\beta) - \alpha_c = \frac{\psi^{\beta}(\theta(\beta)) - \psi^{\beta}(\alpha_c)}{\partial_\theta \psi^{\beta}(\tilde{\theta})}$$

Using the mean value theorem, and noting that $\lim_{\tilde{\beta}\to 1}\psi^{\tilde{\beta}}(\alpha_c) = \lim_{\tilde{\beta}\to 1}\psi^{\tilde{\beta}}(\theta(\beta)) = \frac{1}{2}$ by continuity,

$$\begin{aligned}\psi^{\beta}(\theta(\beta)) - \psi^{\beta}(\alpha_c) &= \lim_{\tilde{\beta}\to 1}\psi^{\tilde{\beta}}(\alpha_c) - \partial_\beta\psi^{\tilde{\beta}_1}(\alpha_c)(1-\beta) - \lim_{\tilde{\beta}\to 1}\psi^{\tilde{\beta}}(\theta(\beta)) + \partial_\beta\psi^{\tilde{\beta}_2}(\theta(\beta))(1-\beta)\\ &= \left(\partial_\beta\psi^{\tilde{\beta}_2}(\theta(\beta)) - \partial_\beta\psi^{\tilde{\beta}_1}(\alpha_c)\right)(1-\beta).\end{aligned}$$

So we obtain

$$\theta(\beta) - \alpha_c = \frac{\left(\partial_\beta\psi^{\tilde{\beta}_2}(\theta(\beta)) - \partial_\beta\psi^{\tilde{\beta}_1}(\alpha_c)\right)(1-\beta)}{\partial_\theta\psi^{\beta}(\tilde{\theta})} = O(1-\beta).$$

Hence, for every $\beta < \tilde{\beta} < 1$ sufficiently close to 1,

$$b_1(\tilde{\beta}-\beta) + O(1-\beta) \le \psi^{\tilde{\beta}}(\alpha_c+\omega) - \psi^{\beta}(\theta) \le b_2(\tilde{\beta}-\beta) + O(1-\beta).$$

Taking limits as $\tilde{\beta}\to 1$, we obtain, for some $a_1 \le a_2$,

$$a_1(1-\beta) \le 1 - \psi^{\beta}(\theta) \le a_2(1-\beta).$$

Recalling that $\delta_{\alpha_c}(\beta) = 1 - \psi^{\beta}(\theta(\beta))$, and $0 < \delta_{\alpha_c}(\beta)$ for $\beta < 1$, we may choose $0 \le a_1 = \inf_{\beta<1}\frac{\delta_{\alpha_c}(\beta)}{1-\beta}$, to obtain

$$a_1(1-\beta) \le \delta_{\alpha_c}(\beta) \le a_2(1-\beta),$$

asymptotically as $\beta \to 1$, for $0 \le a_1 \le a_2$. $\square$

Remark. An easier approach would be to directly compute the integral

$$\int_{\beta}^{1} \partial_\beta\psi^{K}(\theta(K))dK = 1 - \psi^{\beta}(\theta(\beta)).$$

However, that would require $\partial_\beta\psi^{K}(\theta(K))$ to be absolutely continuous, requiring in particular the continuity of $\partial_\beta\psi^{K}(\theta)$ with respect to θ, which is beyond the results in the present paper (we haven't proved mixed continuity of the derivatives!).

We now restate the main theorem, which follows from corollary 5.4 and propositions 5.5, 5.9 and 5.13,

Main Theorem. *Assume that ω satisfies the Diophantine condition $(DC)_{\kappa,\tau}$ for some $\kappa > 0$ and $\tau \ge 1$. Then for all sufficiently large $\lambda > 0$, there is a parameter value $\alpha = \alpha_c$ such that the following holds for the map $\Phi_{\alpha,\beta}$:*

i) When $\beta = 1$, there is a strange attractor, the graph of a nowhere continuous measurable function $\psi : \mathbb{T} \to [0,1]$, which attracts points (θ, x), for a.e. $\theta \in \mathbb{T}$, and every $x \in (0,1)$.
ii) When $0 \le \beta < 1$, there is a curve, the graph of a C^1 function $\psi : \mathbb{T} \to [0,1]$, which attracts every point $(\theta, x) \in \mathbb{T} \times (0,1)$.

iii) The (minimum) distance $\delta_{\alpha_c}(\beta)$ *between the attractor and the repelling set, considered as a function of* β*, is asymptotically bounded by linear functions as* $\beta \to 1$*, that is*

$$a_1(1-\beta) \leq \delta_{\alpha_c}(\beta) \leq a_2(1-\beta),$$

for some constant $0 \leq a_1 \leq a_2$ *as* $\beta \to 1$.

iv) The system (and hence the attractor) is nonchaotic for $0 \leq \beta \leq 1$*, since* $\overline{\gamma_x}(\theta_0, x_0) < \frac{1}{2}\log(3/5) < 0$ *for (almost, when* $\beta = 1$*) every* $\theta \in \mathbb{T}$ *and for every* $x \in (0,1)$.

LIST OF NOTATION

α The alignment parameter of the peaks in the forcing map $c(\theta)$, page 10

α_c The critical value $\alpha = \alpha_c$ in [2] for which $\lim_{n\to\infty} \pi_2 \circ \Phi^{M_n}(\alpha_c - M_n\omega, \frac{1}{3}) = \frac{1}{2}$, page 25

$\delta_\alpha(\beta)$ The minimum distance between the attractor and the repelling set $\mathbb{T}\times\{0,1\}$, for a fixed α, as a function of β, page 31

$\gg$ $x \gg y$ means that x is much larger than y, in the sense that $x \geq Cy$ for every constant C that is at least as large as we will need in any of our inequalities involving both quantities, page 19

β The scaling parameter of the peaks in the forcing map $c(\theta)$, page 10

λ Controls how narrow the peaks are, and is assumed to be very large, page 10

$\mathbb{T}$ The circle $\mathbb{R}\backslash\mathbb{Z}$, page 2

$\mathcal{A}_0$ $\mathcal{A}_0 = [\omega - \lambda^{-2/5}/2, \omega - 2\lambda^{-2/3}]$, page 11

$\Phi_{\alpha,\beta}(\theta,x)$ $\Phi_{\alpha,\beta} : \mathbb{T}\times[0,1] \to \mathbb{T}\times[0,1] : (\theta,x) \mapsto (\theta+\omega, c_{\alpha,\beta}(\theta)\cdot p(x))$, page 10

π_1 Projection onto the first coordinate, page 11

π_2 Projection onto the second coordinate, page 11

Θ_n $\Theta_n = \bigcup_{i=0}^{n} \bigcup_{m=-M_i}^{M_i} (I_i + m\omega)$, for $n \geq 0, \Theta_{-1} = \mathbb{T}\backslash(I_0 \cup (I_0+\omega))$, page 17

θ_n Short-hand notation for the projection onto the first coordinate $\theta_n = \pi_1\Phi^n(\theta_0,x_0)$, or the n-th iterate of θ_0, page 11

B_n $B_n = \{\beta : M_C(\beta) \leq 2K_n - 7\}$, page 17

$c(\theta)$ $c(\theta) = c_{\alpha,\beta}(\theta) = \frac{3}{2} + \beta\frac{5}{2}\left(\frac{1}{1+\lambda(\cos 2\pi(\theta-\alpha/2)-\cos\pi\alpha)^2}\right)$ is the forcing map in Φ, page 10

$g(\theta,\alpha)$ $g(\theta,\alpha) = \cos 2\pi(\theta - \alpha/2) - \cos\pi\alpha$, page 10

G_n $\bigcup_{i=0}^{n} \bigcup_{m=0}^{3K_i} (I_i + m\omega)$, for $n \geq 0, G_{-1} = \emptyset$, page 17

I_0 $I_0 = [-\lambda^{-1/7}, \lambda^{-1/7}]$, page 11

I_k Zoomed in interval at step k, page 18

K_0 $K_0 = [\lambda^{1/(28\tau)}]$, $K_0 \approx N^{1/4}$ where N is the minimal return time to I_0 in lemma 3.1, page 11

K_k $K_k \approx \sqrt{M_k} \approx N_k^{1/4}$, where N_k is the minimal return time to I_k, as in lemma 3.1, page 18

M_0 $M_0 = [\lambda^{1/(14\tau)}]$, $M_0 \approx \sqrt{N}$ where N is the minimal return time to I_0 in lemma 3.1, page 11

$M_C(\beta)$ An upper bound on the time it takes for an orbit to re-stabilize after coming close to the peaks, page 15

M_k $M_k \approx \sqrt{N_k}$, where N_k is the minimal return time to I_k, as in lemma 3.1, page 18

$N(\theta_0;I)$ $N(\theta_0;I)$ is the smallest non-negative integer N such that $\theta_N = \theta_0 + N\omega \in I$, page 11

N_k The minimal return time to I_k given by lemma 3.1, page 19

$p(x)$ The quadratic map $p(x) = x(1-x)$, which is symmetric around the maximum $\frac{1}{4}$ at $x = \frac{1}{2}$, page 10

x_n Short-hand notation for the projection onto the second coordinate $x_n = \pi_2\Phi^n(\theta_0,x_0)$, or the n-th iterate of x_0, page 11

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